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A tour of the uncharted with Effective Field Theories

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I am sure many would be surprised to understand how much physics turns out very often to be not exclusively about computations and ideas that work or not as much as it is about people. For this reason, I think it is only fitting that the first actual words one is presented with when reading this work are about people, too. Specifically, about the people that have been around me during the years that from which this work emerges as an attempt of a synthesis, and without whom both this thesis and the results behind it would surely not be there.

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Considerate la vostra semenza:
fatti non foste a viver come bruti,
ma per seguir virtute e canoscenza

Dante Alighieri
Divina Commedia, Inferno, Canto XXVI

Eidesstattliche Versicherung / Declaration on oath

Hiermit versichere ich an Eides statt, die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt zu haben.

Santa Barbara, den 11.09.2022

Unterschrift des Doktoranden

Zusammenfassung

Die in den letzten Jahren mit dem Large Hadron Collider (LHC) gesammelten Daten haben keine Anzeichen für neue, direkt beobachtete Freiheitsgrade bis zu TeV-Skalen gezeigt, die über die vom Standardmodell der Teilchenphysik (SM) vorhergesagten hinausgehen. Diese Erkenntnis lässt den Schluss zu, dass eine beträchtliche Masselücke die bekannten Teilchen von der Neuen Physik (NP) trennt, wie auch immer sie aussehen könnte. In diesem Kontext haben sich die Werkzeuge der Effektiven Feldtheorien (EFT) als am besten geeignet erwiesen, um die möglichen Abweichungen vom etablierten Modell zu berücksichtigen und zu untersuchen. Unter der Annahme, dass die neuen Freiheitsgrade von der Physik bei den Energien entkoppelt werden können, zu denen wir an den Collidern Zugang haben, ermöglichen die EFTs die Parametrisierung jeder Andeutung von NP auf eine Weise, die eine große Vielfalt möglicher Modelle zulässt. Ein solcher Bottom-up-Ansatz hat jedoch den Nachteil, dass wir einen riesigen Parameterraum aufspannen müssen, der durch eine Fülle von freien Koeffizienten gekennzeichnet ist, die wir im Prinzip durch direkte Messung bestimmen müssen. Aus diesem Grund ist es von entscheidender Bedeutung, jede mögliche Information zu nutzen, die wir sammeln können, um den von den EFTs aufgespannten Parameterraum einzuschränken und so viel Wissen wie möglich aus dem Verständnis der Auswirkungen von Symmetrien auf ihre Struktur zu gewinnen. Aufgrund dieser zwingenden Argumente konzentrieren wir uns in dieser Arbeit auf die Untersuchung bemerkenswerter Eigenschaften der effektiven Feldtheorie des Standardmodells (SMEFT), d. h. das Ergebnis der Anwendung des EFT-Ansatzes auf das SM.

Deshalb müssen wir zunächst genau verstehen, wie eine Effektive Feldtheorie aufgebaut ist, und zu diesem Zweck werden wir eine gründliche Einführung in die Instrumente der EFT geben. Insbesondere werden wir die Bedeutung der Hilfsmittel hervorheben, die uns von der Mathematik und Geometrie geschenkt werden, wie z. B. die Hilbert-Serie (HS), und wie man sie einsetzen kann.

Wir werden dann sehen, wie es möglich ist, Beschränkungen für einige der Koeffizienten aufzuzeigen, die den EFT-Parameterraum charakterisieren, indem man nur fordert, dass die Theorie im UV immer noch eine ziemlich lockere Reihe von Eigenschaften respektiert, wie

Kausalität und Lokalität. Diese Schranken weisen eine immer komplexere und interessantere Struktur auf, je mehr Symmetrie zur Theorie selbst hinzugefügt wird. Hier werden wir uns speziell auf das Zusammenspiel zwischen den Schranken und der Flavor-Symmetrie der SMEFT konzentrieren und als konkreten Bezugswert ihre Kompatibilität mit dem Minimal Flavor Violation-Ansatz untersuchen.

Die Beziehung zwischen EFTs und Symmetrien ist jedoch nicht auf ihre Auswirkungen auf die Koeffizientenschranken begrenzt. In der Tat wissen wir, dass die besondere Struktur des SM zur Folge hat, dass einige Symmetrien *fast* konserviert sind, d.h. dass ihre Verletzung extrem gedämpft wird. Dies gilt zum Beispiel für die CP-Symmetrie, die im elektroschwachen Sektor durch den CKM-Mechanismus über ein einzigartiges Wechselspiel zwischen allen drei Quark-Generationen gebrochen wird. Eine solche Besonderheit wird nicht direkt auf die SMEFT übertragen, und wir werden uns mit der korrekten Art und Weise befassen, die Verletzung der CP-Symmetrie in diesem Zusammenhang mit Hilfe von Objekten zu untersuchen, die invariant bleiben, auch wenn wir die Flavor-Parametrisierung wechseln.

Summary

The recent years of data collected by the Large Hadron Collider (LHC) have shown no sign of new directly observed excitations, up to TeV scales, beyond those predicted by the Standard Model of Particle Physics (SM). This knowledge allows us to infer that a sizable mass gap separates the known particles from the New Physics (NP), whatever shape it may take. In this context, the tools of Effective Field Theories (EFTs) have emerged as the most suitable to accommodate and study what the possible deviations from the established model will look like. Indeed, with the assumption that the new degrees of freedom can be decoupled from the physics at the energies we can access at colliders, EFTs allow to parametrize any hint of NP in a way that accommodates a huge variety of possible models. Such a bottom-up approach, however, has the disadvantage of leaving us with a vast parameter space to explore, characterized by a plethora of free coefficients that we need in principle to fix by direct measurement. For this reason, it is crucial to exploit every possible piece of information we can gather to constrain such parameter space, and gain as much knowledge as possible from understanding the impact that symmetries have on its structure. Driven by such compelling arguments, we focus in this thesis on the study of specific properties of the Standard Model Effective Field Theory (SMEFT), which is the result of applying the EFT approach to the SM.

Accordingly, we will first need to understand precisely how an Effective Field Theory is built, and a thorough introduction to the tools of EFTs will be provided to this end. Particularly, we will emphasize the importance of the machinery handed to us by Mathematics and Geometry, such as the Hilbert Series (HS), and how to put it to use.

We will then see how it is possible to exhibit constraints on some of the coefficients characterizing the EFT parameter space, just by requiring that the theory in the UV still respects a quite loose set of properties, such as causality and locality. Particularly, these bounds turn out to exhibit structure of growing complexity and interest the more symmetries are added to the theory itself. Here, we will focus specifically on the interplay between the bounds and flavor symmetry of the SMEFT, and study as a concrete benchmark their compatibility with the Minimal Flavor Violation ansatz.

The relationship between EFTs and symmetries is not however limited to their impact on the coefficients bounds. Indeed, we know that the peculiar structure of the SM has within its consequence that some symmetries are *almost* conserved, i.e. that their violation is extremely suppressed. This happens, for example, for the CP symmetry, broken in the electroweak sector through the CKM mechanism via a unique interplay between all three quark generations. Such a distinctive feature is not strictly carried over to the SMEFT, and we will address the correct way to study the breaking of CP in this context through the language of flavor-reparametrization invariant objects.

List of publications

[1] Q. Bonnefoy, E. Gendy, and C. Grojean, *Positivity bounds on Minimal Flavor Violation*, *JHEP* **04** (2021) 115, [arXiv:2011.12855].

[2] Q. Bonnefoy, E. Gendy, C. Grojean, and J. T. Ruderman, *Beyond Jarlskog: 699 invariants for CP violation in SMEFT*, *JHEP* **08** (2022) 032, [arXiv:2112.03889].

The present work is mainly based on the references in [1] and [2]. Chapter 3 contains the results published in [1], while Chapter 4 refers to the work of [2]. Some original work is also contained in Chapter 2, where indicated.

Contents

CHAPTER 1

Introduction

With the benefit of hindsight, the whole history of modern physics can be summarized as a history of scales and specifically of scales separations.

Being part of this history, and being a high energy physicist at the beginning of the 21st centuries is certainly a challenging and at the same time privileged position. Challenging, because the last decade has been characterized by a lack of indisputable experimental signatures of New Physics (NP) at colliders, and privileged, because we have the advantage of being able to look back at the progress that has come before us, and observe each of its steps with the awareness of their impact. In such a time of seeming impasse, we have to make use of this privilege to try and identify the patterns that have lead to the success of the scientific method, to build a systematic way of applying them, and to spot the points where the biases that plague our perspective both as individuals and as a community have led us astray. A clear structure that emerges when undertaking this attitude is, precisely, that of scales separations. First of all, thanks to Special Relativity, we now understand that (inverse) time and space lengths and energies are different realization of the same concept. With this in mind, we can retrospectively place all the most successful and predictive theories that the brilliant minds of the past and the present have worked out on a rather single dimensional axis, where they occupy a range of some extension and have to give way to their neighboring, more fundamental companion when their range is exhausted. This is how Classical Mechanics had to concede to its Quantum version, which had itself to make room to the Quantum Theory of Fields to accommodate the laws of Special Relativity.

Such a seemingly clear pattern poses the physicist in front of a crossroad: on one hand, the ideal goal is the enticing perspective that this single dimensional axis actually stops somewhere, and a final Theory of Everything can be reached. On the other hand, there is, up to this point, no clear indication that the end of this axis is within reach. On the contrary, the single point in this axis indicating that at least a paradigm shift is necessary, namely the Planck scale

$M_{Pl} \approx 10^{19}$GeV, where quantum effects of gravity are supposed to become strong, is well beyond any experimentally testable region. Setting aside, for the time being, the discussion around the Final Theory, which surely deserves a deeper investigation than the few lines invested here, we set ourselves to try and continue the journey of understanding what are the phenomena, and theories describing them, that lie on the axis of scales by starting with where we are now and try to figure out what the next step will look like.

This is hardly a drawback, and for a number of reasons. Most importantly, it is not even guaranteed that this task is strictly separated from the one above. To better justify this statement, we need to go a bit deeper and give more details about the setting we are dealing with. Let us start from the symbolic end: as it is widely known, in 2012 the ATLAS and CMS collaborations announce the discovery of a scalar particle with properties compatible with those of the Standard Model Higgs boson [1, 2]. It was the last particle of the Standard Model of Particle Physics (SM in the following) that had still not been observed. Its discovery was the final confirmation that the SM is the most suitable model to describe all the interactions between the known particles in a quantum mechanical framework, with the notable exception of the gravitational one, who still evades a quantum description. This model makes use of the most refined theoretical framework at our disposal, namely Quantum Field Theory, and it is a Gauge Theory based on the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group. The SM has been introduced in various pieces and steps going back to the beginning of the '60s [3–12]. As it has been around and established for all our life, it is easy for those belonging to the younger generations of physicists to take it for granted and thus underestimate its incredible value. However, it is actually really hard to overestimate how remarkable this rather simple looking model is. With as few as 19 free parameters, we can describe all non-gravitational microscopic interactions, and thus, scrolling back the axis of scales, virtually all phenomena. Rather, the problem may be the fact that the SM works *too* well. Indeed, as of now, no experimental or theoretical challenge has been able to convincingly point to its breakdown, although some hints coming from both sides have emerged throughout the years. Experimentally, one can mention, for example, that neutrino oscillations [13–15] are not included in the SM predictions. However, they could be still explained with the degrees of freedom of the SM if one allows for dimension-five operators, at the price of sacrificing the renormalizability of the theory [16]. As long as particle physics goes, however, this is the only phenomenon that lies outside of the SM, although some small deviations in various experiments hint at possible discrepancies. However, as of know they are still not significant enough, statistically speaking [17–24] (also see Ref. [25] for a review). If we include observations outside of the realm of particle physics, as we should, however, there is still a lot the SM cannot explain: the nature and origin of Dark Matter (DM) [26, 27] and Dark Energy (DE) [28], as well the observed asymmetry between matter and anti-matter. Finally, and perhaps most frustratingly, the SM fails to give an interpretation that goes beyond the classical level to the fourth (known) force, gravity, whose best theoretical model, Einstein's General Relativity, still withstands any attempt to describe it quantum mechanically in a satisfying way.

The experimental challenges are, in a sense, the clearest ones we have to confront ourselves with, as they stand on the solid legs of being the manifestation of Nature. On the other hand, the theoretical ones are more subtle and more diverse in nature. This is due to the SM lacking here, too, any indisputably weak point. One of its most striking feature, for example, is the fact that it can be extrapolated up to infinitely high energies [29]. So, were it not for the experimental questions raised above, we could be content with it forever. In a way, this is a first timer for physics: knowingly or not, every theory the physicists have worked out in the past has had to deal with some internal breakdown at some length or energy scale, be it the electric potential being divergent at $r \to 0$ or Fermi's theory needing a Ultraviolet (UV) completion at $E \sim 100$ GeV. Without a definitive internal breakdown of this kind, investigation from theoretical physicists has focused on more subtle aspects of the SM, whose status as problems ranges from unsatisfactory feature to challenging issue. Most of these concerns revolve around various realization of the question: *why does this parameter have the value it has?*

First, we cannot explain why the parameters of the Yukawa sector of the SM, namely the quark and lepton masses and the parameters of the CKM matrix, are so widely spread in value, spanning a range of roughly 10^6, an accident known as the Flavor Puzzle [30–32]. In addition, although maybe less popular now than it was 10 years ago, the issue of the smallness of the Higgs mass is still cause for fierce discussions at conferences and in journals alike [33–37]. Finally, there is the less controversial issue of why does CP appear to be conserved in the strong interactions, since Nature does not seem to care particularly about this symmetry, given that it is broken in the Yukawa sector [38–41].

Perhaps encompassing, and probably underlying all of these issues, there is the following matter: it is true that the SM allows, upon having measured 19 parameters, to make an infinite number of predictions. That kind of cost-benefit ratio would be enticing even for the most careful wager. However, physicists are still not completely satisfied, as we would like a theory where all these parameters, as well as the new ones that will be required when DM, DE, gravity, neutrino masses, ecc. are included, are calculable, and ideally only descend from the dynamics of the theory. The archetype of this behavior is Quantum Chromodynamics (QCD), where the strong scale Λ_{QCD} where perturbativity breaks down is set by the anomalous breaking of the scale symmetry by quantum corrections and the subsequent dimensional transmutation.

What makes this task all the more harder is also the apparent circumstance that the SM is a very carefully built system, and it is not a trivial task to modify it without disturbing this delicate balance. The parameter values seem calibrated as to achieve this. In fact, one could actually argue that all the problems stated above, the Flavor Puzzle, the Higgs hierarchy problem and the Strong CP problem, could all be reformulated like this: it is a very non-trivial task to build a model where the SM parameters are calculable that is not itself plagued with inconsistencies or that will not at least exhibit the same issues the SM has. For example, the stability of the Higgs potential [42–44] seems to be extremely sensitive to any modifications coming from any UV sector [42, 45]. Similarly, and more relevant to what we will explore in this thesis, adding to the SM any degree of freedom sensitive to flavor makes it very hard to

avoid new sources of Flavor Changing Neutral Currents [46–48] and CP violation [49–54], both greatly constrained by experiments.

It should be clear, at this point, what our starting point looks like. It is way less clear, however, what the best way to proceed looks like. As we said, we wish to understand, from where we are, what the next step on the axis of scales looks like. Unfortunately, like we explained, the model we have now is so good that we cannot know where any evidence of this next step, NP, will appear. It may well be possible that the confusion we are in is deemed to be solved only at energies way outside the experimental reach humanity will gather even beyond our lifetime.

It is time, then, to make a calculated choice, and to try and justify it. In this thesis, we make use of the powerful instrument of Effective Field Theories (EFTs) [55–64] applied to the Standard Model, and mainly concern ourselves with studying the properties of the resulting object, namely the Standard Model Effective Field Theory (SMEFT) [65]. This means, roughly speaking, taking the most general and consistent extension of the Standard Model, while remaining almost completely agnostic about any knowledge of the UV completion.

As advertised, however justified, this is a choice. Some criticisms to it, and some attempts to balance them, will be addressed in the body of this work. However, there is one that, for its quite general and philosophic nature, we will like to address here, and so conclude this introduction. In the views of Thomas Kuhn, as expressed in The Structure of Scientific Revolutions [66], it could be argued that what we are doing is limiting ourselves to Normal Science and to working within the current existing paradigm, while its shortcomings are starting to pile up and a paradigm shift is called for. In insisting down this road, we may subject to the accusation of relying too much on induction, and are only treading the path that has worked for physics in the last almost 50 years. With the words of Russell [67]:

> A horse which has been often driven along a certain road resists the attempt to drive him in a different direction. Domestic animals expect food when they see the person who usually feeds them. We know that all these rather crude expectations of uniformity are liable to be misleading. The man who has fed the chicken every day throughout its life at last wrings its neck instead, showing that more refined views as to the uniformity of nature would have been useful to the chicken.

This is certainly a risk, and the argument is not one to be dismissed lightly. Rather than an accusation, however, it should be interpreted as a call to be on the lookout and always question our own work, without sweeping inconsistencies and shortcomings under the carpet, and rather being ready to acknowledge them when necessary. The thesis is organized as follows: in Chapter 2 we introduce the tools of Effective Field Theories, often relying on the example of the Fermi theory as a benchmark to better expose our arguments. In particular, we spend some time on the description of how a non-redundant EFT can be built, and how the tools of the Hilbert Series can be of help in this process. In Chapter 3, we address how some consistency requirements linked to the possibility of completing the theory in the UV result in EFTs constraints. In particular, we will describe in detail what these constraints look like in

some specific set of the SMEFT operators, when an assumption is made on its particular flavor structure. In Chapter 4, we will turn to study the realization of a rather important discrete symmetry, CP, in the SMEFT, highlighting the relative impact of its violation with respect to the Standard Model alone. Finally, we provide some concluding discussion and remarks in Chapter 5. The work of Chapters 3 and 4, in particular, is based on Ref. [68] and Ref. [69], respectively.

Conventions

Throughout this work, we adopt the *mostly minus* signature $(+,-,-,-)$ for the metric. All internal indices are raised and lowered rather for real representations, and Einstein summation convention is assumed, unless otherwise stated. The vacuum state in Fock space is labeled $|0\rangle$, while expectation values for observables are written $\langle\mathcal{O}\rangle$. Units are chosen such that $c = \hbar = 1$, except in Section 2.1.3, where units of $\hbar$ are explicitly reinstated. We follow Ref. [70] for the conventions on phases and hypercharges normalization in the SM and the SMEFT, and Ref. [71] for factors of 2π and i. When dealing with the SM and SMEFT lagrangian, we will omit L and R chirality subscripts on the fields, and indicate with lowercase u, d and e the right-handed up, down quark and electron respectively, and with uppercase Q and L the quark and lepton doublets.

CHAPTER 2

Effective Field Theories

As we mentioned in the Introduction, Effective Field Theories, and more specifically the Standard Model Effective Field Theory and its properties, are going to be the main focus of this thesis. This Chapter, then, is going to be devoted to reviewing the theoretical tools pertaining these subjects, and to try and justify their use.

2.1 Effective Field Theories

At the basis of Effective Field Theories there is the idea that, to describe a physical phenomenon, we do not need the details about physics at all scales, but only the degrees of freedom and the laws of their interactions in the range of scales relevant for that process. This rather generic concept is actually applied in every branch of science, and it is the reason why a biologist can study cells and mitochondria without needing to know about the Yukawa couplings, or an engineer build a house ignoring the Higgs mass. More relevantly to physics, we know we can treat the problem of how charges distribute between two conducting spheres just by using Maxwell Equations, without resorting to Quantum Electrodynamics (QED). The common principle is that, once the correct mathematical framework and the appropriate degrees of freedom are identified, there will be a set of parameters that needs to be measured, be it the elastic coefficient of a metal pole to build a house or the vacuum permittivity to use Maxwell's Equations. In the full, microscopic theory, these parameters are functions of other, more fundamental ones, and can be computed in terms of those. However, once the macroscopic parameters we care about have been measured, we can do without the microscopic theory.

Effective Field Theories are the result of applying such a paradigm to Quantum Field Theory. The recipe is simple, and can be summarized in the following steps:

1. Identify the correct degrees of freedom, translated into light fields with masses within the energy ranges we are interested in.

2. Identify the relevant symmetries.

3. Write down the lagrangian by adding all possible operators built with the fields at point 1 and compatible with the symmetries of point 2, with arbitrary coefficients.

4. Rescale the the coefficients of operators of dimension $d > 4$ by Λ^{4-d}, so as to make them dimensionless. The quantity Λ has the dimensions of a mass[1], $[\Lambda] = m$, and it is interpreted as the cutoff of the theory, where its predictivity breaks down.

5. Organize the operators by their dimension in increasing order.

This procedure outputs a lagrangian that we can schematically write as

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{d\leq 4} + \sum_{d>4}^{\infty} \sum_{i} \frac{c_i}{\Lambda^{d-4}} \mathcal{O}_i^{(d)} \ . \tag{2.1.1}$$

Conventionally, the operators of dimension $d \leq 4$ are grouped separately from the rest, in what is referred to as the *renormalizable* part of the lagrangian. Indeed, taken alone, this part of the lagrangian leads to a renormalizable theory, i.e. one where, at any fixed loop order, we can absorb all infinities by shifting the coefficients of just the operators already present in $\mathcal{L}_{d\leq 4}$. On the other hand, to renormalize an operator in $\mathcal{L}_{d>4}$, we would need to add at each step an infinite tower of operators, affecting the whole EFT expansion. For this reason, $\mathcal{L}_{\text{EFT}}$ is referred to as a *non-renormalizable* lagrangian. We will briefly come back on this issue later.

When introducing EFTs, the first example that is brought forward is usually the one of the Fermi theory of β-decay, both because of its conceptual cleanliness and of its historical importance. We will do the same here, and use it to introduce the concepts we will need in the rest of the thesis. In 1933 [72], Fermi was trying to write down a theory that could explain the β-decay, where a neutron decays into a proton, emitting an electron and an (anti)neutrino $\bar{\nu}_e$, i.e.

$$n^0 \to p^+ + e^- + \bar{\nu}_e \tag{2.1.2}$$

At the time, neutrinos had been just proposed by Pauli, in 1930, to explain the missing energy and spin in the process, although they would not be observed until 1956 [73]. To account for this process, Fermi proposed a theory with a Hamiltonian containing a field for each of the involved particles, interacting through a potential term containing all four fermionic fields. Although deemed by Nature to be "too remote from reality to be of interest to the reader" [74], the theory turns out to be a great approximation of the weak interactions at energies below the masses of the W and Z bosons, i.e. $E < 80$ GeV. In its original formulation, adapted to the language of Quantum Field Theory, the interaction term of the lagrangian looks like [71]:

$$\mathcal{L}_{\text{Fermi}} = G_F \bar{\psi}_p \psi_n \bar{\psi}_e \psi_\nu + \text{ h.c. } , \tag{2.1.3}$$

[1] Actually, it has the dimensions of a scale, which is a slightly different thing. More on this later.

where

$$G_F = 1.1663787(6)^{-5}\ \text{GeV}^{-2} \approx \left(\frac{1}{292.9\ \text{GeV}}\right)^2 \ . \tag{2.1.4}$$

The great thing about this EFT, and of EFTs in general, is that Eq. (2.1.3) contains, conceptually, all we need. Here we can see how the recipe enumerated earlier is followed quite faithfully: first we identified the field relevant to the process, i.e. the neutron, proton, electron and (anti)neutrino fields. The relevant symmetries, here, are electric charge and lepton (family) number $U(1)_{L_e}$, acting as

$$\begin{aligned} &\psi_e,\ \psi_p \to e^{-i\theta_q}\psi_e,\ e^{i\theta_q}\psi_p \\ &\psi_e,\ \psi_\nu \to e^{i\theta_{L_e}}\psi_e,\ e^{i\theta_{L_e}}\psi_\nu \ , \end{aligned} \tag{2.1.5}$$

as well as Lorentz symmetry, forcing the Dirac indices to be all contracted. Moreover, since the four-fermions operator has dimension 6, we multiply it with the coefficient G_F, with dimension $[G_F] = m^{-2}$. Looking at Eq. (2.1.3), one may be however a bit disappointed of its appearance being quite different from Eq. (2.1.1), which we claim was our reference. Indeed, to be precise, one should include the whole infinite tower of operators compatible with the symmetries in Eq. (2.1.5), and the full lagrangian should look like[2] [71]:

$$\mathcal{L} = G_F\bar{\psi}\psi\bar{\psi}\psi + a_1 G_F^2\bar{\psi}\psi\Box\bar{\psi}\psi + a_2 G_F^3\bar{\psi}\partial\!\!\!/\psi\Box\bar{\psi}\partial\!\!\!/\psi + \dots \ , \tag{2.1.6}$$

(where we parametrized all dimensionful coefficients in terms of G_F). However, the higher order terms will affect the (cross section of the) β-decay, i.e. the process we care about, with terms proportional to $(G_F E^2)^{d-4}$, where E is some scale characterizing the energy involved in the process. Since the decay obviously happens at the energy of neutron mass $m_n \approx 1$ GeV, then for our process $E \ll G_F^{-1/2}$, and we can safely ignore the terms with $d > 6$ and be content with Eq. (2.1.3). So, to be precise, one should add to the points 1-5 listed above another one:

6. Truncate the expansion to keep only the terms relevant to the process at hand.

This philosophy, in its simplicity, gives us in a sense the possibility to parametrize our ignorance, and it will be the one we will try to follow in the rest of this thesis.

2.1.1 EFT breakdown

We already stressed multiple times that all theories we consider are going to be only valid within some energy range, while they will need to be completed by a more fundamental theory when the scale becomes large enough. We now have the tools to make the meaning of this statement more precise with the language of EFTs, again relying on the example of the Fermi theory outlined above. Conceptually, we can understand where and why our theory stops

[2] we will lose the lower index on the fermionic fields in the following

working from slightly different perspectives. The first, perhaps most naive way of doing this is by considering the following: we know that 4-point amplitudes are dimensionless objects. Thus, like we saw for the β-decay, an operator of dimension d will contribute to such amplitudes, by dimensional analysis, as

$$i\mathcal{M}_{\text{tree}} \propto \left(\frac{E}{\Lambda}\right)^{d-4} , \tag{2.1.7}$$

where E is some energy characterizing the process. If we truncate at, e.g., $d = 6$,

$$i\mathcal{M}_{\text{tree}} \propto \left(\frac{E}{\Lambda}\right)^{2} . \tag{2.1.8}$$

Since observable quantities expressing probabilities, like cross-sections and decay widths, are proportional to $|\mathcal{M}|^2$, if we insist to trust this result up to large values of E, we end up with probabilities larger than 1, i.e. a violation of unitarity. This argument is, however, a little sloppy at best. Indeed, if we saw a mathematician taking the function,

$$f(x) = \frac{1}{1+x} \quad x > 0 \tag{2.1.9}$$

which is bounded for $x \in [0, +\infty)$, expanding it around $x \to 0^+$ as

$$\frac{1}{1+x} \approx 1 - x + \mathcal{O}\big(x^2\big) \tag{2.1.10}$$

and then complain because the right hand side diverges when $x \to \infty$, we would certainly have something to say about it. Here, like for the mathematician's $f(x)$, the issue is rather, obviously, that the expansion we performed cannot be trusted anymore.

Actually, two related but different perturbative expansion breakdowns occur: the first is strictly dependent on how we built the EFT lagrangian: point 2.1 of our recipe requires us to truncate the series and only retain the operators deemed relevant. However, when $E \sim \Lambda$, the powers in Eq. (2.1.7) become all comparable with each other, and the truncation makes no sense anymore. The second breakdown happens at the level of perturbation theory. Let us focus on dimension 6 operators. Since their couplings are suppressed as Λ^{-2}, a diagram built with them and containing L loops will behave as

$$i\mathcal{M}_{\text{L-loops}} \propto \left(\frac{E}{\Lambda}\right)^{2L} . \tag{2.1.11}$$

Thus, as soon as $E \sim \Lambda$, all loops will become comparable to the tree level contribution and to each other, and we see that it is also perturbation theory in the usual sense that breaks down.

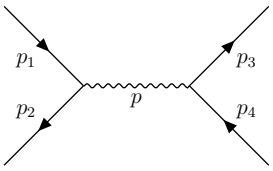
2.1.2 Matching

What the mathematics of the EFT expansion is telling us, here, is that at the cutoff the theory *has to* break down. However, it *could* actually break down way earlier than that. If the energy E is enough to produce a new particle, whose field is not within those we picked at point 1, that means, by definition, that we are not using the correct tools to describe the phenomena at hand anymore.

Let us explain what happens in this case relying, again, on the Fermi theory of interactions. Suppose we start from a lagrangian containing the fermionic field interacting with a massive vector boson W_μ

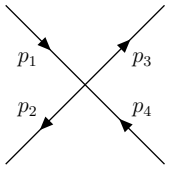
$$\mathcal{L}_{UV} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 W_\mu W^\mu + \bar{\psi}(i\partial\!\!\!/ + gW\!\!\!/)\psi \ , \tag{2.1.12}$$

where $F^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu$ and g is some coupling. Then, we can compute the matrix element for $\psi\psi \to \psi\psi$ as [71]:

$i\mathcal{M} =$ 

$\sim (ig)^2 \bar{v}_2 \gamma^\mu u_1 \dfrac{-1(g^{\mu\nu} - \frac{p^\mu p^\nu}{M^2})}{s - M^2} \bar{u}_3 \gamma^\nu v_4 \ .$ (2.1.13)

Here, the Dirac equation guarantees that the piece $\propto p^\mu p^\nu$ in the propagator vanishes[3], and at energies $s \ll M$ we can approximate the matrix element as

$i\mathcal{M} =$ 

$= -i\dfrac{g^2}{M^2} \bar{v}_2 \gamma^\mu u_1 \bar{u}_3 \gamma_\mu v_4 \ .$ (2.1.14)

This is the same matrix element we would get from a 4-Fermi lagrangian with a dimension-six term of the form

$$\mathcal{L}_{\text{4-Fermi}} = G_F \bar{\psi}\gamma^\mu \psi \bar{\psi}\gamma_\mu \psi \tag{2.1.15}$$

if we set $G_F = \frac{g^2}{M^2}$. Visually, we can interpret this result as the W boson propagating over distances of order M^{-1}, so that it cannot be seen if we observe the process at a large distance, and the interaction reduces to a point-like contact term, as in Eq. (2.1.14). The theory in Eq. (2.1.12) is then referred to as the *UV completion* of the 4-Fermi theory. It should be said that, in the real world, the fields actually responsible for the β-decays are the quarks inside the nucleons, so that the corresponding 4-Fermi lagrangian should be built using their fields. There, the UV completion is in fact the electroweak sector of the SM, where the W boson is one of the gauge bosons of the $SU(2)_L \times U(1)_Y$ gauge group, which acquires a mass through the Higgs mechanism, which induces the spontaneous symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_{\text{e.m.}}$

[3] it can be seen by noticing that $p\!\!\!/ = p\!\!\!/_1 + p\!\!\!/_2$ acts on $\bar{v}_2$ and on u_1.

The procedure we just outlined is an example of what goes under the name of *matching*. We could have gone further in Taylor-expanding Eq. (2.1.13) in powers of $\frac{s^2}{M^2}$, and we would have needed more and more higher dimensional operators in the EFT to reproduce the matrix element, with coefficients that could all be expressed in terms of monomials of g and M.

In this short example, we actually limited ourselves to the tree-level matching. A generalization to a universal recipe is immediate: compute a process involving fields contained both in the UV theory and in the EFT, and fix the coefficients of the latter so that the two results agree, up to some order in the expansion. This is also referred to as the *on-shell* method, to distinguish it from another way of performing the matching, which happens directly at the level of the path integral, and is thus dubbed *off-shell* [75, 76]. We can also go beyond tree-level and perform the matching at one (or more) loops. This is where the EFT approach shows some of its greatest strengths, in combination with the tools of the renormalization group equation[4].

2.1.3 Masses and scales

We stressed above that the scale where the EFT stops working because a new excitations enters the game can be met way earlier than the scale where the EFT expansion breaks down. To explain this, it is useful to understand the difference that lies here between masses and scales. A useful trick, to this end, is to reintroduce in the action explicit factors of $\hbar$, while keeping $c = 1$. For example, for a scalar field $\phi(x)$, $\hbar$ enters the equal time commutation relation between $\phi(t, \vec{x})$ and its canonically conjugated momentum $\pi(t, \vec{y}) = \partial_t \phi(t, \vec{y})$ as

$$[\phi(t, \vec{x}), \pi(t, \vec{y})] = i\hbar \delta^3(\vec{x} - \vec{y}) \ , \tag{2.1.16}$$

which is the QFT analogue of the Quantum Mechanics $[\hat{x}, \hat{p}] = i\hbar$. We can then distinguish between two different units, units of energy E and of length L, so that $[\hbar] = EL$. Then, we can use that the action S has $[S] = [\hbar]$, and that $S = \int \mathrm{d}^4x \, \mathcal{L}$, together with the canonical commutation relations for all the fields normalized as Eq. (2.1.16), to get the dimensionality of all the relevant quantities:

$$\begin{aligned} &[\hbar] = EL \qquad [\mathcal{L}] = EL^{-3} \qquad [\phi] = [A_\mu] = E^{1/2}L^{-1/2} \qquad [\psi] = E^{1/2}L^{-1} \\ &[\partial] = L^{-1} \qquad [g] = [y] = E^{-1/2}L^{-1/2} \qquad [\lambda] = E^{-1}L^{-1} \ , \end{aligned} \tag{2.1.17}$$

where A_μ and ψ indicate generic spin 1 and fermionic fields, g a gauge coupling, y a Yukawa coupling and λ a quartic coupling. To make the discussion more transparent, let us define units of mass and coupling as [78] $\tilde{M} \equiv L^{-1}$ and $C \equiv E^{-1/2}L^{-1/2}$. Then, consider a generic operator of dimension d

$$\frac{1}{\Lambda^{d-4}} \partial^{n_D} \Phi^{n_B} \psi^{n_F} \ , \tag{2.1.18}$$

[4]For a nice review, see e.g. Ref. [77]

where n_D is the number of derivatives, n_B the number of bosonic fields ($\Phi = \phi, A_\mu$), and n_F the number of fermions, with $n_D + n_B + \frac{3}{2}n_F = d$. In units of C and $\tilde{M}$,

$$[\phi] = [A_\mu] = \tilde{M}C^{-1} \qquad [\psi] = \tilde{M}^{3/2}C^{-1} \qquad [\mathcal{L}] = \tilde{M}^4 C^{-2} \ , \tag{2.1.19}$$

whence

$$[\Lambda] = \frac{\tilde{M}}{C^{\frac{n-2}{d-4}}} \ , \tag{2.1.20}$$

with $n = n_F + n_B$ the total number of fields in the operator. As $n > 2$ and $d > 4$ for the operators in the EFT, the exponent of C in Eq. (2.1.20) will always be positive and there will always be some power of the couplings separating mass and scale. In other terms, *masses and scales are incommensurable*. Reconnecting to the discussion we were leading above, we can understand the two different physical meanings of the two quantities in that $\tilde{M}$ is associated to the energy where new degrees of freedom appear, while Λ to the energy where the EFT expansion and perturbation theory break down.

Let us go back to our example of the lagrangian in Eq.(2.1.12) of a massive vector boson interacting with a fermion field and to its low-energy counterpart, the 4-Fermi theory in Eq. (2.1.15). Looking at the latter, we could infer that the theory breaks down at energies of the order of the cutoff scale $E \sim G_F^{-1/2} = M/g$. However, we know that the theory needs to be completed as soon as the W boson enters the game, i.e. for $E \sim M = gG_F^{-1/2}$. If [5] $g \ll 1$ this would happen way earlier than the failure of the EFT perturbative expansion. This result is quite general, and it is worth keeping in mind when dealing with the construction of EFTs.

2.1.4 Coset construction and chiral lagrangian

We now wish to make a short departure from the line of discussion followed up to this point to introduce another fundamental tool in developing Effective Theories. Such is the so called coset or CCWZ construction (from the name of their authors) [79, 80]. Although there will not be particular focus on this construction in the rest of the work, it is certainly useful to have in mind a complete picture of the set of tools a theoretical physicist can count on when building a theory, and we will often reference it later. In addition, its historical importance and the theoretical cleanliness and conciseness of the arguments (the two original papers are 12 pages in total), are themselves enough to make it worth discussing.

The CCWZ construction deals with the situation where some compact symmetry Lie group G is spontaneously broken down to one of its subgroups H, allowing an effective description of the relevant low energy modes, i.e. the Goldstone bosons. Let us label $\hat{T}^i$ the generators of the subgroup H and X^α all the other generators of G. Since H is a subgroup, the $\hat{T}^i$'s will form a

[5] in the real world $G_F = \sqrt{2}g_w^2/8m_W^2$, where $m_W \approx 80.4$ GeV and $g_w \approx 0.65$, and the $\sqrt{2}/8$ factor is a consequence of the particular structure of the EW sector.

subalgebra, and the commutation relations look like:

$$\left[\hat{T}^i, \hat{T}^j\right] = iC_{ijk}\hat{T}^k \qquad \left[\hat{T}^i, X^\alpha\right] = iC_{i\alpha\beta}X^\beta \qquad \left[X^\alpha, X^\beta\right] = iC_{\alpha\beta\gamma}X^\gamma + iC_{\alpha\beta i}\hat{T}^i \ . \tag{2.1.21}$$

We can define the operator

$$\Omega(x) = e^{i\pi_\alpha(x)X^\alpha} \ , \tag{2.1.22}$$

parametrized in terms of the fields $\pi_\alpha(x)$ that are going to represent our Goldstone bosons. Suppose the breaking happens through some order parameter v, and the theory finds itself in some vacuum $|v\rangle$, with

$$\hat{T}^i |v\rangle = 0 \qquad\qquad X^\alpha |v\rangle \neq 0 \ . \tag{2.1.23}$$

If we pick constant values for the $\pi_\alpha(x) \to \bar{\pi}_\alpha$, $\partial_\mu \bar{\pi}_\alpha = 0$, the action of $\Omega(\bar{\pi})$ takes us from a vacuum state $|v\rangle$ to another vacuum $|v'\rangle$. Since G is a symmetry of the theory by assumption, $[X^\alpha, \mathcal{H}] = 0$, so $|v\rangle$ and $|v'\rangle$ are degenerate, but inequivalent. In other words, Ω spans the *coset space* manifold G/H. Then, we can imagine the operator $\Omega(x)$ as acting on $|v\rangle$ and the Goldstone bosons parametrizing the excitations with respect to it in the direction of the broken generators X^α.

Importantly, under the action of a generic $g \in G$, this object transforms as

$$g\Omega(\pi(x)) = \Omega(\pi'(x))h(\pi(x), g) \qquad\qquad h \in H \ . \tag{2.1.24}$$

Eq. (2.1.24) can be interpreted as $\Omega(x)$ being the quantity that allows us to trade a transformation in G for one in H, although the latter depends on the former. From Ω, we can define the Maurer-Cartan 1-form $\Omega^{-1}\partial_\mu\Omega(x)$, which belongs to the tangent bundle of the group, i.e. to the Lie algebra. As such, it can be written as a combination of the generators

$$\Omega^{-1}\partial_\mu\Omega(x) = iD_\mu\pi_\alpha X^\alpha + iA_{i\mu}\hat{T}^i \tag{2.1.25}$$

Now, by direct computation, we can show that $D_\mu\pi_\alpha$ transforms under the *full* group G with some matrices that belong to a linear representation of H, while the $A_{i\mu}$'s transform like a connection

$$X^\alpha(D_\mu\pi_\alpha) \xrightarrow{G} X^\alpha(D_\mu\pi_\alpha)' = h(\pi, g)\left(X^\beta D_\mu\pi_\beta\right)h^{-1}(\pi, g) \tag{2.1.26}$$

$$\hat{T}^i A_{i\mu} \xrightarrow{G} \hat{T}^i A'_{i\mu} = h(\pi, g)\Big(\hat{T}^j A_{j,\mu}\Big)h^{-1}(\pi, g) + i[\partial_\mu h(\pi, g)]h^{-1}(\pi, g) \tag{2.1.27}$$

The interpretation is then the following: the Goldstone bosons, and possibly other light fields also present in the lagrangian, all live in the curved coset manifold, parametrized by the Goldstones themselves. The expansion in Eq. (2.1.25) provides us with the tools to deal with this curved space: $D_\mu\pi_\alpha$ can be seen as the covariant derivative of the π_α's, while we can use $A_{i\mu}$

to build a covariant derivative

$$\mathcal{D}_\mu \equiv \partial_\mu + iA_{i\mu}\hat{T}^i \ , \tag{2.1.28}$$

that we can use to consistently treat any light field we could want to add to the theory. We can at this point build the most general lagrangian $\mathcal{L}$ with these building blocks, and just be sure to make it invariant under the linearly realized subgroup H. By virtue of Eqs. (2.1.26) and (2.1.27), we can then rest assured that $\mathcal{L}$ will also be invariant under the full group G.

Surely, the most prominent example of an application of such construction is the systematization of the Chiral lagrangian of mesons. Consider the lagrangian of massless QCD limited to two or three quarks[6]

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_{i=1}^{N_f} i\bar{\psi}_i \not{D}\psi_i \ , \tag{2.1.29}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, $\not{D} = \gamma^\mu(\partial_\mu - igT^a A^a_\mu)$, $N_f = 2, 3$ and T^a, $a = 1, \dots, 8$ are the generators of the color gauge group $SU(N_c)$. Since there is no mass term, $\mathcal{L}_{QCD}$ enjoys a global flavor symmetry acting separately on the right- and left-handed components of the Dirac fermions ψ_i,

$$G = U(N_f)_L \times U(N_f)_R \ . \tag{2.1.30}$$

At low energies, as it is well known, the theory undergoes spontaneous symmetry breaking by means of a quark condensate acquiring non-zero expectation value

$$\left\langle \bar{\psi}_i \psi_j \right\rangle \propto \delta_{ij} \neq 0 \ , \tag{2.1.31}$$

where i, j are flavor indices. The full group G is then broken to its diagonal subgroup

$$G = U(N_f)_L \times U(N_f)_R \xrightarrow{\left\langle \bar{\psi}_i \psi_j \right\rangle \neq 0} H = SU(N_f)_V \times U(1)_A \ . \tag{2.1.32}$$

The chiral $U(1)_A$ is broken by anomalies [81, 82], so we will not care about it in the following. To follow the CCWZ recipe, then, Eq. (2.1.32) is all we need. Here, there are $N_f^2 - 1$ broken generators, so we can parametrize the coset manifold with

$$\Omega(x) \equiv \exp\left(i\frac{\pi^a}{F_\pi}X^a\right) \qquad\qquad a = 1, \dots, N \ , \tag{2.1.33}$$

where we have factored out the dimensionful quantity F_π so that $[\pi^a] = m$ as expected for a

[6] While the up and down masses are $\mathcal{O}(\text{MeV}) \ll \Lambda_{QCD}$, so they can be taken to be massless to a very good degree, the same approximation is less clean for the strange quark with $m_s = \mathcal{O}(100\ \text{MeV})$.

scalar field. We can extend this rescaling to all involved quantities as

$$D_\mu \pi^a \to \frac{1}{F_\pi^2} D_\mu \pi^a \qquad\qquad \mathcal{D}_\mu \to \frac{1}{F_\pi} \mathcal{D}_\mu \ . \tag{2.1.34}$$

Then, the Chiral lagrangian parametrizing the interactions of the mesons π^a within themselves can be written, at the lowest order, as

$$\mathcal{L}_\chi^{(2)} = \frac{1}{2} D_\mu \pi^a D^\mu \pi^a \ . \tag{2.1.35}$$

Actually, this lagrangian is usually written in terms of Ω as [83]

$$\mathcal{L}_\chi^{(2)} = \frac{F_\pi^2}{4} \operatorname{Tr}\big(\partial_\mu \Omega \partial^\mu \Omega^\dagger\big) \ , \tag{2.1.36}$$

which can be brought in the form of Eq. (2.1.35) using that Ω is unitary, $\Omega^\dagger = \Omega^{-1}$, and

$$\operatorname{Tr}\big(\partial_\mu \Omega \partial^\mu \Omega^\dagger\big) = \operatorname{Tr}\left((\Omega^{-1}\partial_\mu \Omega)^\dagger (\Omega^{-1}\partial_\mu \Omega)\right) \tag{2.1.37}$$

and using $\operatorname{Tr}\big[X^a X^b\big] = 2\delta^{ab}$, if X^a are the Pauli or Gell-Mann matrices.

The construction provided is by all means a specific recipe for the building of an Effective Field Theory. Indeed, Eq. (2.1.35) can be supplemented with an infinite tower of operators of the schematic form

$$\mathcal{O}^{(d)} = \frac{c}{\Lambda^{d-4}} \mathcal{D}_{\mu_1} \dots \mathcal{D}_{\mu_n} D_{\nu_1} \pi^{a_1} \dots D_{\nu_m} \pi^{a_m} \ , \tag{2.1.38}$$

with $d = n + 2m$ and all the indices are contracted to form a Lorentz and $SU(N_f)_V$ singlet. More remarkable, however, is the fact that the $D_\mu \pi^a$'s can be themselves expanded in an infinite series in the fields π^a and their derivatives, with the appropriate suppression. For example, for $N_f = 2$, we can expand the quadratic term in Eq.(2.1.35) as

$$\mathcal{L}_\chi^{(2)} = \frac{1}{2} D_\mu \vec{\pi} \cdot D^\mu \vec{\pi} = \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \frac{1}{6F_\pi^2} \left(\vec{\pi} \times \partial_\mu \vec{\pi}\right) \cdot \left(\vec{\pi} \times \partial^\mu \vec{\pi}\right) + \mathcal{O}\big(F_\pi^{-4}\big) \ , \tag{2.1.39}$$

where we adopted the vector notation $\vec{\pi} = \{\pi_1, \pi_2, \pi_3\}$. The notable feature is here that symmetry fixes the coefficient of the second term, and of all higher order terms coming from the expansion, so that they can all be expressed in terms of the dimensionful quantity F_π. In other words, we can think of the coset construction as of an improvement of the simplest EFT construction, where all terms would have arbitrary coefficients, allowed by the knowledge of the symmetry breaking pattern. We conclude by mentioning that the coset construction, as we built it here, is fit to address the spontaneous breaking of internal symmetries. However, it can be generalized to include the breaking of some Poincaré generators, in which case some interesting behavior, e.g. the presence of massive Goldstone-like excitations in the spectrum, is observed [84–88].

2.1.5 Spurions

Another useful concept when building EFTs is that of spurions, which we will use quite extensively in the following and thus wish to introduce here [78]. Consider a theory invariant under a global symmetry G but for one operator $\mathcal{O}^{(\mathbf{r})}$ transforming in some representation $\mathbf{r}$ of G. Let us call C the coefficient of this operator. Imagine, then, that instead of C being a number it were the vacuum expectation value of some heavy scalar field we have integrated out, transforming in the conjugate representation $\bar{\mathbf{r}}$ with respect to $\mathcal{O}^{(\mathbf{r})}$, with indices contracted so that $C\mathcal{O}^{(\mathbf{r})}$ is a singlet under G and the full lagrangian is now completely G-invariant. Such a C is called *spurion*.

Thinking of symmetry-breaking coefficients in this way has a number of advantages. The first one, which we will only mention briefly, is that they can be used in the definition of *technical naturalness* [89]. If we take the limit $C \to 0$, the symmetry group G is fully restored, so that any diagram that is sensitive to the breaking of G must have at least one insertion of $\mathcal{O}^{(\mathbf{r})}$ and so be accompanied by C. This is valid, in particular, for the quantum corrections that C receives from loop diagrams. The usual lore is then that if C has a small value at tree level, its quantum corrections will be small and it will stay small at loop level, too. Consider e.g. the lagrangian of a complex scalar field ϕ with mass M_ϕ.

$$\mathcal{L} = \left|\partial_\mu \phi\right|^2 - M_\phi^2 |\phi|^2 \ . \tag{2.1.40}$$

In the limit $M_\phi \to 0$, the theory recovers a shift symmetry $\phi \to \phi + a$ with constant a. Thus, it is technically natural for M_ϕ to be small. This would also be valid if we added to the lagrangian in Eq. (2.1.40) interactions of the form $\partial_\mu \phi \mathcal{O}^\mu$, with $\mathcal{O}^\mu$ any operator with an open vector index. However, imagine we now add to the lagrangian in Eq. (2.1.40) a fermionic field ψ, with mass M_ψ, interacting with ϕ via a Yukawa-like term

$$\mathcal{L} = \left|\partial_\mu \phi\right|^2 + \bar{\psi}(i\partial\!\!\!/ - M_\psi)\psi - M_\phi^2 |\phi|^2 - y\phi\bar{\psi}\psi \ . \tag{2.1.41}$$

Now, the scalar mass term will also receive corrections from the coupling y, which, too, breaks the shift symmetry. This is basically what happens in the SM, where the Higgs mass term receives corrections from the Yukawa as well as the Higgs quartic term, so it is not technically natural for it to be as small as it turns out to be. This technical naturalness interpretation of the Hierarchy problem, however, can be rather misleading. Indeed, one can argue that, being the Higgs mass the only scale in the SM[7], all other terms being forbidden by gauge symmetry, all corrections it receives are, again, proportional to itself.

The problem arises, rather, when UV completing the SM. If the UV completion is introduced at a scale $\Lambda_{UV} \gg v \sim 246$ GeV, where v is the Higgs vev, then the Higgs mass will receive contributions proportional to this scale, too. Then, to obtain the observed value at

[7] At least in the electroweak sector. As we mentioned, QCD condensation produces a dimensionful quantity Λ_{QCD}, which plays however no role at the scales of concern here.

our energies, some nontrivial fine-tuning has to happen in the UV[8]. This will be the case unless some mechanism intervenes to protect the Higgs mass itself, and a plethora of models have been proposed exploring different instances of such possibility, some examples including Little-Higgs [90, 91], Composite Higgs [92–95], Twin Higgs [96], extra dimensions [97–102] and obviously Supersymmetry (see Ref. [103] for a review).

Spurions, however, can also be applied in a clever way to get information linking the UV to the IR phase of a theory. The most successful example of this use is, again, linked to the Chiral lagrangian of mesons [83, 104, 105]. The treatment presented in Section 2.1.4 is based on the assumption that the quarks can be taken to be massless. In the real world, this is not the case, so the problem arises of how to consistently account for this mass term and understand its implications for the Chiral lagrangian. Consider the QCD lagrangian in Eq. (2.1.29) with the addition of a mass term

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_{i=1}^{N_f}\bar{\psi}_i\left(i\not{D}\delta_{ij} - M_{ij}\psi_j\right) \ , \tag{2.1.42}$$

where M_{ij} is hermitian and, upon a change of flavor basis, can always be taken to be diagonal. We can rewrite the mass term as

$$-\mathcal{L}_M = \sum_{i=1}^{N_f}\bar{\psi}_i M_{ij}\psi_j = \sum_{i=1}^{N_f}\left[\bar{\psi}_{R,i}S_{ij}\psi_{L,j} + \bar{\psi}_{L,i}S^\dagger_{ij}\psi_{R,j}\right] \quad \text{with} \quad M_{ij} = \begin{pmatrix} S_{ij} & 0 \\ 0 & S^\dagger_{ij}\end{pmatrix} \tag{2.1.43}$$

to separate the two helicities. Clearly, M_{ij} breaks the flavor symmetry $U(N_f)_L \times U(N_f)_R$. However, we can think of it as a spurion, i.e. we can promote it to a field $\tilde{M}_{ij}$, whose vev is M_{ij}, and so that, under $U(N_f)_L \times U(N_f)_R$

$$\tilde{M}_{ij} \to \tilde{M}' = \begin{pmatrix} S' & 0 \\ 0 & S^{\dagger\prime}\end{pmatrix} = \begin{pmatrix} V_R S V_L^\dagger & 0 \\ 0 & V_L S^\dagger V_R^\dagger\end{pmatrix} \quad \text{with} \quad V_{R,L} \in U(N_f)_{R,L} \tag{2.1.44}$$

This transformation is exactly the one needed to counterbalance that of the fermions, so that the mass term results invariant under the full group G. Suppose we want then to add this new field to the Chiral lagrangian. Since global symmetries must be also realized in the IR theory (although sometimes non-linearly, has we have seen), and the QCD lagrangian with the spurion $\tilde{M}$ is invariant under $U(N_f)_L \times U(N_f)_R$, we also have to add this spurion field to the Chiral lagrangian in a way that respects this invariance. The first nontrivial singlet one can build is then

$$\mathcal{L}_{\chi M} = \frac{F_\pi^2}{2}\left[B\,\mathrm{Tr}\Big(\tilde{M}\Omega(x)\Big) + B^*\,\mathrm{Tr}\Big(\tilde{M}^\dagger\Omega(x)^\dagger\Big)\right] \ . \tag{2.1.45}$$

[8] For a clear treatment of this interpretation of the Hierarchy problem, see e.g. Refs. [37, 77]

After setting $\tilde{M}$ to its vev, and imposing parity

$$\pi(x) \xrightarrow{P} -\pi(x') \Longrightarrow \Omega(x) \xrightarrow{P} \Omega^\dagger(x') \ , \tag{2.1.46}$$

we arrive at

$$\mathcal{L}_{\chi M} = \frac{F_\pi^2 B}{2} \operatorname{Tr}\left[M\left(\Omega(x) + \Omega^\dagger(x)\right)\right] \ . \tag{2.1.47}$$

Here B is a dimensionful quantity that needs to be fixed by measurement[9], and that can be related to the meson masses M_π by expanding Eq. (2.1.47) to second order in the field π. For $N_f = 2$, for example, we have

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \ , \tag{2.1.48}$$

and

$$\mathcal{L}_{\chi M} = \frac{1}{2}F_\pi^2 B(m_u + m_d) - \frac{1}{2}B(m_u + m_d)\vec{\pi}^2 + \mathcal{O}\left(F_\pi^{-2}\right) \ . \tag{2.1.49}$$

We can relate pion masses $M_\pi^2 \equiv B(m_u + m_d)$ to the chiral condensate by imposing

$$\left\langle \bar{\psi}_i \psi_j \right\rangle = -\left\langle \frac{\delta}{\delta \tilde{M}_{ij}} \mathcal{L}_{QCD} \right\rangle \stackrel{!}{=} -\left\langle \frac{\delta}{\delta \tilde{M}_{ij}} \mathcal{L}_\chi \right\rangle \ , \tag{2.1.50}$$

where $\mathcal{L}_\chi$ indicates the full Chiral lagrangian, and obtaining

$$\frac{\delta}{\delta \tilde{M}_{ij}} \mathcal{L}_\chi = \frac{1}{2} F_\pi B \left(\Omega(x) + \Omega^\dagger(x)\right)_{ij} \ . \tag{2.1.51}$$

Using $\langle \Omega(x)_{ij} \rangle = \left\langle \Omega^\dagger(x)_{ij} \right\rangle = \delta_{ij}$, we finally get

$$\left\langle \bar{\psi}_i \psi_j \right\rangle = -F_\pi^2 B \tag{2.1.52}$$

and

$$(m_u + m_d)\left\langle \bar{\psi}_i \psi_j \right\rangle = -F_\pi^2 M_\pi^2 \ , \tag{2.1.53}$$

i.e. the so called Gell-Mann–Oakes–Rènner relation [106]. More applications of spurions analysis can be found e.g. in Refs [107–109]. This discussion shows how the concept of spurion can be a powerful tool both make clear physical predictions like Eq. (2.1.53) and for conceptual prescriptions like the technical naturalness.

[9] We could have obviously written $F_\pi^2 B = F_\pi^3 \tilde{B}$ for some dimensionless $\tilde{B}$. However, we present $\mathcal{L}_{\chi M}$ here as it usually appears, see Ref. [105].

2.1.6 Accidental symmetries

Another important concept that will be accompanying us is that of *accidental symmetries*. These are symmetries which arise accidentally at a given order in the EFT expansion without being imposed on the theory as a first principle. This happens because, at each fixed dimension, there is only a finite number of operators we can build with a given set of fields and the required gauge and global symmetries. It can occur, then, that we have built all the possible operators compatible the imposed symmetries and we notice that they happen in addition to be invariant under some other global symmetry. This concept is important because it also gives us hints about when we expect the accidental symmetry to be broken. Imagine we build a EFT which is accidentally symmetric under a group G (usually some $U(1)$ factors) up to dimension $d-1$, so that at dimension d a bunch of operators

$$\frac{c_{\mathscr{G},i}}{\Lambda^{d-4}}\mathcal{O}_{\mathscr{G},i} \tag{2.1.54}$$

finally break G. Then we know that G-violating observables will be suppressed at least by a factor of $(E/\Lambda)^{d-4}$, with E some energy scale characterizing the process at hand.

We stumble into this accident rather often in Quantum Field Theory. The most prominent examples are probably the two accidental symmetries arising in the Standard Model. Indeed, the renormalizable $d \leq 4$ part of the SM lagrangian possesses conservation of baryon number and lepton family number as accidental symmetries. The former is a $U(1)_B$ global symmetry under which (anti-)quark have charges $(-)\frac{1}{3}$, while leptons are uncharged, while the latter is composed of three $U(1)_{L_i}$ factors under which one of the three lepton families has charge 1, while the other two and the quarks are inert. Both of these accidental symmetries are broken when going further in the EFT expansion. Lepton family number, for example, is broken already at dimension-five by the so called Weinberg operator [16]:

$$\mathcal{L}^{(5)} = \frac{C_{ij}}{\Lambda}\left(H\bar{L}_i^c\right)\left(L_j H\right) + \text{ h.c. }. \tag{2.1.55}$$

Provided the coefficient C_{ij} has a generic structure, this operators breaks all three generators of $U(1)_{L_i}$. After Electroweak Symmetry Breaking (EWSB), when the Higgs field is set to its vev, this operator provides a Majorana mass for the left-handed neutrinos, and can reproduce the observed neutrino oscillations [110–112]. As mentioned, we can use symmetry breaking interactions to infer the magnitude of the suppression scale Λ. Since neutrino oscillations are a lepton-number violating observable, we can use it to estimate the suppression scale Λ. Assuming $\mathcal{O}(1)$ values for the C_{ij} entries, we get $\Lambda \sim 10^{14}$ GeV. This is clearly an extremely large value, and were it the general scale of suppression we had to deal with, it would clearly make the EFT quite hard to use practically. However, due to the particular nature of neutrinos, it is plausible that some mechanism intervenes to mitigate this prediction [110, 113, 114]. For example, one can think that some new resonances appear at some energy $E \ll 10^{14}$ GeV, but that whatever model we need to describe the physics at that energy still has lepton family number as an

accidental symmetry.

Baryon number, instead, is broken first at dimension-six. Experimentally, baryon number violation would have the remarkable implication of allowing the proton to decay [115], via

$$p^+ \to \pi^0 e^+ \ . \tag{2.1.56}$$

The very strong bounds on the proton lifetime, then, $\tau > 10^{33}$ years [116], imply that baryon number violating operators are suppressed by a scale $\Lambda \gtrsim 10^{15}$ GeV, again with the assumption of $\mathcal{O}(1)$ coefficients. Here, similar mechanisms to the case of the Weinberg operator can intervene to lighten this bound. Additional suppression can be gained in this case by remembering that the proton is composed by up and down quarks. If there is some fundamental explanation to the Yukawa couplings being so hierarchical, it is plausible that the same mechanism that causes the suppression of the first generation Yukawa couplings also suppresses the proton decay [112].

In addition to being broken at $d > 4$, both baryon number and lepton family numbers are anomalous symmetries, so that their violation can be also mediated non-perturbatively by topological effects such as instantons [71]. Their combination $B - L$, where L is intended as the diagonal subgroup of $U(1)_{L_i}$ where all the three lepton families transform with the same phase, is however anomaly-free, and can be in principle gauged [117–120]. This would mean that this subgroup of the original symmetry would need to be imposed at each order in the EFT expansion. Actually, a conjecture from quantum gravity states that any quantum theory that wishes to include the description of gravity cannot have global symmetries [121–125]. If taken seriously, this means that any global symmetry should be interpreted as an accidental one, since it would have to be broken at least by gravitational effects, i.e. higher dimensional operators suppressed by powers of M_P.

2.1.7 Renormalizability

Let us also briefly comment on the applicability of renormalization to EFTs. We said that a theory in 4 dimensions with only operators of dimension $d \leq 4$ is said to be renormalizable, while it is called non-renormalizable if operators of $d > 4$ are also included. This nomenclature is due to the property of renormalizable theories of making an infinite number of predictions by measuring a finite number of parameters at a given energy scale. More concretely, this means that all UV divergences appearing in loop diagrams can be reabsorbed by shifting a finite number of parameters in the lagrangian. Such a feature is related to the BPHZ theorem [126–128], stating that all divergences can be removed by counterterms corresponding to superficially divergent 1PI amplitudes. Renormalizable theories, then, are those with only a finite number of such divergent 1PI amplitudes [129]. On the contrary, non-renormalizable theories produce an infinite number of them, and will need an infinite number of counterterms to balance them [130]. A generic loop diagram in 4 dimension with n_f external fermions, n_b external bosons and with

n_i insertions of couplings with dimension Δ_i will have a superficial degree of divergence [71]:

$$D_{n_f,n_b,n_i} = 4 - \frac{3}{2}n_f - n_b - \sum_i n_i \Delta_i \ . \tag{2.1.57}$$

So, interactions with $\Delta_i < 0$, i.e. those multiplying operators of $d > 4$, correspond to an infinite number of possible n_i and thus of $n_{f,b}$ that make $D_{n_f,n_b,n_i} > 0$, implying an infinite number of divergent 1PI amplitude. This is why couplings of negative dimension are called *non-renormalizable*, while those of $\Delta_i = 0$ are called *marginal*, and *super-renormalizable* the ones with $\Delta_i > 0$.

The property of renormalizability has been regarded for quite some time as a fundamental one for a healthy theory. More recently, however, it has been understood in a more transparent way that, as far as physical predictions are concerned, a non-renormalizable theory can be just as useful as a renormalizable one [71]. As a matter of fact, despite requiring an infinite number of derivatives, non-renormalizable theories can be used to make predictions. At tree level, this descends from the lists of operators in the EFT becoming less and less important with increasing dimension, so that for each observable we can retain the relevant subset in the E/Λ expansion. However, they are also predictive at loop level. Non-renormalizable theories, indeed, can be renormalized, but the only way to do it is by continually adding terms to the lagrangian to provide counterterms to cancel divergences. Crucially, this is always possible since divergences coming from loop diagram are always local, i.e. they will always multiply polynomials in the external momenta. Thus, there always exists a local operator that can be built to cancel them[10].

Lastly, it is worth stressing that renormalizable theories allow to make infinite predictions with a finite number of parameters only formally. In actuality, when one needs to concretely perform computations, this may not be possible. For example, n-points amplitudes become more and more complicated to compute with increasing n, and the same goes for the loop order. More profoundly, there can be obstructions that fundamentally preclude the theory from being predictive, such as it becoming strongly coupled. In conclusion, although renormalizable theories have a number of theoretical and practical advantages, unless some strong reasoning suggests otherwise, or if we are dealing with a would-be theory of everything, we should interpret every renormalizable theory just as the leading order of an EFT expansion.

2.1.8 SMEFT

Let us finally get to the object whose properties will represent the main focus of this thesis, i.e. the Standard Model Effective Field Theory (SMEFT) [65]. It stems from generalizing the SM lagrangian to include higher dimensional operators, as required by the EFT prescription. To avoid confusion in the following, we will refer to the renormalizable part of the SM lagrangian,

[10] See Ref. [71], Chapter 21 for a nice explanation.

i.e. the set of SM operators with $d \leq 4$, as SM_4. To fix the notation, we show it here

$$\mathcal{L}_{\mathrm{SM}_4} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^I_{\mu\nu}W^{I\mu\nu} - \frac{1}{4}G_{a\mu\nu}G^{a\mu\nu} + (D_\mu H)^\dagger(D_\mu H) + m^2 H^\dagger H - \frac{1}{2}\lambda(H^\dagger H)^2 + \\ + \sum_{\psi=L,Q,e,u,d} i\bar{\psi}\not{D}\psi - \left(H\bar{L}_i Y_{e,ij} e_j + H\bar{Q}_i Y_{d,ij} d_j + \tilde{H}\bar{Q}_i Y_{u,ij} u_j + \text{ h.c.}\right) \ , \quad (2.1.58)$$

where H is the Higgs doublet, $\tilde{H} = i\sigma_2 H^*$, with σ_2 the second Pauli matrix. Q and L are the left-handed components of the quarks and leptons fields, while e, u and d are the right handed ones. We will always omit helicity indices unless necessary. We mostly follow Ref. [70] for the sign and i conventions, so for example the covariant derivative acting on the left-handed quark field Q looks like

$$(D_\mu Q)^{\alpha j} = \left[\delta_{\alpha\beta}\delta_{jk}(\partial_\mu + ig' Y_Q B_\mu) + ig\delta_{\alpha\beta}\tau^I_{jk} W^I_\mu + ig_s \delta_{jk} T^A_{\alpha\beta} G^A_\mu\right] Q^{\beta k} \ , \quad (2.1.59)$$

with $\tau^I = \frac{\sigma^I}{2}$ the $SU(2)$ generators, $T^A = \frac{\lambda^A}{2}$ the $SU(3)_c$ generators, σ^I are the Pauli matrices and λ^A the Gell-Mann matrices. Finally, Y_Q indicates the hypercharge.

Now, we wish to extend $\mathcal{L}_{\mathrm{SM}_4}$ by supplementing it with all possible higher dimensional operators. At dimension-five, we already saw that the only possibility is represented by the Weinberg operator, Eq. (2.1.55). With increasing dimension, the number grows quite fast, and at dimension-six we already have to deal with 59 operators. A complete set for $d = 6$ has been presented for the first time in Ref. [70], and we will in the following take this basis as reference. Such basis is usually referred to as the *Warsaw basis* . It may be surprising that this set was only first presented some 12 years ago. However, this circumstance is the consequence of two contingencies: first of all, one does not *need* a complete basis for most practical purposes. If we are studying an observable, say the electron EDM, we will only need a subset of these operators [131,132], and a convenient basis for them can be picked case by case. Secondly, there is a large number of redundancies relating operators within each other, be it by integration by parts, Fierz identities or field redefinitions. We will discuss this issue in great detail in the next Section. Although it is not technically wrong to pick a basis containing redundant operators, they will produce flat directions in fits, since they will always appear in some combination. Finally, notice that the number 59 we flashed above only refers to the SMEFT with only one generation of quarks and leptons. Including all three of them actually pushes that number up to an astounding 2499 [133], although most of them are just different entries of the same flavorful objects.

It is then easy to understand why the classifications to some higher dimensions has only been completed quite recently. At the time of writing, complete bases have been shown for the SMEFT at dimension-seven [134], dimension-eight [135, 136], and nine [137, 138]. By looking at their number at each dimension, one can notice a distinct feature [139]: operators of even mass dimension are usually more than in the odd case. This is intuitively understood since symmetry favors building blocks of even dimension, e.g. Lorenz indices contracted two by two, or the Higgs field in the combination $H^\dagger H$, and so on.

We already discussed the effect of the single operator present at dimension-five, the Weinberg operator. Dimension-six terms, then, represent the next leading order in the E/Λ expansion, and are thus expected to be the ones phenomenologically most relevant. Consistently with this picture, we will dedicate most of our attention to the SMEFT at dimension-six. However, dimension-eight operators could be disentangled from dimension-six ones thanks to them having a specific impact on the angular dependence of the differential cross-section in some processes [140, 141], or could become relevant in processes where dimension-six ones do not interfere with the leading order SM_4 thanks to some selection rules [142, 143], so that one could even think of testing them [144]. Moreover, they are the first ones that can be subject to positivity bounds, which we are going to discuss extensively in the following.

2.2 How to build an EFT and the Hilbert series

In Section 2.1 we outlined the recipe one should follow when building an Effective Field Theory to describe some phenomena at hand. After fixing the fields and the gauge and global symmetries for the theory, point 3 requires us to write down all possible independent operators built with the former and compatible with the latter. This is easier said than done. A number of issues conspire to make this step quite nontrivial and subject to mistakes. First of all, there is not a unique way to define these operators, as some of them are related to others. Moreover, the number of possible combinations grows exponentially with the dimension, as we have seen in some examples for the SMEFT. In this Section we will try to address these issues, and provide the description of a tool that can help in this quest, the so called Hilbert Series.

2.2.1 Operator redundancies

Let us imagine we followed the recipe for the building of an EFT we outlined in Section 2.1 for some theory at some dimension, and we ended up with a bunch of operators defining our lagrangian. By looking at them more carefully, however, we notice that some of them can be arranged to form a total derivative, or that some other can be removed via suitable field redefinitions. In short, we have reached a point where we may wonder where did our recipe fail, what can we do to obtain a sensible EFT lagrangian and how to justify it.

As a first step, we need to give a more solid footing to the doubts emerged above. As it frequently happens in physics, the problem here is the mismatch between what we do computations with and what we can actually observe, and the fact that the tools of our description are not perfectly adherent to the quantities that we need to describe. In this case, the issue is with the redundancies inherent to the lagrangian description. Let us make a step back: what we can observe are scattering amplitudes, i.e. elements of the S-matrix $\langle f|S|i\rangle$. To get from the lagrangian to these elements, we need to go through two steps. First, we need to compute

time-ordered correlation functions, which can be done in the path integral formalism as

$$\langle 0|T\{\phi(x_1)\dots\phi(x_n)\}|0\rangle = \frac{1}{Z[J]}\frac{\delta}{\delta J(x_1)}\dots\frac{\delta}{\delta J(x_n)}Z[J]\bigg|_{J=0} , \tag{2.2.1}$$

where $Z[J]$ is the generating functional

$$Z[J] = \int \mathcal{D}\phi \exp\left\{ i\int \mathrm{d}^4x\,\mathcal{L} + i\int \mathrm{d}^4x\, J(x)\phi(x)\right\} , \tag{2.2.2}$$

where ϕ indicates a generic kind of field and possible indices have been omitted. Then, we have to rely on the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula [145] to convert these off-shell correlation functions to on-shell S-matrix elements

$$\langle f|S|i\rangle = \mathcal{Z}_\phi^{-\frac{n}{2}}\left[i\int \mathrm{d}^4x_1\, e^{-ip_1x_1}(\Box_{x_1}+m^2)\right]\dots\left[i\int \mathrm{d}^4x_n\, e^{+ip_nx_n}(\Box_{x_n}+m^2)\right]\times \\ \times\ \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\dots\phi(x_n)\}|0\rangle \ , \tag{2.2.3}$$

where the $-i$ in the exponent applies to initial states, while the $+i$ to final ones. Here $\sqrt{\mathcal{Z}_\phi}$ is the wave-function renormalization factor, defined as the probability amplitude that the field $\phi(x)$ produces the particle ϕ with momentum p when acting on the vacuum

$$\sqrt{\mathcal{Z}_\phi} = \langle p_\phi|\phi|0\rangle \ . \tag{2.2.4}$$

It is then in these two passages that some information from the lagrangian may be lost, so that two different lagrangians produce the same S-matrix elements. Let us indicate with $\langle f|S|i\rangle_{\mathcal{L}}$ a matrix element obtained with the lagrangian $\mathcal{L}$. What we are after is then an equivalence class, i.e. the set of lagrangians defined by

$$I = \{\text{lagrangians built following points 1-5 in Section 2.1}\}\,/\sim , \tag{2.2.5}$$

where

$$\mathcal{L}_1\sim\mathcal{L}_2 \iff \langle f|S|i\rangle_{\mathcal{L}_1} = \langle f|S|i\rangle_{\mathcal{L}_2} \quad \forall\,|f\rangle\,,|i\rangle \ . \tag{2.2.6}$$

We are used to this kind of reasoning e.g. from the theory of gauge fields, where gauge symmetry manifests itself as a redundancy of the lagrangian. To account for this redundancy, we need to treat the lagrangian with a gauge-fixing term. In the end, the physical results have to be independent on the choice of this term, so that we actually work with the equivalence class of the gauge-fixed lagrangians.

At this point, we need to identify the possible sources of redundancies relevant to our construction, in order to define a quotient operation. These are of two kinds: *integration by parts (IBP)* and *field redefinitions*. Under the latter category we also include *equations of motion (EOM)* redundancies, as we will explain.

Integration by parts Consider two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ of dimension d and such that there exists an operator $\mathcal{O}_3^\mu$ of dimension $d-1$ in the vector representation of the Lorenz group and fulfilling

$$\mathcal{O}_1 = \mathcal{O}_2 + \partial_\mu \mathcal{O}_3^\mu \ . \tag{2.2.7}$$

Then, when inserted in the action, this operator will give a contribution that looks like:

$$S \supset \int \mathrm{d}^4x \, \mathcal{O}_1 = \int \mathrm{d}^4x \, \mathcal{O}_2 + \int \mathrm{d}^4x \, \partial_\mu \mathcal{O}_3^\mu \ . \tag{2.2.8}$$

The term $\int \mathrm{d}^4x \, \partial_\mu \mathcal{O}_3^\mu$ is the integral of a total derivative and thus can give no contribution at any order in perturbation theory. Intuitively, this can be seen as the Feynman rule for $\partial_\mu \mathcal{O}_3^\mu$ would feature a sum of the incoming momenta minus the outgoing ones, always adding up to zero because of momentum conservation. Thus, if two lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ only differ because they contain $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, they will produce the same S-matrix elements and belong to the same equivalence class.

Field redefinitions and EOMs Consider a perturbative field redefinition, i.e. a redefinition of the form [146, 147]

$$\phi = F[\phi'] = \phi' + \lambda G[\phi'] \ , \tag{2.2.9}$$

where $G[\phi']$ is local, i.e. a polynomial in ϕ' and its derivatives, $G[\phi'] = \mathcal{O}\big(\phi'^2\big)$, and λ is some small parameter. Actually, to keep the correct dimensions, we can think of λ as Λ^{-1} if ϕ is bosonic, and as $\Lambda^{-3/2}$ if fermionic. Then, the lagrangian becomes

$$\mathcal{L}[\phi] = \mathcal{L}\left[F[\phi']\right] \equiv \mathcal{L}'[\phi'] \ . \tag{2.2.10}$$

We can then compute the correlation function obtained with ϕ' and $\mathcal{L}'$:

$$\begin{aligned} Z'[J] = \int \mathcal{D}\phi' \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}'[\phi'] + i \int \mathrm{d}^4x \, J(x)\phi'(x) \right\} = \\ = \int \mathcal{D}\phi \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}'[\phi] + i \int \mathrm{d}^4x \, J(x)\phi(x) \right\} \ , \end{aligned} \tag{2.2.11}$$

where we used that ϕ' is an integration variable and relabled $\phi' \to \phi$. On the other hand, the original generating functional $Z[J]$ becomes

$$Z[J] = \int \mathcal{D}\phi' \det\left(\frac{\delta F}{\delta \phi'}\right) \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}'[\phi'] + i \int \mathrm{d}^4x \, J(x) F[\phi'] \right\} \ , \tag{2.2.12}$$

where $\det\left(\frac{\delta F}{\delta \phi'}\right)$ is the Jacobian of the the transformation. With the usual trick, we can exponentiate it by introducing ghost fields $\bar{c}, c$:

$$\det\left(\frac{\delta F[\phi'(y)]}{\delta \phi'(x)}\right) = \det\left(\delta(x-y) + \lambda \frac{\delta G[\phi'(y)]}{\delta \phi'(x)}\right) = \int \mathcal{D}c\mathcal{D}\bar{c} \exp\left\{ i \int \bar{c}\left(1 + \lambda \frac{\delta G[\phi']}{\delta \phi'}\right) c \right\} . \tag{2.2.13}$$

Since $G[\phi'] = \mathcal{O}\left(\phi'^2\right)$, the functional derivative will contain at least a term linear in ϕ, so that $\lambda \bar{c} \frac{\delta G[\phi']}{\delta \phi'} c$ only contains ghost interactions with ϕ. As a consequence, the ghost propagator is given only by the $\bar{c}c$ term, i.e. their propagator is the identity. Then, every loop containing them will vanish in dimensional regularization (dim-reg) [148]. For this reason, in dim-reg, the Jacobian of a transformation like Eq. (2.2.9) is equal to the identity and the ghosts can be ignored [147]. The change in the source term, while relevant for off-shell quantities, has no impact on the S-matrix. For its computation, it only matters that the fields we deal with create the relevant particle from the vacuum, i.e. that

$$\langle p_\phi | \phi | 0 \rangle \neq 0 \qquad \text{and} \qquad \langle p_\phi | \phi' | 0 \rangle \neq 0 \ . \tag{2.2.14}$$

For a perturbative field redefinition like Eq. (2.2.9) this is the case. All operators respecting Eq. (2.2.14), moreover, will produce two point functions, i.e. propagators, with poles at points corresponding to the same mass, thus giving the same contribution to the S-matrix element in the LSZ formula.

Using what we proved here, we can clarify what we mean by eliminating equations of motion redundancies. In general, we wish to remove operators of the form $\mathcal{O} = \tilde{\mathcal{O}} E[X]$, where $E[X]$ is the term proportional to EOM, i.e.

$$E[X] = \begin{cases} E[\phi] = D^2\phi & \text{for scalars,} \\ E[\psi] = \not{D}\psi & \text{for fermions,} \\ E[F] = D^\mu F^a_{\mu\nu} & \text{for field strength tensors,} \end{cases} \tag{2.2.15}$$

with a some group index if $F^a_{\mu\nu}$ corresponds to some non-abelian group, while $\tilde{\mathcal{O}}$ is a generic combination of the fields in the lagrangian and their derivatives so that combined with $E[X]$ produces a singlet of the Lorentz, gauge and global symmetries. Operators of this kind can always be removed via a perturbative field redefinition. Here, we only show an example, which can be easily generalized. A complete proof can be found e.g. in Ref. [147]. Consider the lagrangian [146]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \frac{c_1}{\Lambda^2}\phi^3\partial^2\phi + \frac{c_2}{\Lambda^2}\phi^6 + \mathcal{O}\left(\frac{1}{\Lambda^4}\right) , \tag{2.2.16}$$

from which we would like to remove the $\phi^3\partial^2\phi$ term. We can perform the perturbative redefi-

nition

$$\phi \to \phi + \frac{c_1}{\Lambda^2}\phi^3 \ . \tag{2.2.17}$$

Then, the kinetic term becomes

$$\partial_\mu\phi\partial^\mu\phi \to \partial_\mu\phi\partial^\mu\phi + \partial_\mu\left(\frac{c_1}{\Lambda^2}\phi^3\right)\partial^\mu\phi + \frac{1}{\Lambda^4} = \tag{2.2.18}$$

$$= \partial_\mu\phi\partial^\mu\phi - \frac{c_1}{\Lambda^2}\phi^3\partial^2\phi + \frac{1}{\Lambda^4} \ , \tag{2.2.19}$$

up to a total derivative which we removed. The additional term exactly cancels the $\phi^3\partial^2\phi$ we wanted to remove, and the new lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \left[\frac{1}{4!}\lambda + \frac{c_1}{\Lambda^2}m^2\right]\phi^4 + \left[\frac{c_2}{\Lambda^2} - \frac{c_1\lambda}{3!\Lambda^2}\right]\phi^6 + \mathcal{O}\left(\frac{1}{\Lambda^4}\right) \ , \tag{2.2.20}$$

Since the operators coefficients are arbitrary and only fixed by measurements, we can absorb the additional terms by a redefinition of λ and c_2. The two lagrangians in Eq. (2.2.16) and (2.2.20) produce the same S-matrix, so they belong to the same equivalence class.

The two cases above represent the only two possibilities where operators can be removed because of their redundancies. Thus, we come down to a rather straightforward addition to our recipe

7. Remove operators that are equal to others up to total derivatives (IBP) and remove operators proportional to equations of motion of the fields (EOM).

2.2.2 Invariant ring and the Hilbert series

For the time being, let us leave the issue of IBP and EOMs aside and focus first on the issue of characterizing invariants of a given symmetry, be it the Lorentz group or any additional gauge and global symmetries of the theory. This problem, in its more generic incarnation, is the main focus of the so called Plethystic Program [149–153] (see also Refs. [154, 155] and references therein, and Ref. [156] for other recent techniques to compute the Hilbert series). This has been the attempt to apply tools which originate from the study of polynomial rings to physics, and in particular where representation and group theory come into play. The most remarkable success of this program has perhaps been the addition of the Hilbert Series and its Plethystic Logarithm to the theoretical physicist's toolbox. In seeing how these tools apply to our problem, we will adopt the logic followed in Refs. [154, 155]. Concretely, given a set of parameters $\vec{x}$ and a symmetry group G that acts on $\vec{x}$ as some representation, i.e $\forall g \in G$, $\exists R(g)$ such that $\vec{x} \to R(g)\vec{x}$, one can define invariants $I(\vec{x})$ as the quantities that obey

$$I(\vec{x}) = I(R(g)\vec{x}) \ . \tag{2.2.21}$$

As the sum and products of invariants still form an invariant, from an algebraic point of view we say they form a *ring*. Within this ring, one can define different notions of dependence within the invariants themselves. First of all, a set of invariants can be *linearly dependent* if one can find a linear combination of them with non-zero coefficients which gives exactly zero. More generally, one can define an invariant I' to be *polynomially dependent* on the set of invariants $\{I_1, I_2, \ldots, I_m\}$ if it can be expressed as a polynomial of them, i.e.

$$I' = P'(I_1, I_2, \ldots, I_m) \ . \tag{2.2.22}$$

The polinomially *independent* invariants of the ring are called generators, and it can be shown that, at least for all the groups relevant to physics, their number is finite [157, 158]. By construction, no relation like Eq. (2.2.22) can exist between the generators. Nevertheless, there could exist a polynomial P such that

$$P(I_1, I_2, \ldots, I_m) = 0 \ . \tag{2.2.23}$$

Relations of this kind are called *syzygies* in the literature, and the invariants that obey a syzygy are *algebraically dependent*. Taking all syzygies into account we can successively remove invariants until we get to the set of algebraically independent ones. To recap, the chain of implication in the three distinct definitions of independence we gave are

$$\text{algebraic independence} \Rightarrow \text{polynomial independence} \Rightarrow \text{linear independence} \ , \tag{2.2.24}$$

while the reversed arrows are not true. In this setting, the Hilbert series is used as a helping hand in finding what independent invariants, in all three senses, look like [159], although usually some additional steps are needed to construct them explicitly. Even though their link to our lagrangian problem is yet not clear, we will see the connection in a moment. First of all, we need to distinguish yet another class of invariants, i.e. those which are linearly independent. The Hilbert Series is defined as a generating function for the linearly-independent invariants:

$$\mathcal{H}(q) = \sum_{k=1}^{\infty} c_{\mathbf{1}}^{(k)} q^k \ , \tag{2.2.25}$$

where $c_{\mathbf{1}}^{(0)} = 1$. $c_{\mathbf{1}}^{(k)}$ denotes precisely the number of linearly-independent invariants at dimension k, and q is an arbitrary spurionic variable satisfying $|q| < 1$, and represents a placeholder for the building blocks of the invariants. Let us make an example. Consider a theory with a coupling m transforming under a $U(1)$ symmetry as

$$m \to e^{i\phi_m} m \ . \tag{2.2.26}$$

Then the basic invariant is obviously $I = mm^*$, which has dimension 2, and all the invariants of this theory will have the form I^n. Hilbert series will thus have the form

$$\mathcal{H}(q) = 1 + q^2 + q^4 + \cdots = \frac{1}{1-q^2} \ , \tag{2.2.27}$$

where q^2 corresponds to I, q^4 to I^2, and so on. It can be shown that, in the general case of a semi-simple Lie algebra, the Hilbert series has the form

$$\mathcal{H}(q) = \frac{N(q)}{D(q)} \ . \tag{2.2.28}$$

The numerator $N(q)$ is a polynomial of degree d_N with non-negative coefficients and with the property of being palindromic, i.e

$$N(q) = 1 + c_1 q + c_2 q^2 + \cdots + c_{d_N-1} q^{d_N-1} + q^{d_N}, \tag{2.2.29}$$

with $c_i = c_{d_N-i}$. The denominator takes the form

$$D(q) = \prod_{r=1}^{p} (1 - q^{d_r}) \ , \tag{2.2.30}$$

and is thus of degree $d_D = \sum_r d_r$. The number of factors is equal to the number of parameters, i.e of physical observables, and coincides with the number of algebraically independent invariants. Moreover, the denominator provides information on what the algebraically independent invariants look like: a factor $(1-q^{d_r})^l$ corresponds to l algebraically independent invariants of degree d_r.

To recap, so far we learned that:

- the Hilbert series will in general look like

$$\mathcal{H}(q) = \frac{1 + c_1 q + c_2 q^2 + \cdots + c_{d_N-1} q^{d_N-1} + q^{d_N}}{\prod_{r=1}^{p} (1 - q^{d_r})} \qquad \text{with} \quad c_i = c_{d_N-i} \ , \tag{2.2.31}$$

- each factor $(1-q^{d_r})^l$ in the denominator corresponds to l *algebraically* independent invariants
- each factor in the Taylor expansion of Eq. (2.2.31) corresponds to *linearly* independent invariants.

In the previous example of Eq. (2.2.27), only one factor is present, corresponding to a single basic (and algebraically independent) invariant mm^*. Indeed, we start with a complex variable m, and we can remove its phase using the $U(1)$, bringing the observables down to 1. On the other hand, each q^{2k} term in the expansion corresponds to a linearly independent $(mm^*)^k$. If

we enlarged our toy-model to have two parameters, m_1 and m_2, transforming under the $U(1)$ symmetry as

$$m_1 \to e^{i\phi_1} m_1 \qquad\qquad m_2 \to e^{i\phi_2} m_2 \ , \tag{2.2.32}$$

we can build an example of the so called *multi-graded* Hilbert series by assigning different spurions to m_1 and m_2, say q_1 and q_2. The invariants in this case are built as all possible products of all possible powers of $I_{1,2} \equiv m_{1,2} m_{1,2}^*$, which means that the multi-graded Hilbert series is

$$h(q_1, q_2) = (1 + q_1^2 + q_1^4 + \dots)(1 + q_2^2 + q_2^4 + \dots) = \frac{1}{(1 - q_1^2)(1 - q_2^2)} \ . \tag{2.2.33}$$

The multi-graded Hilbert series can give more information about the structure of the invariants, but does not have in general the properties for the numerator and denominator cited for its ungraded version. The latter can here be easily obtained by setting $q_1 = q_2 = q$, i.e $H(q) = h(q, q)$. In the examples we showed until now, the numerator has always taken the trivial form $N(q) = 1$, and the set of generators coincided with the algebraically independent invariants. When this happens, the invariant ring is said to be *free*. However, this turns out not to be always the case for more complicated groups and representations.

As one would expect, the computation we could perform straightforwardly by hand in the simple cases above quickly becomes unfeasible when larger groups are involved. Thus a general formula to compute the Hilbert series is called for. The solution is provided by the so called Molien–Weyl formula, which is the focus of the next section.

2.2.3 The Molien–Weyl formula

The Molien–Weyl formula [160] allows to compute the Hilbert Series of a set of operators starting from the character χ of the representations they belong to. Because of its central importance in the following, a careful introduction of the character is in order. This is a rather central feature of a group G, carrying important informations on the structure of the group itself. In a representation $\mathbf{r}$, the character is defined as the trace of a group element g in $\mathbf{r}$, i. e. $\chi_{\mathbf{r}} \equiv \mathrm{Tr}_{\mathbf{r}}(g)$, $g \in G$ [161]. In a connected Lie Group, any g can be conjugated into its maximal torus, that is the maximal compact, connected, abelian subgroup of G, $T = U(1)^r$, with $r = \mathrm{rank}(G)$. This means that $\forall g$, $\exists h$ s.t. $h^{-1} g h \in T$. For example, the maximal torus of $SO(3)$ is just $U(1) \cong SO(2)$, which can be taken, for instance, to be the rotations around the z-axis. This is just the familiar idea that any 3d-rotation can be performed by first rotating the z-axis with a matrix $h \in SO(3)$ to make it coincide with the axis of rotation, then performing the rotation around z, and finally rotating the axis back with h^{-1}. This property is expressed in mathematical language by saying that the character is a *class function*. This means the

following: we can first define an equivalence relation by conjugation via

$$g_1, g_2 \in G, \quad g_1 \sim g_2 \text{ iff } \exists h \in G \text{ s.t. } h^{-1} g_2 h = g_1 \ . \tag{2.2.34}$$

Then, the value of $\chi_{\mathbf{r}}$ is constant on a given conjugacy class, and, in particular, it is sufficient to evaluate it on the maximal torus T. A generic matrix representing an element $t \in T$ will have the form

$$t = \begin{pmatrix} \begin{matrix} \cos(\theta_1) & \sin(\theta_1) \\ -\sin(\theta_1) & \cos(\theta_1) \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} \cos(\theta_2) & \sin(\theta_2) \\ -\sin(\theta_2) & \cos(\theta_2) \end{matrix} & 0 \\ 0 & 0 & \ddots \end{pmatrix} , \tag{2.2.35}$$

with as many block matrices as the rank r of the group G, and an additional 1 in the (n, n), with n the dimension of the representation $\mathbf{r}$, if n is odd. Further, we can always diagonalize these matrices to

$$t^D = \text{diag}\left\{e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_r}, e^{-i\theta_r}\right\} \equiv \text{diag}\left\{x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}\right\} \ , \tag{2.2.36}$$

(again, with possibly a 1 eigenvalue at the end), where we defined $x_i \equiv e^{i\theta_i}$. So the character depends on the eigenvalues x_i defining the position on the maximal torus.

All the machinery we can build is centered around three properties of the character. Given two representations $\mathbf{r}_1$ and $\mathbf{r}_2$ of G, we have the following three properties

1. character orthogonality:

$$\int [\mathrm{d}\mu_G(x)] \chi^*_{\mathbf{r}_1}(x) \chi_{\mathbf{r}_2}(x) = \delta_{\mathbf{r}_1 \mathbf{r}_2} \ , \tag{2.2.37}$$

   where $\mathrm{d}\mu_G$ is the Haar measure of the group [162], and we explicitly indicated that both the character and the measure depend on the eigenvalues x_i, here collectively labeled x,

2. character of a tensor product of representations: $\chi_{\mathbf{r}_1 \otimes \mathbf{r}_2}(x) = \chi_{\mathbf{r}_1}(x) \chi_{\mathbf{r}_2}(x)$,

3. character of a direct sum of representations: $\chi_{\mathbf{r}_1 \oplus \mathbf{r}_2}(x) = \chi_{\mathbf{r}_1}(x) + \chi_{\mathbf{r}_2}(x)$.

Consider a group G and an object M transforming in some representation $\mathbf{r}$. We want to build all possible invariants composed of a fixed number n of the object M. What we do, then, is take n tensor products of M with itself, i.e. products where all the indices are left free, decompose this product into irreducible representations (irreps), and then pick the singlets from the result. In other words

$$M \to \underbrace{M \otimes M \otimes \cdots \otimes M}_{n\text{-times}} \Rightarrow \text{Sym}^n(\mathbf{r}) = c^{(n)}_{\mathbf{r}_1} \mathbf{r}_1 \oplus \cdots \oplus c^{(n)}_{\mathbf{1}} \mathbf{1} \tag{2.2.38}$$

where the last equality sketches the decomposition in irreps. Since we are always using the same building block M, we actually need to take only the symmetric part of the tensor product, as we indicated explicitly.

What we are after, ideally, would be a way to explicitly understand what are all the possible ways of contracting the indices as to form a singlet. The next best thing, however, and what we will manage to obtain, is a way to extract the number of singlets appearing in the decomposition, $c_{\mathbf{1}}^{(n)}$. Given the character's properties, we can use it to reproduce Eq. (2.2.38) as

$$M \to \underbrace{M \otimes M \otimes \cdots \otimes M}_{n\text{-times}} \Rightarrow \chi_{\mathbf{r}} \to \chi_{\mathrm{Sym}^n(\mathbf{r})} = \chi_{\left[c_{\mathbf{r}_1}^{(n)}\mathbf{r}_1 \oplus \cdots \oplus c_{\mathbf{1}}^{(n)}\mathbf{1}\right]} = c_{\mathbf{r}_1}^{(n)}\chi_{\mathbf{r}_1} + \ldots c_{\mathbf{1}}^{(n)}\chi_{\mathbf{1}} \ . \quad (2.2.39)$$

We see that we need a way to compute not just the character of the product of representations, but specifically that of the *fully symmetric* product. To this end, let us define a convenient quantity, the *Plethystic Exponential*, as the generating functional of these objects

$$\mathrm{PE}[q; x; \mathbf{r}] = \sum_{n=0}^{\infty} q^n \chi_{\mathrm{Sym}^n(\mathbf{r})}(x) \ , \quad (2.2.40)$$

so that

$$\chi_{\mathrm{Sym}^n(\mathbf{r})}(x) = \left.\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}q^n}\mathrm{PE}[q; x; \mathbf{r}]\right|_{q=0} \ . \quad (2.2.41)$$

It turns out that $\mathrm{PE}[q; x; \mathbf{r}]$ can be computed as

$$\mathrm{PE}[q; x; \mathbf{r}] = \frac{1}{\det(\mathbb{1} - q g_{\mathbf{r}})} = \exp\left(\sum_{m=1}^{\infty} \frac{q^m}{m} \mathrm{Tr}(g_{\mathbf{r}}^m)\right) = \exp\left(\sum_{m=1}^{\infty} \frac{q^m}{m} \chi_{\mathbf{r}}(x^m)\right) \ , \quad (2.2.42)$$

where $g_{\mathbf{r}}$ is a matrix in the representation and the second equality follows from $\log(\det(A)) = \mathrm{tr}(\log(A))\ \forall A \in \mathbb{C}_{l\times l}, \forall l$. The third equality follows from the fact that $\mathrm{Tr}(g_{\mathbf{r}}^m)$ can be computed after diagonalizing $g_{\mathbf{r}}$ into $t_{\mathbf{r}}^D$, so it amounts to just taking the trace of such diagonal matrix with eigenvalues to the power m.

With this result, we can extract the number of singlets at each mass dimension by projecting them out of the expansion in Eq. (2.2.39). To this end, we just need to use the orthogonality property and take the scalar product of this expression with the character of a singlet, $\chi_{\mathbf{1}} = 1$,

$$c_{\mathbf{1}}^{(n)} = \int [\mathrm{d}\mu_G(x)] \chi_{\mathbf{1}}^*(x) \cdot \chi_{\mathrm{Sym}^n(\mathbf{r})}(x) = \left.\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}q^n}\int [\mathrm{d}\mu_G(x)]\mathrm{PE}[q; x; \mathbf{r}]\right|_{q=0} \ . \quad (2.2.43)$$

From the definition of Hilbert Series we gave in Eq.(2.2.25), we also have

$$c_{\mathbf{1}}^{(n)} = \left.\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}q^n}\mathcal{H}(q)\right|_{q=0} \ . \quad (2.2.44)$$

By comparison, we arrive at the Molien–Weyl formula

$$\mathcal{H}(q) = \int [\mathrm{d}\mu_G(x)] \, \frac{1}{\det(\mathbb{1} - q g_{\mathbf{r}})} = \int [\mathrm{d}\mu_G(x)] \, \mathrm{PE}[q; x; \mathbf{r}] \ . \tag{2.2.45}$$

* * *

Now we can prove Eq. (2.2.42), that we showed above without proof. This part is not necessary to understand the rest of the work, and the uninterested reader can safely skip to the next $***$. Call $g_{\mathbf{r}}$ the element of G acting on our building block M and belonging to the representation $\mathbf{r}$ of G with dimension l, so that $g_{\mathbf{r}} \in \mathbb{C}_{l\times l}$. Let us label with v_i, $i = 1, \ldots, l$ its eigenvectors, and with x_i the corresponding eigenvalues

$$g_{\mathbf{r}} v_i = x_i v_i \ . \tag{2.2.46}$$

For all the groups relevant for physics[11], i.e. $U(1)$, $SU(n)$, $SO(n)$, Spin(n), ecc., the v_i's are all independent and can be taken as a basis of $\mathbb{C}^l$. If we take n tensor products of M, they will be acted upon by symmetric n-tensor products of $g_{\mathbf{r}}$. For $n = 2$, for example, this is $g_{\mathbf{r}} \otimes_{\mathrm{Sym}} g_{\mathbf{r}}$. The eigenvectors of this product can be taken to be

$$v_i \otimes_{\mathrm{Sym}} v_j \equiv \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i) \ , \tag{2.2.47}$$

so that

$$(g_{\mathbf{r}} \otimes_{\mathrm{Sym}} g_{\mathbf{r}}) \, (v_i \otimes_{\mathrm{Sym}} v_j) = x_i x_j \, (v_i \otimes_{\mathrm{Sym}} v_j) \ . \tag{2.2.48}$$

Then, since the character can just be computed as sum of eigenvalues, we find it to be

$$\chi_{\mathrm{Sym}^2(g_{\mathbf{r}})}(x) = \mathrm{Tr} \, (g_{\mathbf{r}} \otimes_{\mathrm{Sym}} g_{\mathbf{r}}) = \sum_{i=1}^{l} \sum_{j=1}^{l} x_i x_j \ . \tag{2.2.49}$$

We can rewrite this as

$$\chi_{\mathrm{Sym}^2(g_{\mathbf{r}})}(x) = \sum_{a_1 + a_2 + \cdots + a_l = 2} x_1^{a_1} x_2^{a_2} \ldots x_l^{a_l} \ , \tag{2.2.50}$$

i.e. as the sum of all possible monomials of degree 2 formed with the eigenvalues. This last rewriting lets us easily generalize to n tensor products

$$\chi_{\mathrm{Sym}^n(g_{\mathbf{r}})}(x) = \sum_{a_1 + a_2 + \cdots + a_l = n} x_1^{a_1} x_2^{a_2} \ldots x_l^{a_l} \ . \tag{2.2.51}$$

[11] more specifically, this holds for the so called reductive groups, i.e. those whose every representation is completely reducible, see Ref. [158] for more details.

Plugging this into Eq. (2.2.40), we get

$$\mathrm{PE}[q;x;\mathbf{r}] = \sum_{n=0}^{\infty} \sum_{a_1+a_2+\cdots+a_l=n} q^n x_1^{a_1} x_2^{a_2} \dots x_l^{a_l} = \sum_{a_1,a_2,\dots,a_l} (qx_1)^{a_1}(qx_2)^{a_2}\dots(qx_l)^{a_l} = \\ = \prod_{i=1}^{l} \frac{1}{1-qx^i} = \frac{1}{\det(\mathbb{1} - qt^D_{\mathbf{r}}(x))} \ , \tag{2.2.52}$$

where $t^D_{\mathbf{r}} = \mathrm{diag}(x_1, \dots, x_n)$. Since the determinant, too, is a class function,

$$\frac{1}{\det(\mathbb{1} - qt^D_{\mathbf{r}}(x))} = \frac{1}{\det(\mathbb{1} - qg_{\mathbf{r}})} \ , \tag{2.2.53}$$

since $g_{\mathbf{r}}$ can be conjugated into $t^D_{\mathbf{r}}$ by assumption, and, finally

$$\mathrm{PE}[q;x;\mathbf{r}] = \frac{1}{\det(\mathbb{1} - qg_{\mathbf{r}})} = \exp\left(\sum_{m=1}^{\infty} \frac{q^m}{m} \mathrm{Tr}(g^m_{\mathbf{r}})\right) = \exp\left(\sum_{m=1}^{\infty} \frac{q^m}{m} \chi_{\mathbf{r}}(x^m)\right) \ . \tag{2.2.54}$$

Example Let us make a simple example using $SO(3)$, the group of rotations in three dimensions, and its fundamental vector representation, $\mathbf{r} = \mathbf{3}$. $SO(3)$ has rank 1, meaning that its maximal torus in this representation has the form

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

which can be further diagonalized on the complex field to

$$t^D_{\mathbf{3}} = \mathrm{diag}(e^{i\theta}, e^{-i\theta}, 1) \ . \tag{2.2.55}$$

Let us call $b = \{v_1, v_2, v_3\}$ the basis in which the matrix is in the form (2.2.55), and $x \equiv e^{i\theta}$. We can build a representation of $\mathrm{Sym}^2(\mathbf{3})$ by defining its action on the basis of the symmetric subspace spanned by:

$$b \otimes_{\mathrm{Sym}} b = \Bigg\{ v_1 \otimes v_1, v_2 \otimes v_2, v_3 \otimes v_3, \frac{v_1 \otimes v_2 + v_2 \otimes v_1}{2}, \\ \frac{v_1 \otimes v_3 + v_3 \otimes v_1}{2}, \frac{v_2 \otimes v_3 + v_3 \otimes v_2}{2} \Bigg\} \ . \tag{2.2.56}$$

On the subspace generated by the base in (2.2.56) ($g^D_{\mathbf{3}} \otimes g^D_{\mathbf{3}}$) is diagonal and takes the form:

$$\mathrm{diag}(x^2, x^{-2}, 1, 1, x, x^{-1}) \ , \tag{2.2.57}$$

whose trace is

$$\chi_{\mathrm{Sym}^2(\mathbf{3})}(x) = x^2 + x^{-2} + 2 + x + x^{-1} \tag{2.2.58}$$

Another way to get the same result, which is closer to the usual notion of irreps decomposition, is the following. We still consider the product of two fundamental **3** representations. We know that we can decompose this product as:

$$\mathbf{3}\otimes\mathbf{3}=\underbrace{\mathbf{5}\oplus\mathbf{1}}_{\text{symm}}\oplus\underbrace{\mathbf{3}}_{\text{antisymm}}\ .$$

If we want to compute the character of just the symmetric part of this tensor product we need to sum the characters from the **5** and **1** of $SO(3)$. The character of **5** can be obtained by looking at what the torus matrix looks like in this representation. With the familiar language of quantum mechanics, this is just a spin-2 rotation around the z-axis, $e^{i\theta\hat{J}_z}$. Its trace is just the sum of all its possible eigenvalues

$$\chi_{\mathbf{5}}(x)=\sum_{\text{all possible }J_z}e^{i\theta J_z}=\sum_{J_z=-2}^{2}e^{i\theta J_z}=x^2+x+1+x^{-1}+x^{-2}\ .$$

Obviously $\chi_{\mathbf{1}}=1$, so we get again

$$\chi_{\mathrm{Sym}^2(\mathbf{3})}(x)=x^2+x+2+x^{-1}+x^{-2}\ . \tag{2.2.59}$$

This is the same as the trace from (2.2.58), and both are to be confronted with the expression coming from (2.2.41). Expanding the exponential in the definition of the PE and retaining only the term in q^2:

$$\mathrm{PE}\left[q;x;\mathbf{3}\right]\supset\frac{1}{2}(q\,\mathrm{Tr}\big(g^D_{\mathbf{3}}(x)\big))^2+\frac{q^2}{2}\,\mathrm{Tr}\big((g^D_{\mathbf{3}}(x))^2\big)=q^2(x^2+x+2+x^{-1}+x^{-2})\ ,$$

as expected.

* * *

We have now arrived at the so called Molien–Weyl formula, Eq. (2.2.45). If G is connected, by using the conjugation invariance of the character, we can reduce the integration to an integration over the maximal torus, which is just the direct product of r_0 copies of the S^1 unit circle (see e.g. Ref. [163]). Thus, the integral is reduced to the computation of residues inside these circles.

The generalization to a multi-graded Hilbert series is straightforward:

$$h(q_1,\ldots,q_n)=\int\left[\mathrm{d}\mu_G(x)\right]\prod_{i=1}^{n}\mathrm{PE}\left[q_i;x;;\mathbf{r}_i\right]\ . \tag{2.2.60}$$

As we will se, an important role is also played by the inverse of the PE, quite fittingly called *plethystic logarithm* (PL) and defined so that

$$f(x_1,\ldots,x_n)=\mathrm{PE}\left[g(x_1,\ldots,x_n)\right]\Leftrightarrow g(x_1,\ldots,x_n)=\mathrm{PL}\left[f(x_1,\ldots,x_n)\right]\ . \tag{2.2.61}$$

It can be proved that

$$\mathrm{PL}\left[f(x_1,\ldots,x_n)\right] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln\left[f(x_1^k,\ldots,x_n^k)\right] \ . \tag{2.2.62}$$

The introduction of the PL allows us to gather information on the notion of independence we have so far left out, i.e. *polynomial* independence. Indeed, the PL of a Hilbert series is a polynomial whose leading positive terms correspond to the basic invariants, i.e to the generators, and whose leading negative terms correspond to the syzygies between them. Remarkably, when the invariant ring is free, this polynomial has a finite number of terms. Some complications arise when the groups and the representations that appear become increasingly non-trivial[12] [150,155]. Finally, as it will be used later, it is useful to introduce the generating functional for the fully antisymmetric product of n representations $\mathbf{r}$. With a computation similar to the one leading to the expression for the Plethystic Exponential, this is found to be

$$\sum_{n=0}^{\infty}(-q)^n \chi_{\wedge^n(\mathbf{r})}(x) = \det(\mathbb{1} - q g_{\mathbf{r}}) = \frac{1}{\mathrm{PE}\left[q;x;\mathbf{r}\right]} \ , \tag{2.2.63}$$

with x indicating collectively the eigenvalues of $g_{\mathbf{r}}$.

2.2.4 IBP and EOMs

Up to this point, we have described a way to count singlets from a generic tensor product of representations of some symmetry. However, as we explained in Section 2.2.1, we need to address operator redundancies due to integration by parts and equations of motion. In this section we focus on this final issue, and provide a way to compute the Hilbert Series that counts non-redundant operators at each mass dimension. We call $\mathcal{K}$ the set of non-redundant operators. To be precise, the set $\mathcal{K}$ is composed by a basis of non-redundant operators, since we do not count linear combination of non-redundant operators as separate contributions, although they cannot technically be eliminated via through IBP and EOMs.

To specify this set, we first focus on how to remove EOM redundancies from the character description, and then turn our attention on fixing the IBP ambiguities. Both matters are addressed in Ref. [139]. In particular, two ways to address the IBP issue are presented there. Here, we will describe the one where the problem is addressed using the language of forms.

EOM It is clear that both the EOM and the IBP subtleties are related to representations of the Lorentz group, as all other symmetries are simply added as tensor products to that one. As the character of tensor products is the product of the characters, we can focus on an object transforming trivially under any internal symmetry, solve the issue there, and then simply add

[12] In Ref. [164] the author argues that some of the assumptions in Ref. [155] are imprecise, as the finiteness of the generating set of invariants is a consequence of the group being reductive, which is supposedly not the case for $U(n)$. However, as explained there and clarified by the same authors of Ref. [155] in Ref. [165], the final result is nonetheless correct, as at least the ring of invariants of $U(n)$ is isomorphic to that of $GL(n,\mathbb{C})$, which is itself reductive. See further discussions in Ref. [166].

internal symmetries as additional character factors. Actually, we will limit ourselves to the case of a scalar field. While generalizations to spins > 0 may not be always straightforward, this case already contains most of the conceptual essence of the issue. For a treatment of fermions and vector fields see Ref. [139], while gravitons are studied in Ref. [167]. First of all, since the machinery we built only works for compact groups, we perform a Wick rotation on the time dimension and, instead of working with the Lorentz group $SO(3,1)$, we focus on its compact Euclidean counterpart $SO(4)$. Among other things, we now do not need to worry about the distinction between upper and lower indices.

Let us label our scalar field as ϕ. It transforms trivially under $SO(4)$, so it has character $\chi_{\mathbf{1}} = 1$. By acting on it with a derivative, we get $\partial_\mu\phi$, transforming in the vector representation $\square$, with its corresponding character $\chi_\square$. Now if we want to build all possible operators with these two objects we can pack them into a single vector $R_\phi = \phi \oplus \partial_\mu\phi$, or, with a column notation

$$R_\phi = \begin{pmatrix} \phi \\ \partial_\mu\phi \end{pmatrix} . \tag{2.2.64}$$

Then, we can take all possible (symmetric) tensor products of n copies of R_ϕ, decompose them into irreps of $SO(4)$, and select the singlets. For example, for $n = 2$

$$R_\phi \otimes R_\phi = \{\phi^2, \phi\partial_\mu\phi, \partial_{(\mu_1}\phi\partial_{\mu_2)}\phi\} \tag{2.2.65}$$

where we the round parentheses denote symmetrization of the $\mu_{1,2}$ indices. In our example in Eq. (2.2.65), the singlets we can extract are clearly ϕ^2 and $\partial_\mu\phi\partial_\mu\phi$ (we still have to remove EOMs!). Obviously, we can add as many derivatives as we like to R_ϕ, which in general looks like

$$R_\phi = \begin{pmatrix} \phi \\ \partial_\mu\phi \\ \partial_{(\mu_1}\partial_{\mu_2)}\phi \\ \vdots \end{pmatrix} , \tag{2.2.66}$$

Using the character properties, we can then compute the character of R_ϕ as

$$\chi_{R_\phi}(x) = 1 + \chi_\square(x) + \chi_{\mathrm{Sym}^2(\square)}(x) + \dots , \tag{2.2.67}$$

where we used the character of the symmetric tensor product since derivatives commute. Since we will want to keep track of the number of derivatives, we add a spurionic variable t counting them, as

$$\chi_{R_\phi}(x,t) = 1 + t\chi_\square(x) + t^2\chi_{\mathrm{Sym}^2(\square)}(x) + \dots . \tag{2.2.68}$$

This is nothing but the definition of the Plethystic Exponential, Eq. (2.2.40), so

$$\chi_{R_\phi}(x,t) = \mathrm{PE}[t;x;\Box] = \exp\left(\sum_{m=1}^{\infty}\frac{t^m}{m}\chi_\Box(x^m)\right) . \tag{2.2.69}$$

If we wish to subtract EOMs here, we just need to remove all possible occurrences of $\partial^2\phi$ in R_ϕ. This is simply done by removing all possible traces from the symmetric products $\partial_{\mu_1}\dots\partial_{\mu_n}\phi$, which we indicate as

$$\partial_{\{\mu_1}\dots\partial_{\mu_n\}}\phi , \tag{2.2.70}$$

and we obtain as a building block

$$\tilde{R}_\phi = \begin{pmatrix} \phi \\ \partial_\mu\phi \\ \partial_{\{\mu_1}\partial_{\mu_2\}}\phi \\ \partial_{\{\mu_1}\partial_{\mu_2}\phi\partial_{\mu_3\}}\phi \\ \vdots \end{pmatrix} . \tag{2.2.71}$$

The representation $\tilde{\mathbf{r}}$ of the traceless symmetric product satisfies

$$\begin{cases} \tilde{\mathbf{r}} \oplus \mathbf{r}_{\mathrm{Sym}^{(m-2)}\Box} = \mathbf{r}_{\mathrm{Sym}^m\Box} & \text{if } m \geq 2 \\ \tilde{\mathbf{r}} = \mathbf{r}_{\mathrm{Sym}^m\Box} & \text{if } m = 0,1 , \end{cases} \tag{2.2.72}$$

since a generic trace transforms as a $\mathbf{r}_{\mathrm{Sym}^{(n-2)}\Box}$ representation. This translates, for the characters, in

$$\begin{cases} \chi_{\tilde{\mathbf{r}}}(x,t) = \chi_{\mathbf{r}_{\mathrm{Sym}^m\Box}}(x,t) - \chi_{\mathbf{r}_{\mathrm{Sym}^{(m-2)}\Box}}(x,t) & \text{if } n \geq 2 \\ \chi_{\tilde{\mathbf{r}}}(x,t) = \chi_{\mathbf{r}_{\mathrm{Sym}^m\Box}}(x,t) & \text{if } m = 0,1 . \end{cases} \tag{2.2.73}$$

The analogue of Eq. (2.2.68) then becomes

$$\begin{aligned} \chi_{\tilde{R}_\phi}(x,t) &= 1 + t\chi_\Box(x) + \sum_{m=2}^{\infty}\left(t^m\chi_{\mathbf{r}_{\mathrm{Sym}^m\Box}}(x) - t^m\chi_{\mathbf{r}_{\mathrm{Sym}^{m-2}\Box}}(x)\right) = \\ &= \sum_{m=0}^{\infty} t^m\chi_{\mathbf{r}_{\mathrm{Sym}^m\Box}}(x) - t^2\sum_{m=0}^{\infty} t^m\chi_{\mathbf{r}_{\mathrm{Sym}^m\Box}}(x) = (1-t^2)\mathrm{PE}[t;x;\Box] . \end{aligned} \tag{2.2.74}$$

So, if we use $\tilde{R}_\phi$ and its character $\chi_{\tilde{R}_\phi}(x,t)$ as building blocks, any term we build will be free of terms proportional to EOMs, as we wanted.

IBP To address the IBP redundancies, we need to introduce some of the jargon borrowed from the theory of differential forms [168, 169]. Let us indicate with $\mathcal{O}_p$ a p-form operator,

defined in components as

$$\mathcal{O}_p \equiv \frac{1}{p!}(\mathcal{O}_p)_{\mu_1\dots\mu_p}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\ , \tag{2.2.75}$$

where $\wedge$ is the antisymmetric wedge product. Then, IBP redundancies can be expressed as

$$\mathcal{O}_0^a \sim \mathcal{O}_0^b \quad \text{iff} \quad \exists\mathcal{O}_1^c \quad \text{such that} \quad \mathcal{O}_0^a = \mathcal{O}_0^b + \partial\cdot\mathcal{O}_1^c\ , \tag{2.2.76}$$

where ∂ is an operator that has to play the role of a divergence. To define it more properly, we need to refer to the Hodge duals of $\mathcal{O}_0^{a,b}$. In d dimensions, the Hodge dual of a p-form ω_p is a $(d-p)$-form $*\omega_p$ defined as

$$\begin{aligned}\widetilde{\omega}_{d-p} \equiv *\omega_p = \frac{1}{p!}(\omega_p)_{\mu_1\dots\mu_p} * (\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}) = \\ = \frac{1}{p!(d-p)!}(\omega_p)_{\mu_1\dots\mu_p}\varepsilon^{\mu_1\dots\mu_p}{}_{\mu_{p+1}\dots\mu_d}\,\mathrm{d}x^{\mu_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{\mu_d}\ .\end{aligned} \tag{2.2.77}$$

In particular, the Hodge dual of a 0-form is a d-form. The exterior derivative d of a form ω_p is a $(p+1)$-form defined in components as

$$\mathrm{d}\omega_p = \frac{1}{p!}(\partial_\mu\omega_p)_{\mu_1\dots\mu_p}\,\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\ . \tag{2.2.78}$$

If a 0-form is defined as a divergence of some 1-form, then its Hodge dual, a d form, is *exact*, i.e. it is the exterior derivative of a $d-1$ form

$$\exists\mathcal{O}_1^b \text{ such that } \mathcal{O}_0^a = \partial\cdot\mathcal{O}_1^b \iff \exists\widetilde{\mathcal{O}}_{(d-1)}^b \text{ such that } \widetilde{\mathcal{O}}_d^a = \mathrm{d}\widetilde{\mathcal{O}}_{(d-1)}^b\ . \tag{2.2.79}$$

This means that we can recast the equivalence in Eq. (2.2.76) as one between the two d-forms dual to $\mathcal{O}_0^{a,b}$:

$$\widetilde{\mathcal{O}}_d^a \sim \widetilde{\mathcal{O}}_d^b \quad \text{iff} \quad \exists\mathcal{O}_1^c \quad \text{such that} \quad \widetilde{\mathcal{O}}_d^a = \widetilde{\mathcal{O}}_d^b + \mathrm{d}\widetilde{\mathcal{O}}_{d-1}^c\ , \tag{2.2.80}$$

i.e. two operators are IBP-equivalent if their Hodge duals differ by an exact d-form. We can write everything without passing through the Hodge dual picture if we can find a suitable generalization of the divergence. To this end, let us define the codifferential $\delta \equiv *\mathrm{d}*$. δ takes a p form into a $p-1$ form, and it reduces to the familiar divergence (up to some sign which plays no role here) when applied to 1-forms, as desired. Analogously to the exterior derivative d, the codifferential δ, too, obeys $\delta^2 = 0$. If $\delta\omega_k = 0$, ω_k is said to be *co-closed*. In addition, given a k-form ω_k, if there exists a $(k+1)$-form β_{k+1}, such that $\omega_k = \delta\beta_{k+1}$, then ω_k is said to be *co-exact*. Trivially, every co-exact form is also co-closed, but not vice-versa.

Then, we can reformulate Eq. (2.2.80) in the original picture as

$$\mathcal{O}_0^a \sim \mathcal{O}_0^b \quad \text{iff} \quad \exists\mathcal{O}_1^c \quad \text{such that} \quad \mathcal{O}_0^a = \mathcal{O}_0^b + \delta\mathcal{O}_1^c\ , \tag{2.2.81}$$

Now the basis of operators $\mathcal{K}$ we are after clearly does not include co-exact forms, which would be operators in the same equivalence class as the zero operator according to Eq. (2.2.81). Thus, $\mathcal{K}$ must be a subset of the set of 0-forms that are not co-exact, a set which we can label as $\widetilde{\mathcal{K}}$, $\widetilde{\mathcal{K}} \supset \mathcal{K}$.

Now, imagine there were two operators $\mathcal{O}_0^a, \mathcal{O}_0^b \in \widetilde{\mathcal{K}}$ that are equivalent according to Eq. (2.2.81), so that we can only pick one to put in $\mathcal{K}$. Since we are only interested in a basis of operators, we can always trade $\mathcal{O}_0^a$ and $\mathcal{O}_0^b$ for their combination $\mathcal{O}_0^a \pm \mathcal{O}_0^b$, which generate the same span and thus still work as a basis. However, now $\mathcal{O}_0^a - \mathcal{O}_0^b \sim 0$, so it cannot be in $\widetilde{\mathcal{K}}$, and we arrive at a contradiction. This implies $\widetilde{\mathcal{K}} = \mathcal{K}$. For the counting of operators in $\mathcal{K}$, this means

$$\#(\text{operators}) = \#(\text{0-forms}) - \#(\text{co-exact 0-forms}) \ . \tag{2.2.82}$$

Co-exact 0-form can only be generated by 1-forms that do not vanish when acted upon by δ, i.e. from 1-forms that are not co-closed

$$\#(\text{co-exact 0-forms}) = \#(\text{1-forms}) - \#(\text{co-closed 1-forms}) = \tag{2.2.83}$$

$$= \#(\text{1-forms}) - \#(\text{co-closed but not co-exact 1-forms}) - \#(\text{co-exact 1-forms}) \ . \tag{2.2.84}$$

If we iterate this logic up to d-forms, which is the maximum degree of a form in d-dimensions, we arrive at

$$\#(\text{operators}) = \left\{ \sum_{p=0}^{d} (-1)^p \#(\text{p-forms}) \right\} + \left\{ \sum_{p=1}^{d} (-1)^{p+1} \#(\text{co-closed but not co-exact } p\text{-form}) \right\} \ . \tag{2.2.85}$$

Following the notation of Ref. [139], we can then split the Hilbert Series into two pieces

$$\mathcal{H}(q,t) = \mathcal{H}_0(q,t) + \Delta\mathcal{H}(q,t) \ , \tag{2.2.86}$$

with

$$\mathcal{H}_0(q,t) = \left\{ \sum_{p=0}^{d} (-1)^p \#(\text{p-forms}) \right\} \tag{2.2.87}$$

$$\Delta\mathcal{H}(q,t) = \left\{ \sum_{p=1}^{d} (-1)^{p+1} \#(\text{co-closed but not co-exact } p\text{-form}) \right\} \ . \tag{2.2.88}$$

where q is the spurion variable labeling the field and t is the one used to keep track of derivatives. It can be shown that $\Delta\mathcal{H}(q,t)$ only contains terms with mass-dimension $\leq d$, which are usually not very interesting for EFTs constructions. So we focus on $\mathcal{H}_0(q,t)$.

The discussion lead above boils down to the fact that removing IBP redundancies requires

that we modify the Molien-Weyl formula in Eq. (2.2.45) when dealing with the (Wick-rotated) space-time $SO(d)$. Instead of simply counting singlets, we have to count p-forms, weighted with an appropriate sign. To do this, for fixed p, we need to use the orthogonality properties of the character and compute the integral not with $\chi_1 = 1$, i.e. the character of the singlet, but with the character of a p-form. Since the latter transforms as a fully antisymmetric product of $\square$ representations of $SO(d)$, its character is $\chi_{\wedge^p(\square)}$. Then, for a field with in the representation $\mathbf{r}$ of $SO(d)$ the counting becomes

$$\#(\text{p-forms}) = \int \left[\mathrm{d}\mu_{SO(d)}(x)\right] \chi_{\wedge^p(\square)}(x) \cdot \mathrm{PE}[q; x; \mathbf{r}] \ . \tag{2.2.89}$$

Thus, to compute the full $\mathcal{H}_0(q,t)$, we have to integrate the PE with the factor

$$\sum_{p=0}^{d} t^p (-1)^p \chi_{\wedge^p(\square)}(x) = \frac{1}{\mathrm{PE}[t; x; \square]} \ . \tag{2.2.90}$$

where we used Eq. (2.2.63). So, finally,

$$\mathcal{H}_0(q,t) = \int \left[\mathrm{d}\mu_{SO(d)}(x)\right] \frac{1}{\mathrm{PE}[t; x; \square]} \mathrm{PE}[q; x; \mathbf{r}] \ . \tag{2.2.91}$$

Now, if we wanted to compute the Hilbert Series for a single scalar field we have all the tools to remove both IBP and EOM redundancies, and the expression boils down to

$$\begin{aligned} \mathcal{H}_0(\phi,t) = \int \left[\mathrm{d}\mu_{SO(d)}(x)\right] \frac{1}{\mathrm{PE}[t; x; \square]} \cdot \mathrm{PE}[\phi; x; R_\phi] = \\ = \int \left[\mathrm{d}\mu_{SO(d)}(x)\right] \frac{1}{\mathrm{PE}[t; x; \square]} \exp\left(\sum_{m=1}^{\infty} \frac{\phi^m}{m} \chi_{\tilde{R}_\phi}(x^m, t^m)\right) \ , \end{aligned} \tag{2.2.92}$$

where $\chi_{\tilde{R}_\phi}(x,t)$ is that of Eq. (2.2.74).

2.2.5 Examples and applications

In this section we will show some examples to demonstrate how the Hilbert series can be used to assist in building EFT lagrangians.

Reals scalar field theory

To familiarize ourselves with the objects at play here, we can start by looking in detail at how to compute the plethystic exponential corresponding to the fundamental representation of the (compact) Lorentz group $SO(d)$, $\mathrm{PE}[t; x; \square]$. Let us specialize to $d = 4$ dimensions. $SO(4)$ has

rank 2, so its maximal torus is $SO(2) \otimes SO(2)$, which can be represented with a matrix

$$g_{SO(2)\otimes SO(2)} = \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & \cos(\theta_2) & \sin(\theta_2) \\ 0 & 0 & -\sin(\theta_2) & \cos(\theta_2) \end{pmatrix} . \tag{2.2.93}$$

We can then define $x_{1,2} \equiv e^{i\theta_{1,2}}$, and diagonalize this matrix as

$$t^D_{SO(2)\otimes SO(2)} = \text{diag}\left\{x_1, x_1^{-1}, x_2, x_2^{-1}\right\} . \tag{2.2.94}$$

We can then use the definition of the PE, Eq. (2.2.42), to write

$$\text{PE}[t; x; \square] = \frac{1}{\det\left(\mathbb{1} - t g^D_{L\otimes R}\right)} = \frac{1}{(1-tx_1)(1-t/x_1)(1-tx_2)(1-t/x_2)} . \tag{2.2.95}$$

Another ingredient we need is the Haar measure of the $SO(4)$ group. A formula due to Weyl [162] states that, for class functions like the character, the Haar measure reduces to

$$\int [\mathrm{d}\mu_G(x)] \to \oint \prod_i \frac{\mathrm{d}x_i}{2\pi i x_i} \prod_{\alpha \in \mathrm{rt}^+(G)} (1 - x^\alpha) \tag{2.2.96}$$

where the integration is performed over the variables x_i that span the maximal torus, and the product over α is performed over the positive roots. The notation $\alpha = (\alpha_1, \ldots, \alpha_r)$, $x = (x_1, \ldots, x_r)$, $x^\alpha = x_1^{\alpha_1} \ldots x_r^{\alpha_r}$, with $r \equiv \text{rank}(G)$, is used. $SO(4)$ has two positive roots, $(1, \pm 1)$, so we get

$$\int [\mathrm{d}\mu_{SO(4)}(x)] \to \oint \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{(2\pi i x_1)(2\pi i x_2)} (1 - x_1 x_2)(1 - x_1/x_2) \tag{2.2.97}$$

Now we have all the ingredients to plug into Eq. (2.2.92)

$$\begin{aligned} \mathcal{H}_0(\phi, t) = \oint \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{(2\pi i x_1)(2\pi i x_2)} (1 - x_1 x_2)(1 - x_1/x_2) \left[(1-tx_1)(1-t/x_1)(1-tx_2)(1-t/x_2)\right] \times \\ \times \exp\left(\sum_{m=1}^{\infty} \frac{\phi^m}{m} \frac{1 - t^{2m}}{(1 - t^m x_1^m)(1 - t^m/x_1^m)(1 - t^m x_2^m)(1 - t^m/x_2^m)}\right) \end{aligned} \tag{2.2.98}$$

where we used Eq. (2.2.74) for the definition of $\chi_{\tilde{R}_\phi}(x, t)$. The integration on the torus can be reduced to simply picking the residues of $x_{1,2}$ around the poles contained in the integrand for $|x_{1,2}| < 1$. Unfortunately, extending our construction to the Lorentz group has the side effect that expressions like Eq. (2.2.98) do not seem to have an expression that reduces it to the closed form of Eq. (2.2.31). However, we can still trust its expansion to give us the number of independent invariants at each dimension, which is of tremendous help in building an EFT. Let us see what this procedure outputs here.

As we mentioned, we can trust $\mathcal{H}_0(\phi, t)$ for operators of dimension > 4, otherwise we also

have to account for the contribution from $\Delta\mathcal{H}(\phi,t)$. For example, if we expand Eq. (2.2.98) from dimension 5 to 8 we get

$$\mathcal{H}_0(\phi,t)|_{5\leq d\leq 8} = \phi^5+\phi^6+\phi^7+\phi^8+t^4\phi^4\ . \tag{2.2.99}$$

We see that, up until mass dimension 8, there are no independent operators except the trivial ones one can build as powers of ϕ. At dimension 8, the first non-trivial contraction of derivatives is possible, as signaled by the monomial $t^4\phi^4$ in the expression. It can be easily checked that this factor corresponds to the operator

$$t^4\phi^4 \to \partial_\mu\phi\partial_\mu\phi\partial_\nu\phi\partial_\nu\phi\ . \tag{2.2.100}$$

SMEFT

If we wanted to extend the example of the previous section to treat the case where ϕ also transforms as some representation $\mathbf{r}$ of an internal symmetry group G, we have to change two things. First of all, the integration should now be performed both on the spacetime $SO(4)$ and on the internal group. Moreover, the character for ϕ should also include the factor coming from the character for G, i.e.

$$\int[\mathrm{d}\mu_{SO(4)}(x)] \to \int[\mathrm{d}\mu_{SO(4)}(x)][\mathrm{d}\mu_G(y)] \quad \text{and} \quad \chi_{\tilde{R}_\phi}(x,t)\to\chi_{\tilde{R}_\phi}(x,t)\chi_{\mathbf{r}}(y) \tag{2.2.101}$$

where we indicated with y the eigenvalues of an element of G in the representation $\mathbf{r}$. Working out the details of fermionic fields as well as gauge bosons and gravitons requires a bit of care [163,167]. Most notably, since the HS can only address objects transforming linearly under the groups at hand, it turn out that it is convenient to use the self- and anti self-dual components of the fields strengths to address gauge groups, instead of the gauge fields themselves. In the end, the hard work is well repaid, and for example one can apply the Hilbert Series to the SM gauge groups to obtain the counting of operators at dimension-6 in the SMEFT [139]:

$$\begin{aligned}
&\mathcal{H}_0^{SMEFT}|_{d-6} = \\
&= H^3H^{\dagger\,3} + u^\dagger Q^\dagger HH^{\dagger\,2} + 2Q^2Q^{\dagger\,2} + Q^{\dagger\,3}L^\dagger + Q^3L + 2QQ^\dagger LL^\dagger + L^2L^{\dagger\,2} + uQH^2H^\dagger \\
&+ 2uu^\dagger QQ^\dagger + uu^\dagger LL^\dagger + u^2u^{\dagger\,2} + e^\dagger u^\dagger Q^2 + e^\dagger L^\dagger H^2H^\dagger + 2e^\dagger u^\dagger Q^\dagger L^\dagger + eLHH^{\dagger\,2} + euQ^{\dagger\,2} \\
&+ 2euQL + ee^\dagger QQ^\dagger + ee^\dagger LL^\dagger + ee^\dagger uu^\dagger + e^2e^{\dagger\,2} + d^\dagger Q^\dagger H^2H^\dagger + 2d^\dagger u^\dagger Q^{\dagger\,2} + d^\dagger u^\dagger QL \\
&+ d^\dagger e^\dagger u^{\dagger\,2} + d^\dagger eQ^\dagger L + dQHH^{\dagger\,2} + 2duQ^2 + duQ^\dagger L^\dagger + de^\dagger QL^\dagger + deu^2 + 2dd^\dagger QQ^\dagger + dd^\dagger LL^\dagger \\
&+ 2dd^\dagger uu^\dagger + dd^\dagger ee^\dagger + d^2d^{\dagger\,2} + u^\dagger Q^\dagger H^\dagger G_R + d^\dagger Q^\dagger HG_R + HH^\dagger G_R^2 + G_R^3 + uQHG_L \\
&+ dQH^\dagger G_L + HH^\dagger G_L^2 + G_L^3 + u^\dagger Q^\dagger H^\dagger W_R + e^\dagger L^\dagger HW_R + d^\dagger Q^\dagger HW_R + HH^\dagger W_R^2 + W_R^3 \\
&+ uQHW_L + eLH^\dagger W_L + dQH^\dagger W_L + HH^\dagger W_L^2 + W_L^3 + u^\dagger Q^\dagger H^\dagger B_R + e^\dagger L^\dagger HB_R \\
&+ d^\dagger Q^\dagger HB_R + HH^\dagger B_RW_R + HH^\dagger B_R^2 + uQHB_L + eLH^\dagger B_L + dQH^\dagger B_L + HH^\dagger B_LW_L \\
&+ HH^\dagger B_L^2 + 2QQ^\dagger HH^\dagger t + 2LL^\dagger HH^\dagger t + uu^\dagger HH^\dagger t + ee^\dagger HH^\dagger t + d^\dagger uH^2t + du^\dagger H^{\dagger\,2}t \\
&+ dd^\dagger HH^\dagger t + 2H^2H^{\dagger\,2}t^2\,,
\end{aligned} \tag{2.2.102}$$

where H stands for the Higgs field, u, d and e for the right-handed up and down quark and electron field, Q and L for the left-handed quark and lepton fields, and $G_{R,L}$, $W_{R,L}$ and $B_{R,L}$ for the self-dual and anti self-dual components of the $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$ gauge field strengths. Remarkably, this counting agrees with the classic result of Ref. [70].

Chiral lagrangian*

Ref. [163] provides the details on how to apply the Hilbert series to the coset construction we explained in Section 2.1.4. Specifically, it turns out that we can obtain the HS by using as building blocks the objects $D_\mu \pi_a$ and the covariant derivative $\mathcal{D}_\mu$ defined in Eq. (2.1.25). To ease the notation, let us define $u_{\mu,a} \equiv D_\mu \pi_a$. In short, one finds that the appropriate character to use for $u_{\mu,a}$ is

$$\chi_u(q, x, y) = [(1 - q^2)P(q; x) - 1]\chi_{H,u}(y) \tag{2.2.103}$$

where the first factor indicates the character under the Lorentz group, while $\chi_{H,u}(y)$ that of the unbroken group H. We can then apply the construction to the Chiral Lagrangian. The results for $N_f = 2, 3$ are displayed in Table 2.1 They can be compared e.g. with Table 7 of [171],

# of $\mathcal{D}_\mu$	# of u_μ	# of parity conserving operators		# of parity violating operators	
		$N_f = 2$	$N_f = 3$	$N_f = 2$	$N_f = 3$
p^2	2	1	1	-	-
p^4	4	2	3	-	-
p^6	4	2	3	-	-
	5	-	1	1	4
	6	3	8	-	3
p^8	4	3	5	-	5
	5	-	4	2	12
	6	9	40	2	20
	7	-	22	2	32
	8	4	20	-	10

Table 2.1: number of operators broken down according to the number of covariant derivatives and of $u_{\mu,a}$ for the $N_f = 2, 3$ Chiral lagrangian.

which we reproduce in Table 2.2 for the parity conserving operators. While for $N_f = 2$ we have complete agreement, for $N_f = 3$ we only match the lines with even number of $u_{\mu,a}$'s. Indeed, the authors of [171] only consider in their construction operators built out of an even number of $u_{\mu,a}$, since that is sufficient to get a parity even object. However, we can also consider so called anomalous operators, formed with an odd number of $u_{\mu,a}$ fields and one insertion of the epsilon tensor $\varepsilon^{\mu\nu\rho\sigma}$. For example, we see that at $\mathcal{O}(p^6)$ there exists one operator containing 5 $u_{\mu,a}$ fields. This can be explicitly built and written as

$$\partial_\alpha u_{\mu,a} u_{\nu,b} u_{\rho,c} u_{\sigma,d} u^d_\alpha \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{abc} \ .$$

*This subsection contains some original work of the author, although partly overlapping with [170].

# of $\mathcal{D}_\mu$	# of u_μ	$N_f = 2$	$N_f = 3$
p^2	2	1	1
p^4	4	2	3
p^6	4	2	3
	6	3	8
p^8	4	3	5
	6	9	31
	8	4	20

Table 2.2: number of parity invariant operators broken down according to the number of covariant derivatives and of $u_{\mu,a}$ for the $N_f = 2, 3$ Chiral lagrangian, taken from [171].

U(3) flavor invariants

Another useful application is in the understanding of the flavor structure of the Standard Model. We will treat this issue in greater detail in Chapter 4, but it is worth giving a taste of its power here. The gauge sector of the SM_4 enjoys a $U(3)^5$ global symmetry that acts on the different fermion fields by mixing flavor families with each other with unitary transformations. The Yukawa sector of the SM_4

$$\mathcal{L}_{\text{Yukawa}} = \bar{Q} Y_d d H + \bar{Q} Y_u u \tilde{H} + \bar{L} Y_e e H + \text{ h.c. }, \tag{2.2.104}$$

breaks this symmetry. Let us focus here specifically on the quark sector. The original $U(3)^3$, acting on the u, d and Q fields, takes us from one basis to another, and the $Y_{u,d}$ matrices will have a different form in each of them, while the physics will be independent on these transformations. Then, the usual procedure is to utilize the full $U(3)^3$ to bring $Y_{u,d}$ in a form that only contains physical parameters. The number of these objects turns out to be 10, i.e. the three up- and three down-quark masses, three mixing angles of the CKM matrix, and 1 complex phase responsible for CP violation.

However, the HS provides a way to get this result bypassing the choice of a basis. First of all, promote $Y_{u,d}$ to spurions of $U(3)^3$ in such a way that the group formally leaves $\mathcal{L}_{\text{Yukawa}}$ invariant, i.e. by choosing $Y_{u,d} \in \mathbf{3}_Q \otimes \bar{\mathbf{3}}_{u,d}$. Then, we now know that the physically meaningful quantities contained in the spurions correspond to the algebraically independent invariants of the ring built with $Y_{u,d}$. Thus, it is sufficient to compute the HS for these two objects. The output is [159]:

$$\mathcal{H}(q) = \frac{1 + q^{12}}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)} \ . \tag{2.2.105}$$

This result has the nice shape of Eq. (2.2.31), and we can directly read off of its denominator that the Yukawa matrices indeed contain 10 independent physical quantities.

2.3 Discussion

We have devoted this Chapter to presenting the power of the EFT picture when applied to physical problems. We have tried to delineate their domain of applicability, how they can be fitted to the issue at hand and when one should expect them to fail and be superseded by a more complete UV model. We argued that as long as one is not looking for the final theory of Nature, they represent the way we should understand Quantum Field Theories, and the Standard Model in particular. An extensive portion has been dedicated to the topic of the Hilbert Series, and for a number of reasons: first of all, it is quite remarkable that a construction that relies on quite deep mathematical concepts can find such a concrete application to physical systems. Secondly, because it allows to bypass, to some extent, the hard work that used to be needed to build an EFT lagrangian at a fixed dimension and then check that the obtained operators form indeed a complete and non-redundant basis. Finally, because we believe that it is a branch where a lot of understanding still has to be done. Specific instances have been touched in the main body of the chapter, for example the fact that there seems to be no closed form resembling Eq. (2.2.31) for the Hilbert series when we refine it to remove EOM and IBP redundancies. In addition, there is still no clear understanding of the role of the Plethystic Logarithm, whose interpretation as a counter for generators and syzygies becomes less and less clear the more the groups we deal with get complicated or the number of distinct fields grows. This latter problem will make a reappearance in Chapter 4.

CHAPTER 3

Positivity bounds on EFTs

In the previous Chapter we have explained how the Hilbert Series can be used to help identify the independent non-redundant operators we can build given some field content and symmetries. Once these operators are identified, we said that we can just then build the EFT by adding all of them to the EFT lagrangian, multiplied by arbitrary coefficients, sometimes referred to as *Wilson coefficients*. In this Chapter[1], we will discuss how the arbitrariness of these coefficients can actually be constrained on the basis of quite general assumptions. These constraints go usually under the name of *positivity bounds*. After describing how they emerge, we turn to the study of their application to the Standard Model EFT. Specifically, we will be interested in their compatibility with Minimal Flavor Violation, an ansatz regarding the flavor structure of higher dimensional SMEFT operators containing fermionic fields. The interest of such pursuit lies in its attempt to try and reverse the usual logic that usually accompanies positivity bounds in the literature. Indeed, instead of just carving out the allowed parameter space for the higher dimensional operators (which we will nevertheless do), we will try to use the knowledge obtained on the parameter space to gather information about renormalizable, dimension-4 parameters.

3.1 Positivity bounds

The positivity bounds are obtained by asking whether all possible EFTs we can write down can be consistently completed in the UV. More specifically, they were derived in Ref. [172][2] by requiring that whatever the theory in the UV is, it should still respect causality of signal propagation, i.e. the fact that no superluminal modes are allowed, as well as unitarity, intended in the quantum mechanic sense of conservation of probabilities. As (micro)causality is strictly related to the analyticity of the S-matrix, i.e. the nice behavior the amplitude displays as

[1] This Chapter is mostly a reproduction of the work contained in [68]

[2] Similar ideas had been applied earlier to the Chiral lagrangian, see Refs. [173–176]

a function of the complexified Mandelstam variables [172, 177–185], this latter requirement is assumed, too.

These features are quite weak requirements on a theory. Among others, they are satisfied by any local QFT, and by perturbative string theory, where a definition of a S-matrix is possible. Let us then see how positivity bounds can arise from them, and what they look like [172].

To this end, we will first illustrate the simple example of a single scalar field ϕ obeying a shift symmetry $\phi \to \phi + c$, with c a constant [172, 186]. Then, except for the kinetic term, all renormalizable operators are forbidden, and the first term appears at $\mathcal{O}(\Lambda^{-4})$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{c}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^2 + \mathcal{O}\left(\Lambda^{-6}\right) . \tag{3.1.1}$$

Then, we can consider the amplitude $\mathcal{M}(s,t)$ for a $2 \to 2$ scattering of the states $|\phi\rangle$ excited by ϕ as a function of the Mandelstam variables $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$, where by convention we take all momenta to be outgoing. From $\mathcal{M}(s,t)$, one can define the forward amplitude $\mathcal{A}(s)$ as its $t \to 0$ limit, with the constraint that the initial and final states be the same. We will want to study the properties of $\mathcal{M}(s,t)$, and specifically of its limit $\mathcal{A}(s)$, when we analytically extend the variables s and t to the complex plane. In particular, we know that $\mathcal{A}(s)$ is analytic everywhere in the plane except on the real s axis [184,187]. There, two different types of discontinuities can appear: simple poles, corresponding to on-shell tree-level exchange of massive intermediate states, and branch cuts, coming from logarithmic terms arising at loop level from multiparticle on-shell production.

We can always expand an analytic function $\mathcal{A}(s)$ as $\mathcal{A}(s) = \sum_{n=0}^{\infty} \lambda_n s^n$. Using Cauchy's formula, we can express the coefficients in the expansions via an integral along a closed circle $\mathcal{C}$ around the origin

$$\lambda_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\mathrm{d}s}{s^{n+1}}\mathcal{A}(s) . \tag{3.1.2}$$

Here, we take $\mathcal{C}$ to be small enough as to encircle the origin without enclosing any discontinuities. Their exact location is determined by the UV completion of the theory. For example, if a particle Φ of mass m can be produced through a trilinear coupling $g\Phi\phi^2$, a pole in $s = \pm m^2$ will appear. If, instead, the lagrangian contains a quartic coupling $g'\Phi^2\phi^2$, a branch cut will be present in the real axis, starting from $s = \pm m^2$ and extending to infinity. Let us assume here for simplicity that the discontinuities, whatever their nature, start at $\pm\bar{s}$ and extend to $\pm\infty$. In the case of exchanged massless particles, which allow the discontinuities to extend down to the origin of the complex plane, a IR mass regulator can be added. After doing the computation, we can send this regulator to 0. One can check that these two operations commute [172] in EFTs that do not include gravity, where some subtleties arise (see e.g. Refs. [188, 189]). Then, we can deform $\mathcal{C}$ to another contour $\mathcal{C}'$ as in Fig. 3.1, i.e. by stretching it to infinity wherever it is not obstructed by the presence of singularities. Since we meet no singularities in this process, the integral in Eq. (3.1.2) is the same whether we evaluate it on $\mathcal{C}$ or $\mathcal{C}'$. A theorem due to Froissart and refined by Martin [180,185] implies that, for large values of s, the amplitude $\mathcal{A}(s)$

is bounded from above by $s^2 \log^2(s)$, so that the integral at infinity vanishes for $n \geq 2$. Thus, the only non-vanishing contributions come from the integral around the discontinuities on the real axis. For $n = 2$, in particular

$$\lambda_2 = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\mathrm{d}s}{s^3} \mathcal{A}(s) = \frac{1}{2\pi i} \left(\int_{-\infty}^{-\bar{s}} + \int_{\bar{s}}^{\infty} \right) \frac{\mathrm{d}s}{s^3} \lim_{\varepsilon \to 0} \left[\mathcal{A}(s + i\varepsilon) - \mathcal{A}(s - i\varepsilon) \right] \ . \tag{3.1.3}$$

Now, we can send $s \to -s$ for the integral in $(-\infty, -\bar{s}]$, and use the $1 \leftrightarrow 3$ symmetry of the amplitude, which in our case implies $\mathcal{A}(s) = \mathcal{A}(-s)$, to obtain

$$\lambda_2 = \frac{1}{i\pi} \int_{\bar{s}}^{\infty} \frac{\mathrm{d}s}{s^3} \lim_{\varepsilon \to 0} \left[\mathcal{A}(s + i\varepsilon) - \mathcal{A}(s - i\varepsilon) \right] = \frac{2}{\pi} \int_{\bar{s}}^{\infty} \frac{\mathrm{d}s}{s^3} \mathrm{Im}\mathcal{A}(s) \ . \tag{3.1.4}$$

Finally, if we assume that the theory respects unitarity in the whole domain spanned by s, i.e. in the UV, too, we can use the optical theorem $\mathrm{Im}\mathcal{A}(s) = s\sigma(s)$, with $\sigma(s)$ the cross section, to get

$$\lambda_2 = \frac{2}{\pi} \int_{\bar{s}}^{\infty} \frac{\mathrm{d}s}{s^2} \sigma(s) \ . \tag{3.1.5}$$

However, we could still compute λ_2 using $\mathcal{C}$. As we can shrink the contour at will, we can make it all fit in the region where s is small enough that the EFT is valid and the lagrangian in Eq. (3.1.1) can be used. At first order in the coupling, we get

$$\mathcal{M}(s,t) = \frac{2c}{\Lambda^4}(s^2 + t^2 + u^2) \ , \tag{3.1.6}$$

where $u = (p_1 + p_4)^2$ is the last Mandelstam variable, related to the other two by $s + t + u = 0$, in this massless case. Then

$$\mathcal{A}(s) = \lim_{t \to 0} \mathcal{M}(s,t) = \frac{4cs^2}{\Lambda^4} \ , \tag{3.1.7}$$

and $\lambda_2 = 4c/\Lambda^2$. Since cross sections are always positive, we conclude

$$c > 0 \ . \tag{3.1.8}$$

Eq. (3.1.8) is the first example of a positivity bound on an EFT coefficient.

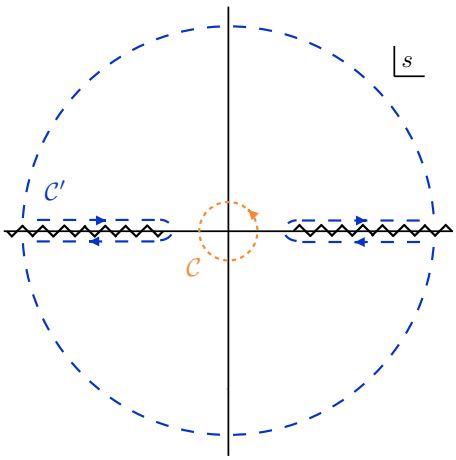


Figure 3.1: Visualization of the contours $\mathcal{C}$ and $\mathcal{C}'$ in the complexified Mandelstam variable s used to compute the s^n coefficient of the forward amplitude $\mathcal{A}(s)$. The branch cut in the real axis are in general superpositions of various poles and branch cuts appearing each time a new threshold is reached. Picture taken from Ref. [186].

Notice that, in our case, crossing symmetry implies that the amplitude is an even function of s. If there were any odd component, it would have canceled out in the subtraction in Eq.(3.1.3). This is the reason why positivity bounds cannot be obtained, at least following this approach, on the coefficients of operators of dimension $6 + 4n$, $n \geq 0$, as opposed to those of dimension $8 + 4n$, $n \geq 0$ (see also Ref. [190] for a more detailed discussion on the implications of crossing symmetry).

Positivity bounds have been used for a number of applications, from pion physics [173,174] to Quantum Gravity [191] and to the derivation of the a-theorem [192], see also Ref. [193] and references therein. Recently, efforts have been made to constrain the coefficients in the SMEFT expansion [190, 194], particularly regarding vector boson scattering [186, 195, 196] (see also Refs. [144, 197–206] for recent developments). In the next Sections, we will focus in particular on the bounds that can be obtained on fermionic dimension-eight operators of the SMEFT, and on their implications on the Minimal Flavor Violation flavor assumption.

3.2 Compatibility with Minimal Flavor Violation

In the work of Ref. [207], positivity bounds on the coefficients of dimension-8 operators composed of 4 fermions and two derivatives appearing in the SMEFT have been derived, with a number of interesting implications. For example, the authors show how the bounds are more involved than the simplest example we presented in the previous Section, and can appear as constraints on combinations of coefficients. Starting from this observation, it is possible to prove e.g. that flavor-violating coefficients are bounded from above by combinations of flavor-conserving ones. In general, they depend on the flavor structure of the dimension-eight operator coefficient. Thus, it is natural to check whether they are compatible with one of the most popular structure that is usually assumed on flavorful coefficients, namely Minimal Flavor Violation (MFV) [208, 209]. MFV is one of the simplest methods of constraining the flavor structure of any higher dimensional operator of the SMEFT that contains fermions, in a way that does not

clash with the stringent experimental bounds on flavor violation. It stems from the observation that the gauge sector of the SM_4 lagrangian enjoys a large $U(3)^5$ global symmetry. Promoting the Yukawa matrices $Y_{u,d,e}$ to spurions of this group, we can then formally extend this symmetry to the Yukawa sector of the lagrangian. Then, MFV is the assumption that this symmetry is formally conserved in the full SMEFT, with $Y_{u,d,e}$ the only spurions parametrizing its breaking. Thus, the flavor structure of any operator involving fermions is fixed by the dimension-4 Yukawas, up to some overall multiplicative factors. This applies in particular to four-fermions dimension-8 operators, whose coefficients are constrained by positivity bounds of Ref. [207]. Consequently, assuming MFV, one can ask whether the positivity bounds yield constraints on the parameters of the dimension-4 lagrangian. For instance, flavor-violating dimension-eight couplings are proportional to entries of the CKM matrix in MFV, and positivity constraints will involve both dimension-four and dimension-eight coefficients. As our initial goal, we will then first check whether one can extract from the EFT consistency some bounds on the Yukawa and CKM parameters of the SM, to be then compared with experimental values.

It is a known fact that, whatever the values of the SM parameters, there exist MFV-compatible UV-completions of the SMEFT four-Fermi operators, an example being a heavy vector coupled to fermions in a flavor-blind way [210, 211]. So surely the positivity bounds cannot be powerful enough as to make some of the values of the SM parameters inconsistent. Rather, their impact will be that of restricting the allowed region for the flavor-blind EFT coefficients that are left unspecified by the MFV assumption. What we will find is that, remarkably, only fermion masses will have such an impact, while the CKM matrix elements completely disappear from the most stringent bounds at leading order, and are only to be considered as having a subleading effect.

To go further, we consider the expectation that the allowed region for the EFT coefficients should enable the flavor-blind MFV factors to be order one. Otherwise, MFV would be cornered by the EFT consistency to unnatural realizations (or specific ones, like the aforementioned case of a flavor-blind heavy vector), which would question its use in the first place. Consequently, we study whether the assumption of order one EFT coefficients could yield interesting constraints on fermion masses and CKM elements, when implemented in the bounds involving dimension-four and dimension-eight coefficients. We study this case analytically when all the coefficients are degenerate and equal to one, and numerically when they vary independently in a neighborhood of unity and find indeed a bound on the largest fermion mass. As it turns out, however, these bounds are ineffective both phenomenologically speaking, as they are by far satisfied by experimental values, and theoretically, as perturbativity breaks down before they can be violated. Nevertheless, they present a good example of how, in some cases, restrictions on higher dimensional operators can get reflected on renormalizable parameters.

3.2.1 Minimal Flavor Violation

First of all, let us define in a more detailed way the Minimal Flavor Violation setting. Any operator built with fermions will contain flavor indices that have to be saturated by its coefficient

to form a flavor singlet. For example, take

$$\mathcal{O}_{Hu} = C_{Hu,ij}(H^\dagger i\overleftrightarrow{D} H)(u_i\gamma^\mu u_j) \ , \tag{3.2.1}$$

which is present in the SMEFT at dimension six [212]. Its coefficient, here $C_{Hu,ij}$, is then also a tensor in flavor space. If no additional assumption is made, all entries of this tensor should be assumed to be of the same order of magnitude, and in particular should be roughly $\mathcal{O}(1)$. However, such an anarchic structure is greatly constrained by observations via measurements of mesons oscillations, electric dipole moments and lepton flavor violation [132, 213–218] with lower bounds on the scale of New Physics (NP) of the order of $\mathcal{O}(10^3$ TeV). Were this the common suppression scale of the EFT, it would render its impact on observables irrelevant for collider physics, as well as imply a strong fine tuning on the scalar mass of the Higgs. On the other hand, we know that, already at dimension four, the flavorful parameters of the Standard Model are far from being anarchic, spanning a range that scans almost 6 orders of magnitude. Thus, different ansatzes have then emerged addressing the flavor structure of the SMEFT [219, 220]. These approaches have the twofold advantage of allowing us to bring the lower bounds on the NP scale down to the TeV region, and of reducing the number of relevant free parameters added by the EFT expansion. They usually proceed by either relating the SMEFT flavor structure to that of SM_4 or deriving it from certain families of UV models.

The archetype of the first kind of approach is represented by the Minimal Flavor Violation (MFV) ansatz [213, 221]. As we have already seen in the previous chapter, if we do not consider the Yukawa operators, the lagrangian of the Standard Model enjoys a $U(3)^5$ global symmetry acting on the quark and lepton flavor space. Its non-abelian subgroup can be split as:

$$\begin{aligned} SU(3)_q^3 &= SU(3)_Q \otimes SU(3)_u \otimes SU(3)_d \\ SU(3)_l^2 &= SU(3)_L \otimes SU(3)_e \ , \end{aligned} \tag{3.2.2}$$

which means that the flavor vector $\psi \in \{Q, u, d, L, e\}$ transforms as a fundamental of $SU(3)_i$. Then the Yukawa sector of the lagrangian

$$\mathcal{L}_{\text{Yukawa}} = \bar{Q}Y_d dH + \bar{Q}Y_u u\tilde{H} + \bar{L}Y_e eH + \text{ h.c.} \ , \tag{3.2.3}$$

can be made formally invariant under this group if we promote the Yukawa matrices, $Y_{u,d,e}$, to spurion fields transforming as in Table 3.1.

	$SU(3)_Q$	$SU(3)_u$	$SU(3)_d$	$SU(3)_L$	$SU(3)_e$
Y_u	**3**	$\bar{\mathbf{3}}$	**1**	**1**	**1**
Y_d	**3**	**1**	$\bar{\mathbf{3}}$	**1**	**1**
Y_e	**1**	**1**	**1**	**3**	$\bar{\mathbf{3}}$

Table 3.1: Transformation properties of the Yukawa matrices treated as spurions under the MFV assumption.

Minimal Flavor Violation is the requirement that any higher dimensional operator has to be built out of Y matrices and Standard Model fields, and must be formally invariant under the flavor group, taking into account the transformation properties in Table 3.1. Notice that the spurions transform under the $U(1)$ abelian factors of $U(3)^5$, too. We do not treat those explicitly, but they turn out to be useful, e.g., to get rid of terms like $\sim \left(Y_u\right)^m \left(Y_u^\dagger\right)^n \left(\bar{Q}Q\right)^p$ with $m \neq n$ (see Appendix 3.A for more details).

After building the lagrangian, we can set the spurion fields to their vacuum expectation values, i.e the physical values for Yukawas. The latter are defined up to the freedom of changing the fermion fields basis, i.e., they are defined up to $U(3)_q^3 \otimes U(3)_l^2$ transformations. In the following, we choose a basis where they read:

$$Y_u = \lambda_u \qquad Y_d = V_{CKM}\lambda_d \qquad Y_e = \lambda_e \ , \tag{3.2.4}$$

where the λ's are diagonal matrices containing the diagonal Yukawa couplings, e.g., $\lambda_u = \text{diag}(y_u, y_c, y_t)$, and V_{CKM} is the CKM matrix. We refer to this as to the *up basis*. Another basis that we will occasionally mention is the following,

$$Y_u = V_{CKM}^\dagger\lambda_u \qquad Y_d = \lambda_d \qquad Y_e = \lambda_e \ , \tag{3.2.5}$$

related to the previous one via a $U(3)_Q$ transformation, and which we label *down basis*. As we will make clear in Section 3.3.1, our discussion cannot and does not depend on the particular choice of basis.

The MFV framework is relevant and particularly convenient from a theoretical point of view, since it drastically reduces the number of free parameters entering the lagrangian at each mass-dimension [222]. In addition, by tying the amount of flavor violation to that already present in the SM_4, it is more easily compatible with observables, bringing the lower bounds on the NP scale down to the TeV region [132].

Dimension 8 independent fermionic operators

As we already mentioned, we will focus our attention on SMEFT operators made of four fermionic fields. However, the lowest order operators of this kind, those appearing at dimension-six, are unaffected by positivity bounds, as they give no s^2 contribution to the forward amplitude. Thus, as anticipated, we will aim our attention at dimension-eight operators formed by 4 fermionic fields and two derivatives, i.e. the lowest order objects affected by the bounds. Bounds for their coefficients have been obtained in Ref. [207], whence we will borrow part of the terminology and conventions. In particular, operators formed with fields of one kind only, i.e., those of the schematic form $\mathcal{O} \sim \partial^2(\bar{\psi}_m\Gamma\psi_n)(\bar{\psi}_p\Gamma\psi_q)$, $\psi \in \{u, d, Q\}$, where Γ is some combination of Dirac and SM gauge matrices and we only made flavor indices explicit, will be dubbed *self-quartic*. We will refer to those formed with two kinds of fields, $\mathcal{O} \sim \partial^2(\bar{\psi}_m\Gamma\psi_n)(\bar{\chi}_p\Gamma\chi_q)$, $\psi, \chi \in \{u, d, Q\}$ and $\psi \neq \chi$, as *cross-quartic* (the way Lorentz indices are contracted is not shown here). Moreover, we restrict for this discussion to the quark sector only. The ex-

Type	Content	Operator	Symmetry
self-quartic	(4-u)	$\mathcal{O}_1[u] = c^{u,1}_{mnpq}\partial_\mu(\bar{u}_m\gamma_\nu u_n)\partial^\mu(\bar{u}_p\gamma^\nu u_q)$ $\mathcal{O}_3[u] = c^{u,3}_{mnpq}\partial_\mu(\bar{u}_m T^a\gamma_\nu u_n)\partial^\mu(\bar{u}_p T^a\gamma^\nu u_q)$	$c_{mnpq} = c_{pqmn}$ $c_{mnpq} = c^*_{nmqp}$
	(4-Q)	$\mathcal{O}_1[Q] = c^{Q,1}_{mnpq}\partial_\mu(\bar{Q}_m\gamma_\nu Q_n)\partial^\mu(\bar{Q}_p\gamma^\nu Q_q)$ $\mathcal{O}_2[Q] = c^{Q,2}_{mnpq}\partial_\mu(\bar{Q}_m\tau^I\gamma_\nu Q_n)\partial^\mu(\bar{Q}_p\tau^I\gamma^\nu Q_q)$ $\mathcal{O}_3[Q] = c^{Q,3}_{mnpq}\partial_\mu(\bar{Q}_m T^a\gamma_\nu Q_n)\partial^\mu(\bar{Q}_p T^a\gamma^\nu Q_q)$ $\mathcal{O}_4[Q] = c^{Q,4}_{mnpq}\partial_\mu(\bar{Q}_m T^a\tau^I\gamma_\nu Q_n)\partial^\mu(\bar{Q}_p T^a\tau^I\gamma^\nu Q_q)$	
	(4-d)	$\mathcal{O}_1[d] = c^{d,1}_{mnpq}\partial_\mu(\bar{d}_m\gamma_\nu d_n)\partial^\mu(\bar{d}_p\gamma^\nu d_q)$ $\mathcal{O}_3[d] = c^{d,3}_{mnpq}\partial_\mu(\bar{d}_m T^a\gamma_\nu d_n)\partial^\mu(\bar{d}_p T^a\gamma^\nu d_q)$	
cross-quartic	(2-u)(2-Q)	$\mathcal{O}_{K1}[u,Q] = -a^{uQ,1}_{mnpq}\,(\bar{u}_m\gamma_\mu\partial_\nu u_q)\left(\bar{Q}_n\gamma^\nu\partial^\mu Q_p\right)$ $\mathcal{O}_{K3}[u,Q] = -a^{uQ,3}_{mnpq}\,(\bar{u}_m T^a\gamma_\mu\partial_\nu u_q)\left(\bar{Q}_n T^a\gamma^\nu\partial^\mu Q_p\right)$	$a^{\psi\chi}_{mnpq} = a^{\chi\psi}_{nmqp}$ $a_{mnpq} = a^*_{qpnm}$
	(2-d)(2-Q)	$\mathcal{O}_{K1}[d,Q] = -a^{dQ,1}_{mnpq}\left(\bar{d}_m\gamma_\mu\partial_\nu d_q\right)\left(\bar{Q}_n\gamma^\nu\partial^\mu Q_p\right)$ $\mathcal{O}_{K3}[d,Q] = -a^{dQ,3}_{mnpq}\left(\bar{d}_m T^a\gamma_\mu\partial_\nu d_q\right)\left(\bar{Q}_n T^a\gamma^\nu\partial^\mu Q_p\right)$	
	(2-d)(2-u)	$\mathcal{O}_{K1}[d,u] = -a^{du,1}_{mnpq}\left(\bar{d}_m\gamma_\mu\partial_\nu d_q\right)(\bar{u}_n\gamma^\nu\partial^\mu u_p)$ $\mathcal{O}_{K3}[d,u] = -a^{du,3}_{mnpq}\left(\bar{d}_m T^a\gamma_\mu\partial_\nu d_q\right)(\bar{u}_n T^a\gamma^\nu\partial^\mu u_p)$	

Table 3.2: List of independent self-quartic and cross-quartic operators. T^a are the $SU(3)_C$ QCD generators and $\tau^I = \frac{\sigma^I}{2}$ are the $SU(2)_L$ EW generators.

tension of our methods to include leptons can then be readily found[3]. The list of independent operators we are interested in, then, can be read off Table 3.2. Operators of the form $\mathcal{O} = \partial_\mu(\bar{\psi}_m\gamma_\nu\psi_n)\partial^\mu(\bar{\chi}_p\gamma^\nu\chi_q)$, $\psi \neq \chi$ are also present, and are independent from the ones listed in Table 3.2. However, since they do not contribute to the forward amplitude, there are no bounds on their Wilson coefficients [207].

Therefore, restricting ourselves to the operators listed in Table 3.2, we can see that there are $2+4+2=8$ independent self-quartic tensors c_{mnpq}. As stated already in Ref. [207], imposing the symmetry requirements $c_{mnpq} = c_{pqmn}$ and $c_{mnpq} = c^*_{nmqp}$ leaves $\frac{1}{2}N_f^2(N_f^2+1)$ independent real entries in each tensor. Indeed, the first condition is a symmetry requirement on the complex $N_f^2 \times N_f^2$ matrix c_{mnpq} whose rows are indexed by (m,n) and columns by (p,q), so that it leaves $2 \times \frac{1}{2}N_f^2(N_f^2+1)$ unconstrained real entries. The second condition further halves them. On the other hand, there are $2+2+2=6$ independent cross-quartic structure of operators. Each a_{mnpq} tensor has only to obey the hermiticity condition $a_{mnpq} = a^*_{qpnm}$, thus each of them contains N_f^4 independent real entries. Since $a^{\psi\chi}_{mnpq} = a^{\chi\psi}_{nmqp}$, fixing one $a^{\psi\chi}_{mnpq}$ tensor automatically fixes the one with $\psi \leftrightarrow \chi$. Overall, we will deal with $6+8=14$ independent types of operators, and $2N_f^2(5N_f^2+2)$ independent operators.

[3]This is true provided one does not include right-handed neutrinos in the discussion. Then, a generalization of MFV accounting for the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) mixing matrix is needed [223].

MFV ansatz for dimension 8 operators

We now wish to enforce the MFV assumption on the list of four-fermion dimension-8 operators. This means that all EFT coefficients in Table 3.2 must be written in terms of Yukawa spurions and flavor-blind EFT coefficients. Since our goal is to study possible bounds on the entries of the Yukawa matrices, which are then to be compared with their phenomenological values, one must in principle depart from the latter and treat the fermion masses and the entries of the CKM matrix as generic. This means, in particular, that all possible hierarchies between them should be allowed. Practically, however, we need a way to perform a proper power counting, meaning an assumption on the magnitude of the entries is needed. Indeed, phenomenological studies of MFV [208, 209] rely on the measured values of the fermion masses or of the CKM elements to define a consistent expansion. Large Yukawas demand further care, but can also be treated consistently [224, 225], as we will show. Here, we choose to follow the same approach, meaning we stick to cases where there exists a Yukawa much larger than the others, so that we can fix all remaining ones to zero at first order in a consistent MFV expansion. Clearly, this means giving up on full generality and on the hope of constraining the smallest Yukawa coupling. This assumption is obviously compatible with the actual, phenomenological values of the Yukawas.

We will consider two simplified scenarios, with respectively 2 and 3 flavors. The largest Yukawa, y_c in the former case and y_t in the latter, will be the only one we take as non-vanishing at leading order. We keep them as free parameters in all the expressions below, with a caveat for y_t discussed at the end of this Section. In keeping only the first relevant order in this expansion, we will see that, at least in the proper realizations of $N_f = 2,\ 3$, there is always a choice of basis in flavor space such that the CKM matrix V_{CKM} makes no appearance in the computations, and no hope of putting any bound on its entries can be retained. This is due to the fact that only the up-Yukawa matrix Y_u will enter our expressions, while we can always pick a basis where V_{CKM} is placed exclusively in Y_d. This basis is nothing but the up basis one of Eq. (3.2.4), which in our approximations and when restricted to the quark sector, becomes, for $N_f = 3$,

$$Y_u = \text{diag}(0, 0, y_t) \ , \quad Y_d = \text{diag}(0, 0, 0) \ . \tag{3.2.6}$$

The expansion that we use depends on the size of the largest Yukawa. By assumption, we neglect any term where Y_d appears, but an expansion in the up-Yukawa matrix Y_u demands that the entries of the matrix are $\ll 1$, to ensure a consistent, non-divergent expansion. While this works for $N_f = 2$ due to the smallness of the charm quark Yukawa, this does not hold for the top, so that the expansion has to be resummed when $N_f = 3$. We start by discussing the naive expansion, and explain at the end of this Section how to modify it to correctly account for the top-Yukawa resummation.

Numerically, for $N_f = 3$, and since the Yukawa matrices will always appear in pairs, these approximations amount to neglecting terms of order $\mathcal{O}\big((y_c/y_t)^2\big) \sim \mathcal{O}\big((y_b/y_t)^2\big) \sim \mathcal{O}(10^{-3})$ at

most, when setting the Yukawas to their real values. Keeping this approximation also serves as a measure of how much we can let y_t vary without spoiling it. For example, if we want to be precise up to at most $\mathcal{O}(10^{-2})$ corrections, we can let y_t take values in $[0.1, 4\pi)$. In addition, focusing only on quarks is justified in the same way, at this level. Indeed, since $(y_\tau/y_t)^2 \sim 10^{-4}$, the only bilinears formed by leptons that would be added to this order are of the form $\bar{L}_m \Gamma L_m$ or $\bar{e}_m \Gamma e_m$, i.e. only diagonal ones. Thus, they only contribute trivially to the flavor tensor structure, and bounds for the operators built with them can be retrieved, e.g., looking at the ones built with d quark fields. All this is somehow weaker for $N_f = 2$. There, the biggest contributions we neglected have an approximate size of $(y_s/y_c)^2 \sim (y_\mu/y_c)^2 \sim 10^{-2}$.

We can now start by asking what the MFV ansatz implies for operators containing 4 right-handed up-type quarks, when we work at order $\mathcal{O}(Y_u^2 Y_d^0)$ in the expansion. There are two possible operators containing 4 up-quark fields:

$$\begin{aligned}\mathcal{O}_1[u] &= c^{u,1}_{mnpq} \partial_\mu (\bar{u}_m \gamma_\nu u_n) \partial^\mu (\bar{u}_p \gamma^\nu u_q) \\ \mathcal{O}_3[u] &= c^{u,3}_{mnpq} \partial_\mu (\bar{u}_m T^a \gamma_\nu u_n) \partial^\mu (\bar{u}_p T^a \gamma^\nu u_q) \ ,\end{aligned}$$

where only flavor indices are shown. To obtain the MFV expansion of the c^u coefficients, it is useful to define two objects[4]:

$$X_u \equiv Y_u Y_u^\dagger \ , \tag{3.2.7}$$

$$\tilde{X}_u \equiv Y_u^\dagger Y_u \ . \tag{3.2.8}$$

Let us study what happens for the physical case $N_f = 3$. Indeed, as invariance under $U(2)$ can be obtained by imposing additional constraints to the $U(3)$ case, $N_f = 2$ can always be enforced at a later moment. The product of quark bilinears $\sim \bar{u}_m u_n \bar{u}_p u_q$ is a $(\bar{\mathbf{3}} \otimes \mathbf{3}) \otimes (\bar{\mathbf{3}} \otimes \mathbf{3})$ of $SU(3)_u$, and can be decomposed as $\mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{8}_1 \oplus \mathbf{8}_2 \oplus \mathbf{27}$, since the $\mathbf{10}$ and the $\overline{\mathbf{10}}$ vanish thanks to the exchange symmetry. Then, at $\mathcal{O}(Y_u^0 Y_d^0)$, $c^{u,i}_{mnpq} = \rho_1^{u,i} \delta_{mn} \delta_{pq} + \rho_3^{u,i} \delta_{mq} \delta_{pn}$. Because of the $SU(3)_Q$ index carried by Y_u, there is no invariant we can build with just one copy of it. However, the contraction $\tilde{X}_u$ defined earlier is a singlet of $SU(3)_Q$ and contains a $\mathbf{1} \oplus \mathbf{8}$ of $SU(3)_u$. Its trace can be reabsorbed through a redefinition of the $\rho_1^{u,i}$ and $\rho_3^{u,i}$ coefficients, while its traceless part can be used to build two further structures,. Thus, at $\mathcal{O}(Y_u^2 Y_d^0)$, we get:

$$\begin{aligned}c^{u,i}_{mnpq} = \rho_1^{u,i}(\delta_{mn}\delta_{pq}) + \rho_2^{u,i}(\tilde{X}_{u,mn}\delta_{pq} + \delta_{mn}\tilde{X}_{u,pq}) + \rho_3^{u,i}(\delta_{mq}\delta_{pn}) + \\ + \rho_4^{u,i}(\tilde{X}_{u,mq}\delta_{pn} + \delta_{mq}\tilde{X}_{u,pn}) \ ,\end{aligned} \tag{3.2.9}$$

where all of the ρ coefficients are unconstrained, and can be taken of $\mathcal{O}(1)$.

In Table 3.3, we generalize the construction just shown for an operator containing four right-handed u-quark fields and list the shape that the MFV ansatz forces on the Wilson coefficients respectively of the self-quartic and cross-quartic kinds of operators previously listed.

The EFT coefficients appearing in Table 3.3 are the objects on which we will soon apply

[4]Obviously, after the spurions freeze to their expectation values, $X_u = \tilde{X}_u$ in our choice of basis.

Type	Content	Operator
self-quartic	(4-u) $_{i=1,3}$	$c^{u,i}_{mnpq} = \rho^{u,i}_1(\delta_{mn}\delta_{pq}) + \rho^{u,i}_2(\tilde{X}_{u,mn}\delta_{pq} + \delta_{mn}\tilde{X}_{u,pq}) + \rho^{u,i}_3(\delta_{mq}\delta_{pn}) + \rho^{u,i}_4(\tilde{X}_{u,mq}\delta_{pn} + \delta_{mq}\tilde{X}_{u,pn})$
	(4-Q) $_{i=1,2,3,4}$	$c^{Q,i}_{mnpq} = \rho^{Q,i}_1(\delta_{mn}\delta_{pq}) + \rho^{Q,i}_2(X_{u,mn}\delta_{pq} + \delta_{mn}X_{u,pq}) + \rho^{Q,i}_3(\delta_{mq}\delta_{pn}) + \rho^{Q,i}_4(X_{u,mq}\delta_{pn} + \delta_{mq}X_{u,np})$
	(4-d) $_{i=1,3}$	$c^{d,i}_{mnpq} = \rho^{d,i}_1(\delta_{mn}\delta_{pq}) + \rho^{d,i}_3(\delta_{mq}\delta_{pn})$
cross-quartic	(2-u)(2-Q) $_{i=1,3}$	$a^{uQ,i}_{mnpq} = \rho^{uQ,i}_1(\delta_{mq}\delta_{np}) + \rho^{uQ,i}_2(\tilde{X}_{u,mq}\delta_{np}) + \rho^{uQ,i}_3(\delta_{mq}X_{u,np}) + \rho^{uQ,i}_4((Y_u)_{nq}(Y^\dagger_u)_{mp})$
	(2-d)(2-Q) $_{i=1,3}$	$a^{dQ,i}_{mnpq} = \rho^{dQ,i}_1(\delta_{mq}\delta_{np}) + \rho^{dQ,i}_2(\delta_{mq}X_{u,np})$
	(2-d)(2-u) $_{i=1,3}$	$a^{du,i}_{mnpq} = \rho^{du,i}_1(\delta_{mq}\delta_{np}) + \rho^{du,i}_2(\delta_{mq}\tilde{X}_{u,np})$

Table 3.3: $\mathcal{O}(Y_u^2 Y_d^0)$-MVF expansion of the self-quartic and cross-quartic operators

positivity bounds. They can be seen to respect all the symmetry properties required by Table 3.2. However, the number of independent coefficients is drastically reduced, as now the only free parameters are flavor-blind overall coefficients, i.e., those we named ρ^i_A, and their number is independent on N_f. In particular, as already stated, the number of independent real coefficients in a unconstrained setting is $2N_f^2(5N_f^2+2)$ (176 for $N_f = 2$ and 846 for $N_f = 3$), while after imposing MFV we are left with 44 independent real coefficients ρ^i_A at the $\mathcal{O}(Y_u^2 Y_d^0)$ in the MFV expansion. This result is independent on the number of flavors as long as $N_f \geq 3$, while for $N_f = 2$, it turns that 6 coefficients are redundant. This can be readily seen as, for any N_f, the bilinear product $\mathbf{fund} \otimes \overline{\mathbf{fund}}$ of a fundamental and anti-fundamental representation, can always be decomposed as $\mathbf{fund} \otimes \overline{\mathbf{fund}} = \mathbf{1} \oplus \mathbf{Adj}$, so the contractions we can use to achieve MFV invariance are always the same. The exceptional nature of$N_f = 2$ is due to $SU(2)$ being pseudoreal, so that $\mathbf{fund} \sim \overline{\mathbf{fund}}$, as we will see.

Let us now pause to comment on the $N_f = 3$ case. We kept y_t as a generic parameter in our analysis, but eventually we will be interested in its phenomenological value, which is $\sim \mathcal{O}(1)$. Therefore, the truncation to $\mathcal{O}(Y_u^2 Y_d^0)$ we perform is not in principle justified for the physical $N_f = 3$ case: higher-order terms such as

$$\left(Y_u Y_u^\dagger\right)^n_{ij} \sim y_t^{2n}\delta_{3i}\delta_{3j} \ . \tag{3.2.10}$$

should be properly resummed. Interestingly, the resummation does not bring new flavor violation beyond the one contained in the first non-trivial contraction $Y_u Y_u^\dagger$. This is more transparent in the basis of (3.2.5), where

$$\left(Y_u Y_u^\dagger\right)^n_{ij} \sim y_t^{2n}\left(V^*_{CKM}\right)_{3i}\left(V_{CKM}\right)_{3j} \ . \tag{3.2.11}$$

Heuristically, this means that the flavor violation structure can be obtained from our *naive* expansion in Table 3.3, up to a redefinition of the parameters to account for the resummation. In the end, it turns out that to account for this resummation one should just remove the explicit

y_t dependence by fixing $y_t = 1$ and turn the EFT coefficients into $\mathcal{O}(1)$ arbitrary functions of y_t, $\rho_i^{x,j} \to \rho(y_t)_i^{x,j}$. An explicit proof of such behavior is given in Appendix 3.A. When we freeze the spurions to their background values, functions of y_t become simple numbers, which means that any explicit y_t-dependence simply disappears from the expansion.

3.3 Analysis of the bounds

At this point, we have all the machinery we need to confront the bounds obtained in Ref. [207] with the MFV hypothesis dictating the expansion of the various dimension-8 operators in powers of the Yukawa, as listed in Table 3.3. First of all, the positivity constraints of Ref. [207] depend not only on the Wilson coefficients c_{mnpq} and a_{mnpq}, but also on some arbitrary external states, dubbed α and β. These consist in generic complex vectors in the space of the internal quantum numbers involved, here just flavor and gauge symmetries, and are normalized to one. Thus, we first need to remove these vectors from the bounds, in order to obtain expressions that depend on the operator coefficients only. Secondly, many of the inequalities contain linear combinations of coefficients coming from distinct independent operators. To simplify the computations, we define new coefficients via suitable linear transformations. Let us carry this latter simplification first, and then proceed to show how we removed the dependencies on the α's and β's.

To begin, let us look at the case of operators containing 4 up fields. The bounds on them are obtained [207] by scattering the following states:

$$\begin{aligned} |\psi_1\rangle &= \alpha_{mi}\,|\bar{u}_{mi}\rangle\,, & |\psi_2\rangle &= \beta_{mi}\,|u_{mi}\rangle\,, \\ |\psi_3\rangle &= \beta^*_{mi}\,|\bar{u}_{mi}\rangle\,, & |\psi_4\rangle &= \alpha^*_{mi}\,|u_{mi}\rangle\,, \end{aligned} \tag{3.3.1}$$

where m and i are flavor and gauge indices respectively. The amplitude then reads:

$$\mathcal{A} = 4s^2\left[\left(c^{u,1}_{mnpq} - \frac{1}{6}c^{u,3}_{mnpq}\right)\alpha^*_{mi}\beta_{ni}\beta^*_{pj}\alpha_{qj} + \frac{1}{2}c^{u,3}_{mnpq}\alpha^*_{mi}\beta_{nj}\beta^*_{pj}\alpha_{qi}\right]. \tag{3.3.2}$$

In the following, we make the simplifying assumption that we can apply the factorization $\alpha_{mi} = \alpha_m a_i$, for some vectors α_m and a_i carrying flavor and gauge indices respectively, and similarly for β_{mi}. This will allow us to marginalize over the gauge indices, with the caveat that the resulting bounds may not be the strongest ones. Performing such marginalization, two bounds are obtained:

$$\begin{aligned} \alpha_m\alpha^*_q\beta_n\beta^*_p\left(c^{u,1}_{mnpq} + \frac{1}{3}c^{u,3}_{mnpq}\right) > 0\,, \\ \alpha_m\alpha^*_q\beta_n\beta^*_p c^{u,3}_{mnpq} > 0\ . \end{aligned} \tag{3.3.3}$$

As we said α_n and β_n parametrize arbitrary external states, since the bounds are obtained by constraining the s^2 coefficient of a $2 \to 2$ scattering of generic superpositions of flavor (and gauge) eigenstates. As such, the inequalities in Eq. (3.3.3) have to be fulfilled for all values of

α_n and β_n.

We perform a linear transformation on Eq. (3.3.3) by defining:

$$\xi_k^{u,1} \equiv \rho_k^{u,1} + \frac{1}{3}\rho_k^{u,3} \quad \text{and} \quad \xi_k^{u,3} \equiv \rho_k^{u,3} \quad \text{for} \quad k = 1,2,3,4 \ , \tag{3.3.4}$$

so that, defining $c(\xi)^{u,i}_{mnpq}$ as in the first line of Table 3.3 but with $\rho \to \xi$, i.e.,

$$\begin{aligned} c(\xi)^{u,i}_{mnpq} = {} & \xi_1^{u,i}(\delta_{mn}\delta_{pq}) + \xi_2^{u,i}(\tilde{X}_{u,mn}\delta_{pq} + \delta_{mn}\tilde{X}_{u,pq}) + \xi_3^{u,i}(\delta_{mq}\delta_{pn}) \\ & + \xi_4^{u,i}(\tilde{X}_{u,mq}\delta_{pn} + \delta_{mq}\tilde{X}_{u,np}) \qquad\qquad i = 1,3 \ , \end{aligned}$$

the bounds become simply

$$\alpha_m\alpha_q^*\beta_n\beta_p^* c(\xi)^{u,i}_{mnpq} > 0 \qquad\qquad i = 1,3 \ . \tag{3.3.5}$$

Since all the bounds are expressed as inequalities on linear combinations of the flavor structure tensors as in Eq. (3.3.3), it is always possible to perform a linear redefinition such as Eq. (3.3.4) to bring them to a form analogous to Eq. (3.3.5). From now on, we will do this on all operators, and show both bounds and flavor tensors as functions of ξ's. Their explicit dependence on the original ρ coefficients is shown in Appendix 3.E.

In conclusion, the bounds we have to study are all of the form:

$$\alpha_m\alpha_q^*\beta_n\beta_p^* c(\xi)^{X,i}_{mnpq} > 0 \ , \qquad X = u, Q, d \ , \tag{3.3.6}$$

$$\alpha_m\alpha_q^*\beta_n\beta_p^* a(\xi)^{X,i}_{mnpq} > 0 \ , \qquad X = uQ, dQ, du \ . \tag{3.3.7}$$

As a first check, it is useful to prove that the bounds do not imply an empty region, and that it is instead always possible to find some values of the ξ's such that the constraints can be satisfied for any α, β. For example, for the self-quartic operators, one can look at how the flavor indices are summed and notice that, choosing $\xi_1^i = \xi_2^i = 0$, the bounds can be expressed as

$$\xi_3^i|\alpha|^2|\beta|^2 + \xi_4^i\left(\alpha_m A_{mq}\alpha_q^*|\beta|^2 + \beta_p^* A_{pn}\beta_n|\alpha|^2\right) > 0 \ , \tag{3.3.8}$$

where $A = \tilde{X}_u$, $A = X_u$ and $A = 0$ for the (4-u), (4-Q) and (4-d) cases respectively. In the former two cases, being the product of an invertible matrix and its hermitian conjugate, A is (semi-)positive definite. Thus $\xi_1^i = \xi_2^i = 0$, $\xi_3^i > 0$ and $\xi_4^i > 0$ is, in this setting, an allowed region in the parameter space, and fulfills the bounds $\forall\alpha, \beta$. In the (4-d) case, $A = 0$, and $\{\xi_1^i = 0, \xi_3^i > 0\}$ is an always allowed region of the parameter space[5]. Similar conclusions can be drawn for the cross-quartic operators. As a consequence, there exists at least one region that is a solution of Eqs. (3.3.6) and (3.3.7), with the coefficients expressed as per Table 3.3.

[5]incidentally, if we integrated out at tree level a weakly coupled heavy vector boson with MFV-compatible interactions with the quarks, this is what the matched coefficients would look like.

The final goal, however, is to study the anatomy of the bounds when all the ρ coefficients of the MVF expansions are of $\mathcal{O}(1)$, which is the natural realization of the MFV ansatz. By fixing the coefficients, the bounds become functions of SM parameters alone. We then verify whether they are strict enough as to impose constraints on the parameters of the dimension-4 lagrangian. In the two flavor case, that means y_c, the charm-quark Yukawa coupling. Naively, that also means y_t when $N_f = 3$, however, as we discussed previously, y_t should be absorbed in the ρ (equivalently, in the ξ) coefficients. In that case, we can only check whether all the EFT coefficients can be consistently $\mathcal{O}(1)$.

3.3.1 Flavor violation and CKM-(in)dependence of the positivity bounds

Before going any further, some clarifications are in order. All along the discussion we made, it looks like there is no place for any flavor violation at all. Indeed, in our approximation, the only matrix involved in building the flavor invariants is Y_u, which we chose to be diagonal, and the only physical parameter shaping the allowed region is y_t, while there is no sign of the CKM matrix. Obviously, our discussion cannot depend on the specific basis that we pick. In this Section we show that this is the case and that the CKM matrix only enters the bounds at subleading order with respect to our approximations. Suppose we had chosen, instead of Eq. (3.2.4), the basis (3.2.5). Even then, we could have still diagonalized Y_u, and consequently X_u and $\tilde{X}_u$, albeit at a later time. We could have done this by exploiting the redundancy contained in the definition of expressions like Eq. (3.3.6) or (3.3.7). Indeed, we can apply the singular value decomposition, valid for any square matrix, to express Y_u as

$$Y_u = U\Sigma\tilde{U}^\dagger \ , \tag{3.3.9}$$

where U and $\tilde{U}$ are unitary matrices and Σ is diagonal. Then:

$$X_u = Y_u Y_u^\dagger = U\Sigma\Sigma^* U^\dagger \tag{3.3.10}$$

$$\tilde{X}_u = Y_u^\dagger Y_u = \tilde{U}\Sigma^*\Sigma\tilde{U}^\dagger \ . \tag{3.3.11}$$

$$\tag{3.3.12}$$

Thus, X_u and $\tilde{X}_u$ are diagonalized by U and $\tilde{U}$ respectively. Then, in Eq. (3.3.6) or (3.3.7), we could have rotated[6] both α and β (and their hermitian conjugates) using U or $\tilde{U}$. This does not modify the space that α and β span, since unitary matrices conserve norms. Therefore we can explore the α's and β's space with the diagonalized version of X_u and $\tilde{X}_u$. In particular,

[6]More precisely, we can multiply by the identity $\mathbb{1}_{N_f} = UU^\dagger$ so that

$$\begin{aligned}
&\alpha_m \alpha_q^* \beta_n \beta_p^* c^{u,i}_{mnpq} = \\
&= \alpha_m \left(U_{mm'} U^\dagger_{m'm''}\right) \left(U_{q''q'} U^\dagger_{q'q}\right) \alpha_q^* \left(U_{n''n'} U^\dagger_{n'n}\right) \beta_n \beta_p^* \left(U_{pp'} U^\dagger_{p'p''}\right) c_{m''n''p''q''} \equiv \\
&\equiv \left(\tilde{\alpha}_{m'} U^\dagger_{m'm''}\right) \left(U_{q''q'} \tilde{\alpha}^*_{q'}\right) \left(U_{n''n'} \tilde{\beta}_{n'}\right) \left(\tilde{\beta}^*_{p'} U^\dagger_{p'p''}\right) c_{m''n''p''q''} \ .
\end{aligned}$$

$\tilde{X}_u = \lambda_u^2$ is diagonal to begin with, while we can rotate $X_u \to \Sigma^*\Sigma = \lambda_u^2$ using $U = V_{CKM}^\dagger$, $\tilde{U} = \mathbb{1}_{N_f}$. These are the same matrices we got when we started with the basis in Eq. (3.2.4) in the first place.

The freedom to absorb unitary matrices in the generic vectors α and β arises from specific properties of the positivity bounds. First, those bounds are obtained in Ref. [207] in a high-energy limit where all SM fermions are considered massless. In this limit, the mass terms disappear and they do not single out anymore the preferred basis that diagonalizes them. In addition, only dimension-8 operators are constrained by the bounds, so that the dimension-8 EFT coefficients are the only spurions that break the flavor symmetry and enter the bound. Thus, the flavor symmetry can be used to absorb *irrelevant* parameters, here in the sense of *not entering the positivity bounds*, among the ones that form the dimension-8 EFT coefficients. In our case, the CKM matrix is precisely such an *irrelevant* parameter. Notice that this statement derives from the use of the full flavor group. Consequently, it does not hold if we only scatter a subset of the flavor states (said differently, if we imposed some conditions on α, β). Indeed, the restriction of the flavor group to those states may not be sufficient to remove all the CKM dependence from the bounds. We will see an example of this in Section 3.3.3, when a two-flavor scenario is embedded in $N_f = 3$. There, by scattering the two first flavors only, we obtain a subset of the $N_f = 3$ bounds which depends on the entries of the CKM matrix. Nevertheless, the full $N_f = 3$ bounds are more stringent and do not depend on $V_{\rm CKM}$, as explained.

The next natural question that arises is then at which order in the expansion in powers of Yukawa matrices does a CKM contribution appear, in a way that cannot be removed via unitary rotations. From what we just saw, this has to happen when a combination containing both up- and down-Yukawa matrices comes into play. If we define $X_{d,mn} \equiv (Y_d Y_d^\dagger)_{mn}$, i.e., the analog of $X_{u,mn}$ for the down Yukawa matrix, we notice that it contains a **8** irrep of $SU(3)_Q$ as well. Thus, if we expand a bit further, we can add for example to the second line of Table 3.3 a term like $\sim \tilde{\rho}_1(X_{d,mn}\delta_{pq} + \delta_{mn}X_{d,pq}) + \tilde{\rho}_2(n \leftrightarrow q)$. With this example we can see that the freedom left by the redundancy in the definition of the external states is larger than the symmetry of the lagrangian alone. Indeed, by exploiting the flavor $U(3)^3$, we can diagonalize either X_d or X_u, but not both. From the point of view of the bounds, the addition of only the aforementioned terms corresponds to a shift $X_u \to X_u + X_d$ in the second line of Table 3.3. This combination, being still hermitian, can also be diagonalized. However, its eigenvalues will now depend explicitly on the entries of the CKM matrix, that will in such way enter the bounds at this level in the expansion. Conversely, the fact that the V_{CKM} entries are relevant only at such sub-leading order means that the bounds are not particularly sensitive to their values, and even relatively large modifications for them do not affect much the structure of the bounds.

3.3.2 Disentangling the external states

As already stated[7], positivity conditions like Eq. (3.3.5) have to be fulfilled for every value of α and β, since they simply label arbitrary in-states. However, to obtain bounds that are purely expression of the EFT coefficients, one has to disentangle the coefficients from the external states. This Section is devoted to show how this can be done in the case under consideration. We can start by removing the dependence of the bounds on either α or β. Suppose we fix β and define $C(\beta)_{mq} \equiv c_{mnpq}\beta_n\beta_p^*$. Notice that this matrix is hermitian, and thus diagonalizable[8]. Then the positivity requirement (3.3.5) takes the form $C(\beta)_{mq}\alpha_m\alpha_q^* > 0$, to be satisfied for any unit N_f-vector α. This is equivalent to asking that the matrix $C(\beta)$ is positive definite, i.e., that its real eigenvalues $r(\beta)_I$, $I = 1, \ldots, N_f$, are all positive. In other words, we can trade

$$\begin{cases} c_{mnpq}\beta_n\beta_p^*\alpha_m\alpha_q^* > 0 \\ \forall \alpha, \beta \text{ with } \|\alpha\| = \|\beta\| = 1 \end{cases} \iff \begin{cases} r(\beta)_I > 0 \quad I = 1, \ldots, N_f \\ \forall \beta \text{ with } \|\beta\| = 1 \ . \end{cases} \tag{3.3.13}$$

The conditions on the r.h.s of Eq. (3.3.13) are necessary and sufficient. They are necessary since, if we find a negative eigenvalue for some $\beta = \hat{\beta}$, we can pick $\alpha = \hat{\alpha}$ to be an eigenvector associated to that eigenvalue and the quartic expression on the l.h.s of Eq. (3.3.13) evaluated at $\hat{\alpha},\ \hat{\beta}$ would be negative. They are sufficient because, if there is some value of α, and β, say $\hat{\alpha},\ \hat{\beta}$, in which the l.h.s of Eq. (3.3.13) is negative, or in other words $C(\hat{\beta})_{mq}\hat{\alpha}_m\hat{\alpha}_q^* < 0$, then $C(\hat{\beta})$ has to have at least one negative eigenvalue. Indeed, one can decompose $\hat{\alpha}$ on the basis $\{v_m^I\}$ of eigenvectors of $C(\hat{\beta})$, i.e., write $\hat{\alpha}_m = \hat{\alpha}_I v_m^I$, and obtain $C(\hat{\beta})_{mq}\hat{\alpha}_m\hat{\alpha}_q^* = \sum_I r(\beta)_I |\hat{\alpha}_I|^2 \left\|v^I\right\|^2 < 0$, which can only happen if at least one of the $r(\beta)_I$ is negative.

Alternatively, we can phrase the conditions in the r.h.s of Eq. (3.3.13) by noticing that expressions like Eq. (3.3.5) can be viewed as quadratic homogeneous polynomials in the complex components of α, with zero linear term. Then, requiring that the polynomial is greater than zero reduces to asking the multidimensional parabola to point upwards in any direction parametrized by α. Were this not the case, we could find an eigendirection with negative hessian eigenvalue, and following said direction we would end up in the negative region.

Since α and α^* always appear in pairs, and so do β and β^*, we can remove a total phase from each of them. Moreover, they are of fixed unit norm. Then, they contain $2N_f - 2$ free real parameters each, and the l.h.s of Eq. (3.3.13) depends on $4N_f - 4$ parameters. We trade it for the N_f conditions on the eigenvalues, each condition depending only on the $2N_f - 2$ real parameters contained in β. This rapidly turns out to be inconvenient for large values of N_f, but it works well for $N_f = 2,\ 3$. In particular, for $N_f = 2$, the two eigenvalues are positive if and only if the trace and the determinant of $C(\beta)_{mq}$ are positive. Notice that the discrepancy in the counting between the two requirements lies only in the number of free parameters we have to marginalize over. In fact, having shown that the r.h.s. and the l.h.s. of Eq. (3.3.13)

[7] We focus here on the c_{mnpq}, as a generalization to the a_{mnpq} is straightforward. For the sake of simplicity, and since this analysis applies everywhere, we also drop in this Section any superscript on c_{mnpq}.

[8] even if it were not, its antihermitian part would drop out of expressions like Eq. (3.3.5)

are equivalent, they have to boil down to the same conditions on the c_{mnpq} after all the α's and β's are removed.

3.3.3 A benchmark case: (4-Q) operators

We start by studying the positivity bounds (3.3.6) for the (4-Q) operators, and work it out step by step, the procedure for the other cases being very similar. Under the MFV assumption, the coefficients of the self-quartic (4-Q) operators take the form:

$$\begin{aligned} c(\xi)^{Q,i}_{mnpq} = \xi_1^{Q,i}(\delta_{mn}\delta_{pq}) + \xi_2^{Q,i}(X_{u,mn}\delta_{pq} + \delta_{mn}X_{u,pq}) + \xi_3^{Q,i}(\delta_{mq}\delta_{pn}) + \\ + \xi_4^{Q,i}(X_{u,mq}\delta_{pn} + \delta_{mq}X_{u,pn}), \qquad\qquad i = 1,2,3,4\ . \end{aligned}$$

In order to see what are the consequences imposed by Eq. (3.3.6) on the coefficients $\xi_A^{Q,i}$, we will first consider for simplicity a 2-flavor setting. This can be implemented in two slightly different ways: first, we will describe a theory of two generations only (Section 3.3.3), and then the restriction of a 3-flavor setting to the lightest two flavors (Section 3.3.3). Later, we proceed to study $N_f = 3$.

Positivity bounds on true $N_f = 2$ ansatz

First of all, if we reduce the symmetry group to be $SU(2)$, one can verify that $(X_{u,mq}\delta_{pn} + \delta_{mq}X_{u,pn})$ is not an independent structure, and its coefficient ρ_4 can be reabsorbed through a redefinition of the remaining three. This can be seen both by counting the allowed singlets in the tensor product, or in a more direct way, as shown in Appendix 3.B. This way, we can remove 4 $\xi_4^{Q,i}$ coefficients. However, since they provide just an innocuous redundancy, we will keep them at first and set them to zero at a later moment. Now, we need to parametrize the generic complex unit vector $\beta \in \mathbb{C}^2$. A possible parametrization is:

$$\beta = \begin{pmatrix} xe^{i\theta_x} \\ ye^{i\theta_y} \end{pmatrix} \equiv e^{i\theta_y} \begin{pmatrix} xe^{i\tilde{\theta}_x} \\ y \end{pmatrix} \qquad \text{with} \quad x^2 + y^2 = 1\ , \tag{3.3.14}$$

where all the parameters are real positive and $\tilde{\theta}_x = \theta_x - \theta_y$. As mentioned, we can remove the total phase and set $\theta_y = 0$. In the flavor basis (3.2.4), the up-Yukawa matrix is simply

$$Y_u = \begin{pmatrix} y_u & 0 \\ 0 & y_c \end{pmatrix} \sim y_c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.3.15}$$

y_u, y_c being the up and charm Yukawa respectively, and in the last step we specifically assumed a mass hierarchy and kept only the leading term. Although we also set y_c to zero in the $N_f = 3$ case, we keep it here since it corresponds to the largest coupling of this two-flavor theory. Now, as anticipated, we can translate the positivity condition (3.3.6) as two conditions on the eigenvalues of $C(\beta)_{mq} = c_{mnpq}\beta_n\beta_p^*$, or, equivalently, on its determinant and trace.

The assumed mass hierarchy, $y_c \gg y_u$ and the subsequent approximation $y_u \to 0$ ensure

that the trace and the determinant of $C(\beta)$ depend only on x and not on y nor on $\tilde{\theta}_x$. We find:

$$\mathrm{Tr}[C(\beta)] = 2x^2\left(\xi_2^{Q,i}+\xi_4^{Q,i}\right)y_c^2+\xi_4^{Q,i}y_c^2+\xi_1^{Q,i}+2\xi_3^{Q,i} \tag{3.3.16}$$

$$\begin{aligned}\det[C(\beta)] = {} & x^4\left(\xi_2^{Q,i}+\xi_4^{Q,i}\right)^2 y_c^4 - x^2y_c^2\left(\xi_2^{Q,i}+\xi_4^{Q,i}\right)\left(y_c^2\left(\xi_2^{Q,i}-\xi_4^{Q,i}\right)-2\xi_3^{Q,i}\right)+ \\ & +\left(\xi_4^{Q,i}y_c^2+\xi_3^{Q,i}\right)\left(\xi_1^{Q,i}+\xi_3^{Q,i}\right)\ .\end{aligned} \tag{3.3.17}$$

The trace is a linear function of x^2 which varies within the interval $[0,1]$. Thus, it is positive for any value of x in this interval if and only if its values at its boundaries are positive. The determinant, on the other hand, is a quadratic polynomial in x^2. One can verify that such parabola is positive in $[0,1]$ if and only if[9]:

- it is positive at the boundaries, and
- one of the following conditions is met:

$$\Delta<0 \text{ or } a<0 \text{ or } b(b+2a)>0\ ,$$

where a, b and c are defined by the parametrization $\det[C(\beta)]\equiv ax^4+bx^2+c$.

Putting everything together, and after some simplifications, we get the full set of conditions:

$$\text{(4-Q) } (N_f{=}2)\text{:}\quad \begin{cases} \xi_4^{Q,i}y_c^2+\xi_3^{Q,i}>0 \\ 2y_c^2\left(\xi_2^{Q,i}+\xi_4^{Q,i}\right)+\xi_1^{Q,i}+\xi_3^{Q,i}>0 \\ \xi_1^{Q,i}+\xi_3^{Q,i}>0 \\ y_c^4(\xi_4^{Q,i}-\xi_2^{Q,i})\left(\xi_2^{Q,i}+3\xi_4^{Q,i}\right)+8\xi_3^{Q,i}\xi_4^{Q,i}y_c^2+4\left(\xi_3^{Q,i}\right)^2>0 \quad \text{or} \\ \quad \left(-4y_c^2\left(\xi_1^{Q,i}\xi_4^{Q,i}+\xi_2^{Q,i}\xi_3^{Q,i}\right)+y_c^4\left(\xi_2^{Q,i}-\xi_4^{Q,i}\right)^2-4\xi_1^{Q,i}\xi_3^{Q,i}\right)<0\ . \end{cases} \tag{3.3.18}$$

The allowed region specified by these bounds is shown in Fig. 3.2 as a function of the unique relevant parameter y_c.

One can notice in particular that the *natural* MFV benchmark point $\xi_{1,2,3}=1$ is compatible with the positivity bound (3.3.6) if and only if $y_c^2<2\left(1+\sqrt{2}\right)$. However, as mentioned already, a consistent MFV expansion in Y_u requires $y_c<1$. So for any consistent MVF expansion, the positivity bounds are easily satisfied.

$N_f=2$ revisited: projected $N_f=3$ onto $N_f=2$

Another approach one could follow to describe the $N_f=2$ case is to take the full $N_f=3$ setting and to restrict all flavor indices to be $\{1,2\}$. In our flavor basis (3.2.4), this turns out to be

[9] This is jusfied like this: assuming the determinant is positive in 0 and 1 we have the following options: if it has negative discriminant, it is positive in the whole interval. Otherwise, if the discriminant is positive, and if the parabola opens downwards, i.e., $a<0$, it is also positive within the interval. If $\Delta>0$ and $a>0$, we then need to make sure that the minimum falls outside $[0,1]$. This is done by requiring $x_{min}^2-x_{min}>0\longrightarrow b(b+2a)>0$.

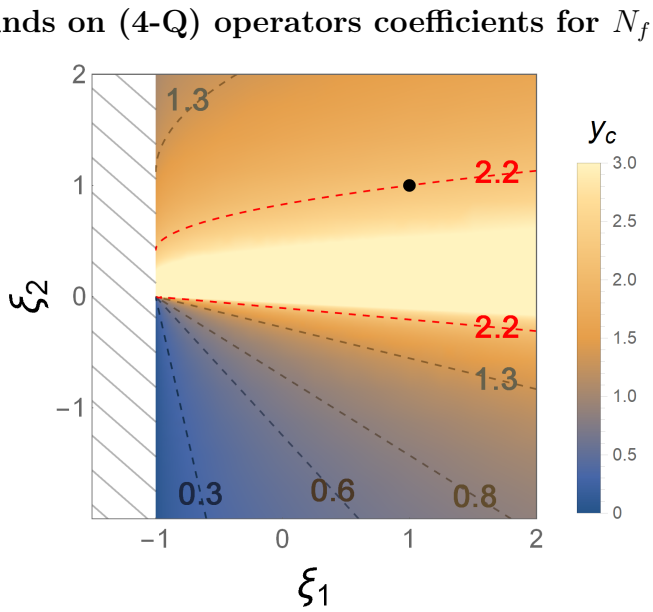


Figure 3.2: Plot showing the allowed region (in color) obtained for the (4-Q) operators restricted to $N_f = 2$ with generic ξ values, as y_c changes. Every region associated to a larger y_c value is contained in the previous ones: for instance, blue and dark orange regions are allowed for any y_c roughly smaller than 4, but forbidden for larger values of y_c. The redundant ξ_4 has been set to 0. In this case $\xi_3 > 0$, and using that the bounds are invariant under a full rescaling, we have set $\xi_3 = 1$, and plot the remaining two independent coefficients. As explained in the text, values of $y_c > 1$ are unphysical and are only plotted for visual reasons. The black point represents the *natural* MFV benchmark point $\xi_{1,2,3} = 1$. The red line contours the region corresponding to the threshold value of $y_c = \sqrt{2(1+\sqrt{2})}$: for bigger values of y_c, the *natural* benchmark point does not belong to the allowed region any more. The region $\xi_1 < -1$ is excluded for any value of y_c. For the physical value $y_c \approx 10^{-2}$, almost all points $(\xi_1 \geq -1, \xi_2)$ are allowed.

trivial, and it leads to the same result as depicted in Fig. 3.2. However, in the present case, one can modify the bounds using a $U(3)^3$ transformation: although the full $N_f = 3$ bounds (to be discussed in the next Section) are basis-independent, what we identify with the first two flavors is a basis dependent statement, and so are the bounds derived using the restricted $N_f = 2$ approach of this Section. As an example, we rotate to the basis of Eq. (3.2.5), and only then perform the projection. Consequently, the restricted version of Y_u is now:

$$X_{u,ij} = \left(Y_u Y_u^\dagger\right)_{ij} \sim (V_{CKM})_{3i}(V_{CKM}^*)_{3j} \ , \qquad\qquad i,j = 1,2 \ ,$$

while $\tilde{X}_y = 0$. Notice that this is different from the usual value of $X_{u,ij} \sim y_t^2 (V_{CKM})_{3i}(V_{CKM}^*)_{3j}$, as we fixed $y_t = 1$ to account for the y_t-resummation, like we explained at the end of Section 3.2.1. At this point, we are left with only a $U(2)^3$ symmetry, part of which, $U(2)_Q$ for the present (4-Q) case, can be used to diagonalize $X_{u,ij}$. This can always be done since $X_{u,ij}$ is still hermitian. Here we parametrize V_{CKM} through the Wolfenstein parametrization up to order $\mathcal{O}(\lambda^5)$, where $\lambda = \sin(\theta_c) \approx 0.225$ [226], θ_c being the Cabibbo angle:

$$V_{CKM} \approx \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda + \frac{1}{2}A^2\lambda^5[1 - 2(\rho + i\eta)] & 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4\left(1 + 4A^2\right) & A\lambda^2 \\ A\lambda^3[1 - (1 - \frac{1}{2}\lambda^2)(\rho + i\eta)] & -A\lambda^2 + \frac{1}{2}A\lambda^4[1 - 2(\rho + i\eta)] & 1 - \frac{1}{2}A^2\lambda^4 \end{pmatrix} .$$

Diagonalizing the 2×2 matrix $X_{u,ij}$, one can see that its only non-zero eigenvalue is:

$$\sigma \equiv A^2\lambda^4 \ , \tag{3.3.19}$$

which again we require to be < 1 to ensure a consistent expansion of the unitary V_{CKM} matrix. One can then easily map the positivity bounds on the $\xi_{1,2}$ parameter space using the results of the previous Section by substituting y_c^2 by σ. Notice that the values of $\xi_{1,2}$ compatible with the positivity bounds now depend on σ, which itself depends explicitly on the CKM entries, contrary to the general property presented in Section 3.3.1. Indeed, when going back to how the bounds where found in the first place, we see that the setting studied in this Section corresponds to a $2 \to 2$ scattering where the initial and final states are restricted to the first two flavors. However, fixing them breaks the flavor symmetry down to $U(2)^3$. The latter is then too small to absorb all the CKM parameters, which consistently enter the bounds. This is different in the full $N_f = 3$ case, as we now discuss.

$N_f = 3$

Now we wish to tackle the $N_f = 3$ setup. We can approximate $X_u = \left(Y_u Y_u^\dagger\right)_{ij} \sim \delta_{3i}\delta_{3j}$, $i, j = 1, 2, 3$, again after fixing $y_t = 1$. The only non-zero eigenvalue of this matrix is obviously 1. Barring a total irrelevant phase, we can parametrize the complex unit vector $\beta \in \mathbb{C}^3$ as

$$\beta = \begin{pmatrix} x e^{i\theta_x} \\ y e^{i\theta_y} \\ z \end{pmatrix}, \quad \text{with} \quad x^2 + y^2 + z^2 = 1 \ . \tag{3.3.20}$$

As before, the positivity bounds (3.3.6) mapped onto the $\xi^{Q,i}_{1,2,3,4}$ space will be obtained by requiring that the eigenvalues of the matrix $C(\beta)$ are all positive. For simplicity, let us first compute these eigenvalues for the *natural* benchmark point with all $\xi = 1$. Again, because of the mass hierarchy, $y_t \gg y_c, y_u$, the characteristic polynomial depends only on z and not on x, y, θ_x, θ_y. It factorizes nicely:

$$p(t) = -\left(t - z^2 - 1\right)\left[t^2 - 4t\left(z^2 + 1\right) + 4\left(1 + z^2 + z^4\right)\right], \tag{3.3.21}$$

so its first eigenvalue is simply $t_1(z) = z^2 + 1$ and it is always positive for $z \in [-1, 1]$. To avoid unpleasant radicals, we can evaluate the sum and product of the remaining two eigenvalues. This is equivalent to taking the trace and the determinant of $C(\beta)$ and subtracting and factoring out t_1 respectively:

$$t_2(z) + t_3(z) = 4\left(z^2 + 1\right) \tag{3.3.22}$$

$$t_2(z)t_3(z) = 4\left(z^4 + z^2 + 1\right) \ , \tag{3.3.23}$$

which both remain positive for any value of $z \in [-1, 1]$. We can then conclude that the benchmark point $\xi^{Q,i}_{1,2,3,4} = 1$ is fully consistent with the positivity bounds (3.3.6).

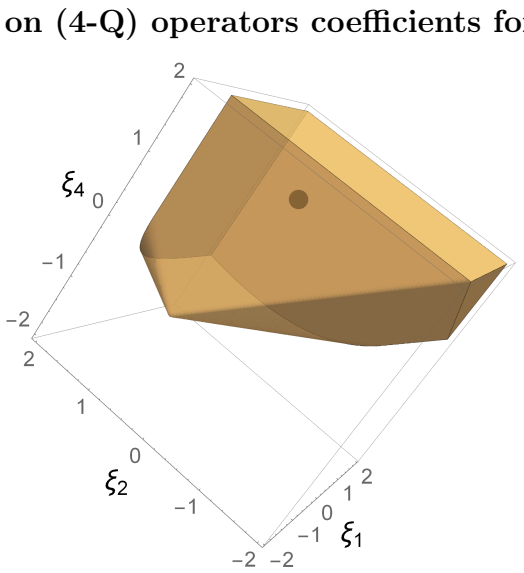


Figure 3.3: Plot showing in yellow the allowed region obtained for the (4-Q) (or, equivalently, (4-u)) operators with generic ξ values. Using the scaling invariance of the bounds and since $\xi_3 > 0$, we have set $\xi_3 = 1$, and plot the remaining three independent coefficients. The red dot indicates the *natural* MFV benchmark point, $\xi_{1,2,3,4} = 1$, that can be seen being inside the allowed region.

We can then extend our analysis to generic values of the ξ coefficients, as we did for $N_f = 2$. The explicit computations are shown in Appendix 3.D, while we report here only the resulting expression:

$$\text{(4-Q) } (N_f{=}3)\text{:} \quad \begin{cases} \xi_3^{Q,i} > 0 \\ \xi_1^{Q,i} + \xi_3^{Q,i} > 0 \\ \xi_3^{Q,i} + \xi_4^{Q,i} > 0 \\ \xi_1^{Q,i} + 2\xi_2^{Q,i} + 2\xi_3^{Q,i} + 3\xi_4^{Q,i} > 0 \\ \xi_1^{Q,i} + 2\xi_2^{Q,i} + \xi_3^{Q,i} + 2\xi_4^{Q,i} > 0 \\ \left((\xi_2^{Q,i} - \xi_4^{Q,i})^2 - 4\xi_3^{Q,i}\xi_2^{Q,i}\right) < 4\xi_1^{Q,i}\left(\xi_4^{Q,i} + \xi_3^{Q,i}\right) \\ \qquad \text{or } \left(\xi_4^{Q,i} - \xi_2^{Q,i} + 2\xi_3^{Q,i}\right)\left(\xi_2^{Q,i} + 3\xi_4^{Q,i} + 2\xi_3^{Q,i}\right) > 0 \ . \end{cases} \tag{3.3.24}$$

A visualization of such constraints is provided in Fig. 3.3.

3.3.4 (4-u), (4-d) self-quartic and cross-quartic operators

Here we continue the discussion for the remaining operators. We will see that, in most cases, we have already done most of the work that was needed, and the bounds for the coefficients of these operators can simply be obtained by taking appropriate limits of the ones in Eqs. (3.3.18) and (3.3.24), or by carefully looking at the order in which the indices are summed. We start by going through the remaining self-quartic operators, and then address the cross-quartic ones.

(4-u) operators As we have seen, the bounds we have found up to this point for the coefficients of the (4-Q) operators have turned out to depend exclusively on the eigenvalues of the

matrix X_u. This is the case for the (4-u) operators, too, provided we exchange $X_u \to \tilde{X}_u$. However, the eigenvalues of these matrices coincide. Although this is clearly a basis-independent statement, it can also just be seen in the basis in Eq. (3.2.5), where the two matrices coincide. As a consequence, the resulting bounds are the same for the (4-u) operators as for the (4-Q) ones, with the simple replacement $\xi_A^{Q,i} \to \xi_A^{u,i}$ in Eqs. (3.3.18) and (3.3.24). Similarly to what we did for the (4-Q) case, we can, in the $N_f = 2$ case, exploit the redundancy of $(\tilde{X}_{u,mq}\delta_{pn} + \delta_{mq}\tilde{X}_{u,pn})$ to remove two $\xi_4^{u,i}$ coefficients.

An exception where the simple exchange $\xi_A^{Q,i} \to \xi_A^{u,i}$ does not work is the case studied in Section 3.3.3. Indeed, in this setting the two matrices differ, and in particular $\tilde{X}_{u,mq} = 0$. The bounds are then retrieved in this case by sending $y_c \to 0$ in Eq. (3.3.18). This gives simply

$$\text{(4-u) } (N_f{=}2 \text{ revisited}): \begin{cases} \xi_3^{u,i} > 0, \\ \xi_1^{u,i} + \xi_3^{u,i} > 0 \ . \end{cases} \tag{3.3.25}$$

(4-d) operators As anticipated, the bounds for the (4-d) operators can be obtained by applying a formal limit to the ones we have already. Indeed, we see that by sending[10] $\tilde{X}_u \to 0$ or $X_u \to 0$ in the first two lines of Table 3.3 respectively, we retrieve the tensor structure associated to the (4-d) operators. Then, the $N_f = 2$ and $N_f = 3$ cases produce the same bounds as the ones we already saw in Eq. (3.3.25), namely:

$$\text{(4-d): } \begin{cases} \xi_3^{d,i} > 0, \\ \xi_1^{d,i} + \xi_3^{d,i} > 0 \ . \end{cases} \tag{3.3.26}$$

Fig. 3.4 shows a plot of the corresponding allowed region. Again, we can note that the *natural* MFV benchmark point, $\xi_{1,3}^{d,i} = 1$, is compatible with the positivity constraints.

Cross-quartic operators All the bounds on the cross-quartic operators give in fact much less information than the ones on the self-quartic ones, in our MFV setting. Indeed, looking at the index disposition in a_{mnpq} for the (2-d)(2-Q) and (2-d)(2-u) operators, we see that the objects in the l.h.s of Eq. (3.3.7) are essentially linear combinations of products of the norms of α and β, meaning terms of the form $\alpha_m A_{mq} \alpha_q^* \beta_n B_{np} \beta_p^*$, where one between A_{mn} and B_{mn} is a δ_{mn}, while the other is either a δ, or X_u, or $\tilde{X}_u$. Then, since all three of these matrices are (semi-)positive definite, taking all $\xi = 1$ means that Eq. (3.3.7) turns into a sum of positive terms and the bounds are trivially satisfied. The (2-u)(2-Q) case, however, has an additional term $\alpha_m \alpha_q^* \beta_n \beta_p^* (Y_u)_{nq} (Y_u^\dagger)_{mp}$. This is nothing but the modulus squared of $\beta_n Y_{u,nq} \alpha_q^*$, which is then also positive. Moreover, we can rest assured that, as long as we pick only positive values for the ξ coefficients, the bounds will be fulfilled, so we can definitely find acceptable $\mathcal{O}(1)$ values for them, independently on the renormalizable lagrangian parameters. This kills any hope of bounding them through these operators. To get a full picture, one can here, too,

[10] This is just a trick to get to the result, so one does not need to worry about spoiling the Yukawa hierarchy that led to the approximation at the beginning of Section 3.3.4.

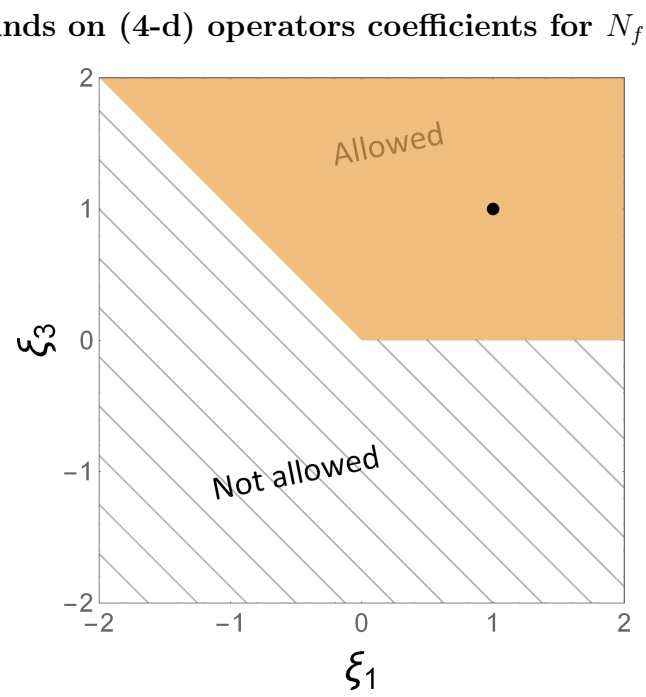


Figure 3.4: Plot showing in yellow the allowed parameter space for $\xi_1^{d,i}$ and $\xi_3^{d,i}$, $i = 1, 2$. The black dot indicates the *natural* MFV benchmark point, $\xi_{1,3} = 1$, that can be seen being inside the allowed region.

allow for generic ξ values, whose shape is dictated by the Yukawa couplings. Results as well as specific plots are shown in Appendix 3.D.

In the end, we have proven that in the *natural* MFV benchmark point where all the flavor-blind factors of the MFV expansion are chosen to be one, the positivity constraints of Ref. [207] are all satisfied.

Moreover, since the constraints themselves are unchanged if multiplied by a positive factor, what we have showed is actually that any configurations where the ξ coefficients are degenerate and positive are compatible with the positivity requirements. Because of this scaling invariance, and of the particular shape of the constraints, when considering generic values for the ξ coefficients, we can always rescale one of them to be 1 or -1, once per every type of operators. Taking this into account, we see that, for physical values of the parameters, at least in the $N_f = 3$ case, the parameter space spanned by the ξ coefficients is at least cut by a factor of two. Since there are a total of 14 independent types of operators under consideration, the overall allowed region is at least 2^{14} times smaller than the one with no positivity restriction[11].

3.4 Discussion and Conclusions

In Ref. [207], using arguments that rely on the analyticity and unitarity of the theory in the UV, the authors obtained positivity constraints on the coefficients of dimension-8 operators with 4 fermions. Starting from that result, we showed that Minimal Flavor Violation, perhaps the simplest way to generalize the Standard Model flavor structure to higher dimensional operators,

[11] This has obviously to be understood as the result of a limit, meaning that if the space of parameters is restricted to a box of volume $\mathcal{V}$, then the allowed region has a volume $\sim 2^{-14}\mathcal{V}$, where $\mathcal{V}$ is eventually sent to infinity.

can be made consistent with those positivity constraints. To show this, we have first identified the bounds on the (flavor-blind) parameters that control the MFV expansion of the EFT coefficients. Such bounds are obtained after we disentangle the physical quantities from other parameters describing initial and final states of the $2 \to 2$ scattering processes. We have shown how this can be done in the specific case where the scattered states contain non-trivial flavor structure. This allowed us to find bounds on the various coefficients that parametrize the dimension-8 operators under the MFV assumption. In the space spanned by these coefficients, the positivity constraints become in general non-linear. This is an example of a more general feature: the linear nature of the positivity bounds exhibited in the simplest cases (e.g. that of a single scalar of Section 3.1 or a single flavor of an uncharged Weyl fermion) is gradually lost when the number of degrees of freedom describing the scattered initial and final states is increased, for instance by considering non-trivial internal quantum numbers, thereby increasing the number of dimension-8 operators to be studied.

We showed in addition how the *natural* benchmark point, where all the flavor-blind parameters that enter the MFV expansion of EFT operators are degenerate and equal to unity, has proven to trivially satisfy the positivity constraints. This is true independently of the parameters of the renormalizable SM lagrangian, i.e. the fermion masses and the entries of the CKM matrix. More generally, every setting where the flavor-blind parameters are degenerate and positive is seen to be compatible with the positivity conditions. Still, the positivity constraints taken in their general form are such that they reduce the full parameter space by a factor of $\sin 2$[14].

Remarkably, MFV is not restraining enough to turn the positivity constraints on the dimension-8 operators into restrictions on the physical input parameters of the SM defined by the dimension-4 operators. An immediate consequence is that, for flavor models which are less restricting than MFV, such as the so called $U(2)^5$ model [222, 227], and which reduce to MFV for some values of the parameters, there exist at least an allowed region of the parameter space where the free coefficients are compatible with the bounds.

It would also be interesting to derive possibly tighter constraints following the approach of Ref. [198] and considering the scattering of states that are no longer SM gauge eigenstates. For instance, after suitable redefinitions accounting for the different operator basis used there, the bounds used in our work can be mapped into the ones in Eqs. (7)–(10) of Ref. [198], while Eqs. (11)–(12) are missing from our analysis[12]. However, in Ref. [198] only one flavor family is taken into account, whereas, due to the non-linearity of the additional bounds obtained, the $N_f \neq 1$ case cannot be straightforwardly tackled using, e.g., the approach we outlined in Section 3.3. Therefore, we sticked for our analysis to the bounds of Ref. [207], and left the study of the most general scatterings to future work.

[12]The precise signs of the bounds differ between Ref. [198] and Ref. [207], which is likely due to different conventions. We chose conventions so that the signs are those given in Ref. [207]. The fact that some bounds are missing in our analysis and in Ref. [207] is anyway independent of these sign conventions.

Appendix to Chapter 3

3.A Flavor-violation and large Yukawas

Our MFV expansion needs a little more justification, in particular concerning the resummation of the top-Yukawa y_t. Indeed, while Y_d has eigenvalues $\ll 1$, and thus allows for the approximation we explained in Section 3.2.1, this is not true for Y_u, whose biggest eigenvalue is $y_t \sim 1$. Consequently there is in principle no clear expansion in powers of Y_u as long as we keep all the coefficients in the operator expansion of $\mathcal{O}(1)$. However, in Ref. [209], for example, it is stated that any $\mathcal{O}(1)$ term in the (4-Q) case has to be of the form $\left(Y_u Y_u^\dagger\right)^n$. In the basis of Eq. (3.2.5), this reads:

$$\left(Y_u Y_u^\dagger\right)^n_{ij} \sim y_t^{2n} (V^*_{CKM})_{3i} (V_{CKM})_{3j} \ , \tag{3.A.1}$$

while the structure for the (4-u) case similarly reads $\left(Y_u^\dagger Y_u\right)^n_{ij} \sim y_t^{2n} \delta_{3i}\delta_{3j}$. Consequently, considering more Yukawa matrices does not change the flavor structure of the couplings, but simply demands to resum the powers of y_t. We dwell a bit on the details of this conclusion below.

3.A.1 Group theory argument

The restriction to Eq. (3.A.1) can be justified like this: in $SU(3)$, the invariant tensors are ε_{abc}, ε^{abc} and δ^a_b. Suppose we want to build a contribution to the (4-Q) case using n powers of Y_u and m powers of $Y_u^\dagger$. Then, since $\bar{Q}Q\bar{Q}Q$ is a singlet under $SU(3)_u$, we need to contract all of the $SU(3)_u$ indices of the various Y_u and $Y_u^\dagger$ using ε's or δ's. If we contract the indices of three Y_u's using an ε, then, for this product not to vanish, the $SU(3)_q$ indices of those matrices need to be fully antisymmetrized as well. However, by doing so we form a singlet under $SU(3)_Q \otimes SU(3)_u$, which just contributes as a redefinition of the coefficient of one operator of order $(Y_u)^{n-3}(Y_u^\dagger)^m$. Similarly for $Y_u^\dagger$. Then, we can only consistently use δ^a_b to contract the $SU(3)_u$ indices to build non-trivial structures, meaning that the building block is actually $Y_u Y_u^\dagger$,

which is a $\bar{\mathbf{3}} \otimes \mathbf{3}$ of $SU(3)_Q$, and that we have to take $n = m$. To proceed, we want eventually to contract the remaining indices with those from $\bar{Q}Q\bar{Q}Q$. We want to prove that this, too, can be done exclusively with δ's. Indeed, suppose we wanted to employ ε_{abc}. Similarly to what happened before, if we use it to contract three upstairs indices coming from three copies of $Y_u Y_u^\dagger$, symmetry imposes that the downstairs indices are antisymmetrized, too, giving rise to an uninteresting singlet. If we contract it with two upstairs indices from two $Y_u Y_u^\dagger$ and one from a Q, then the two downstairs indices form the two $Y_u Y_u^\dagger$ need to be antisymmetrized with an ε^{abc}. However the product of two epsilon is but a sum of products of deltas. Similarly, if we contract ε_{abc} with the two fundamental indices of the two Q, then the antifundamental ones of the two $\bar{Q}$ need to be contracted with an ε^{abc}, again giving products of δ's. In conclusion, a series in $\left(Y_u^\dagger Y_u\right)^n$ includes all allowed contractions. Similar reasoning holds for the (4-u) case, with the exchange of $SU(3)_Q$ and $SU(3)_u$ indices, and $Y_u Y_u^\dagger \to Y_u^\dagger Y_u$. For the (4-d) operators, there are simply no possible non-trivial insertions of Y_u. (2-d)(2-Q) and (2-d)(2-u) are similar to the former cases, with the exception that there are now only one index in the fundamental and one in the antifundamental of $SU(3)_Q$ in the structure coming from the spinors.

Finally, (2-Q)(2-u) requires a bit of attention. Again, contracting three Y_u's or three $Y_u^\dagger$'s with an epsilon tensor eventually produces a $SU(3)_Q \otimes SU(3)_u$ singlet. However, the spinor structure provides an index in the fundamental and one in the antifundamental for both $SU(3)_u$ and $SU(3)_Q$. So we can contract two Y_u or two $Y_u^\dagger$ with one index coming from the spinors. We cannot use an upstairs and a downstairs epsilon from the same group as that would reduce to sum of products of deltas. To sum up, we can use two epsilon tensors, one for each group, each with one index contracted to one coming from the spinor structure. Let us take the first one to be a $SU(3)_u$ ε^{abc}. Its remaining two indices can only be contracted with two downstairs $SU(3)_u$ indices from two Y_u's. Then, the $SU(3)_Q$ fundamental indices that these two matrices carry have to be antisymmetrized. This has to be done with the only remaining possible epsilon tensor, giving a structure as:

$$(\bar{u}u)_{\bar{u}_1 u_1}(\bar{Q}Q)_{\bar{q}_1 q_1}(Y_u)_{q_2\bar{u}_2}(Y_u)_{q_3\bar{u}_3}\varepsilon_{\bar{u}_1\bar{u}_2\bar{u}_3}\varepsilon_{q_1 q_2 q_3}(Y_u^\dagger)_{u_1\bar{q}_1} \ . \tag{3.A.2}$$

The other case, i.e., picking a $SU(3)_u$ ε_{abc}, gives the hermitian conjugate of (3.A.2). However, this term is subleading, as it can be immediately seen by plugging the leading contribution $(Y_u)_{q\bar{u}} \sim y_t \delta_{3\bar{u}}\delta_{3q}$. Moreover, it is not $U(1)_u$ invariant. In conclusion, only δ factors can be used to contract indices consistently. This forces the operators to be of the form already contained in the (2-u)(2-Q) line of Table 3.3, times an arbitrary number of $Y_u^\dagger Y_u$ of $Y_u Y_u^\dagger$, suitably contracted in. The latter can anyhow be reabsorbed in a redefinition of the overall ρ coefficients, as we show in the next Section.

3.A.2 Non-linear realization

One can also phrase the argument in favor of the single flavor-violating structure in Eq. (3.A.1) in a non-linear language. Indeed, when y_t is $\mathcal{O}(1)$, the necessary resummation of the expansion

in powers of y_t means that the flavor group is non-linearly realized [224,225]. In the basis (3.2.5) where the up-type Yukawa is diagonal, the EFT is an expansion around the vev

$$\langle Y_u \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \end{pmatrix} \ , \tag{3.A.3}$$

breaking $U(3)_Q \times U(3)_u$ down to $U(2)_Q \times U(2)_u \times U(1)_3$. Following Ref. [225], the building blocks for the EFT are found as follows. We first identify the Goldstone modes,

$$Y_u = e^{i\hat{\rho}_Q} \begin{pmatrix} \phi_u & 0 \\ 0 & y_t \end{pmatrix} e^{-i\hat{\rho}_u} \ , \text{ with } \hat{\rho}_i = \begin{pmatrix} 0 & \rho_i \\ \rho_i^\dagger & \theta_i \end{pmatrix} \ , \tag{3.A.4}$$

with ρ_i a complex 2-vector and $\theta_Q = -\theta_u \equiv \theta$, with θ a real field. The fields transform as

$$e^{i\hat{\rho}_i} \to V_i e^{i\hat{\rho}_i} U_i^\dagger(\hat{\rho}_i, V_i) \ , \quad \begin{pmatrix} \phi_u & 0 \\ 0 & y_t \end{pmatrix} \to U_Q(\hat{\rho}_Q, V_Q) \begin{pmatrix} \phi_u & 0 \\ 0 & y_t \end{pmatrix} U_u^\dagger(\hat{\rho}_u, V_u) \ , \tag{3.A.5}$$

where U_i are functions of V_i and of the Goldstones that belong to $U(2)_i \times U(1)_3$:

$$U_i = \begin{pmatrix} U_i^{2\times 2} & 0 \\ 0 & e^{i\phi_3} \end{pmatrix} \ . \tag{3.A.6}$$

We can also dress the down-type Yukawa matrix as well as some of the quark fields to obtain fields that transform as linear representations of $U(2)_Q \times U(2)_u \times U(1)_3$ under the full flavor group:

$$\tilde{Y}_d \equiv e^{-i\hat{\rho}_Q} Y_d \ , \quad \tilde{Q} \equiv e^{-i\hat{\rho}_Q} Q \ , \quad \tilde{u} \equiv e^{-i\hat{\rho}_u} u \ . \tag{3.A.7}$$

The new fields can be split as follows:

$$\tilde{Y}_d = \begin{pmatrix} \phi_d \\ \phi_d'^\dagger \end{pmatrix} \ , \quad \tilde{Q} = \begin{pmatrix} \tilde{Q}^{(2)} \\ \tilde{t}_L \end{pmatrix} \ , \quad \tilde{u} = \begin{pmatrix} \tilde{u}^{(2)} \\ \tilde{t}_R \end{pmatrix} \ , \tag{3.A.8}$$

where ϕ_d is a 2×3 matrix, ϕ_d' a 3-vector, $\tilde{Q}^{(2)}$ a doublet of $U(2)_Q$, $\tilde{t}_L$ a singlet, and similarly for $\tilde{u}$. The components transform as

$$\phi_d \to U_Q^{2\times 2} \phi_d V_d^\dagger \ , \quad \phi_d' \to e^{-i\phi_3} V_d \phi_d' \ , \quad \tilde{Q}^{(2)} \to U_Q^{2\times 2} \tilde{Q}^{(2)} \ , \tag{3.A.9}$$

$$\tilde{t}_L \to e^{i\phi_3} \tilde{t}_L \ , \quad \tilde{u}^{(2)} \to U_u^{2\times 2} \tilde{u}^{(2)} \ , \quad \tilde{t}_R \to e^{i\phi_3} \tilde{t}_R \ . \tag{3.A.10}$$

The fields above, together with d_R, ϕ_u, y_t and the invariance under $U(2)_Q \times U(2)_u \times U(1)_3 \times U(3)_d (\times U(3)_L \times U(3)_e)$, are the building blocks for the EFT. One should in principle also use

the covariant derivatives obtained from the Maurer-Cartan form

$$e^{-i\hat{\rho}_Q}\partial_\mu e^{i\hat{\rho}_Q} \ , \tag{3.A.11}$$

but they are identically zero when we freeze the Yukawa spurions to their background values. Summing up, the different fields and their representations are:

	$U(2)_Q$	$U(2)_u$	$U(1)_3$	$U(3)_d$
Fermions				
$\tilde{Q}^{(2)}$	**2**	**1**	0	**1**
$\tilde{t}_L$	**1**	**1**	+1	**1**
$\tilde{u}^{(2)}$	**1**	**2**	0	**1**
$\tilde{t}_R$	**1**	**1**	+1	**1**
$\tilde{d}$	**1**	**1**	0	**3**
Spurions				
ϕ_u	**2**	$\bar{\mathbf{2}}$	0	**1**
y_t	**1**	**1**	0	**1**
ϕ_d	**2**	**1**	0	$\bar{\mathbf{3}}$
ϕ'_d	**1**	**1**	-1	**3**

The background values of the spurions are obtained from $Y_u = \text{diag}(y_u, y_c, y_t)$, $Y_d = V_{CKM}\text{diag}(y_d, y_s, y_b)$. We see that all spurions but y_t are small, so that there exists an expansion in terms of small Yukawas and CKM elements. Every EFT term constructed from ϕ_u, ϕ_d, ϕ'_d up to a given order is then completed by multiplying it by an arbitrary function of y_t. In particular, the approximation we have been discussing in this appendix is the one where all ys but y_t are zero. At this order, the only fermion bilinears that can enter the dimension-8 coefficients in Table 3.2 in a flavor-invariant way are

$$\overline{\tilde{Q}^{(2)}}\gamma^\mu\tilde{Q}^{(2)} \ , \quad \overline{\tilde{t}_L}\gamma^\mu\tilde{t}_L \ , \quad \overline{\tilde{u}^{(2)}}\gamma^\mu\tilde{u}^{(2)} \ , \quad \overline{\tilde{t}_R}\gamma^\mu\tilde{t}_R \ , \quad \overline{\tilde{d}}\gamma^\mu\tilde{d} \ . \tag{3.A.12}$$

When we freeze the spurions to their background values, the Goldstone fields $\hat{\rho}$ are zero in the basis where Y_u is diagonal, one can simply remove the tildes in the expression above and rename $\tilde{t}_L = Q_3, \tilde{t}_R = u_3$. In the basis where $Y_u = V_{CKM}^\dagger\text{diag}(y_u, y_c, y_t)$ and $Y_d = \text{diag}(y_d, y_s, y_b)$, Eq. (3.A.12) becomes

$$\begin{gathered}\left[\delta_{ij} - (V_{CKM}^*)_{3i}\,(V_{CKM})_{3j}\right]\overline{Q_i}\gamma^\mu Q_j \ , \quad (V_{CKM}^*)_{3i}\,(V_{CKM})_{3j}\,\overline{Q_i}\gamma^\mu Q_j \ , \\ \overline{u^{(2)}}\gamma^\mu u^{(2)} \ , \quad \overline{u_3}\gamma^\mu u_3 \ , \quad \overline{d}\gamma^\mu d \ .\end{gathered} \tag{3.A.13}$$

This is consistent with Eq. (3.A.1): the two terms in the first line can be combined to reconstruct $\delta_{ij}\overline{Q_i}\gamma^\mu Q_j$ and $(V_{CKM}^*)_{3i}\,(V_{CKM})_{3j}\,\overline{Q_i}\gamma^\mu Q_j$, which are the flavor structures that are obtained from $\left(Y_u Y_u^\dagger\right)^n$. The u-quark terms in the second line can be combined to reconstruct $\delta_{ij}\overline{u_i}\gamma^\mu u_j$ and $\delta_{3i}\delta_{3j}\overline{u_i}\gamma^\mu u_j$, which are the flavor structures that are obtained from $\left(Y_u^\dagger Y_u\right)^n$. The d-quark terms are flavor diagonal, as they should at order Y_d^0.

3.B Redundancy of the ρ_4 structure in $N_f = 2$

Here we provide a proof showing that, in $SU(2)$, $\bar{u}_m u_n \bar{u}_p u_q (\tilde{X}_{u,mq}\delta_{pn} + \delta_{mq}\tilde{X}_{u,pn})$ is redundant with respect to the other structures contained in the first line of Table 3.3, and the corresponding coefficient, which we called ρ_4, can be reabsorbed in the definitions of the remaining three. An analogous discussion can be done for $\bar{Q}_m Q_n \bar{Q}_p Q_q (X_{u,mq}\delta_{pn} + \delta_{mq}X_{u,pn})$ and the second line of Table 3.3. First of all, define $M_{mn} \equiv \bar{u}_m u_n$. $SU(2)$ does not distinguish between fundamental and antifundamental indices, and all summations need to be performed with ε_{ab}, the only invariant tensor. Then

$$\begin{aligned} M_{ij}M_{kl}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\tilde{X}_{u,nm} &= \left(M_{\{ij\}} + \frac{1}{2}M_{ab}\varepsilon_{ab}\varepsilon_{ij}\right)\left(M_{\{kl\}} + \frac{1}{2}M_{cd}\varepsilon_{cd}\varepsilon_{kl}\right)\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\tilde{X}_{u,nm} = \\ &= M_{\{ij\}}M_{\{kl\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\tilde{X}_{u,nm} + M_{ab}\varepsilon_{ab}\varepsilon_{nk}\varepsilon_{lm}M_{\{kl\}}\tilde{X}_{u,nm} + \\ &\quad - \frac{1}{4}\varepsilon_{ab}M_{ab}\varepsilon_{cd}M_{cd}\varepsilon_{nm}\tilde{X}_{u,nm} \ . \end{aligned} \tag{3.B.1}$$

The second piece can be absorbed by a shift in ρ_1, while the third one with a shift on ρ_2. The first piece, instead, splits as:

$$\begin{aligned} M_{\{ij\}}M_{\{kl\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\tilde{X}_{u,nm} &= M_{\{ij\}}M_{\{kl\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in}\left(\tilde{X}_{u,\{nm\}} + \frac{1}{2}\varepsilon_{nm}\varepsilon_{ef}\tilde{X}_{u,ef}\right) = \\ &= M_{\{ij\}}M_{\{kl\}}\tilde{X}_{u,\{nm\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in} + \frac{1}{2}M_{\{ij\}}M_{\{kl\}}\varepsilon_{jk}\varepsilon_{il}\varepsilon_{ef}\tilde{X}_{u,ef} \ . \end{aligned} \tag{3.B.2}$$

Here, the second term is reabsorbed through a shift on ρ_3, while the first one vanishes:

$$\begin{aligned} M_{\{ij\}}M_{\{kl\}}\tilde{X}_{u,\{nm\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in} &\overset{j\leftrightarrow k}{=} -M_{\{ij\}}M_{\{kl\}}\tilde{X}_{u,\{nm\}}\varepsilon_{kj}\varepsilon_{lm}\varepsilon_{in} \overset{(ij)\leftrightarrow(kl)}{=} \\ &= -M_{\{ij\}}M_{\{kl\}}\tilde{X}_{u,\{nm\}}\varepsilon_{il}\varepsilon_{jm}\varepsilon_{kn} \overset{m\leftrightarrow n}{=} \\ &= -M_{\{ij\}}M_{\{kl\}}\tilde{X}_{u,\{nm\}}\varepsilon_{il}\varepsilon_{jn}\varepsilon_{km} \overset{i\leftrightarrow j}{=} \\ &= -M_{\{ij\}}M_{\{kl\}}\tilde{X}_{u,\{nm\}}\varepsilon_{jl}\varepsilon_{in}\varepsilon_{km} \overset{l\leftrightarrow k}{=} \\ &= -M_{\{ij\}}M_{\{kl\}}\tilde{X}_{u,\{nm\}}\varepsilon_{jk}\varepsilon_{lm}\varepsilon_{in} \ . \end{aligned} \tag{3.B.3}$$

In conclusion, the examined term provides no new structure and can be set to 0.

3.C Contribution of other SMEFT operators

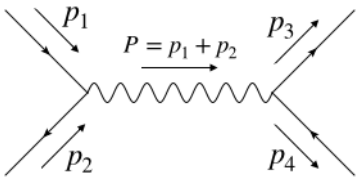
One comment should also be made about the contributions of other SMEFT operators to the positivity bounds. Indeed, in the usual version of the latter that we consider in this paper, they are associated to the s^2 coefficient of a $2 \to 2$ forward amplitude. By dimensional analysis only, one sees that such a s^2 growth can be obtained from a dimension-8 four-fermions contact term, as we considered in the main text, but also from the product of two coefficients, both of dimension-6, or one of dimension-5 and the other of dimension-7, or one of dimension-8 and at

least one of dimension-4.

Let us discuss first the case of dimension-6 operators. We mentioned already that dimension-6 four-fermions operators do not enter the positivity bounds, due to their softer UV behaviour when compared to dimension-8 operators. Nevertheless, there are operators at dimension-6 which, when combined together, contribute at tree level to the four-fermions amplitudes with a UV behaviour similar to that of dimension-8 four-fermions operators. A simple example is the following: consider the dimension-6 operator that couples the photon to the up-type right-handed quark current,

$$\mathcal{L} \supset \frac{c}{\Lambda^2}\partial_\mu J_\nu F^{\mu\nu} \ , \tag{3.C.1}$$

where $J_\mu = \overline{u}_m \gamma_\mu u_m$. This coupling generates a four-fermion amplitude whose s-channel component is depicted below and reads



$$\begin{aligned}\mathcal{A} &= i\frac{c^2}{\Lambda^4}\overline{v_2}\gamma_\nu u_1 P^2 \left(\frac{\eta^{\nu\sigma}}{P^2} - (1-\xi)\frac{P^\nu P^\sigma}{(P^2)^2}\right) P^2 \overline{u_3}\gamma_\sigma v_4 \\ &\approx i\frac{c^2}{\Lambda^4} P^2 \overline{v_2}\gamma_\mu u_1 \overline{u_3}\gamma^\mu v_4 \ ,\end{aligned} \tag{3.C.2}$$

where u_i, v_i are (anti)particle polarizations and we used the fact that the particles are effectively massless at high energies, so that $\not{p}u(p), \not{p}v(p) \approx 0$. We would have obtained the same result with the four fermion interaction $\frac{c^2}{\Lambda^4}\partial_\mu J_\nu \partial^\mu J^\nu$, which is of the kind subject to positivity bounds. Thus, c^2 should be added to the combination of dimension-8 coefficients that are constrained to be positive, modifying the bound. However, Eq. (3.C.1) is a redundant operator that can consistently be ignored in the first place. Indeed, enforcing the photon equation of motion derived from Eq. (3.C.1),

$$\partial^\mu F_{\mu\nu} - \frac{c}{\Lambda^2}\Box J_\nu = 0 \ , \tag{3.C.3}$$

we find that $\frac{c}{\Lambda^2}\partial_\mu J_\nu F^{\mu\nu} = \frac{c^2}{\Lambda^4}\partial_\mu J_\nu \partial^\mu J^\nu$, consistently with our previous analysis. Thus, we can set $c = 0$ at no cost. Using the Warsaw basis for the dimension-6 SMEFT [212], and considering the high-energy phase where all particles are massless and the electroweak symmetry unbroken, it is straightforward to check that there are no dimension-6 contributions to the four-fermion positivity bounds at tree level. This differs from the case of $2 \to 2$ gauge bosons scattering, where non-redundant trilinear couplings enter the bounds at tree-level and strenghthen the bounds on dimension-8 operators [195, 196, 199].

Similarly, contributions that could spoil our bounds come from combinations of dimension-5 and dimension-7, or of dimension-4 and dimension-8 terms. Since in the SMEFT the only dimension-5 term is the Weinberg operator HHL_iL_j, which does not contain quarks, the former possibility is irrelevant. Instead, combining one dimension-4 and one dimension-8 operator could give rise to a diagram with the same shape as the one in Eq. (3.C.2) or an analogous one with an intermediate Higgs boson. The former could arise from combining a gauge interaction and a dimension-8 operator of the schematic form $\mathcal{O} \sim \partial^3 X \bar{\psi}\psi$, with X any of the SM field strengths, and the latter from combining a Yukawa interaction and an operator $\mathcal{O} \sim \partial^4 H \bar{\psi}\psi$.

However, in both cases the dimension-8 operator is proportional to some equations of motion [228, 229], and can thus be reabsorbed by redefining the coefficients of lower-dimensional operators, as just shown for the dimension-6 ones. Such conclusions can be actually verified in a number of ways, including Hilbert series techniques and on-shell methods. In conclusion, it is useful to have in mind that specifying the basis of operators is necessary when writing positivity bounds.

3.D Bounds for generic ξ values

3.D.1 Self-quartic

We report here the bounds we obtain when we allow for generic values of the ρ coefficients in the c_{mnqp} and a_{mnpq} tensors. The procedure is exactly the same as in the (4-Q) case with $N_f = 2$, which was shown in Section 3.3.3. We keep explicit factors of y_t^2 to make it easier to identify the contributions of contracted Yukawa matrices, but we remind that they should be put to 1 for consistency of the expansion.

Let us see for example how the (4-Q) case changes. Again, we impose the first eigenvalue of the matrix $C(\beta)$, as well as the sum and product of the remaining two, to be positive:

$$t_1(z) = \xi_3^{Q,i} + \xi_3^{Q,i} y_t^2 z^2 \tag{3.D.1}$$

$$t_2(z) + t_3(z) = \xi_1^{Q,i} + 2\xi_3^{Q,i} + \xi_4^{Q,i} y_t^2 + 2y_t^2 z^2 (\xi_2^{Q,i} + \xi_4^{Q,i}) \tag{3.D.2}$$

$$\begin{aligned} t_2(z)t_3(z) = {} & (\xi_1^{Q,i} + \xi_3^{Q,i})(\xi_3^{Q,i} + \xi_4^{Q,i} y_t^2) \\ & + z^2 y_t^2 (\xi_2^{Q,i} + \xi_4^{Q,i})(y_t^2(\xi_4^{Q,i} - \xi_2^{Q,i}) + 2\xi_3^{Q,i}) \\ & + z^4 y_t^4 (\xi_2^{Q,i} + \xi_4^{Q,i})^2 \ . \end{aligned} \tag{3.D.3}$$

The first two are linear objects in z^2, which varies within the interval $[0, 1]$. Thus, they are positive $\forall z$ if and only if their value at the boundaries of the interval is also positive. However, the third expression is a quadratic polynomial in z^2, which we can parametrize as $az^4 + bx^2 + c$. Then, as already seen, we require this polynomial to be positive at the boundaries and to satisfy $\Delta < 0$ or $a < 0$ or $b(b + 2a) > 0$. Putting everything together, and after some simplifications, we get the full set of conditions:

$$\text{(4-Q):} \begin{cases} \xi_3^{Q,i} > 0 \\ \xi_1^{Q,i} + \xi_3^{Q,i} > 0 \\ \xi_3^{Q,i} + \xi_4^{Q,i} y_t^2 > 0 \\ \xi_1^{Q,i} + 2\xi_2^{Q,i} y_t^2 + 2\xi_3^{Q,i} + 3\xi_4^{Q,i} y_t^2 > 0 \\ \xi_1^{Q,i} + 2\xi_2^{Q,i} y_t^2 + \xi_3^{Q,i} + 2\xi_4^{Q,i} y_t^2 > 0 \\ y_t^2 \left(y_t^2 (\xi_2^{Q,i} - \xi_4^{Q,i})^2 - 4\xi_3^{Q,i}\xi_2^{Q,i} \right) < 4\xi_1^{Q,i} \left(\xi_4^{Q,i} y_t^2 + \xi_3^{Q,i} \right) \\ \qquad \text{or} \quad \left(y_t^2 (\xi_4^{Q,i} - \xi_2^{Q,i}) + 2\xi_3^{Q,i} \right) \left(y_t^2 (\xi_2^{Q,i} + 3\xi_4^{Q,i}) + 2\xi_3^{Q,i} \right) > 0 \ . \end{cases} \tag{3.D.4}$$

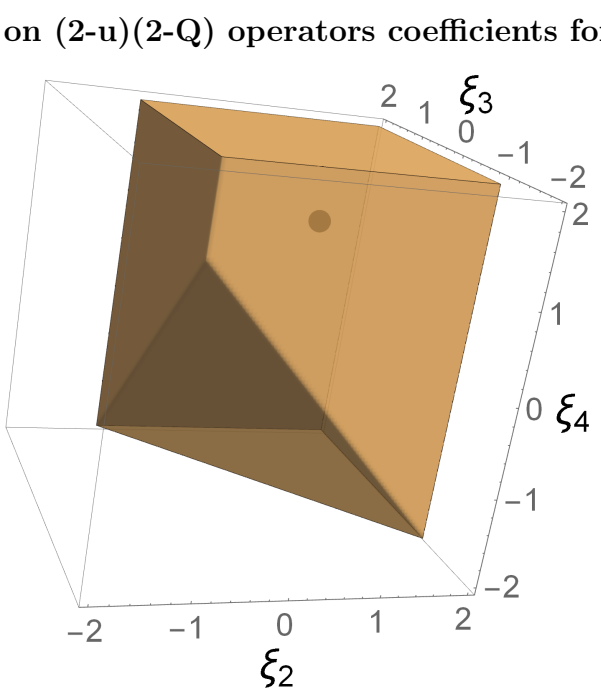


Figure 3.D.1: Plots showing in yellow the allowed region obtained for the (2-u)(2-Q) operators, with generic ξ values. Here $\xi_1 > 0$, so we have rescaled it to 1, and we show the allowed region for the remaining three. y_t is set to 1. The red dot indicates the *naturak* MFV benchmark point, $\xi_{1,2,3,4} = 1$, which can be seen being inside the allowed region.

Again, since within our basis choice X_u and $\tilde{X}_u$ are the same, identical results apply to the $\xi_k^{u,i}$, $k = 1, 2, 3, 4$.

3.D.2 Cross-quartic operators

In the case of cross-quartic operators, the $C(\beta)$ matrix turns out to have two equal eigenvalues, and the conditions are simply:

$$\text{(2-u)(2-Q)}: \begin{cases} \xi_1^{uQ,i} > 0 \\ \xi_1^{uQ,i} + \xi_2^{uQ,i} y_t^2 > 0 \\ \xi_1^{uQ,i} + \xi_3^{uQ,i} y_t^2 > 0 \\ \xi_1^{uQ,i} + (\xi_2^{uQ,i} + \xi_3^{uQ,i} + \xi_4^{uQ,i}) y_t^2 > 0 \ . \end{cases} \tag{3.D.5}$$

A plot of the results is reported in Fig. 3.D.1. Here we also report the bounds for the (2-d)(2-Q) and the (2-d)(2-u) operators, which can both be computed directly or obtained from the previous ones with appropriate limits. Again, since X_u and $\tilde{X}_u$ coincide, so do the bounds:

$$\text{(2-d)(2-Q)}: \begin{cases} \xi_1^{dQ,i} > 0, \\ \xi_1^{dQ,i} + \xi_2^{dQ,i} y_t^2 > 0 \ , \end{cases} \tag{3.D.6}$$

$$\text{(2-d)(2-u):} \quad \begin{cases} \xi_1^{du,i} > 0, \\ \xi_1^{du,i} + \xi_2^{du,i} y_t^2 > 0 \ . \end{cases} \tag{3.D.7}$$

3.E ρ dependence of ξ coefficients

We report here the relations defined between the ξ and ρ coefficients for every operator under consideration.

Self-quartic For the operators built out of (4-Q) fields, the coefficients look like:

$$\begin{aligned} c_{mnpq}^{Q,i} = \rho_1^{Q,i}(\delta_{mn}\delta_{pq}) + \rho_2^{Q,i}(M_{mn}\delta_{pq} + \delta_{mn}M_{pq}) + \rho_3^{Q,i}(\delta_{mq}\delta_{pn}) + \\ + \rho_4^{Q,i}(M_{mq}\delta_{pn} + \delta_{mq}M_{pn}) \ . \end{aligned} \tag{3.E.1}$$

The conditions coming from Eq. (12) of Ref. [207] are:

$$\begin{cases} \alpha_m\alpha_q^*\beta_n\beta_p^* \left(c_{mnpq}^{Q,1} + \frac{1}{4}c_{mnpq}^{Q,2} + \frac{1}{3}c_{mnpq}^{Q,3} + \frac{1}{12}c_{mnpq}^{Q,4}\right) > 0 \\ \alpha_m\alpha_q^*\beta_n\beta_p^* \left(c_{mnpq}^{Q,2} + \frac{1}{3}c_{mnpq}^{Q,4}\right) > 0 \\ \alpha_m\alpha_q^*\beta_n\beta_p^* \left(c_{mnpq}^{Q,3} + \frac{1}{4}c_{mnpq}^{Q,4}\right) > 0 \alpha_m\alpha_q^*\beta_n\beta_p^* c_{mnpq}^{Q,4} > 0 \ . \end{cases} \tag{3.E.2}$$

Then, we can define the linearly transformed coefficients $\xi(\rho)$:

$$\begin{cases} \xi_k^{Q,1} & \equiv \rho_k^{Q,1} + \frac{1}{4}\rho_k^{Q,2} + \frac{1}{3}\rho_k^{Q,3} + \frac{1}{12}\rho_k^{Q,4} \\ \xi_k^{Q,2} & \equiv \rho_k^{Q,2} + \frac{1}{3}\rho_k^{Q,4} \\ \xi_k^{Q,3} & \equiv \rho_k^{Q,3} + \frac{1}{4}\rho_k^{Q,4} \\ \xi_k^{Q,4} & \equiv \rho_k^{Q,4} \end{cases} \qquad k = 1,2,3,4 \ , \tag{3.E.3}$$

to turn the bounds into

$$\alpha_m\alpha_q^*\beta_n\beta_p^* c(\xi)_{mnpq}^{Q,i} > 0 \qquad i = 1,2,3,4 \ , \tag{3.E.4}$$

i.e., in the form of Eq. (3.3.6). For the (4-d) operators, the bounds are the same as in the (4-u) case:

$$\begin{aligned} \alpha_m\alpha_q^*\beta_n\beta_p^* \left(c_{mnpq}^{d,1} + \frac{1}{3}c_{mnpq}^{d,3}\right) > 0 \ , \\ \alpha_m\alpha_q^*\beta_n\beta_p^* c_{mnpq}^{u,3} > 0 \ . \end{aligned} \tag{3.E.5}$$

Thus, with a redefinition identical to (3.3.4), we recast them into the desired form in Eq. (3.3.6).

Cross-quartic The bounds on the (2-u)(2-Q) operators are:

$$\alpha_m \alpha_q^* \beta_n \beta_p^* \left(a_{mnpq}^{uQ,1} + \frac{1 \pm 3}{12} a_{mnpq}^{uQ,3} \right) > 0 \ . \tag{3.E.6}$$

Upon defining:

$$\begin{cases} \xi_k^{uQ,1} & \equiv \rho_k^{uQ,1} + \frac{1}{3}\rho_k^{uQ,3} \\ \xi_k^{uQ,3} & \equiv \rho_k^{uQ,1} - \frac{1}{6}\rho_k^{uQ,3} \end{cases} \qquad k = 1,2,3,4 \ , \tag{3.E.7}$$

they are recast as in Eq. (3.3.7). The shape of the bounds on the remaining cross quartic operators is the same, therefore identical redefinitions

$$\begin{cases} \xi_k^{dQ,1} & \equiv \rho_k^{dQ,1} + \frac{1}{3}\rho_k^{dQ,3} \\ \xi_k^{dQ,3} & \equiv \rho_k^{dQ,1} - \frac{1}{6}\rho_k^{dQ,3} \end{cases} \qquad k = 1,2 \ , \tag{3.E.8}$$

$$\begin{cases} \xi_k^{du,1} & \equiv \rho_k^{du,1} + \frac{1}{3}\rho_k^{du,3} \\ \xi_k^{du,3} & \equiv \rho_k^{du,1} - \frac{1}{6}\rho_k^{du,3} \end{cases} \qquad k = 1,2 \ , \tag{3.E.9}$$

are needed to recast the bounds in the form of Eq. (3.3.7).

CHAPTER 4

CP violation in the SMEFT

In Chapter 3, we studied some properties of the SMEFT at dimension-eight, showing how not all the parameter space spanned by the EFT coefficients is allowed if we want to complete our theory in the UV while still respecting locality and causality. As we have explained in Chapter 2 when introducing the basics of EFTs, the higher the dimension of an operator, the smaller its contribution to observables turns out to be, at least naively. Indeed, although it has been argued that the impact of dimension-eight operators can be disentangled in observations in some specific cases [140, 142–144], we still expect dimension-five and six operators to have more phenomenological relevance. In this Chapter[1], we focus on dimension-six operators of the SMEFT, and characterize the breaking of CP symmetry they may carry. We will explain why CP violation is such an important feature of the Standard Model, and especially why precise measurements of its magnitude can potentially hide hints of NP. We argue that SMEFT represents the best framework to study such hints, and bring forward the idea that the most fitting way of parametrizing CP violation in this context is one that clearly reflects the invariance of physical quantities under the choice of basis.

4.1 A first brief look into CP violation

We mentioned already how much the Standard Model is the result of a delicate equilibrium, and how this fragility makes it a non-trivial task to build UV models solving some issues without introducing disruptions somewhere else. On the other hand, this fragility can be exploited experimentally, as even a slight deviation from this delicate structure can be a hint pointing to New Physics. This point is maybe best highlighted through an example. The most prominent one is probably represented by the search for Flavor Changing Neutral Currents (FCNC). As the name suggests, these are interactions where a quark changes its flavor without altering its

[1] This Chapter is mostly a reproduction of the work contained in [69]

electric charge. In the Standard Model alone, these interactions are forbidden at tree-level, as there is no vertex allowing for them, and are extremely suppressed at loop-level thanks to the so called GIM mechanism [4]. As a consequence, for example, the branching ratios of some meson decays are extremely suppressed with respect to some other ones (see for example Ref. [46] for a review). This behavior, however, is very specific of the SM, and an example of the delicate equilibrium characterizing it. Indeed, this feature is quite easily disrupted in numerous extensions of the SM, like Two Higgs Doublet Models [48] or SUSY-inspired ones [47]. Then, it is not a surprise that the search for such FCNC is a goal obstinately pursued by experiments [230–234]. Another outstanding example is CP violation (CPV). When building the SM lagrangian, CP is not imposed as a symmetry. Let us then look at the Yukawa couplings for the quarks:

$$\mathcal{L}_{\mathrm{SM}_4} \supset -\tilde{H}\bar{Q}_i Y_{u,ij} u_j - H\bar{Q}_i Y_{d,ij} d_j + \text{ h.c. } . \tag{4.1.1}$$

A priori, $Y_{u,d}$ are arbitrary *complex* matrices. However, we have seen that the rest of the SM_4 lagrangian enjoys a $U(3)^5$ flavor symmetry, acting on Q, u and d as

$$Q \to U_q Q \qquad\qquad u \to U_u u \qquad\qquad d \to U_d d \ , \tag{4.1.2}$$

where the U_i are 3×3 unitary matrices. Then, we have also seen how under this transformation

$$Y_u \to U_Q^\dagger Y_u U_u \qquad\qquad Y_d \to U_Q^\dagger Y_d U_d \ . \tag{4.1.3}$$

It can be checked that, under CP

$$\mathcal{L}_{\mathrm{SM}_4} \overset{CP}{\to} \mathcal{L}_{\mathrm{SM}_4} \quad \text{iff} \quad Y_d = Y_d^* \text{ and } Y_u = Y_u^* \ . \tag{4.1.4}$$

More precisely, because of the symmetry in Eq. (4.1.2), this means that CP is conserved in the SM_4 if there exist at least a transformation like the one of Eq. (4.1.3) that makes the Yukawa matrices real. Indeed, making use of Eq. (4.1.3), we can get very close to such a feature, and remove all complex phases in $Y_{u,d}$ but a single one, δ_{CKM}.

Thus, as it turns out, CP is indeed broken in the quark sector, albeit by a single object. Remarkably, this feature is specific of the presence of 3 distinct generations of quarks. Indeed, where we dealing with 2×2 Yukawa matrices, the $U(2)_i$ transformations would have been enough to remove all phases from them, and a basis where they are all real would exist [235]. A main consequence of this feature is, then, that any process where CP-violation from the SM can be observed needs the simultaneous presence of all three quark generations. In terms of Feynman diagrams, this means that all three Yukawa couplings must appear, together with appropriate powers of the CKM matrix. Since those couplings are quite small, being all $\ll 1$ but the top-quark one, it follows that processes sensitive to CP-violation are extremely suppressed. This property is obviously reflected on observables. Prominent probes of CP violations are found, most notably, in the Electric Dipole Moments (EDMs) of quarks and leptons [236]. For

example, it has been shown that, considering the SM alone, the EDM of the electron receives its first non-zero contribution at four-loops level [237], and in general all SM fermions EDMs are extremely suppressed [132, 236]. This is, again, a feature very specific to the SM, and various UV models introduce additional sources of CPV (see e.g. Refs. [235, 236]). It is no surprise, then, that experimental searches focus so much on the precise measurement of CP-violating quantities, such as the electron [238], muon [239], tau [240] and neutron [241] EDMs, as well as on CP-violating coupling of the Higgs boson to electroweak gauge bosons [242–244] and to the top quark [245–247]. Consistently with the approach of this work, it is certainly worth understanding how deviations from the SM pattern of CPV can appear in the framework of EFTs, which guarantees a certain degree of model-independence. Previous attempts in evaluating the impact of dimension-6 CPV couplings on observables, mostly on EDMs, can be found in Refs. [131, 248–256].

While such works focus on the contribution from specific flavors, often only accounting for the dominant top quark contributions to the observables, we here wish to carry out a systematic study that takes into account the fact that CPV possesses a flavor-independent meaning. To this end, first of all, we need to be more careful in the definition of CPV in the SM_4 alone. As we have seen, the presence of a single phase in the CKM matrix is the result of exhausting all the transformations contained in the flavor group to bring the Yukawa matrix in a specific form. However, since CP-violation is a physical quantity, it cannot depend on the choice of flavor basis. As it turns out, there is indeed a single, flavor-invariant quantity that characterizes CPV in SM_4,

$$J_4 \equiv \text{Im Tr}\left[Y_u Y_u^\dagger, Y_d Y_d^\dagger\right]^3 = 3\ \text{Im Det}\left[Y_u Y_u^\dagger, Y_d Y_d^\dagger\right] \ . \tag{4.1.5}$$

The quantity J_4 goes under the name of Jarlskog invariant [257–259], and vanishes iff CP is conserved. The structure of J_4 is such that it is not modified by unitary reshuffling of the quark fields, which means that it corresponds to a physical quantity. In addition, its expression shows that CPV in the SM_4 is not a feature of Y_u or Y_d alone, but a feature of the whole model which can only be assessed with the knowledge of both matrices (and in particular, of the fact whether they can be simultaneously diagonalized). This "collective" property of CPV, namely the fact that it depends on several lagrangian parameters at once, is a key property of the SM_4, as well as of its extensions. This also holds for strong CPV, whose order parameter is given by $\theta_{QCD} - \arg\det(Y_u Y_d)$, and resides simultaneously within the θ-term of QCD and within the quark Yukawa matrices (see Appendix 4.F for more details).

As we mentioned, while J_4 is the only order parameter of CPV in the fermionic sector of the SM_4, additional ones need to be specified whenever the SM_4 is extended, see e.g. Ref. [154, 156, 260–263] for multi-Higgs doublet models, Ref. [264–266] for the case of supersymmetric extensions of the SM_4, Ref. [259, 267–269] for the case of additional generations of matter, Ref. [155, 159, 270–276] for the inclusion of neutrino masses, Ref. [277] for vector-like extensions, Ref. [278] for CP-violating ALP EFTs or Ref. [279] for models of spontaneous CP breaking. In this Chapter, we assume that there are no new light degrees of freedom (d.o.f.) below, or close to, the weak scale, but we remain agnostic about the presence of heavy states. In that

case, the SM_4 should be understood as the low-energy approximation of some fundamental UV dynamics and we can extend it into an effective field theory (EFT). Under the assumption that a decoupling limit can be consistently taken, the adequate description is then the SMEFT. In particular, if we look at the expansion of its lagrangian in terms of operators of growing dimension,

$$\mathcal{L} = \mathcal{L}_{\mathrm{SM}_4} + \sum_i \frac{C_i}{\Lambda^{d_i-4}} \mathcal{O}_i \ , \tag{4.1.6}$$

we notice that the coefficients C_i are generically complex and introduce a large number of new sources of CPV in SMEFT (see the counting in Ref. [133] at dimension $d_i \leq 6$). In the rest of the Chapter, we will be in particular interested in CPV associated to flavorful Wilson coefficients, whose analysis requires the careful extraction of basis-independent physical parameters, which account for the collective properties of CPV. Therefore, the new CPV phases should be captured by CP-odd flavor-invariants, similar to J_4. We focus here on the CPV phases found in the fermionic sector, since the bosonic sources of CPV in the dimension-six SMEFT are trivially flavor-invariant[2]. Moreover, we focus on the limit of vanishing neutrino masses throughout the rest of the Chapter.

The requirement of looking for invariant quantities needs to be more precisely justified. For instance, it is common to associate a complex top-quark Yukawa coupling with a new source of CPV, without referring to invariants. However a complex top Yukawa only signals CPV if one works in a given flavor basis where the top is a mass eigenstate of real mass. One may wonder how to describe CPV in SMEFT, without any specific reference to the IR physics, such as the masses or the electroweak vacuum. This picture is for instance justified if one cares about new sources of CPV which arise from the matching to a given UV model, which should be analyzed at an energy scale above the electroweak vacuum expectation value (vev) and should not depend on the details of the IR dynamics, such as specific flavor bases motivated by low energy considerations. Flavor-invariants are well-suited to answer such questions, as they allow one to capture physical and collective properties of the model, to parametrize CP-odd observables in a basis-independent way, and also to make the matching with UV models and their properties easier, by decorrelating the parametrization of CPV quantities from flavor bases connected to the IR properties of SM_4 particles.

Let us use the aforementioned example of the top-quark complex Yukawa coupling to offer a preview of the flavor-invariants considered in this paper. As said above, a top-Higgs lagrangian with a complex Yukawa coupling

$$-\mathcal{L} \supset m_t \bar{t} t + \frac{m_t}{v} \bar{t} (\kappa_t + i\tilde{\kappa}_t \gamma_5) t h = m_t \bar{t}_L t_R + \frac{m_t}{v} \bar{t}_L (\kappa_t + i\tilde{\kappa}_t) t_R h + \text{h.c.} \ , \tag{4.1.7}$$

[2]For the 6 CP-odd bosonic operators appearing in the basis in Ref. [70], indeed, the condition for CP conservation is simply for their coefficient to vanish.

violates CP. It can originate in SMEFT from the dimension-four and six Yukawa couplings,[3]

$$-\mathcal{L} \supset y_t \bar{Q}_L t_R \tilde{H} + \frac{C_{tH}}{\Lambda^2} \bar{Q}_L t_R \tilde{H} |H|^2 + \text{h.c.} \quad . \tag{4.1.8}$$

The above expression generalizes to three generations of matter in SMEFT upon replacing $Q_L, t_R, y_t, C_{tH} \to Q_{L,i}, t_{R,j}, Y_{u,ij}, C_{uH,ij}$. Focusing on the diagonal entries of C_{uH} in a basis where Y_u is diagonal and real (with non-degenerate entries, as is experimentally relevant), their three imaginary parts Im $C_{uH,ii}$ violate CP. They can be captured by three independent flavor-invariants, whose expressions read

$$L_{k=1,2,3} = \text{Im Tr}\Big(X_u^{k-1} C_{uH} Y_u^\dagger\Big) \tag{4.1.9}$$

where $X_u = Y_u Y_u^\dagger$ as in Chapter 3. In the basis where $Y_u = \text{diag } y_{u,i}$ is diagonal and real, one has $L_{k=1,2,3} = y_{u,i}^{2k-1} \text{Im } C_{uH,ii}$, with an implicit sum over i. Therefore, at fixed, non-vanishing and non-degenerate $y_{u,i}$, the set of three $L_{k=1,2,3}$ maps to that of three $C_{uH,ii=1,2,3}$ in a bijective fashion, hence those three invariants capture the three new sources of CPV associated to up quark complex Yukawa couplings, as in Eq. (4.1.7). In the generic case however, C_{uH} has off-diagonal entries even in the basis where Y_u is diagonal, all of which can be complex, such that one needs nine flavor-invariants to capture the nine new sources of CPV in C_{uH}. Although there is no unique choice, one possible set of invariants reads

$$L_{k=1,\ldots,9} = \left\{ \begin{array}{lll} \text{Im Tr}\left(C_{uH} Y_u^\dagger\right) & \text{Im Tr}\left(X_u X_d C_{uH} Y_u^\dagger\right) & \text{Im Tr}\left(X_d^2 X_u^2 C_{uH} Y_u^\dagger\right) \\ \text{Im Tr}\left(X_u C_{uH} Y_u^\dagger\right) & \text{Im Tr}\left(X_d X_u C_{uH} Y_u^\dagger\right) & \text{Im Tr}\left(X_u X_d^2 X_u^2 C_{uH} Y_u^\dagger\right) \\ \text{Im Tr}\left(X_d C_{uH} Y_u^\dagger\right) & \text{Im Tr}\left(X_u^2 X_d^2 C_{uH} Y_u^\dagger\right) & \text{Im Tr}\left(X_d X_u^2 X_d^2 C_{uH} Y_u^\dagger\right) \end{array} \right\} \tag{4.1.10}$$

where again $X_d = Y_d Y_d^\dagger$ as in Chapter 3.[4]

Naively, the number of new flavor-invariants should match that of the new sources of CPV. However, observables in SMEFT are truncated at a given order in inverse powers of Λ, according to the SMEFT power counting, and it happens that not all sources of CPV contribute to physical observables at this given order as a result of non-interference. In this Chapter, we illustrate this fact by discussing CPV observables truncated at the leading $1/\Lambda^2$ order, to which several of the new sources of CPV at dimension-six cannot contribute.[5] We therefore

[3] At order $1/\Lambda^2$ (and for one generation only) the correspondence between the different coefficients reads [254, 280]

$$m_t = \frac{y_t v}{\sqrt{2}} \left(1 + \frac{1}{2} \frac{\text{Re } C_{tH} v^2}{y_t \Lambda^2}\right) \quad , \quad \kappa_t + i\tilde{\kappa}_t = 1 + \frac{\text{Re } C_{tH}}{y_t} \frac{v^2}{\Lambda^2} + i \frac{\text{Im } C_{tH}}{y_t} \frac{v^2}{\Lambda^2}$$

provided we start in a basis where y_t is already real.

[4] The careful reader may note that the former L_3 in Eq. (4.1.9) does not appear anymore in Eq. (4.1.10). We have removed L_3 because it is not independent of the first two L's when two up-type quark masses are degenerate. This does not happen for any of the invariants in Eq. (4.1.10), which are therefore preferred (see the following sections for a systematic treatment).

[5] Such CP-odd observables are at most linear in the dimension-six SMEFT coefficients, and correspond to the interference between the SM_4 and the leading SMEFT contributions to a given amplitude. A more thorough characterization of the observables we consider can be found in Section 4.3.1.

carefully differentiate between the power counting for observables, which we truncate at order $1/\Lambda^2$, and that of SMEFT operators, which we only include up to dimension-six, i.e. also up to order $1/\Lambda^2$. As we will explain, not all associated SMEFT coefficients can interfere with the SM_4 contribution to a given observable, and therefore they cannot all contribute at leading $1/\Lambda^2$ order to observables. We dub those which can *primary coefficients*, while we refer to the others as *secondary coefficients*. We perform the counting of the number of (both CP-even and CP-odd) SMEFT primary coefficients. Among those, the CP-odd fermionic ones, whose number is 699, are captured by flavor-invariants linear with respect to the dimension-six SMEFT coefficients, and an explicit and complete set of such flavor invariants is built. Consequently, we present a necessary and sufficient set of flavor-invariants, such that CP is conserved at $\mathcal{O}(1/\Lambda^2)$ iff they vanish, together with J_4, the strong-CP phase, and the 6 CP-violating dimension-6 bosonic operators, so that they form a set of $699(+1+1+6)$ order parameters of CPV.

4.2 The collective nature of CP breaking in the SM(EFT)

In order to motivate why we define CP-odd invariants, it is useful to review first one important and interesting aspect of CP breaking in SMEFT: it is collective. Indeed, it relies on the simultaneous presence of several complex parameters in the lagrangian, which cannot all be made simultaneously real, even using the freedom to redefine fields (or equivalently, to define appropriately the CP transformation). In this section we review CP violation in SM_4, in order to establish our conventions and present several of the claims related to CP violation which will be repeatedly encountered in this Chapter.

4.2.1 CP-violation and complex parameters

The usual lore is that complex parameters in the lagrangian violate CP. At the level of the fermionic lagrangian, this claim leaves implicit crucial subtleties related to field redefinitions. The correct statement is instead that *the lagrangian is CP-symmetric iff one can redefine the fields so as to make all couplings real.*[6] In the SM_4, this explains why only one phase out of the six naively contained in the CKM matrix is physical and breaks CP. For instance, were the CKM matrix equal to the following unitary matrix

$$V_{\rm CKM} = \begin{pmatrix} \frac{72-21i}{325} & \frac{4}{13} & -\frac{12i}{13} \\ -\frac{12}{13} & \frac{576+168i}{1625} & \frac{49-168i}{1625} \\ -\frac{96-28i}{325} & -\frac{57}{65} & -\frac{24i}{65} \end{pmatrix} , \tag{4.2.1}$$

[6] This is strictly speaking only true for models with continuous internal symmetries. When discrete symmetries are present, there exists the possibility that the couplings are pseudo-real, namely related to their complex conjugates via flavor transformations. Then one would get a CP-symmetric lagrangian iff there exists a flavor transformation which sends all couplings to their complex conjugates at once. See Ref. [281] for an example, or the section 4.3 of Ref. [282] for more details and references. In this text, we focus on the bulk of the SM(EFT) parameter space where any discrete symmetry is embedded into a continuous one (this is for instance automatic for non-degenerate spectra, see Section 4.4.2 and Appendix 4.A).

it would not violate CP, although it is explicitly complex. Indeed, one can write

$$\begin{pmatrix} \frac{72-21i}{325} & \frac{4}{13} & -\frac{12i}{13} \\ -\frac{12}{13} & \frac{576+168i}{1625} & \frac{49-168i}{1625} \\ -\frac{96-28i}{325} & -\frac{57}{65} & -\frac{24i}{65} \end{pmatrix} = \begin{pmatrix} \frac{3-4i}{5} & 0 & 0 \\ 0 & \frac{4-3i}{5} & 0 \\ 0 & 0 & \frac{3-4i}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{13} & \frac{4}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{24}{65} & \frac{7}{65} \\ -\frac{4}{13} & -\frac{57}{65} & \frac{24}{65} \end{pmatrix} \begin{pmatrix} \frac{4+3i}{5} & 0 & 0 \\ 0 & \frac{3+4i}{5} & 0 \\ 0 & 0 & \frac{4-3i}{5} \end{pmatrix} , \tag{4.2.2}$$

and absorb all the factorized diagonal phases into the fermion fields, in order to obtain a real orthogonal CKM matrix. Such a manipulation cannot be done for the following matrix,

$$V_{\rm CKM} = \begin{pmatrix} \frac{2172-5004i}{8125} & -\frac{1784+432i}{8125} & -\frac{2427+5196i}{8125} \\ -\frac{3747+3996i}{8125} & \frac{3324+912i}{8125} & \frac{4772-1164i}{8125} \\ -\frac{308+144i}{1105} & -\frac{4389+2052i}{5525} & \frac{1848+864i}{5525} \end{pmatrix} . \tag{4.2.3}$$

However, whether it yields a CPV SM_4 depends on the fermion spectrum. Indeed, were two quark masses equal, the kinetic lagrangian would have a $U(2)$ flavor symmetry, allowing for more general fermion field redefinitions. For instance, if $m_u = m_c$, we can redefine the first two flavors of up-type quarks in order to absorb the 2×2 unitary matrix which appears at the top left of the first factor on the right-hand-side of the following equality:

$$\begin{pmatrix} \frac{2172-5004i}{8125} & -\frac{1784+432i}{8125} & -\frac{2427+5196i}{8125} \\ -\frac{3747+3996i}{8125} & \frac{3324+912i}{8125} & \frac{4772-1164i}{8125} \\ -\frac{308+144i}{1105} & -\frac{4389+2052i}{5525} & \frac{1848+864i}{5525} \end{pmatrix} = \begin{pmatrix} -\frac{176+468i}{625} & -\frac{9-12i}{25} & 0 \\ \frac{351-132i}{625} & \frac{16+12i}{25} & 0 \\ 0 & 0 & \frac{77+36i}{85} \end{pmatrix} \begin{pmatrix} \frac{3}{13} & \frac{4}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{24}{65} & \frac{7}{65} \\ -\frac{4}{13} & -\frac{57}{65} & \frac{24}{65} \end{pmatrix} , \tag{4.2.4}$$

obtaining again a real orthogonal CKM matrix.

As is clear from these numerical examples, and as we will repeatedly illustrate, it is more convenient to phrase the condition for CP-violation in a way which does not require scanning over all possible field redefinitions. If the theory preserves CP, the following CP transformation of the (non-degenerate) fermionic mass eigenstates ψ (together with those of bosonic fields [235]) leaves the lagrangian invariant in some field basis

$$(\mathcal{CP})\psi(t,\vec{x})(\mathcal{CP})^{-1} = \gamma^0 C \overline{\psi}^T(t,-\vec{x}) , \tag{4.2.5}$$

where C is the (antisymmetric) charge conjugation matrix such that $\gamma^\mu C = -C(\gamma^\mu)^T$. As we anticipated, this implies that the lagrangian couplings are real (in this field basis). For instance, if we assume that there exists the following coupling in the theory,

$$\mathcal{L} \supset c_{1212} \left(\overline{\psi}_1 \gamma^\mu \psi_2\right) \left(\overline{\psi}_1 \gamma_\mu \psi_2\right) + \text{h.c.} \quad , \tag{4.2.6}$$

we learn from the invariance under the CP transformation in Eq. (4.2.5) that c_{1212} is real. However, the opposite statement is that the theory violates CP iff the transformation in Eq. (4.2.5)

is never a symmetry, whatever the field basis chosen. This is not equivalent to saying that c_{1212} is complex in some basis, but that *whatever the basis chosen, there exists at least one lagrangian parameter which is genuinely complex*[7] (which usually depends on the basis). Therefore, the condition for CPV which we look for takes the following schematic form

$$\text{CPV} \iff \text{Im}\,(\text{something independent of the field basis}) \neq 0. \tag{4.2.7}$$

Such a basis-independent object precisely defines a CP-odd flavor invariant. Within the framework of SMEFT, we can define flavor invariants order-by-order in the power counting. At leading order, the condition for CPV reads:

$$\text{CPV at } \mathcal{O}(1/\Lambda^2) \iff \text{Im}\,(\text{something of } \mathcal{O}(1/\Lambda^2) \text{ independent of the field basis}) \neq 0\ .$$

4.2.2 Flavor transformations and flavor bases

As we just discussed, a meaningful statement about CP violation in the SM(EFT) must account for the possibility of field redefinitions. In addition, the SM(EFT) lagrangian is naturally written in the unbroken electroweak phase, which does not differentiate between the three fermionic generations. Therefore, it should be possible to characterize CP violation without referring to any specific flavor labeling, in particular without identifying which combinations correspond to the mass eigenstates.

As we reviewed when discussing MFV in Section 3.2, when the lagrangian is written in terms of the gauge multiplets relevant in the unbroken phase, the kinetic lagrangian in the fermion sector (including the gauge couplings) is invariant under a $U(3)^5 = U(3)_{Q_L} \times U(3)_{u_R} \times U(3)_{d_R} \times U(3)_{L_L} \times U(3)_{e_R}$ flavor group, where each factor acts on the flavor indices of the associated fermion field. This group is the largest under which all SMEFT coefficients can be assigned a spurious transformation so as to leave the full SM(EFT) lagrangian unchanged.

Similarly to our conclusion in the previous section, the lagrangian is CP-symmetric iff one can redefine the fields so as to make all couplings real. When redefining the fermion fields by means of a $U(3)^5$ flavor transformation, the precise values of all flavored couplings in SMEFT are mixed up, and the real and imaginary parts get scrambled. Consequently, any order parameter of CP breaking cannot correspond to the imaginary part of a given coefficient, but instead should map to the imaginary part of a flavor-invariant combination of coefficients.

In order to build such invariants, it is useful to notice that the flavored SMEFT couplings transform under the flavor group as spurions with transformation properties which depend on the operator they couple to. For the Yukawa couplings at dimension-four, the transformations were already presented in Chapter 3, and we list them again in Table 1. Each (anti)fundamental representation has a charge $(-)1$ under the associated abelian group in the decomposition $U(3)_X = SU(3)_X \times U(1)_X$, where $X = Q, u, d, L, e$.[8] Performing field redefinitions which belong

[7] For pseudo-real couplings, the statement is rather that all complex couplings cannot be turned simultaneously into their conjugates via the same change of basis.

[8] Out of the 5 $U(1)$ factors only the gauged $U(1)_Y$ and the combinations $U(1)_{B-L}$ are conserved at the quantum level, while the other are broken by anomalies.

to the flavor group, this set of spurious charges allows one to easily compute the couplings in the redefined theory, and to easily identify objects which are independent of such redefinitions.

	$SU(3)_Q$	$SU(3)_u$	$SU(3)_d$	$SU(3)_L$	$SU(3)_e$
Y_u	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
Y_d	$\mathbf{3}$	$\mathbf{1}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$
Y_e	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\bar{\mathbf{3}}$

Table 1: flavor transformation properties of the Yukawa matrices treated as spurions

Using flavor transformations, one can reach flavor bases where the Yukawa matrices have a specific form, and which we will sometimes use to explicitly evaluate invariants. Two possibilities are those shown in Eqs. (3.2.4) and (3.2.4), namely the up basis

$$Y_u = \mathrm{diag}(y_u, y_c, y_t) \ , \quad Y_d = V_{\rm CKM} \cdot \mathrm{diag}(y_d, y_s, y_b) \ , \quad Y_e = \mathrm{diag}(y_e, y_\mu, y_\tau) \ , \tag{4.2.8}$$

where all y's are real and positive and $V_{\rm CKM}$ is the Cabibbo–Kobayashi–Maskawa (CKM) matrix, and the down basis, where

$$Y_u = V_{\rm CKM}^\dagger \cdot \mathrm{diag}(y_u, y_c, y_t) \ , \quad Y_d = \mathrm{diag}(y_d, y_s, y_b) \ , \quad Y_e = \mathrm{diag}(y_e, y_\mu, y_\tau) \ . \tag{4.2.9}$$

Fixing this shape for the Yukawa couplings exhausts all flavor transformations but some diagonal ones.[9] If, in addition, a phase choice is made on the CKM matrix (for instance imposing that all its phases are given in terms of a single one as in usual parameterizations), no flavor freedom remains but the conserved baryon and lepton number symmetries $U(1)_B \times U(1)_L$. When we make such a choice below, we use the following parametrization of the CKM matrix [283],

$$V_{\rm CKM} = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{-i\delta_{\rm CKM}} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta_{\rm CKM}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{\rm CKM}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\rm CKM}} & -c_{12}s_{23} - c_{23}s_{12}s_{13}e^{i\delta_{\rm CKM}} & c_{13}c_{23} \end{pmatrix} \ , \tag{4.2.10}$$

where $c_X,\ s_X = \cos(\theta_X),\ \sin(\theta_X)$.

4.2.3 The collective nature of CP breaking in the SM_4

We now review in some detail the (well known) collective nature of CP breaking in the SM_4, which is useful for our purpose.

The fact that CP breaking is collective in the SM_4 can be understood from the fact that CP is preserved with up to two fermionic generations [284,285], so that one needs the simultaneous presence of three generations to be sensitive to CP violation. A question is then: what is the order parameter of CP-breaking in the SM_4?

[9] Their precise form depends on the basis. In the down basis, they are of the form $\mathrm{diag}(e^{i\alpha_X^1}, e^{i\alpha_X^2}, e^{i\alpha_X^3})$, such that $\alpha_Q^i = \alpha_d^i, \alpha_L^i = \alpha_e^i$ (RH up-quark phases are unconstrained).

In order to answer this question unambiguously, one needs to mod out the impact of flavor transformations. Besides the use of invariants, a way to exhaust all flavor transformations is to work in a well-defined flavor basis, for instance in the up or down basis defined earlier. Using the remaining phase rotations allowed in such bases removes all complex parameters but one, which fully specifies the flavor basis. The only leftover complex quantity can be written in a way which is independent of the phase rotations [257–259, 267, 286, 287],

$$\mathcal{J} = \mathrm{Im}\left(V_{\mathrm{CKM},us} V_{\mathrm{CKM},cb} V^*_{\mathrm{CKM},ub} V^*_{\mathrm{CKM},cs}\right) = s_{12} c_{12} s_{13} c_{13}^2 s_{23} c_{23} \sin(\delta_{\mathrm{CKM}}) \ , \tag{4.2.11}$$

where the last equality uses Eq. (4.2.10). It is straightforward to check that $\mathcal{J} = 0$ for the matrix in Eq. (4.2.1).

As we saw in Section 4.2.1, when two masses (of same-type quarks) are degenerate, there is a larger degeneracy within the up or down bases, and one can further remove the leftover complex parameter from the SM_4 lagrangian. Therefore, the genuine order parameter of CP breaking in the SM, J_4, is proportional to[10]

$$J_4 \propto (m_t^2 - m_c^2)(m_t^2 - m_u^2)(m_c^2 - m_u^2)(m_b^2 - m_s^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2)\mathcal{J} \ . \tag{4.2.12}$$

One can show that there are no additional factors to J_4 [257].

We have constructed J_4 in a specific flavor basis, but it is useful to have expressions valid in all bases. In that respect, instead of looking for complex quantities invariant under mere phase rotations, one would rather consider invariants under the full flavor group. As we anticipated in the introduction, J_4, which goes under the name of Jarlskog invariant, then corresponds to [257]

$$J_4 \equiv \mathrm{Im\ Tr}\left[Y_u Y_u^\dagger, Y_d Y_d^\dagger\right]^3 = 3\ \mathrm{Im\ Det}\left[Y_u Y_u^\dagger, Y_d Y_d^\dagger\right] \ . \tag{4.2.13}$$

Defined as above, J_4 is independent of the choice of flavor basis, as can be checked from the transformations in Table 1. Evaluating Eq. (4.2.13) for instance in the up or down basis, the connection with Eq. (4.2.12) is made obvious,

$$J_4 = 6(y_t^2 - y_c^2)(y_t^2 - y_u^2)(y_c^2 - y_u^2)(y_b^2 - y_s^2)(y_b^2 - y_d^2)(y_s^2 - y_d^2)\mathcal{J} \ . \tag{4.2.14}$$

It can be shown that the statement that CP is broken in the SM_4 is equivalent to saying that J_4 does not vanish [257], therefore it corresponds to the genuine order parameter for CP breaking in the SM_4.

[10] This expression depends on differences of squared masses and not, e.g., on $m_t - m_u$. This is due to the fact that only the modulus of a fermion mass is physical: any quark mass can be made complex by an appropriate rephasing of the associated RH field without changing other SM_4 couplings, hence we must consider the rephasing-invariant quantity $m_\psi m_\psi^* = |m_\psi|^2$ for any fermion ψ, which reduces to the mass squared in field bases where the mass is real.

4.2.4 The collective nature of CP breaking beyond the SM_4

The search for flavor-invariant order parameters for CP breaking beyond the SM_4 is subject to very similar discussions. As an example, let us consider the SM_4 with a single generation of fermions, extended by a dimension-six Yukawa coupling for the up-type quark:

$$\mathcal{L} = \mathcal{L}_{\mathrm{SM}_4} + \frac{C_{uH}}{\Lambda^2}|H|^2\overline{Q}_L u_R \tilde{H} + \mathrm{h.c.} \tag{4.2.15}$$

It is well known that such a coupling can generate a two-loop contribution to the electron EDM, a CPV observable, which reads at $\mathcal{O}(1/\Lambda^2)$ [131, 288, 289]

$$\frac{d_e}{e} = -\frac{1}{48\pi^2}\frac{v m_e m_u}{m_h^2}\frac{\mathrm{Im}(C_{uH})}{\Lambda^2} F_1\left(\frac{m_u^2}{m_h^2}, 0\right) \ , \tag{4.2.16}$$

where for conciseness we only kept the dominant contribution due to photons in the loop, where m_h and v are the Higgs mass and vev, respectively, and where

$$F_1(a,0) = \int_0^1 dx \frac{\ln\left(\frac{a}{x(x-1)}\right)}{a - x(x-1)} \ . \tag{4.2.17}$$

The result in Eq. (4.2.16) may suggest that $\mathrm{Im}(C_{uH})$ acts as an order parameter of CP breaking in this theory.[11] However, this imaginary part could be rotated away by a chiral transformation of the up quark field (for instance by redefining $u_R \to e^{-i\arg(C_{uH})}u_R$), so one could wonder if there remains an observable electron EDM. The resolution to this puzzle is due to the implicit assumption that the computation is performed in a basis where the up quark has a real mass. In a generic flavor basis, the mass is complex,

$$\mathcal{L} \supset -m_u \overline{u}_L u_R - m_u^* \overline{u}_R u_L + \frac{v^2 C_{uH}}{\sqrt{2}\Lambda^2}\overline{u}_L u_R h + \frac{v^2 C_{uH}^*}{\sqrt{2}\Lambda^2}\overline{u}_R u_L h \ , \tag{4.2.18}$$

where we wrote the lagrangian in the broken phase, and a careful evaluation of the two-loop diagram (i.e. using propagators featuring complex fermion matrices) yields

$$\frac{d_e}{e} = -\frac{1}{48\pi^2}\frac{v m_e}{m_h^2}\frac{\mathrm{Im}(m_u^* C_{uH})}{\Lambda^2} F_1\left(\frac{|m_u|^2}{m_h^2}, 0\right) \ , \tag{4.2.19}$$

which matches Eq. (4.2.16) when m_u is real, as it should. This expression allows us to identify a more satisfactory order parameter of CP-breaking, $\mathrm{Im}(m_u^* C_{uH})$, which does not depend on

[11] As we mentioned in the first part of the Chapter, in the SM_4 the various contributions to the electron EDM are all proportional to $\mathcal{J}$, consistently with the collective nature of CP breaking. Short-distance quark-level perturbative contributions arise at four-loop order and allow one to identify the whole structure of J_4 (see Ref. [290] for a parametric expression and Ref. [291] for a recent appraisal identifying an additional m_b^2 factor), but the dominant source comes from long-distance hadronic contributions sensitive to CPV four-fermion operators [292]. In addition, such hadronic contributions often dominate CPV observables, e.g. in the case of paramagnetic systems (see Ref. [293] for the identification of a new contribution which increases the previous result by 5 order of magnitude).

the phase convention for the up quark. Similarly to what was discussed for the SM_4 previously, what matters here for CP breaking is not that the Yukawa coupling has an imaginary part, but that there is an irreducible imaginary part due to the simultaneous presence of both the coupling to the Higgs of the up quark and its non-zero mass. This provides also a qualitative argument for why the result in Eq. (4.2.16) had to be explicitly proportional to m_u.

The take-away message of this section is that real or imaginary parts of coefficients are only meaningful with respect to CP breaking when the flavor basis is completely determined. In a general basis, what matters are the imaginary parts of invariant combinations of coefficients.[12] CP-odd invariants in SMEFT, especially with three flavors, are the subject of this Chapter.

4.3 Characterizing CP violation at dimension six

In this section, we discuss the number of primary parameters in SMEFT, as well as the parametrization of those which are CP-odd.

4.3.1 Primary parameters in the SMEFT

First, we count the number of flavorful primary SMEFT parameters. We remind that they are defined to be the dimension-six SMEFT parameters which generate BSM amplitudes which can interfere with SM_4 amplitudes (other parameters being called secondary). Indeed, observables computed in SMEFT are subject to a power expansion, with respect to which we focus on the leading BSM order, *i.e. we include contributions to observables up to order* $1/\Lambda^2$. For instance, in cross-section computations, we only consider the SM_4 amplitude squared and the interference with the leading BSM amplitude. Schematically, we can express a generic amplitude as

$$\mathcal{A} = \mathcal{A}^{(4)} + \mathcal{A}^{(6)} + \dots \tag{4.3.1}$$

where $\mathcal{A}^{(4)}$ is the leading order amplitude built with renormalizable operators, $\mathcal{A}^{(6)}$ is the next-to-leading order one, accompanied by a $1/\Lambda^2$ suppression, and the dots indicate higher order terms that we ignore. Then, observables such as cross sections, which are proportional to the amplitude squared, will receive contributions by

$$|\mathcal{A}|^2 = |\mathcal{A}^{(4)}|^2 + 2\mathrm{Re}\left(\mathcal{A}^{(4)}\mathcal{A}^{(6)*}\right) + \dots \tag{4.3.2}$$

[12] Examples of this also exist in the SM_4. The quantity ε, which encodes indirect CP violation in kaon decays, is sometimes written [117]

$$\varepsilon \approx \frac{e^{i\pi/4}}{\sqrt{2}} \frac{\mathrm{Im}(M_{12})}{\Delta m} , \tag{4.2.20}$$

where M_{12} is associated to $K^0 \leftrightarrow \overline{K^0}$ mixing and Δm is the mass difference between kaon mass eigenstates. This formula actually assumes that $\lambda_u \equiv V_{ud}V_{us}^*$ is real. The expression which is valid independently of the phase conventions reads [235]

$$\varepsilon \approx \frac{e^{i\pi/4}}{\sqrt{2}} \frac{\mathrm{Im}(M_{12}\lambda_u^2)}{\Delta m\ |\lambda_u|^2} . \tag{4.2.21}$$

Our goal in this section is then to determine the primary parameter space that characterizes the first two terms in Eq. (4.3.2).

This counting does not lead to a mere repetition of that of Ref. [133], which counts primary and secondary parameters indifferently and whose results are reviewed in the first double column of Table 2, due to the fact that several of the dimension-six parameters are charged under lepton numbers, unlike SM_4 parameters. Given that physical observables cannot be charged, such parameters can only interfere with themselves (or other charged BSM parameters) to form a neutral object, and are therefore secondary according to our classification.

More precisely, the free fermionic lagrangian in the broken phase of the SMEFT and in the mass basis has abelian flavor symmetries $U(1)_{u_i,d_i,L_i}$, acting on each mass eigenstate independently. By definition, those do not affect the spectrum of asymptotic states and therefore do not affect physical predictions. They simply correspond to irreducible flavor ambiguities in a basis where mass matrices are diagonal and real, which must be fixed by further specifications (for instance, phase prescriptions in the CKM matrix). A consequence is that any observable must be expressed in terms of quantities which are invariant under these $U(1)$ phase rotations,[13] and therefore, any coefficient which is not invariant on its own must enter observables multiplied by another $U(1)$-charged coefficient in order to form a neutral object. This story is known to readers familiar with the notion of rephasing-invariants of the CKM matrix [294]: $V_{\mathrm{CKM},ij}$ being charged under $U(1)_{d_j} - U(1)_{u_i}$, physical predictions can only depend on the moduli and quartets,[14]

$$\left|V_{\mathrm{CKM},ij}\right|^2 , \quad V_{\mathrm{CKM},ij}V_{\mathrm{CKM},kl}V^*_{\mathrm{CKM},il}V^*_{\mathrm{CKM},kj} \ . \tag{4.3.3}$$

For the SMEFT at leading order, that implies that $U(1)$-charged dimension-six coefficients must multiply $U(1)$-charged dimension-four coefficients. In the quark sector, the CKM matrix is the only such object, and there does not exist any in the lepton sector, since $U(1)_{L_i}$ is a symmetry of the SM_4 lagrangian for each i (remember that we work in the limit of vanishing neutrino masses). Therefore, all "off-diagonal" lepton coefficients in the first double column of Table 2, i.e. those which are charged under $U(1)_{L_i} - U(1)_{L_j}$, correspond to secondary parameters. This requirement reduces the number of parameters to the ones in the second double column of Table 2.

A similar reasoning explains why adequate entries of the CKM matrix must multiply dimension-six coefficients charged under $U(1)_{u_i}$ and/or $U(1)_{d_i}$. Those coefficients must therefore contribute to observables with an additional suppression due to the smallness of the off-diagonal CKM entries. For instance, $C_{uB,13}$ (expressed in the up basis) can only enter observables as

$$V^*_{CKM,11}V_{CKM,31}C_{uB,13} \text{ or } V^*_{CKM,12}V_{CKM,32}C_{uB,13} \ , \tag{4.3.4}$$

where the unitarity of the CKM matrix allows us to not consider $V^*_{CKM,13}V_{CKM,33}C_{uB,13}$. Due

[13] This is a slightly different statement than the one that CP violation should be characterized in a flavor-invariant way. Although they are restricted by flavor-invariant statements, amplitudes squared with flavored external states are not flavor-invariant, but they are always invariant under phase rotations.

[14] Notice that sextets and monomial with more entries of the CKM matrix can be expressed in terms of moduli and quartets using the unitarity of V_{CKM}.

	Type of op.	# of ops	# real	# im.	inv. under $U(1)_{L_i} - U(1)_{L_j}$ # real	inv. under $U(1)_{L_i} - U(1)_{L_j}$ # im.
bilinears	Yukawa	3	27	27	21	21
	Dipoles	8	72	72	60	60
	current-current	8	51	30	42	21
	all bilinears	19	150	129	123	102
4-Fermi	LLLL	5	171	126	99	54
	RRRR	7	255	195	186	126
	LLRR	8	360	288	246	174
	LRRL	1	81	81	27	27
	LRLR	4	324	324	216	216
	all 4-Fermi	25	1191	1014	774	597
	all		1341	1143	897	699

Table 2: Number of flavorful real and imaginary parameters in SMEFT at dimension-six (see Tables 4.C.1 and 4.C.2 for the explicit forms of the operators). The first double column counts the number of physical parameters, the second one (highlighted in gray) counts those which are also primary (see the text).

to the gauge anomalies of the abelian symmetries $U(1)_{u_i,d_i,L_i}$, the θ-angles of the different gauge factors of the SM are also charged parameters, which can interfere (non-perturbatively) with appropriate SMEFT coefficients (see Appendix 4.F for a discussion of $\theta_{\rm QCD}$). Nevertheless, $U(1)_{L_i} - U(1)_{L_j}$ is anomaly-free in the SM, therefore the statements made previously about SMEFT coefficients in the lepton sector also hold non-perturbatively.

In this discussion, we ignored the specific cases of observables in the neutrino sector since we work in the limit of vanishing neutrino masses. Examples of observables which we allow include electric dipole moments (EDMs) [131, 254, 289, 295] or the CPV parameters ε_K and ε' in kaon physics [296–298]. We also assume that the leading BSM contribution indeed corresponds to the interference at $\mathcal{O}(1/\Lambda^2)$, and not to a dimension-six contribution squared (or the interference between the SM_4 and a dimension-eight coefficient, etc) due to some accidental suppression of the $\mathcal{O}(1/\Lambda^2)$ term. Below, we will also study the SM_4 parameter space as a whole (i.e., beyond values relevant for phenomenology), which includes points where several entries of the CKM matrix become unphysical and can be redefined away, turning additional dimension-six coefficients in the quark sector into secondary ones. This would for instance happen if all down-type quarks were massless: barring observables which are ill-defined when $m_d \to 0$ within our leading order observables, we find a further reduction of the relevant dimension-six coefficients (see Table 4.A.2 in Appendix 4.A.2). Ref. [299] performs a similar counting of primary parameters (focusing on the kinematic situation where all fermions masses, but the top and bottom quark, are neglected).

4.3.2 CP conservation at leading order and minimal sets of CP-odd invariants

We now turn to the characterization of CP violation in SMEFT at leading order using flavor-invariants. Specifically, in the spirit of the discussion which leads to the introduction of J_4, we ask

> Which flavor invariants vanish iff CP is conserved at leading order in SMEFT? We call such a set of CP-odd invariants of minimal cardinality a *minimal set*.

The notion of minimal cardinality implies that there are no redundancies: the vanishing of each invariant in a minimal set provides an independent condition. The number of invariants in a minimal set must be larger or equal than the number of new primary sources of CPV[15]. In the case of the SMEFT at $\mathcal{O}(1/\Lambda^2)$, we find the non-trivial result that the two numbers agree for all operators (see below).

Before going further, let us discuss one subtlety associated to our definition of a minimal set of CP-odd flavor-invariants, which has to do with the parameter space considered. In the way our definition is stated, it suggests that one aims at characterizing CP-conserving points for all possible choices of parameters, i.e. quark masses as well as mixing angles and the CKM phase $\delta_{\rm CKM}$. However, one could also try to characterize CP-conserving points *within a given parameter subspace*, for instance for values of the quark masses which are non-vanishing. This is the choice we make in the main body of this Chapter: we build flavor invariants which vanish iff CP is conserved at leading order in SMEFT *under the assumption that quark masses are non-vanishing*. Our methods also allow one to identify minimal sets of flavor-invariants when one considers vanishing quark masses, but since the expression of the required invariants is more intricate than for the simpler case of non-vanishing quark masses, we leave the resolution of this question to Appendix 4.A. One could finally restrict to characterizing CP-conservation for a smaller set of parameters, e.g. fixing the values of the quark masses or taking them non-degenerate. Our sets of invariants work in such restricted cases, but there usually exist simpler ones (which do not correspond to minimal sets on larger sets of parameters). We will encounter explicit examples in the next section.

Due to the SMEFT power counting, the conservation of CP at zeroth order first demands that $J_4 = 0$, so that the J_4 is part of any minimal set. Then, in order to build the rest of the minimal set, we look for invariants which are *linear with respect to the SMEFT dimension-six coefficients*, consistently with the goal of characterizing CPV in observables up to the first non-leading order, i.e. up to the $\mathcal{O}(1/\Lambda^2)$ interference term in the R.H.S. of Eq. (4.3.2). This linearity is also valuable to check that we indeed have a necessary and sufficient condition for CP conservation, as it does not suffer from subtleties associated to non-linear invariants, found e.g. in neutrino physics [273]. Indeed, the question of whether vanishing invariants really implies

[15] It may need to be larger: as we detail below, we count an invariant as independent if there exists at least one point in parameter space (in terms of fermion masses or CKM entries) where it cannot be expressed in terms of other invariants of the set.

vanishing CP phases is trivially answered here. We found that there does exist a minimal set of invariants linear with respect to the SMEFT dimension-six coefficients. This set is presented in subsequent sections, and represents the main result.

The linearity implies that the minimal sets are decomposed in minimal sets of invariants $\mathcal{I}_a(C^{(6)})$, defined for each SMEFT dimension-six operator $\mathcal{O}^{(6)}$ and its associated matrix-valued coefficient $C^{(6)}$ independently, and where the index a labels the new primary sources of CPV present in $C^{(6)}$. Therefore, we can define the notion of minimal set at the level of each SMEFT operator independently. The invariants have the following form:

$$\mathcal{I}_a(C^{(6)}) = \mathrm{Im}\left(\text{flavor-invariant linear in } C^{(6)}\right) = T^R_{ai}\left(\mathrm{Re}C^{(6)}\right)_i + T^I_{ai}\left(\mathrm{Im}C^{(6)}\right)_i \equiv \mathcal{T}_{ai}\overrightarrow{C}^{(6)}_i \,, \tag{4.3.5}$$

where the last equality describes the result of evaluating the invariant in a given flavor basis. Here we define

$$\overrightarrow{C}^{(6)}_i \equiv \left(\left(\mathrm{Re}C^{(6)}\right)_1, \left(\mathrm{Re}C^{(6)}\right)_2, ..., \left(\mathrm{Im}C^{(6)}\right)_1, ...\right) \tag{4.3.6}$$

as the vector in flavor space composed by the vectors $\mathrm{Re}/\mathrm{Im}C^{(6)}$, not necessarily of same length. The matrix $\mathcal{T}_{ai}$ will be referred to as the *transfer matrix*, and it is defined to take the block form

$$\mathcal{T} = \begin{pmatrix} T^R & T^I \end{pmatrix} . \tag{4.3.7}$$

By linearity, T^R_{ai} is the imaginary part of a linear combination of products of entries of dimension-four Yukawas (which are the only flavored objects at dimension-four), while T^I_{ai} is the real part of a similar combination, albeit in general different in its specific shape. Those matrices T^R and T^I depend explicitly on the operator $\mathcal{O}^{(6)}$ considered. A very convenient feature of such invariants is that they automatically project out any secondary coefficient, which cannot be arranged into invariants in a linear fashion by definition.

Showing that the set of invariants is minimal can be phrased as a condition on the matrices $T^{R/I}$:

> A set of flavor invariants is a minimal set iff the rank of the transfer matrix $\mathcal{T}$ equals the number of new primary sources of CPV in $C^{(6)}$ when $J_4 = 0$ and never does for all sets with strictly smaller cardinality.

Note that the last part of this characterization is automatic when the number of invariants, the number of primary sources of CPV and the rank are all equal. We use this condition below to check that the sets of invariants we present below are indeed minimal. We stress that the meaning of "when $J_4 = 0$" encompasses a large subset of the whole parameter space spanned by the masses and the mixing angles, as seen from the expression in Eq. (4.2.12): it is achieved when $\theta_{ij} = 0$ or $\pi/2$, or when $m_{u,i} = m_{u,j}$ or $m_{d,i} = m_{d,j}$, for any pair i, j. In addition, setting $J_4 = 0$ via one of these choices still leaves a large freedom for the remaining parameters. For instance, one may have $\theta_{ij} = 0$ *and* $m_{u,k} = m_{u,l}$ for some $i, j, k, l = 1, ..., 3$. A set of flavor invariants is a minimal set only if the rank of its transfer matrix corresponds to the number of new sources of CPV *within the whole parameter space where* $J_4 = 0$ (up to the restriction

of non-vanishing quark masses which we adopt in the main text of this paper and relax in Appendix 4.A). We will come back to this point in Section 4.4.

The transfer matrix $\mathcal{T}$ acts on the flavor-space vector made out of real and imaginary entries of $C^{(6)}$ (the precise order in the labeling as well as the order between real and imaginary part is unimportant). Note that the rank does not change under the action of flavor transformations (which reshuffle real and imaginary parts, as well as the entries of $T^{R/I}$).

4.4 Minimal set of CP-odd invariants

In this section, we present the minimal set of leading order CP-odd invariants in SMEFT at dimension-six, under the aforementioned assumption that all fermion masses are non-vanishing, which has an impact on how many sources of CPV are expected and which invariants correctly capture them. We treat the cases of vanishing masses in Appendix 4.A.

4.4.1 Examples

Let us present some parts of our minimal set of invariants for SMEFT at dimension-six. As we explained previously, the linearity with respect to the Wilson coefficients of the dimension-six CP-odd observables allows one to treat the different SMEFT operators independently. The study of all SMEFT operators proceeds along identical lines, and the full set of invariants is presented in Appendices 4.D and 4.E.

We begin by considering SMEFT operators which are bilinear in fermion fields and hermitian, and therefore have the simplest non-trivial flavor structure. Invariants under unitary groups with bi-fundamental representations must feature the invariant tensor δ^a_b, therefore they correspond to linear combinations of traces of products of matrices, arranged so that indices of a given fundamental representation and its conjugate are contracted in the trace, as seen for instance in Eqs. (4.1.9)-(4.1.10). In addition, there are relations between powers of 3×3 matrices, and/or between their traces, derived from the Cayley–Hamilton theorem, which reduce the candidate invariants to a finite set. We explicitly present such properties in Appendix 4.B. The relevant single-trace invariants linear with respect to a SMEFT coefficient C, for a fermion bilinear operator, take the universal form[16]

$$L_{abcd}(\tilde{C}) \equiv \mathrm{Im}\,\mathrm{Tr}\Big(X_u^a X_d^b X_u^c X_d^d \tilde{C}\Big)\ , \text{ with } a,b,c,d = 0,1,2 \text{ and } a \neq c, b \neq d\ , \tag{4.4.1}$$

where $\tilde{C} = C, CY^\dagger_{f=u,d,e}$ or $Y_f C Y_f^\dagger$, depending on the chiral structure of the operator under study (see below for explicit formulae). We first choose $C = C_{Hu}$ for definiteness, and we find that the following property holds:

$$\mathcal{L} = \mathcal{L}_{\mathrm{SM}_4} + \tfrac{C_{Hu,ij}}{\Lambda^2}\left(iH^\dagger \overleftrightarrow{D}_\mu H\right)\overline{u_{i,R}}\gamma^\mu u_{j,R}$$

[16] As we will see, those structures are also the only ones needed for 4-Fermi operators.

preserves CP at $\mathcal{O}(1/\Lambda^2)$ iff

$$J_4 = L_{1100}\left(Y_u C_{Hu} Y_u^\dagger\right) = L_{2200}\left(Y_u C_{Hu} Y_u^\dagger\right) = L_{1122}\left(Y_u C_{Hu} Y_u^\dagger\right) = 0 \qquad (4.4.2)$$

Indeed, $J_4 = 0$ is necessary so that the leading order SM_4 contribution to any CP-odd observable vanishes. Once enforced, this makes the SM_4 lagrangian CP-symmetric, and there remains generically three new primary sources of CPV in C_{Hu}. Indeed, C_{Hu} is a (3×3) hermitian matrix (which transforms as a $\mathbf{3} \otimes \bar{\mathbf{3}}$ representation of $U(3)_u$). Therefore, the minimal set for C_{Hu} should at least contain three invariants. As we explained in Section 4.3.2, in order to show that the three invariants in Eq. (4.4.2) capture the three necessary conditions, it is sufficient to compute the transfer matrix $\mathcal{T}$ such that,

$$\begin{pmatrix} L_{1100} \\ L_{2200} \\ L_{1122} \end{pmatrix} = \begin{pmatrix} T^R & T^I \end{pmatrix} \begin{pmatrix} \mathrm{Re}C_{Hu,11} \\ \mathrm{Re}C_{Hu,12} \\ \cdots \\ \mathrm{Im}C_{Hu,12} \\ \mathrm{Im}C_{Hu,13} \\ \mathrm{Im}C_{Hu,23} \end{pmatrix} , \qquad (4.4.3)$$

and show that it has rank 3. Parametrically, and in some basis, the generic case corresponds to taking $\delta_{\mathrm{CKM}} \to 0$ in Eq. (4.2.10) while holding all mixing angles different from $0, \pi/2$ and all quark masses non-denegerate, which is what we assume in the current section (we treat more general cases below). Then, $T^R = 0_{3\times 6}$ and the determinant of T^I is found to be non-vanishing. Therefore, $L_{1100} = L_{2200} = L_{1122} = 0$ implies that $\mathrm{Im}C_{Hu,ij} = 0$ in the basis of Eq. (4.2.10) (or any other basis where the Yukawa matrices are real), i.e. CP is conserved. Conversely, the conservation of CP, or equivalently $\mathrm{Im}C_{Hu,ij} = 0$, implies that all L's vanish since $T^R = 0$. This proves the equivalence announced above.

One may be surprised by the fact that some simple invariants, in the sense that they feature low powers of the Yukawa matrices, are not part of the set in Eq. (4.4.2). For instance, for C_{Hu}, the set formed by

$$L_{1100}\left(Y_u C_{Hu} Y_u^\dagger\right) \ , \ L_{1200}\left(Y_u C_{Hu} Y_u^\dagger\right) \ , \ L_{2100}\left(Y_u C_{Hu} Y_u^\dagger\right) \qquad (4.4.4)$$

would pass the test performed in this section, namely the associated transfer matrix would generically have rank 3. Such invariants have been studied beyond SMEFT, and exist generally for any set of three hermitian matrices in the same adjoint representation, as explained in Ref. [265]. However, they would not be a valid choice of invariants, since they would not correspond to sufficient conditions whatever the values of the fermion masses and the CKM matrix. For instance, $m_t = m_c$ is another way to get $J_4 = 0$ other than that discussed above. In this situation, there remain three conditions necessary for CP to be conserved (see Section 4.4.2 for details). Therefore, imposing that a given minimal set vanishes should be equivalent to three

independent conditions also when $m_t = m_c$. However (when $m_t = m_c$) we find that

$$(m_u^2 + m_c^2) L_{1100} = L_{2100} \ , \tag{4.4.5}$$

where the mass-dependent factor can be expressed in terms of invariant quantities,

$$m_u^2 + m_c^2 = \frac{18 \det X_u - 3 \left(\mathrm{Tr}\, X_u\right)^3 + 7 \,\mathrm{Tr}\, X_u \,\mathrm{Tr}\, X_u^2}{6 \,\mathrm{Tr}\, X_u^2 - 2 \left(\mathrm{Tr}\, X_u\right)^2} \ . \tag{4.4.6}$$

Therefore, imposing that the set in Eq. (4.4.4) vanishes only amounts to two conditions. Instead, one can check that the vanishing of the set in Eq. (4.4.2) yields three independent conditions[17], even when $m_t = m_c$. We expand on what happens however the limit $J_4 \to 0$ is taken in Section 4.4.2, and show that the set in Eq. (4.4.2) above yields a satisfactory minimal set in all cases (as long as all quark masses are non-vanishing).

Similar reasoning applies to all SMEFT operators. Let us present the results in two more cases with slightly more complicated flavor structures, the non-hermitian bilinear operator $\mathcal{O}_{uH}$ and the hermitian symmetric 4-Fermi operator $\mathcal{O}_{uu}$. The Wilson coefficient C_{uH} contains nine new primary sources of CPV, since it is an arbitrary (3×3) complex matrix (which transforms as a $(\mathbf{3}, \bar{\mathbf{3}})$ representation of $U(3)_Q \times U(3)_u$). C_{uu} contains eighteen new CPV parameters ($C_{uu,ijkl}$ is "hermitian", i.e. $C^*_{uu,ijkl} = C_{uu,jilk}$, symmetric, $C_{uu,ijkl} = C_{uu,klij}$, and it transforms in the $(\mathbf{3} \otimes \bar{\mathbf{3}})^2$ of $U(3)_u$).

For $\mathcal{O}_{uH}$, we find that

$$\mathcal{L} = \mathcal{L}_{\mathrm{SM}_4} + \tfrac{C_{uH,ij}}{\Lambda^2} \overline{Q_{i,L}} \tilde{H} u_{j,R} |H|^2 + \mathrm{h.c.}$$

$$\text{preserves CP at } \mathcal{O}(1/\Lambda^2) \text{ iff}$$

$$\begin{aligned} J_4 &= L_{0000}\left(C_{uH} Y_u^\dagger\right) = L_{1000}\left(C_{uH} Y_u^\dagger\right) = L_{0100}\left(C_{uH} Y_u^\dagger\right) \\ &= L_{1100}\left(C_{uH} Y_u^\dagger\right) = L_{0110}\left(C_{uH} Y_u^\dagger\right) = L_{2200}\left(C_{uH} Y_u^\dagger\right) \\ &= L_{0220}\left(C_{uH} Y_u^\dagger\right) = L_{1220}\left(C_{uH} Y_u^\dagger\right) = L_{0122}\left(C_{uH} Y_u^\dagger\right) = 0 \ . \end{aligned} \tag{4.4.7}$$

We now turn to $\mathcal{O}_{uu}$. For 4-Fermi operators, we generically define

$$\mathrm{Tr}_A\left(M^{(1)}, M^{(2)}, C\right) \equiv M^{(1)}_{ji} M^{(2)}_{lk} C_{ijkl} \ , \quad \mathrm{Tr}_B\left(M^{(1)}, M^{(2)}, C\right) \equiv M^{(1)}_{li} M^{(2)}_{jk} C_{ijkl} \ , \tag{4.4.8}$$

and

$$\begin{aligned} A^{abcd}_{efgh}(C) =& \mathrm{Im}\,\mathrm{Tr}_A\left(X_u^a X_d^b X_u^c X_d^d, X_u^e X_d^f X_u^g X_d^h, C\right) \ , \\ B^{abcd}_{efgh}(C) =& \mathrm{Im}\,\mathrm{Tr}_B\left(X_u^a X_d^b X_u^c X_d^d, X_u^e X_d^f X_u^g X_d^h, C\right) \ . \end{aligned} \tag{4.4.9}$$

We further define

$$C_{\tilde{u}\tilde{u}uu,ijkl} \equiv \sum_{m,n} Y_{u,im} Y^\dagger_{u,nj} C_{uu,mnkl} \ , \tag{4.4.10}$$

[17] The interplay between degenerate cases and flavor-invariants has also been studied in Ref. [274] for massive Majorana neutrinos.

and similarly for $C_{\tilde{u}uu\tilde{u}}, C_{u\tilde{u}\tilde{u}u}, C_{uu\tilde{u}\tilde{u}}$, $C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}$, and for the down quark versions. We then find that

$$\mathcal{L} = \mathcal{L}_{\mathrm{SM}_4} + \frac{C_{uu,ijkl}}{\Lambda^2}\overline{u_{i,R}}\gamma^\mu u_{j,R}\ \overline{u_{k,R}}\gamma_\mu u_{l,R}$$

preserves CP at $\mathcal{O}(1/\Lambda^2)$ iff

$$\begin{aligned} J_4 &= A^{0000}_{1100}\left(C_{uu\tilde{u}\tilde{u}}\right) = A^{1000}_{1100}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) = A^{0100}_{1100}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) \\ &= A^{0000}_{2200}\left(C_{uu\tilde{u}\tilde{u}}\right) = A^{1100}_{1100}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) = A^{0200}_{1100}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) \\ &= A^{0100}_{2200}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) = A^{0000}_{1122}\left(C_{uu\tilde{u}\tilde{u}}\right) = A^{1100}_{2200}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) \\ &= A^{1000}_{1122}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) = A^{0100}_{1122}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) = A^{1100}_{0122}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) \\ &= A^{1200}_{2200}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) = B^{0000}_{1100}\left(C_{u\tilde{u}\tilde{u}u}\right) = B^{0100}_{1100}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) \\ &= B^{0200}_{2100}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) = A^{1200}_{1122}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) = B^{1000}_{1200}\left(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}\right) = 0\ . \end{aligned} \tag{4.4.11}$$

The proofs of these equivalences follow from the same logic as for C_{Hu}: compute T^I and check that it has maximal rank, which means that the origin in invariant space is uniquely mapped to the origin in imaginary-coefficient space. Therefore, the vanishing of the invariants is equivalent to the conservation of CP (at leading order).

Finally, let us consider the leptonic case. As we argued in Section 4.3.1, the fact that the SM_4 lagrangian is symmetric under the lepton numbers $U(1)_{L_i}$ makes several SMEFT dimension-six coefficients in the lepton sector secondary. For the specific example of C_{eH}, this means that all off-diagonal entries are secondary, since they are charged under $U(1)_{L_i} - U(1)_{L_j}$ for suitable i, j. Therefore, although all imaginary parts of the nine entries of C_{eH} violate CP when the full SMEFT expansion is considered, the only ones which can contribute to CP-odd observables at $\mathcal{O}(1/\Lambda^2)$ are the diagonal ones. The same reasoning applies to C_{He}, which therefore does not violate CP at $\mathcal{O}(1/\Lambda^2)$. Consequently, a minimal set for C_{eH} contains three invariants, and is empty for C_{He}. Indeed, we find that (defining $X_e \equiv Y_e Y_e^\dagger$)

$$\mathcal{L} = \mathcal{L}_{\mathrm{SM}_4} + \frac{C_{He,ij}}{\Lambda^2}\left(iH^\dagger \overleftrightarrow{D}_\mu H\right)\overline{e_{i,R}}\gamma^\mu e_{j,R} + \left(\frac{C_{eH,ij}}{\Lambda^2}\overline{L_{i,L}}He_{j,R}|H|^2 + \text{h.c.}\right)$$

preserves CP at $\mathcal{O}(1/\Lambda^2)$ iff

$$J_4 = \mathrm{Im\,Tr}\left(C_{eH}Y_e^\dagger\right) = \mathrm{Im\,Tr}\left(X_e C_{eH}Y_e^\dagger\right) = \mathrm{Im\,Tr}\left(X_e^2 C_{eH}Y_e^\dagger\right) = 0\ . \tag{4.4.12}$$

Let us end this preview by saying that the above invariant sets are not unique: there are different sets of invariants which would equally well capture the necessary and sufficient conditions for CP-conservations at order $1/\Lambda^2$. Our construction of the above sets requires that the invariants have the lowest possible degree with respect to Yukawa matrices, and that they be as large as possible when the observed values of the fermion masses and CKM entries are plugged in.

4.4.2 Expected ranks when $J_4 = 0$

We now discuss the rank of the transfer matrices, related to the validity of the minimal sets presented above, in non-generic cases, namely when some fermion masses are degenerate and/or when there are texture zeros in the CKM matrix (which happens when some mixing angles take special values).

To apply consistently the definition of minimal sets from Section 4.3.2, we need to carefully determine how many new primary sources of CPV there are when $J_4 = 0$, or equivalently what is the expected rank of the transfer matrix $\mathcal{T}$, irrespective of how we take $J_4 \to 0$. In our generic situation of the previous section, $J_4 \to 0$ meant $\delta_{\rm CKM} \to 0$ in Eq. (4.2.10) while holding all mixing angles different from $0, \pi/2$ and the quark masses non-degenerate. This ensures that there are no texture zeros in the CKM matrix, so that the number of CP-breaking quantities were identified with the number of imaginary parts (in the quark sector). However, that is not the only situation captured by the ambiguous "$J_4 = 0$" condition, as we anticipated in Section 4.3.2. Indeed, mass degeneracies and/or texture zeros in the CKM matrix may lead to a conserved flavor symmetry of the SM_4 lagrangian larger than $U(1)_B$, which has an impact on the number of CP-odd quantities at order $1/\Lambda^2$.

The reason is identical to that discussed in Section 4.3.1: observables should be invariant under any symmetry of the spectrum of asymptotic states. Consequently, at order $1/\Lambda^2$, SMEFT coefficients must combine with SM_4 parameters to form invariant objects, and in particular, when the SM_4 lagrangian has a flavor symmetry (which is therefore part of the symmetry group of the spectrum), only SMEFT coefficients invariant under this flavor symmetry can generate amplitudes which interfere with SM_4 ones.

In the generic case of a CP-preserving SM_4 lagrangian (i.e. taking $\delta_{\rm CKM} \to 0$ in Eq. (4.2.10) with generic values of mixing angles and quark masses, or said differently for a real CKM matrix without texture zeros), there is no flavor symmetry beyond the baryon and lepton numbers $U(1)_B \times U(1)_{L_i}$. Therefore, any additional B- and L_i-preserving SMEFT coupling in the lagrangian is primary, and its imaginary part is a primary source of CPV. We leave to Appendix 4.A the systematic discussion of all flavor symmetries of the SM_4 and their relation to mass degeneracies and/or texture zeros in the CKM matrix, but, for the sake of illustration, we discuss here two specific cases.

In the first one, the CKM matrix is non-generic and has texture zeros:

$$V_{\rm CKM} = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} . \tag{4.4.13}$$

This texture can be achieved from Eq. (4.2.10) by taking $s_{23} = s_{12} = 0$ (in particular, $J_4 = 0$ for such a texture). The flavor symmetry of the dimension-four action is now enlarged to $U(1)^2$, corresponding to two independent abelian transformations of $(Q_1, Q_3, u_1, u_3, d_1, d_3)$ and

(Q_2, u_2, d_2) respectively (in the up or down basis), acting as follows:

$$u_{i,R} \to e^{i\xi_{q_i}} u_{i,R} \ , \tag{4.4.14}$$

with $\xi_{q_1} = \xi_{q_3}$, and similarly for other quarks.

The second example is that of a degenerate fermion spectrum. We take for definiteness $m_t = m_c$ (which again implies $J_4 = 0$). With this degeneracy, the symmetry of the mass terms becomes non-abelian, while the phase in the CKM matrix, as well as one mixing angle, is no longer physical. Indeed, we can perform a flavor transformation (here in the up basis),

$$Q_L \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} Q_L \ , \quad u_R \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} u_R \ , \tag{4.4.15}$$

such that

$$V_{\rm CKM} \to \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{4.4.16}$$

Therefore, the values of θ_{23} and $\delta_{\rm CKM}$ have no physical impact. For generic values of θ_{12} and θ_{13}, the CKM matrix in Eq. (4.4.16) has no texture zeros, and the flavor symmetry of the SM_4 lagrangian in the quark sector still corresponds to the baryon number.

All possible cases are treated similarly, and the full discussion is presented in Appendix 4.A. The summary of this analysis (assuming no vanishing mass, see Appendix 4.A for the more general case) is presented in Table 3 where, for each non-generic case of interest, we present the flavor symmetry group. The discussion in the lepton sector is similar, but features at least a $U(1)^3$ flavor freedom in the SM_4, since the PMNS matrix is taken here to be the identity. When two or three charged lepton masses are degenerate, this $U(1)^3$ increases to $U(2) \times U(1)$ and $U(3)$, respectively.

This discussion allows us to work out the number of new primary sources of CPV at order $\mathcal{O}(1/\Lambda^2)$, in each (non-)generic case for the CKM matrix. Let us focus again on the two previous examples, which generalize easily to all other cases.

When the baryon or lepton numbers are the only flavor symmetries at dimension-four (in the up or down basis), all imaginary parts of the Wilson coefficients at dimension-six in the quark sector (and in the lepton sector, all imaginary parts which are not charged under the lepton numbers) can interfere with the SM_4. Instead, when the flavor symmetry increases to $U(1)^2$, several SMEFT coefficients become secondary. For instance, C_{Hu} transforms, in the up or down basis, as

$$C_{Hu,ij} \to e^{i\left(\xi_{q_j} - \xi_{q_i}\right)} C_{Hu,ij} \ , \tag{4.4.17}$$

where $\xi_{q_1} = \xi_{q_3}$ and arbitrary ξ_{q_2} for the texture of Eq. (4.4.13), and similarly for other textures. Therefore, for the texture of Eq. (4.4.13), only $C_{Hu,13}$ is primary and C_{Hu} only provides one

Parameter values		Flavor symmetries of the SM_4 lagrangian
$m_u \neq m_c \neq m_t$ $m_d \neq m_s \neq m_b$	Generic $V_{\rm CKM}$	$U(1)_B$
	$\lvert V_{{\rm CKM},i_0j_0}\rvert = 1\ ,\quad V_{{\rm CKM},ij_0} = V_{{\rm CKM},i_0j} = 0$ $i \neq i_0,\ j \neq j_0$	$U(1)^2$
	$\lvert V_{{\rm CKM},i_1j_1}\rvert = \lvert V_{{\rm CKM},i_2j_2}\rvert = \lvert V_{{\rm CKM},i_3j_3}\rvert = 1$ for $\begin{matrix} i_1 \neq i_2 \neq i_3 \\ j_1 \neq j_2 \neq j_3 \end{matrix}$ $V_{{\rm CKM},ij} = 0$ elsewhere	$U(1)^3$
$m_u \neq m_c = m_t$ $m_d \neq m_s \neq m_b$	Generic $V_{\rm CKM}$ (see Eq. (4.4.16))	$U(1)_B$
	$\lvert V_{{\rm CKM},i_0j_0}\rvert = 1\ ,\quad V_{{\rm CKM},ij_0} = V_{{\rm CKM},i_0j} = 0$ $i \neq i_0,\ j \neq j_0$	$U(1)^2$
	$\lvert V_{{\rm CKM},i_1j_1}\rvert = \lvert V_{{\rm CKM},i_2j_2}\rvert = \lvert V_{{\rm CKM},i_3j_3}\rvert = 1$ for $\begin{matrix} i_1 \neq i_2 \neq i_3 \\ j_1 \neq j_2 \neq j_3 \end{matrix}$ $V_{{\rm CKM},ij} = 0$ elsewhere	$U(1)^3$
$m_u \neq m_c \neq m_t$ $m_d = m_s \neq m_b$	Same as the previous case with $V_{\rm CKM} \leftrightarrow V_{\rm CKM}^\dagger$	
$m_u \neq m_c = m_t$ $m_d = m_s \neq m_b$	Generic $V_{\rm CKM}$	$U(1)^2$
	$\lvert V_{{\rm CKM},11}\rvert = \lvert V_{{\rm CKM},22}\rvert = \lvert V_{{\rm CKM},33}\rvert = 1$ $V_{{\rm CKM},ij} = 0$ elsewhere	$U(1)^3$
	$\lvert V_{{\rm CKM},13}\rvert = \lvert V_{{\rm CKM},22}\rvert = \lvert V_{{\rm CKM},31}\rvert = 1$ $V_{{\rm CKM},ij} = 0$ elsewhere	$U(2) \times U(1)$
$m_u = m_c = m_t$	$m_d \neq m_s \neq m_b$	$U(1)^3$
	$m_d = m_s \neq m_b$	$U(2) \times U(1)$
	$m_d = m_s = m_b$	$U(3)$
$m_d = m_s = m_b$	$m_u \neq m_c \neq m_t$	$U(1)^3$
	$m_u \neq m_c = m_t$	$U(2) \times U(1)$
	$m_u = m_c = m_t$	$U(3)$

Table 3: Flavor symmetry of the SM_4 lagrangian as a function of special values for quark masses (assumed to be non-vanishing, see Appendix 4.A for the general case) and entries of the CKM matrix. Conditions on the right are understood to be imposed on top of those on their left. Here only some of the possible combinations of mass degeneracies are treated. The other mass degeneracies lead to the same flavor symmetries provided the corresponding non-generic $V_{\rm CKM}$ are multiplied by appropriate matrices exchanging flavor labels (see footnote 19).

primary source of CPV, Im $C_{Hu,13}$.[18] Thus, in this case, a single invariant in the minimal set for C_{Hu} is sufficient.

This exercise can be performed for all non-generic cases for the CKM matrix and for all Wilson coefficients. This results in a set of conditions for CP conservation at leading order, whose number is in one-to-one correspondence to the number of independent CP-odd invariants in a minimal set. As we just saw, those numbers depend on the flavor symmetry of the SM_4 lagrangian, and are given for all SMEFT operators in Table 4.

We can now come back to the statement that the set of invariants in Eq. (4.4.4) is not a satisfying one for C_{Hu}. As seen in Table 4, all its off-diagonal entries are primary when $m_t = m_c$, and all their imaginary parts violate CP (in an appropriate basis), hence we need

[18] One can construct non-linear invariant quantities from $C_{Hu,12/23}$, an example being $C_{Hu,12}C^*_{Hu,23}$. At leading order, however, $C_{Hu,12/23}$ cannot contribute linearly to observables.

	Bilinears				4-Fermi						
Flavour symmetries of the quark sector of the SM	C_{eH} C_{eW} C_{eB}	C_{uH} C_{uG} C_{uW} C_{uB} C_{dH} C_{dG} C_{dW} C_{dB} C_{Hud}	$C_{HL}^{1,3}$ C_{He}	$C_{HQ}^{1,3}$ C_{Hu} C_{Hd}	C_{LL} C_{ee}	C_{Le}	$C_{QQ}^{1,3}$ C_{uu} C_{dd}	$C_{LQ}^{1,3}$ C_{Qe} C_{Lu} C_{eu} C_{Ld} C_{ed}	$C_{ud}^{1,8}$ $C_{Qu}^{1,8}$ $C_{Qd}^{1,8}$	C_{LedQ} $C_{LeQu}^{1,3}$	$C_{QuQd}^{1,8}$
$U(1)_B$	3	9	0	3	0	3	18	9	36	27	81
$U(1)^2$	3	5	0	1	0	3	5	3	12	15	33
$U(1)^3$	3	3	0	0	0	3	0	0	3	9	15
$U(2) \times U(1)$	3	2	0	0	0	3	0	0	1	6	7
$U(3)$	3	1	0	0	0	3	0	0	0	3	2
Two degenerate electron-type leptons	$\times\frac{2}{3}$	$\times 1$		$\times 1$		$\times\frac{2}{3}$	$\times 1$	$\times\frac{2}{3}$	$\times 1$	$\times\frac{2}{3}$	$\times 1$
All electron-type leptons degenerate	$\times\frac{1}{3}$	$\times 1$		$\times 1$		$\times\frac{1}{3}$	$\times 1$	$\times\frac{1}{3}$	$\times 1$	$\times\frac{1}{3}$	$\times 1$

Table 4: Numbers of new primary sources of CPV contained in each dimension-six SMEFT coefficient, for each of the possible flavor groups of the quark sector of the SM_4 at dimension-4 (restricting to situations where fermion masses are non-vanishing). The last two rows indicate which multiplicative coefficient should be applied to all numbers of the same column for special values of the electron-type lepton masses.

three independent invariants to capture the conditions for CP conservation.

Let us stress again at this point that the fact that we found a set of invariants of minimal size (i.e. three invariants for the case of C_{Hu}) which captures the necessary and sufficient conditions for CP conservation in all non-generic cases listed in Table 3 is a non-trivial result. Nevertheless, it turns out that it can be done for all SMEFT coefficients at dimension-six, as we explicitly showed.

4.5 Conclusions and future directions

In this Chapter, we have investigated the collective properties of SMEFT at dimension-six. Their first implication which we have discussed is that only a subset of lagrangian parameters (dubbed primary parameters) can contribute linearly to observables upon interfering with the dimension-four SM_4 amplitude. This is due to the existence of flavor transformations which leave the SM spectrum and thus any observable invariant, thereby demanding that covariant lagrangian parameters combine to form invariant objects. This applies for instance to field-rephasings associated to mass-eigenstates, implying that lagrangian coefficients must combine into rephasing-invariant objects. Associated to the SMEFT power counting, this implies that several coefficients related to dimension-six SMEFT operators cannot contribute to observables at order $1/\Lambda^2$, since they are charged under individual lepton numbers unlike all SM_4 parameters (in the limit of zero neutrino mass). We therefore refined the usual counting of dimension-six SMEFT parameters so as to include this effect, which resulted in the counting of primary parameters in Table 2.

Then, we focused on collective effects related to new sources of CP violation in the dimension-six SMEFT, both of which are captured thanks to CP-odd flavor-invariants. To respect the SMEFT power counting, we restricted to invariants linear in dimension-six coefficients, and we presented minimal sets of invariants which map in a one-to-one basis to all new primary sources of CPV. We proved this by showing that the points in parameter space where CP is conserved (at leading BSM order) are exactly the points where our new invariants vanish. This holds for all parts of the SM_4 parameter space, including degenerate cases. A complete list of CP-odd linear invariants can be found in Appendices 4.D and 4.E. We remind the reader that this list is not unique.

There are several directions along which this construction can be extended. For example, one can numerically evaluate the invariants, which encode collective effects and the suppression they induce, using the observed values of the SM_4 parameters. This illustrates how accidentally small the absolute strength of each new source of CPV is, or if there are hierarchies among them, etc. This in turn explains (part of) the suppression in CPV observables [132, 300–302]. One can also similarly probe the effects of specific UV hypotheses on the SMEFT coefficients, such as textures derived from flavor symmetries. A given UV model may have its own CP-odd flavor invariants, and these can be matched onto our IR invariants, inducing possible correlations among the IR invariants. In other directions, it would be interesting to consider other operator bases than the Warsaw basis we used here, and to check which expressions our invariants map to, given that the overall number of independent sources of CPV must be conserved. One could also extend our construction beyond $\mathcal{O}(1/\Lambda^2)$, e.g. to capture squared dimension-six or interfering dimension-eight SMEFT contributions. Finally, one could consider the RG evolution of invariants [155]. We believe that flavor invariants, at dimension-six and beyond, are essential tools for illuminating the rich CP structure of SMEFT.

Appendix to Chapter 4

4.A Flavor symmetries of the SM_4

In this appendix, we present the details behind Table 3, and identify the possible flavor symmetries of the SM_4 lagrangian in terms of textures in the CKM matrix. We remind that the flavor symmetries act on the quark fields as follows:

$$u_R \to U_u u_R \ , \tag{4.A.1}$$

where $U_u \in U(3)_u$, and similarly for all other fermionic fields d, Q, L, e. Which matrices U lead to genuine symmetries of the SM_4 lagrangian depends on the values of the masses and of the entries of the CKM matrix. In all cases of non-trivial flavor symmetries, we find that $J_4 = 0$, so that there exists at least one combination of CP and flavor symmetries which yield a symmetry of the SM_4 lagrangian in any basis.

In what follows, we work for definiteness in the up basis of Eq. (4.2.8).

4.A.1 Non-vanishing quark masses

The condition for the flavor invariance of Y_u and Y_d reads

$$Y_u = U_Q^\dagger Y_u U_u \ , \quad Y_d = U_Q^\dagger Y_d U_d \ . \tag{4.A.2}$$

For non-vanishing quark masses, the Yukawa matrices are full rank and one can use Eq. (4.A.2) to solve for $U_{u,d}$ as a function of U_Q and the Yukawa matrices:

$$U_u = Y_u^{-1} U_Q Y_u \ , \quad U_d = Y_d^{-1} U_Q Y_d \ . \tag{4.A.3}$$

Therefore, only one matrix determines the two others, and the flavor symmetry group is at most $U(3)$. Imposing that $U_u^\dagger U_u = U_d^\dagger U_d = \mathbb{1}$ implies (in the up basis) that

$$\left[U_Q, m_u m_u^\dagger\right] = \left[V_{\rm CKM}^\dagger U_Q V_{\rm CKM}, m_d m_d^\dagger\right] = 0 \ , \tag{4.A.4}$$

where $m_{u/d} \equiv \text{diag}(m_{u/d_i})$. In the up basis, quark masses are positive and real, therefore

$$\left[U_Q, m_u\right] = \left[V_{\rm CKM}^\dagger U_Q V_{\rm CKM}, m_d\right] = 0 \tag{4.A.5}$$

and (using the explicit expression of the Yukawa matrices in the up basis)

$$U_u = U_Q \ , \quad U_d = V_{\rm CKM}^\dagger U_Q V_{\rm CKM} \ . \tag{4.A.6}$$

The commutation relations in Eq. (4.A.5) are additional constraints to fulfil which depend on the spectrum, as we now explore.

Non-degenerate quark masses

If all up-type quarks are non-degenerate, the first condition in Eq. (4.A.5) implies that $U_Q = \text{diag}\left(e^{i\xi_i}\right)$ and the second that $U_Q = V_{\rm CKM}\text{diag}\left(e^{i\tilde{\xi}_i}\right) V_{\rm CKM}^\dagger$, therefore

$$V_{{\rm CKM},ij} = e^{i\left(\xi_i - \tilde{\xi}_j\right)} V_{{\rm CKM},ij} \ .$$

Consequently, all ξ's are equal and equal to the $\tilde{\xi}$'s (i.e. the flavor symmetry is given by the baryon number $U(1)_B$) unless the CKM matrix has some vanishing entries. For instance, one finds a $U(1)^2$ flavor symmetry when the CKM matrix has the following texture:

$$V_{\rm CKM} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \ , \tag{4.A.7}$$

corresponding to the constraints $\xi_2 = \xi_3 = \tilde{\xi}_2 = \tilde{\xi}_3$, for arbitrary $\xi_1 = \tilde{\xi}_1$. More generally, a $U(1)^2$ flavor symmetry is obtained for any texture such that, given two integers (i_0, j_0), $|V_{{\rm CKM},i_0 j_0}| = 1$ and $V_{{\rm CKM},ij} = 0$ for $i = i_0$ or $j = j_0$. By comparing with the explicit parametrization in Eq. (4.2.10), one finds that a mixing angle has to be equal to 0 or $\pi/2$ in all those cases, hence $J_4 = 0$ and there exists a basis where all SM_4 couplings are real.

A flavor symmetry $U(1)^3$ is obtained for all textures of $V_{\rm CKM}$ such that there is a single number of unit modulus in each row and column, such as e.g. $V_{\rm CKM} = \mathbb{1}$ or

$$V_{\rm CKM} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \ . \tag{4.A.8}$$

Degenerate quark masses

In cases with quark mass degeneracies, $J_4 = 0$ automatically and there exists a basis where all SM_4 couplings are real.

$\mathbf{m_t = m_c}$ Let us start with the case of two degenerate quarks of the same type, which we take to be $m_t = m_c$ for definiteness, all other masses being non-degenerate.[19] The first relation in Eq. (4.A.5) implies that

$$U_Q = \begin{pmatrix} e^{i\xi_1} & 0 \\ 0 & U_u^{(2)} \end{pmatrix} \text{ with } U_u^{(2)} \in U(2) \ , \tag{4.A.9}$$

while the second implies that $U_Q = V_{\rm CKM} e^{i\tilde{\xi}_i} V_{\rm CKM}^\dagger$, therefore

$$V_{\rm CKM} = \begin{pmatrix} e^{i\xi_1} & 0 \\ 0 & U_u^{(2)} \end{pmatrix} \cdot V_{\rm CKM} \cdot \text{diag}\left(e^{-i\tilde{\xi}_i}\right) \ . \tag{4.A.10}$$

Upon solving this equation, one finds that, similarly to the case of non-degenerate masses, one can only obtain the flavor groups $U(1)_B$, $U(1)^2$, and $U(1)^3$. Flavor symmetries beyond baryon numbers are obtained for the textures discussed above, but the generic case for the CKM matrix when $m_t = m_c$ is given in Eq. (4.4.16) and does not have a specific texture.

$\mathbf{m_t = m_c}$ **and** $\mathbf{m_s = m_d}$ Let us now turn to the case where $m_t = m_c$ and $m_s = m_d$, all other masses being non-degenerate. The relations in Eq. (4.A.5) imply that

$$U_Q = \begin{pmatrix} e^{i\xi_1} & 0 \\ 0 & U_u^{(2)} \end{pmatrix} = V_{\rm CKM} \cdot \begin{pmatrix} U_d^{(2)} & 0 \\ 0 & e^{i\tilde{\xi}_3} \end{pmatrix} \cdot V_{\rm CKM}^\dagger \text{ with } U_u^{(2)}, U_d^{(2)} \in U(2) \ , \tag{4.A.11}$$

[19] The equivalent case where down quark masses are degenerate is treated identically after the exchange $V_{\rm CKM} \leftrightarrow V_{\rm CKM}^\dagger$. We also consider the specific case $m_t = m_c$ (and $m_s = m_d$ in the case of down quarks later on), since the formulae are simpler given our parametrization of $V_{\rm CKM}$. Nevertheless, the discussion (and the parametrization) can be adapted to any other quark mass degeneracy. In particular, the remarkable textures $V_{\rm CKM}^{(ji)}$ leading to a given flavor symmetry when $m_{u_j} = m_{u_i}, i < j$ are related to those when $m_t = m_c$ by

$$V_{\rm CKM}^{(ji)} = R_{i2} R_{j3} V_{\rm CKM}^{(tc)} \ ,$$

where R_{ab} is the matrix which exchanges rows (or columns) a and b. Similarly, textures obtained when $m_{d_j} = m_{d_i}, i < j$ are related to those when $m_s = m_d$ by

$$V_{\rm CKM}^{(ji)} = V_{\rm CKM}^{(sd)} R_{i1} R_{j2} \ .$$

For instance, the texture which leads to a flavor symmetry $U(2) \times U(1)$ (see below) for $m_{u_j} = m_{u_i}, i < j$ and $m_{d_l} = m_{d_k}, k < l$ is

$$V_{\rm CKM} = R_{i2} R_{j3} \cdot \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \cdot R_{k1} R_{l2} \ .$$

hence

$$V_{\rm CKM} = \begin{pmatrix} e^{i\xi_1} & 0 \\ 0 & U_u^{(2)} \end{pmatrix} \cdot V_{\rm CKM} \cdot \begin{pmatrix} U_d^{(2)\dagger} & 0 \\ 0 & e^{-i\tilde{\xi}_3} \end{pmatrix} \ . \tag{4.A.12}$$

Starting from the CKM matrix in Eq. (4.4.16), which is generic when $m_t = m_c$, we can further rotate θ_{12} away by performing

$$d_R \to \begin{pmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} d_R \ . \tag{4.A.13}$$

Recall that we work in the *up basis*, where $V_{\rm CKM}$ appears in Y_d and can be affected by right-handed down-quark flavor transformations which commute with the down-quark mass matrices. We then obtain

$$V_{\rm CKM} \to \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \ , \text{ i.e. with a texture } \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \ . \tag{4.A.14}$$

Therefore, the generic CKM matrix for the present case has a texture which allows for a flavor symmetry at least as large as $U(1)^2$. Possible larger flavor symmetries are $U(1)^3$ or $U(2)\times U(1)$, obtained for the following respective textures,[20]

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \ . \tag{4.A.15}$$

$\mathbf{m_u = m_c = m_t}$ When the degeneracy is maximal, the CKM matrix can be fully absorbed by a redefinition of the RH up-quarks:

$$u_R \to V_{\rm CKM} u_R \ , \quad V_{\rm CKM} \to \mathbb{1} \ . \tag{4.A.16}$$

The flavor symmetry is therefore at least as large as $U(1)^3$. With such a CKM matrix, one gets that $U_Q = U_u = U_d$ and the flavor group is determined by the relations in Eq. (4.A.5):

[20] Let us present some details regarding the second case to illustrate the derivation. With such a texture, one can phase-rotate the fields so as to get

$$V_{\rm CKM} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ ,$$

and one finds from Eq. (4.A.12) that

$$\mathbb{1} = \begin{pmatrix} O U_u^{(2)} O U_d^{(2)\dagger} & 0 \\ 0 & e^{i(\xi_1 - \tilde{\xi}_3)} \end{pmatrix} \ ,$$

with $O \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, one obtains $U_d^{(2)} = O U_u^{(2)} O, \tilde{\xi}_3 = \xi_1$ and no further constraint, hence the group is $U(2) \times U(1)$.

the flavor symmetry group is $U(1)^3$, $U(2) \times U(1)$, or $U(3)$, respectively when no, two, or three down-quark masses are degenerate.

4.A.2 Vanishing masses at dimension-four

When some masses vanish at dimension-four, the flavor symmetry can contain axial phases. This case complicates the power-counting, since dimension-six Yukawa couplings now yield the leading contribution to the masses, but it can be treated as suggested in Section 4.3.1 for the case of neutrino masses or vanishing Y_d.

Whenever a quark mass goes to zero, the flavor symmetry is enlarged since the LH and RH components now describe independent particles. A flavor symmetry $U(1)_B$ would then be ugraded to $U(1)_B \times U(1)_{u_{R,1}}$ when $m_u \to 0$, all other parameters being kept fixed. For non-degenerate quark masses, the flavor symmetry is always abelian, and taking one mass to zero simply adds a RH $U(1)$ factor to the flavor symmetry, as discussed just above. On the other hand, when two or three masses are degenerate, taking them to zero together adds a RH factor $U(2)$ or $U(3)$ to the flavor symmetry (even if it was abelian for non-zero masses due to some structure in the CKM matrix which distinguishes the different flavors of LH quarks). We list in Table 4.A.1 the relevant cases, and present in Table 4.A.2 the associated numbers of new primary sources of CPV[21]. Those new sources of CPV can still be captured

If in addition to the values in Table 3:	Add to the flavor group the factor:
$m_u = 0$	$U(1)_{u_R}$
$m_u = m_d = 0$	$U(1)_{u_R} \times U(1)_{d_R}$
$m_u = m_c = 0$	$U(2)_{u_R}$
$m_u = m_c = m_d = 0$	$U(2)_{u_R} \times U(1)_{d_R}$
$m_u = m_c = m_d = m_s = 0$	$U(2)_{u_R} \times U(2)_{d_R}$
$m_u = m_c = m_t = 0$	$U(3)_{u_R}$
$m_u = m_c = m_t = m_d = 0$	$U(3)_{u_R} \times U(1)_{d_R}$
$m_u = m_c = m_t = m_d = m_s = 0$	$U(3)_{u_R} \times U(2)_{d_R}$
$m_u = m_c = m_t = m_d = m_s = m_b = 0$	$U(3)_{u_R} \times U(3)_{d_R}$

Table 4.A.1: Additional factor to the flavor symmetry of the SM_4 lagrangian when quark masses vanish.

by flavor invariants, however there are subtleties to take into account when masses vanish. Namely, the sets of invariants we present in the main text and in Appendices 4.D and 4.E have transfer matrices that do not maintain maximal rank in the limit of vanishing masses. Let us

[21]The counting is performed as in Section 4.4.2, i.e. one counts how many complex linear invariants (under the SM_4 flavor symmetry) there are. Let us give some examples. In the case of a symmetry $U(1)^3 \times U(3)_{u_R}$ and focussing on C_{Hu}, one finds that the only linear invariant reads $\delta^{ij} C_{Hu,ij}$ (summed over i, j). However, C_{Hu} being hermitian, this combination is real in all bases and does not violate CP. In the case of a symmetry $U(1)^2 \times U(1)_{u_R}$, the $U(1)^2$ factor indicates that, in some appropriate flavor basis, the SM_4 lagrangian possesses two independent quark number symmetries (one of which is the usual baryon number), singling out a quark flavor. Assuming without loss of generality that this flavor is the third one and focussing on C_{QuQd}, linear $U(1)^2$-invariants read $C_{QuQd,ijkl}$, $C_{QuQd,33ij}$, $C_{QuQd,3ji3}$, $C_{QuQd,i33j}$, $C_{QuQd,ij33}$, $C_{QuQd,3333}$ (where $i, j, k, l = 1, 2$), which are 33 complex coefficients. Then, one needs to know whether the massless quark is the one which is singled out by the quark number, i.e. $u_{R,3}$ (otherwise, without loss of generality we take the massless quark to be the first flavor). If so one must discard $C_{QuQd,33ij}$, $C_{QuQd,i33j}$ (reducing the 33 coefficients to 25), if not one must instead discard $C_{QuQd,i1kl}$, $C_{QuQd,31i3}$, $C_{QuQd,i133}$ (reducing to 21 coefficients).

	Bilinears				4-Fermi						
Flavour symmetries of the quark sector of the SM	C_{eH} C_{eW} C_{eB}	C_{uH} C_{uG} C_{uW} C_{uB} C_{dH} C_{dG} C_{dW} C_{dB} C_{Hud}	$C_{HL}^{1,3}$ C_{He}	$C_{HQ}^{1,3}$ C_{Hu} C_{Hd}	C_{LL} C_{ee}	C_{Le}	$C_{QQ}^{1,3}$ C_{uu} C_{dd}	$C_{LQ}^{1,3}$ C_{Qe} C_{Lu} C_{eu} C_{Ld} C_{ed}	$C_{ud}^{1,8}$ $C_{Qu}^{1,8}$ $C_{Qd}^{1,8}$	C_{LedQ} $C_{LeQu}^{1,3}$	$C_{QuQd}^{1,8}$
$U(1)_B$	3	9	0	3	0	3	18	9	36	27	81
$U(1)_B \times U(1)_{u_R}$	3	$6^4, 9^4, 6$	0	3,1,3	0	3	18,5,18	$9^2, 3^2, 9^2$	$18^2, 36$	27,18	54
$U(1)_B \times U(1)_{u_R} \times U(1)_{d_R}$	3	$6^8, 4$	0	$3,1^2$	0	3	$18,6^2$	$9^2, 3^4$	$8, 18^2$	18^2	36
$U(1)_B \times U(2)_{u_R}$	3	$3^4, 9^4, 3$	0	3,0,3	0	3	18,0,18	$9^2, 0^2, 9^2$	$6^2, 36$	27,9	27
$U(1)_B \times U(2)_{u_R} \times U(1)_{d_R}$	3	$3^4, 6^4, 2$	0	3,0,1	0	3	18,0,6	$9^2, 0^2, 3^2$	$2, 6, 18$	18,9	18
$U(1)^2$	3	5	0	1	0	3	5	3	12	15	33
$U(1)^2 \times U(1)_{u_R}$	3	$(3 \text{ or } 4)^4, 5^4$,3 or 4	0	1,0 or 1,1	0	3	5,0 or 5,5	3^2,$(0 \text{ or } 3)^2$,3^2	$(5 \text{ or } 8)^2$,12	15,9 or 12	21 or 24
$U(1)^2 \times U(1)_{u_R} \times U(1)_{d_R}$	3	$(3 \text{ or } 4)^8$,2 or 4	0	1,$(0 \text{ or } 1)^2$	0	3	5,$(0 \text{ or } 5)^2$	3^2,$(0 \text{ or } 3)^4$	1 or 3 or 8,$(5 \text{ or } 8)^2$	9 or 12	12 or 13 or 16
$U(1)^2 \times U(2)_{u_R}$	3	$(1 \text{ or } 2)^4$,5,1 or 2	0	1,0,1	0	3	5,0,5	$3^2, 0^2, 3^2$	2^2,12	15,3 or 6	12
$U(1)^2 \times U(2)_{u_R} \times U(1)_{d_R}$	3	$(1 \text{ or } 2)^4$,$(3 \text{ or } 4)^4$,0 or 1 or 2	0	1,0,0 or 1	0	3	5,0,0 or 5	$3^2, 0^2$,$(0 \text{ or } 3)^2$	0 or 2,2,5 or 8	9 or 12,3	8
$U(1)^2 \times U(2)_{u_R} \times U(2)_{d_R}$	3	$(1 \text{ or } 2)^8$,0 or 1	0	$1,0^2$	0	3	$5,0^2$	$3^2, 0^4$	$0,2^2$	3 or 6	4
$U(1)^3$	3	3	0	0	0	3	0	0	3	9	15
$U(1)^3 \times U(1)_{u_R}$	3	$2^4, 3^4, 2$	0	0	0	3	0	0	$1^2, 3$	9,6	10
$U(1)^3 \times U(1)_{u_R} \times U(1)_{d_R}$	3	2^8,1 or 2	0	0	0	3	0	0	0 or $1,1^2$	6	7 or 8
$U(1)^3 \times U(2)_{u_R}$	3	$1^4, 3^4, 1$	0	0	0	3	0	0	$0^2, 3$	9,3	5
$U(1)^3 \times U(2)_{u_R} \times U(1)_{d_R}$	3	$1^4, 2^4$,0 or 1	0	0	0	3	0	0	$0^2, 1$	6,3	3 or 4
$U(1)^3 \times U(2)_{u_R} \times U(2)_{d_R}$	3	1^8,0 or 1	0	0	0	3	0	0	0	3	2
$U(1)^3 \times U(3)_{u_R}$	3	$0^4, 3^4, 0$	0	0	0	3	0	0	$0^2, 3$	9,0	0
$U(1)^3 \times U(3)_{u_R} \times U(1)_{d_R}$	3	$0^4, 2^4, 0$	0	0	0	3	0	0	$0^2, 1$	6,0	0
$U(2) \times U(1)$	3	2	0	0	0	3	0	0	1	6	7
$U(2) \times U(1) \times U(1)_{u_R}$	3	$1^4, 2^4, 1$	0	0	0	3	0	0	$0^2, 1$	6,3	4
$U(2) \times U(1) \times U(1)_{u_R} \times U(1)_{d_R}$	3	1	0	0	0	3	0	0	0	3	1 or 2
$U(2) \times U(1) \times U(2)_{u_R}$	3	$1^4, 2^4, 1$	0	0	0	3	0	0	$0^2, 1$	6,3	3
$U(2) \times U(1) \times U(2)_{u_R} \times U(1)_{d_R}$	3	$1^8, 0$	0	0	0	3	0	0	0	3	2
$U(2) \times U(1) \times U(2)_{u_R} \times U(2)_{d_R}$	3	1	0	0	0	3	0	0	0	3	1
$U(2) \times U(1) \times U(3)_{u_R}$	3	$0^4, 2^4, 0$	0	0	0	3	0	0	$0^2, 1$	6,0	0
$U(2) \times U(1) \times U(3)_{u_R} \times U(1)_{d_R}$	3	$0^4, 1^4, 0$	0	0	0	3	0	0	0	3,0	0
$U(2) \times U(1) \times U(3)_{u_R} \times U(2)_{d_R}$	3	$0^4, 1^4, 0$	0	0	0	3	0	0	0	3,0	0
$U(3)$	3	1	0	0	0	3	0	0	0	3	2
$U(3) \times U(3)_{u_R}$	3	$0^4, 1^4, 0$	0	0	0	3	0	0	0	3,0	0
$U(3) \times U(3)_{u_R} \times U(3)_{d_R}$	3	0	0	0	0	3	0	0	0	0	0
Two degenerate electron-type leptons	$\times\frac{2}{3}$	$\times 1$		$\times 1$		$\times\frac{2}{3}$	$\times 1$	$\times\frac{2}{3}$	$\times 1$	$\times\frac{2}{3}$	$\times 1$
All electron-type leptons degenerate	$\times\frac{1}{3}$	$\times 1$		$\times 1$		$\times\frac{1}{3}$	$\times 1$	$\times\frac{1}{3}$	$\times 1$	$\times\frac{1}{3}$	$\times 1$
One vanishing electron-type mass	$\times\frac{2}{3}$	$\times 1$		$\times 1$		$\times\frac{1}{3}$	$\times 1$	$\times 1$	$\times 1$	$\times\frac{2}{3}$	$\times 1$
Two vanishing electron-type masses	$\times\frac{1}{3}$	$\times 1$		$\times 1$		0	$\times 1$	$\times\frac{2}{3}$	$\times 1$	$\times\frac{1}{3}$	$\times 1$
All electron-type masses vanishing	0	$\times 1$		$\times 1$		0	$\times 1$	$\times\frac{1}{3}$	$\times 1$	0	$\times 1$

Table 4.A.2: Numbers of new primary sources of CPV contained in each dimension-six SMEFT coefficient. When a single number appears, it applies to all operators at the top of the concerned column. When several numbers are needed, they appear as a list, where the integer power refers to the multiplicity of a given number. An entry "i or j" means that the answer depends on the details of the flavor charges. The last five rows indicate which multiplicative coefficient should be applied to all numbers of the same column for remarkable values of the electron-type lepton masses. The situation where $m_d = 0$, relevant for approximations in high-energy observables, generically corresponds to the line $U(1)^3 \times U(3)_{u_R}$, after a suitable replacement $u \leftrightarrow d$ (note that there is a single $\mathcal{O}_{LeqQ}$ operator in the Warsaw basis, while there are two $\mathcal{O}_{LeQu}^{1,3}$ operators).

illustrate what we mean by this with an example. From Table 4.A.2, one learns that the three phases in $C_{HQ}^{(1,3)}$ remain primary when $m_u = m_c = 0$. However, the invariants presented in the associated set in Table 4.D.1 are of the form $\mathrm{Im}\,\mathrm{Tr}\left(X_u M C_{HQ}^{(1,3)}\right)$ for some matrix M built out of the Yukawas. Working in the up basis with a vanishing CKM phase, and focusing on the contribution proportional to $C_{HQ,12}^{(1,3)}$, we find

$$\mathrm{Im}\,\mathrm{Tr}\left(X_u M C_{HQ}^{(1,3)}\right) \supset \mathrm{Im}\left(C_{HQ,12}^{(1,3)}\right)\left(m_u^2 M_{21} - m_c^2 M_{12}\right) \tag{4.A.17}$$

which vanishes when $m_u = m_c = 0$. Therefore, the set of invariants we consider does not allow us to capture the three phases in $C_{HQ}^{(1,3)}$ in such limits. Another example is that of C_{uH}. When $Y_d = 0$, one finds only two invariants in the associated set in Table 4.D.1, whereas three sources of CPV remain as shown in Table 4.A.2. One could therefore conclude that one of the invariants in the set should be replaced by the missing

$$\mathrm{Im\,Tr}\left(X_u^2 C_{uH} Y_u^\dagger\right) \ . \tag{4.A.18}$$

However, this choice would not allow us to retain a sufficient rank for the set, as one finds

$$\mathrm{Im\,Tr}\left(X_u^2 C_{uH} Y_u^\dagger\right) = (m_u^2 + m_t^2)\mathrm{Im\,Tr}\left(X_u C_{uH} Y_u^\dagger\right) - m_u^2 m_t^2 \mathrm{Im\,Tr}\left(C_{uH} Y_u^\dagger\right) \tag{4.A.19}$$

when $m_u = m_c$ (one can use formulae like Eq. (4.4.6) to express the mass factors in terms of invariants), whereas all nine sources of CPV in C_{uH} remain primary and independent in this case, as per Tables 3 and 4. Therefore, it may seem that one needs strictly more than nine invariants to capture the nine CPV phases in C_{uH}, and more generally that the necessary and sufficient conditions presented in Section 4.4.1 are not sufficient anymore when masses can vanish. However, this is a consequence of our assumption that invariants should correspond to traces of a monomial of degree one in SMEFT coefficients, and arbitrary degree in Yukawa matrices. Instead, one could enlarge the set of invariants and include traces of sums over monomials of various degrees. For instance, defining instead $X_u \equiv 1 + Y_u Y_u^\dagger$, and similarly for other fermions, without changing the expression of the invariants, is sufficient to ensure that the vanishing of our sets is a necessary and sufficient condition for the conservation of CP at leading order.

4.B Generalities about invariants

4.B.1 Properties of 3×3 matrices

Here we discuss some properties of generic 3×3 matrices, which we use throughout the Chapter and will refer to later on. We will follow mostly Ref. [159]. The starting point is the Cayley–Hamilton theorem, which allows one to rewrite the n-th power of a $n \times n$ matrix A in terms of the powers $< n$, and that for $n = 3$ takes the form

$$A^3 = A^2\,\mathrm{Tr}(A) - \frac{1}{2}A\left[\mathrm{Tr}(A)^2 - \mathrm{Tr}\big(A^2\big)\right] + \frac{1}{6}\left[\mathrm{Tr}(A)^3 - 3\,\mathrm{Tr}\big(A^2\big)\,\mathrm{Tr}(A) + 2\,\mathrm{Tr}\big(A^3\big)\right]\mathbb{1}_{3\times 3} \ . \tag{4.B.1}$$

Multiplying by A and taking the trace results in

$$\mathrm{Tr}\big(A^4\big) = \frac{1}{6}\,\mathrm{Tr}(A)^4 - \mathrm{Tr}\big(A^2\big)\,\mathrm{Tr}(A)^2 + \frac{4}{3}\,\mathrm{Tr}\big(A^3\big)\,\mathrm{Tr}(A) + \frac{1}{2}\,\mathrm{Tr}\big(A^2\big)^2 . \tag{4.B.2}$$

Shifting $A \to A + B + C$ in Eq. (4.B.2), with B and C some other generic 3 matrices, and taking the terms of order A^2BC, one obtains

$$\begin{aligned}0 = {} & \mathrm{Tr}(A)^2\,\mathrm{Tr}(B)\,\mathrm{Tr}(C) - \mathrm{Tr}(BC)\,\mathrm{Tr}(A)^2 - 2\,\mathrm{Tr}(AB)\,\mathrm{Tr}(A)\,\mathrm{Tr}(C) + \\ & - 2\,\mathrm{Tr}(AC)\,\mathrm{Tr}(A)\,\mathrm{Tr}(B) + 2\,\mathrm{Tr}(ABC)\,\mathrm{Tr}(A) + 2\,\mathrm{Tr}(ACB)\,\mathrm{Tr}(A) + \\ & - \mathrm{Tr}\big(A^2\big)\,\mathrm{Tr}(B)\,\mathrm{Tr}(C) + 2\,\mathrm{Tr}(AB)\,\mathrm{Tr}(AC) + \mathrm{Tr}\big(A^2\big)\,\mathrm{Tr}(BC) + \\ & + 2\,\mathrm{Tr}(C)\,\mathrm{Tr}\big(A^2B\big) + 2\,\mathrm{Tr}(B)\,\mathrm{Tr}\big(A^2C\big) - 2\,\mathrm{Tr}\big(A^2BC\big) - 2\,\mathrm{Tr}\big(A^2CB\big) - 2\,\mathrm{Tr}(ABAC)\ .\end{aligned} \tag{4.B.3}$$

This property is useful for our purpose of building sets of invariants, as it implies that we only need to draw from a finite set. Let us focus on invariants relevant for this paper, such as those related to bi-fermion SMEFT operators, which are single-trace and linear with respect to the associated Wilson coefficient. To build such invariants, flavor-invariance imposes that we only use X_u, X_d and C, where C is the Wilson coefficient under study (up to a specific multiplication by a Yukawa matrix for operators of LR chiral structure). In principle, any invariant of the form

$$\mathrm{Tr}\big(X_u^{a_1} X_d^{b_1} X_u^{a_2} X_d^{b_2} \dots C\big) \tag{4.B.4}$$

is allowed. However, the formulae above imply that traces with third or higher powers of X_q can be redefined away as they are redundant, and that the same happens for traces with more than one occurrence of X_q or of X_q^2. These conditions reduce the possible single-trace invariants to a finite set (see Appendix 4.B.3 for explicit examples).

Finally, we mention that the Cayley–Hamilton theorem also allows us to write the determinant of a 3×3 matrix as

$$\mathrm{Det}(A) = \frac{1}{6}\left(\mathrm{Tr}(A)^3 - 3\,\mathrm{Tr}(A)\,\mathrm{Tr}\big(A^2\big) + 2\,\mathrm{Tr}\big(A^3\big)\right)\ . \tag{4.B.5}$$

4.B.2 Different types of invariants

In Ref. [265], the author presents a discussion of CP-violating invariants in supersymmetric models, in order to find basis independent conditions for CP conservation, as done here. In that context, three types of invariants built using three 3×3 matrices A, B, and C are proposed, namely

$$\begin{aligned} J_{AB} &\equiv \mathrm{Im}\,\mathrm{Tr}\left([A,B]^3\right)\ , & K_{ABC}(p,q,r) &\equiv \mathrm{Im}\,\mathrm{Tr}\left([A^p,B^q]C^r\right)\ , \\ L_C(p) &\equiv \mathrm{Im}\,\mathrm{Tr}\left(C^p - \text{h.c.}\right)\ , \end{aligned} \tag{4.B.6}$$

where A and B are hermitian and C generic. These are dubbed $J-$, $K-$ and $L-$invariants, respectively. In this Chapter we adopted a similar notation, but we only employed L-invariants for our set. However, we can show that this choice is general, as the remaining two types can

be written in terms of the last one.[22] To prove this, let us start from J_{AB}. First of all, using Eq. (4.B.5), it can be shown to be equivalent to a Jarlskog-like invariant, i.e.

$$\begin{aligned}\text{Im Det}\left([A,B]\right) &= \frac{1}{6}\text{Im}\left(\text{Tr}\left([A,B]\right)^3 - 3\,\text{Tr}\left([A,B]\right)\text{Tr}\left([A,B]^2\right) + 2\,\text{Tr}\left([A,B]^3\right)\right) + \\ &= \frac{1}{3}\text{Im}\,\text{Tr}\left([A,B]^3\right)\ ,\end{aligned} \tag{4.B.7}$$

as the trace of a commutator vanishes. This also proves Eq. (4.1.5). Then, by expanding $[A,B]^3$ and using the cyclic property of the trace we can show

$$\text{Im}\,\text{Tr}\left([A,B]^3\right) = 3\text{Im}\,\text{Tr}\left(A^2B^2AB - BAB^2A^2\right) = L_{A^2B^2AB}\ . \tag{4.B.8}$$

Next, we prove that any K-invariant can also be expressed in terms of the L ones. It is enough to show a proof for $K_{ABC}(1,1,1)$, as the other cases can be obtained by redefining A, B or C. Let us split C in its hermitian and anti-hermitian parts, i.e.

$$C_h \equiv \frac{C+C^\dagger}{2} \qquad\qquad C_a \equiv \frac{C-C^\dagger}{2}\ . \tag{4.B.9}$$

and

$$K_{ABC}(1,1,1) = \text{Im}\,\text{Tr}\left([A,B]C\right) = \text{Im}\,\text{Tr}\left([A,B]C_h\right) + \text{Im}\,\text{Tr}\left([A,B]C_a\right)\ . \tag{4.B.10}$$

Now

$$\begin{aligned}\text{Im}\,\text{Tr}\left([A,B]C_h\right) =& \text{Im}\,\text{Tr}\left(ABC_h - BAC_h\right) = \\ =& \frac{1}{2i}\left[\text{Tr}\left(ABC_h\right) - \text{Tr}\left(ABC_h\right)^* - \text{Tr}\left(BAC_h\right) + \text{Tr}\left(BAC_h\right)^*\right] = \\ =& \frac{1}{2i}\left[\text{Tr}\left(ABC_h\right) - \text{Tr}\left(C_hBA\right) - \text{Tr}\left(BAC_h\right) + \text{Tr}\left(C_hAB\right)\right] = \\ =& \frac{1}{i}\left[\text{Tr}\left(ABC_h\right) - \text{Tr}\left(C_hBA\right)\right] = 2L_{ABC_h} = 2L_{ABC}\ .\end{aligned} \tag{4.B.11}$$

With similar steps, one can see that the piece proportional to C_a vanishes, so that

$$K_{ABC}(1,1,1) = 2L_{ABC}\ . \tag{4.B.12}$$

4.B.3 Finding polynomial relations between invariants

In the main body of this Chapter, the logic we have followed to build invariants stemmed from knowing that the relative Wilson coefficient $C^{(6)}$, in a given basis, has a certain number of phases. Then we found as many independent invariants as possible, in order to obtain a transfer matrix whose rank matched the new sources of CPV in $C^{(6)}$ when $J_4 = 0$. Now, however, we could be tempted to pursue a different line of reasoning and find the relevant invariants by

[22] also notice that J_{AB} would not suit our scopes as it is not linear in any of the two matrices in the argument.

applying to our case the power of the Hilbert series and its Plethystic logarithm, developed in Section 2.2. Let us restrict to the case of quark bilinear operators. Their Wilson coefficients are generic 3×3 complex matrices that we can multiply by an appropriate number of $Y_{u,d}^{(\dagger)}$ to turn them into a $\mathbf{3} \otimes \bar{\mathbf{3}}$ representation of $SU(3)_Q$. We refer to this combination as $C^{(6)}$ here. Then, we can identify the building blocks of our invariants as

$$C^{(6)}, (C^{(6)})^\dagger, X_{u,d} \in \mathbf{3} \otimes \bar{\mathbf{3}} \ . \tag{4.B.13}$$

and we can build the multi-graded Hilbert series

$$h(c, c^\dagger, x_u, x_d) = \int [\mathrm{d}\mu]_{SU(3)} \prod_{i=\left\{c, c^\dagger, x_u, x_d\right\}} \mathrm{PE}(\vec{z}; c) \ , \tag{4.B.14}$$

with obvious associations between a spurion and the corresponding building block. The resulting expression is quite long and not particularly illuminating, so we will refrain from presenting it here. However, we can look at its ungraded version

$$H(q) = h(q, q, q, q) = \frac{N(q)}{D(q)} \ , \tag{4.B.15}$$

with

$$\begin{aligned} N(q) = &+ q^{34} + 14q^{31} + 31q^{30} + 56q^{29} + 165q^{28} + 354q^{27} + 660q^{26} + 1256q^{25} + 2097q^{24} + \\ &+ 3184q^{23} + 4720q^{22} + 6404q^{21} + 7992q^{20} + 9536q^{19} + 10510q^{18} + 10744q^{17} + \\ &+ 10510q^{16} + 9536q^{15} + 7992q^{14} + 6404q^{13} + 4720q^{12} + 3184q^{11} + 2097q^{10} + \\ &+ 1256q^9 + 660q^8 + 354q^7 + 165q^6 + 56q^5 + 31q^4 + 14q^3 + 1 \ , \end{aligned} \tag{4.B.16}$$

and

$$D(q) = (1-q)^4(1-q^2)^{10}(1-q^3)^{10}(1-q^4)^4 \ . \tag{4.B.17}$$

We can see that the numerator has the correct palindromic structure we expected, and more importantly the denominators contain 28 factors, correctly matching the 10 observables from the Standard Model and the 18 new (9 real + 9 imaginary) observables contained in $C^{(6)}$. This is already quite remarkable. However, to this point we have neither an idea of how the algebraically independent invariants look like, nor a way to extract the ones that are linear in $C^{(6)}$, which is the subset we are really interested in. To gain some more insight, let us look at

the Plethystic logarithm of the multi-graded Hilbert series:

$$\begin{aligned} PL\left[h(c,c^\dagger,x_u,x_d)\right] =&(x_u+x_d)+(x_u^2+x_dx_u+x_d^2)+(x_u^3+x_u^2x_d+x_ux_d^2+x_d^3)+x_u^2x_d^2+\\ &+x_u^3x_d^3-x_u^6x_d^6+\\ &+\left(c+c^\dagger\right)\left[1+x_u+x_d+(x_u^2+x_d^2+2x_ux_d)+(2x_u^2x_d+2x_ux_d^2)+\right.\\ &\left.+(x_u^3x_d+2x_u^2x_d^2+x_d^3x_u)+(x_u^3x_d^2+x_u^2x_d^3)\right]+\\ &+\mathcal{O}\left(c^2,(c^\dagger)^2,(c+c^\dagger)x_u^3x_d^4,(c+c^\dagger)x_u^4x_d^3\right)\ . \end{aligned} \tag{4.B.18}$$

Since we are interested in invariants that are linear in $C^{(6)}$, we stopped the expansion at $\mathcal{O}\left(c,c^\dagger\right)$. The $\mathcal{O}\left(c^0(c^\dagger)^0\right)$ terms in this expansion correspond to the invariants that can be obtained from the quark sector of the Standard Model. This case has been treated in Ref. [159], and the resulting algebraically independent invariants are

$$\begin{aligned} I_{1,0}&=\mathrm{Tr}(X_u) & I_{0,1}&=\mathrm{Tr}(X_d)\\ I_{2,0}&=\mathrm{Tr}\left(X_u^2\right) & I_{1,1}&=\mathrm{Tr}(X_uX_d)\\ I_{0,2}&=\mathrm{Tr}\left(X_d^2\right) & I_{3,0}&=\mathrm{Tr}\left(X_u^3\right)\\ I_{2,1}&=\mathrm{Tr}\left(X_u^2X_d\right) & I_{1,2}&=\mathrm{Tr}\left(X_uX_d^2\right)\\ I_{0,3}&=\mathrm{Tr}\left(X_d^3\right) & I_{2,2}&=\mathrm{Tr}\left(X_u^2X_d^2\right)\ , \end{aligned} \tag{4.B.19}$$

Notice their number is 10, correctly matching the 6 masses + 3 angles + 1 phase of the Standard Model. The generating set, i.e. the set of invariants which cannot be expressed as polynomials of other ones, contains one additional invariant,

$$I_{3,3}^{(-)}=\mathrm{Tr}\left(X_u^2X_d^2X_uX_d\right)-\mathrm{Tr}\left(X_d^2X_u^2X_dX_u\right)\ , \tag{4.B.20}$$

corresponding to the $x_u^3x_d^3$ term in Eq. (4.B.18) and which is nothing but J_4. In the language adopted here, this invariant does not contain any additional observable, and is just needed to capture the sign of the SM_4 phase δ. The negative term $-x_u^6x_d^6$ at the end signals that there is a syzygy at degree 12, which in this case corresponds to the fact that $\left(I_{3,3}^{(-)}\right)^2$ can be expressed in terms of the remaining 10 invariants, as expected.

The part linear in c and $c^\dagger$ of Eq. (4.B.18) points us at the basic invariants linear in $C^{(6)}$. We see that in this case the set of basic invariants is composed by 34 element, 17 each for $C^{(6)}$ and $(C^{(6)})^\dagger$, which is larger than the basic set. Indeed, the latter is expected to have 18 elements, corresponding to the 9 new complex observables contained in $C^{(6)}$. To try and build the invariants in the generating set, we will make use of the relations showed in Section 4.B.1. Given a generic matrix $C^{(6)}\in\mathbf{3}\otimes\bar{\mathbf{3}}$ of $SU(3)_Q$, we want to contract it with as many $X_{u,d}$'s as needed to form all the possible independent invariants. Using Eq.(4.B.1) on $X_{u,d}$, we can show that all invariants written using $X_{u,d}^n$, with $n\geq 3$, are redundant, and we only need $X_{u,d}^2$ and $X_{u,d}$ as building blocks. Moreover, using Eq. (4.B.3), one can eliminate any invariant where a matrix is repeated. Taking into account these simplifications, one can see that the set of

possible invariants is finite, and is formed by these 29 objects:

$$
\begin{array}{lll}
\mathrm{Tr}(C^{(6)}) & \mathrm{Tr}(X_u C^{(6)}) & \mathrm{Tr}(X_d C^{(6)}) \\
\mathrm{Tr}\left(X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d^2 C^{(6)}\right) & \mathrm{Tr}\left(X_u X_d C^{(6)}\right) \\
\mathrm{Tr}\left(X_u X_d^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d X_u C^{(6)}\right) & \mathrm{Tr}\left(X_d^2 X_u C^{(6)}\right) \\
\mathrm{Tr}\left(X_d X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_u^2 X_d C^{(6)}\right) & \mathrm{Tr}\left(X_u^2 X_d^2 C^{(6)}\right) \\
\mathrm{Tr}\left(X_d^2 X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_u X_d X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d X_u X_d^2 C^{(6)}\right) \\
\mathrm{Tr}\left(X_u^2 X_d X_u C^{(6)}\right) & \mathrm{Tr}\left(X_d^2 X_u X_d C^{(6)}\right) & \mathrm{Tr}\left(X_u X_d^2 X_u^2 C^{(6)}\right) \\
\mathrm{Tr}\left(X_u^2 X_d^2 X_u C^{(6)}\right) & \mathrm{Tr}\left(X_d X_u^2 X_d^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d^2 X_u^2 X_d C^{(6)}\right) \\
\mathrm{Tr}\left(X_u X_d X_u^2 X_d^2 C^{(6)}\right) & \mathrm{Tr}\left(X_u X_d^2 X_u^2 X_d C^{(6)}\right) & \mathrm{Tr}\left(X_d X_u X_d^2 X_u^2 C^{(6)}\right) \\
\mathrm{Tr}\left(X_d X_u^2 X_d^2 X_u C^{(6)}\right) & \mathrm{Tr}\left(X_u^2 X_d X_u X_d^2 C^{(6)}\right) & \mathrm{Tr}\left(X_u^2 X_d^2 X_u X_d C^{(6)}\right) \\
\mathrm{Tr}\left(X_d^2 X_u X_d X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d^2 X_u^2 X_d X_u C^{(6)}\right) , &
\end{array}
\tag{4.B.21}
$$

and the same for $C^{(6)} \to (C^{(6)})^\dagger$. Now, since the generating set only includes 17 elements, this means that 12 invariants of Eq. (4.B.21) can be expressed as polynomials of the remaining ones and can thus be eliminated. To do this, we employ a numerical algorithm adapted from Appendix C of Ref. [155] (see also Ref. [154]). The logic is as follows: by assigning a dummy dimension to the building blocks, i.e. $[X_{u,d}] = 1$ and $\left[C^{(6)}\right] = 1$, we can assign a dimension to all the invariants listed above. Then, we fix some given dimension n. Picking one of the invariants in Eq. (4.B.21), one can then take its product with as many traces from Eq. (4.B.19) as needed to form a monomial M_i of dimension n. Repeating this for all instances of Eq. (4.B.21), we find the set $\{M_i\}$ of all the possible monomials that are dimension n and linear in C^6. For example, at dimension $n = 2$ one can obtain the monomials

$$
\{M_i\} = \left\{\mathrm{Tr}\left(C^{(6)}\right) I_{1,0}, \mathrm{Tr}\left(C^{(6)}\right) I_{0,1}, \mathrm{Tr}\left(X_u C^{(6)}\right), \mathrm{Tr}\left(X_d C^{(6)}\right)\right\} . \tag{4.B.22}
$$

Then we set a linear combination of these monomials to zero, i.e.

$$
\sum_i a_i M_i = 0 , \tag{4.B.23}
$$

where the a_i's are integer coefficients. We then plug random integer values for the entries of the matrices $X_{u,d}$ and C^6. This produces a linear equation for the a_i's. Repeating this last step as many times as there are M_i's, one builds a linear system for the a_i's with zero constant term. The number of independent directions of the null space of the corresponding matrix matches the number of possible relations between the M_i's. The first nontrivial result is found

at dimension 5, where we get the two relations

$$\begin{aligned}
&\mathrm{Tr}\left(X_d X_u X_d^2 C^{(6)}\right) + \mathrm{Tr}\left(X_d^2 X_u X_d C^{(6)}\right) + I_{0,1}\left(\mathrm{Tr}\left(X_d^2 X_u C^{(6)}\right) + \mathrm{Tr}\left(X_u X_d^2 C^{(6)}\right)\right) + \\
&+ (-I_{0,1}I_{1,0} - I_{1,1})\mathrm{Tr}\left(X_d^2 C^{(6)}\right) - I_{0,1}^2(\mathrm{Tr}\left(X_d X_u C^{(6)}\right) + \mathrm{Tr}\left(X_u X_d C^{(6)}\right) + \\
&+ \frac{1}{3}\left(2I_{0,1}^3 - 3I_{0,2}I_{0,1} + I_{0,3}\right)\mathrm{Tr}\left(X_u C^{(6)}\right) + \left(I_{0,1}^2 I_{1,0} - I_{1,2}\right)\mathrm{Tr}\left(X_d C^{(6)}\right) + \\
&+ \mathrm{Tr}(C^{(6)})\left(-\frac{2}{3}I_{0,1}^3 I_{1,0} + I_{0,1}^2 I_{1,1} + I_{0,2}I_{0,1}I_{1,0} - I_{0,1}I_{1,2} - \frac{1}{3}I_{0,3}I_{1,0}\right) = 0 \ , \qquad (4.B.24)
\end{aligned}$$

and the same with $X_u \leftrightarrow X_d$, which we can use to remove $\mathrm{Tr}\left(X_d X_u X_d^2 C^{(6)}\right)$ and $\mathrm{Tr}\left(X_u X_d X_u^2 C^{(6)}\right)$ from the set. At dimension 6 we obtain two more relations, and 8 more are obtained at dimension 7. With these 12 expressions, we can reduce the set to

$$\begin{array}{lll}
\mathrm{Tr}\left(C^{(6)}\right) & \mathrm{Tr}\left(X_u C^{(6)}\right) & \mathrm{Tr}\left(X_d C^{(6)}\right) \\
\mathrm{Tr}\left(X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d^2 C^{(6)}\right) & \mathrm{Tr}\left(X_u X_d C^{(6)}\right) \\
\mathrm{Tr}\left(X_u X_d^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d X_u C^{(6)}\right) & \mathrm{Tr}\left(X_d^2 X_u C^{(6)}\right) \\
\mathrm{Tr}\left(X_d X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_u^2 X_d C^{(6)}\right) & \mathrm{Tr}\left(X_u^2 X_d^2 C^{(6)}\right) \\
\mathrm{Tr}\left(X_d^2 X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_u X_d X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d X_u X_d^2 C^{(6)}\right) \\
\mathrm{Tr}\left(X_u X_d^2 X_u^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d X_u^2 X_d^2 C^{(6)}\right) . &
\end{array} \qquad (4.B.25)$$

The same can be repeated for $(C^{(6)})^\dagger$. The objects in Eq. (4.B.25) are, quite remarkably, exactly in correspondence with the relative terms in Eq. (4.B.18).

However, we now wish to find the additional relations that help us express the 8 too many (complex) invariants we have in Eq (4.B.25) in terms of the 9 we know are sufficient to express all the physical observables, i.e the algebraically independent ones. If we expand a bit further in Eq. (4.B.18), we see that the next two terms are degree 8 and are negative, $-(c+c^\dagger)x_u^4 x_d^3 - (c+c^\dagger)x_u^3 x_d^4$. They should then correspond to the number of syzygies at dimension 8. To obtain them explicitly, we just run again the described algorithm at dimension 8, obtaining indeed two syzygies of the expected degree. They include 107 terms out of the possible 808 one can build at this dimension, and allow us to remove $\mathrm{Tr}\left(X_u X_d^2 X_u^2 C^{(6)}\right)$ and $\mathrm{Tr}\left(X_d X_u^2 X_d^2 C^{(6)}\right)$. Running this argument at degree 9, however, we run into a mismatch. Indeed, even though the next term in Eq. (4.B.18) would call for 4 syzygies, we only find 1, symmetric under the exchange $X_u \leftrightarrow X_d$. This is probably due to the complications that arise when the groups and representations one has to deal with start becoming less and less trivial, as in our case, and that forbid us from reading the syzygies from the negative terms directly. For a deeper discussion of this topic, see in particular Ref. [150, 155] and references therein. One thing to notice, in addition, is that requiring the building blocks to be linear in $(C^{(6)})$, although justified from a physical point of view, breaks the ring structure of the invariant ring, as obviously the set we consider is no longer closed under multiplications.

In any case, even without the Plethystic logarithm as a guide, we can just run our algorithm at increasingly higher dimensions, until no more relations are found. Indeed, upon going up to dimension $n = 13$, one manages to reduce the set down to 9 independent invariants, which we

can pick to be

$$\begin{array}{lll} \mathrm{Tr}(C^{(6)}) & \mathrm{Tr}\left(X_u C^{(6)}\right) & \mathrm{Tr}\left(X_d C^{(6)}\right) \\ \mathrm{Tr}\left(X_u X_d C^{(6)}\right) & \mathrm{Tr}\left(X_d X_u C^{(6)}\right) & \mathrm{Tr}\left(X_d^2 X_u^2 C^{(6)}\right) \\ \mathrm{Tr}\left(X_u^2 X_d^2 C^{(6)}\right) & \mathrm{Tr}\left(X_d X_u^2 X_d^2 C^{(6)}\right) & \mathrm{Tr}\left(X_u X_d^2 X_u^2 C^{(6)}\right) \ , \end{array} \tag{4.B.26}$$

and that, upon taking their imaginary parts, match the minimal set for a non-hermitian fermion bilinear operator in Table 4.D.1.

4.C List of dimension-6 fermionic operators and parameter counting with generic N_f

In Tables 4.C.1 and 4.C.2 we reproduce the subset of operators from Ref. [70] we are interested in in this work, namely dimension-6 operators in SMEFT containing fermions, split between bilinear and 4-Fermi operators. For each of the considered operators we list the number of real and imaginary entries and compare them with the number of (primary) real and imaginary parameters that can appear in observables at order $1/\Lambda^2$, as explained in the main text.

In Table 4.C.3 the counting of independent primary parameters is generalized to an arbitrary number of flavors N.

4.D Complete minimal set of invariants for 2-Fermi operators

We list in Table 4.D.1 a valid choice of minimal sets of CP-odd flavor invariants for all dimension-six Wilson coefficients associated to operators that are bilinear in fermion fields. It can be shown that they provide independent conditions matching the numbers presented in Table 4, in the generic and non-generic cases listed in Table 3.

4.E Complete minimal set of 4-Fermi invariants

We list in Tables 4.E.1, 4.E.2, and 4.E.3 a valid choice of minimal sets of CP-odd flavor invariants for all dimension-six Wilson coefficients associated to operators quartic in fermion fields. It can be shown that they provide independent conditions matching the numbers presented in Table 4, in the generic and non-generic cases listed in Table 3.

4.F Invariants featuring $\theta_{\mathbf{QCD}}$

In the main text of the paper, we focused on quantities which matter for perturbative computations in SMEFT. However, this left out an important contribution to CPV in the SM, the

Bilinears

Label		Operator	# real entries	# imaginary entries	# primary real entries	# primary imaginary entries
Modified Yukawas	Q_{eH}	$(H^\dagger H)(\bar{L}_i e_j H) + \text{h.c.}$	9	9	3	3
	Q_{uH}	$(H^\dagger H)(\bar{Q}_i u_j \widetilde{H}) + \text{h.c.}$	9	9	9	9
	Q_{dH}	$(H^\dagger H)(\bar{Q}_i d_j H) + \text{h.c.}$	9	9	9	9
Dipole	Q_{eW}	$(\bar{L}_i \sigma^{\mu\nu} e_j)\tau^I H W^I_{\mu\nu} + \text{h.c.}$	9	9	3	3
	Q_{eB}	$(\bar{L}_i \sigma^{\mu\nu} e_j) H B_{\mu\nu} + \text{h.c.}$	9	9	3	3
	Q_{uG}	$(\bar{Q}_i \sigma^{\mu\nu} T^A u_j)\widetilde{H}\ G^A_{\mu\nu} + \text{h.c.}$	9	9	9	9
	Q_{uW}	$(\bar{Q}_i \sigma^{\mu\nu} u_j)\tau^I \widetilde{H}\ W^I_{\mu\nu} + \text{h.c.}$	9	9	9	9
	Q_{uB}	$(\bar{Q}_i \sigma^{\mu\nu} u_j)\widetilde{H}\ B_{\mu\nu} + \text{h.c.}$	9	9	9	9
	Q_{dG}	$(\bar{Q}_i \sigma^{\mu\nu} T^A d_j) H\ G^A_{\mu\nu} + \text{h.c.}$	9	9	9	9
	Q_{dW}	$(\bar{Q}_i \sigma^{\mu\nu} d_j)\tau^I H\ W^I_{\mu\nu} + \text{h.c.}$	9	9	9	9
	Q_{dB}	$(\bar{Q}_i \sigma^{\mu\nu} d_j) H\ B_{\mu\nu} + \text{h.c.}$	9	9	9	9
Current-current	$Q^{(1)}_{HL}$	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{L}_i \gamma^\mu L_j)$	6	3	3	0
	$Q^{(3)}_{HL}$	$(H^\dagger i \overleftrightarrow{D}^I_\mu H)(\bar{L}_i \tau^I \gamma^\mu L_j)$	6	3	3	0
	Q_{He}	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{e}_i \gamma^\mu e_j)$	6	3	3	0
	$Q^{(1)}_{HQ}$	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{Q}_i \gamma^\mu Q_j)$	6	3	6	3
	$Q^{(3)}_{HQ}$	$(H^\dagger i \overleftrightarrow{D}^I_\mu H)(\bar{Q}_i \tau^I \gamma^\mu Q_j)$	6	3	6	3
	Q_{Hu}	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{u}_i \gamma^\mu u_j)$	6	3	6	3
	Q_{Hd}	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{d}_i \gamma^\mu d_j)$	6	3	6	3
	Q_{Hud}	$i(\widetilde{H}^\dagger D_\mu H)(\bar{u}_i \gamma^\mu d_j) + \text{h.c.}$	9	9	9	9

Table 4.C.1: The list of dimension-6 fermionic bilinear operators of SMEFT, as given in Ref. [70], together with the number of real and imaginary entries they each contain, as well as the number of primary parameters (highlighted in gray, see the text for more details). When +h.c. is specified, the hermitian conjugate of the operator must be included, too. We indicate with i, j, k, l the flavor indices and with a, b indices in the fundamental of $SU(2)_L$. T^A, $A = 1, \ldots, 8$ are the generators of the gauge $SU(3)_c$, while $\tau^I = \frac{\sigma^I}{2}$, $I = 1, 2, 3$ are the generators of $SU(2)_L$, with σ^I the Pauli matrices.

4-Fermi

Label		Operator	# real entries	# imaginary entries	# primary real entries	# primary imaginary entries
LLLL	Q_{LL}	$(\bar{L}_i\gamma_\mu L_j)(\bar{L}_k\gamma^\mu L_l)$	27	18	9	0
	$Q_{QQ}^{(1)}$	$(\bar{Q}_i\gamma_\mu Q_j)(\bar{Q}_k\gamma^\mu Q_l)$	27	18	27	18
	$Q_{QQ}^{(3)}$	$(\bar{Q}_i\gamma_\mu \tau^I Q_j)(\bar{Q}_k\gamma^\mu \tau^I Q_l)$	27	18	27	18
	$Q_{LQ}^{(1)}$	$(\bar{L}_i\gamma_\mu L_j)(\bar{Q}_k\gamma^\mu Q_l)$	45	36	18	9
	$Q_{LQ}^{(3)}$	$(\bar{L}_i\gamma_\mu \tau^I L_j)(\bar{Q}_k\gamma^\mu \tau^I Q_l)$	45	36	18	9
RRRR	Q_{ee}	$(\bar{e}_i\gamma_\mu e_j)(\bar{e}_k\gamma^\mu e_l)$	21	15	6	0
	Q_{uu}	$(\bar{u}_i\gamma_\mu u_j)(\bar{u}_k\gamma^\mu u_l)$	27	18	27	18
	Q_{dd}	$(\bar{d}_i\gamma_\mu d_j)(\bar{d}_k\gamma^\mu d_l)$	27	18	27	18
	Q_{eu}	$(\bar{e}_i\gamma_\mu e_j)(\bar{u}_k\gamma^\mu u_l)$	45	36	18	9
	Q_{ed}	$(\bar{e}_i\gamma_\mu e_j)(\bar{d}_k\gamma^\mu d_l)$	45	36	18	9
	$Q_{ud}^{(1)}$	$(\bar{u}_i\gamma_\mu u_j)(\bar{d}_k\gamma^\mu d_l)$	45	36	45	36
	$Q_{ud}^{(8)}$	$(\bar{u}_i\gamma_\mu T^A u_j)(\bar{d}_k\gamma^\mu T^A d_l)$	45	36	45	36
LLRR	Q_{Le}	$(\bar{L}_i\gamma_\mu L_j)(\bar{e}_k\gamma^\mu e_l)$	45	36	12	3
	Q_{Lu}	$(\bar{L}_i\gamma_\mu L_j)(\bar{u}_k\gamma^\mu u_l)$	45	36	18	9
	Q_{Ld}	$(\bar{L}_i\gamma_\mu L_j)(\bar{d}_k\gamma^\mu d_l)$	45	36	18	9
	Q_{Qe}	$(\bar{Q}_i\gamma_\mu Q_j)(\bar{e}_k\gamma^\mu e_l)$	45	36	18	9
	$Q_{Qu}^{(1)}$	$(\bar{Q}_i\gamma_\mu Q_j)(\bar{u}_k\gamma^\mu u_l)$	45	36	45	36
	$Q_{Qu}^{(8)}$	$(\bar{Q}_i\gamma_\mu T^A Q_j)(\bar{u}_k\gamma^\mu T^A u_l)$	45	36	45	36
	$Q_{Qd}^{(1)}$	$(\bar{Q}_i\gamma_\mu Q_j)(\bar{d}_k\gamma^\mu d_l)$	45	36	45	36
	$Q_{Qd}^{(8)}$	$(\bar{Q}_i\gamma_\mu T^A Q_j)(\bar{d}_k\gamma^\mu T^A d_l)$	45	36	45	36
LRRL	Q_{LedQ}	$(\bar{L}_i^a e_j)(\bar{d}_k Q_{la}) + \text{h.c.}$	81	81	27	27
LRLR	$Q_{QuQd}^{(1)}$	$(\bar{Q}_i^a u_j)\varepsilon_{ab}(\bar{Q}_k^b d_l) + \text{h.c.}$	81	81	81	81
	$Q_{QuQd}^{(8)}$	$(\bar{Q}_i^a T^A u_j)\varepsilon_{ab}(\bar{Q}_k^b T^A d_l) + \text{h.c.}$	81	81	81	81
	$Q_{LeQu}^{(1)}$	$(\bar{L}_i^a e_j)\varepsilon_{ab}(\bar{Q}_k^b u_l) + \text{h.c.}$	81	81	27	27
	$Q_{LeQu}^{(3)}$	$(\bar{L}_i^a \sigma_{\mu\nu} e_j)\varepsilon_{ab}(\bar{Q}_s^k \sigma^{\mu\nu} u_t) + \text{h.c.}$	81	81	27	27

Table 4.C.2: The list of dimension-6 4-Fermi operators of SMEFT, as given in Ref. [70], together with the number of real and imaginary entries they each contain, as well as the number of primary parameters (highlighted in gray, see the text for more details). When +h.c. is specified, the hermitian conjugate of the operator must be included, too. We indicate with i, j, k, l the flavor indices and with a, b indices in the fundamental of $SU(2)_L$. T^A, $A = 1, \ldots, 8$ are the generators of the gauge $SU(3)_c$, while $\tau^I = \frac{\sigma^I}{2}$, $I = 1, 2, 3$ are the generators of $SU(2)_L$, with σ^I the Pauli matrices.

	Type of op.	# ops	# real	# im.
bilinears	Yuk.	3	# of entries at $\mathcal{O}(1/\Lambda^2)$	
			$3N^2$	$3N^2$
			# of primary parameters entering observables at $\mathcal{O}(1/\Lambda^2)$	
			$2N^2+N$	$2N^2+N$
	Dipole	8	$8N^2$	$8N^2$
			$6N^2+2N$	$6N^2+2N$
	curr-curr	8	$\frac{1}{2}N(9N+7)$	$\frac{1}{2}N(9N-7)$
			$N(3N+5)$	$N(3N-2)$
	all bilinears	19	$\frac{1}{2}N(31N+7)$	$\frac{1}{2}N(31N-7)$
			$N(11N+8)$	$N(11N+1)$
4-Fermi	LLLL	5	$\frac{1}{4}N^2\left(7N^2+13\right)$	$\frac{7}{4}N^2\left(N^2-1\right)$
			$\frac{1}{2}N^2\left(N^2+2N+7\right)$	$\frac{1}{2}N^2\left(N^2+2N-3\right)$
	RRRR	7	$\frac{1}{8}N\left(21N^3+2N^2+31N+2\right)$	$\frac{1}{8}N(21N+2)(N^2-1)$
			$\frac{1}{2}N\left(3N^3+2N^2+8N+1\right)$	$\frac{1}{2}N^2\left(3N^2+2N-5\right)$
	LLRR	8	$4\ N^2\left(N^2+1\right)$	$4N^2\left(N^2-1\right)$
			$\frac{1}{2}N\left(4N^3+3N^2+9N+2\right)$	$\frac{1}{2}N\left(4N^3+3N^2-6N-1\right)$
	LRRL	1	N^4	N^4
			N^3	N^3
	LRLR	4	$4N^4$	$4N^4$
			$2N^3(N+1)$	$2N^3(N+1)$
	all 4-Fermi	25	$\frac{1}{8}N\left(107N^3+2N^2+89N+2\right)$	$\frac{1}{8}N\left(107N^3+2N^2-67N-2\right)$
			$\frac{1}{2}N\left(12N^3+13N^2+24N+3\right)$	$\frac{1}{2}N\left(12N^3+13N^2-14N-1\right)$
	all	44	$\frac{1}{8}N\left(107N^3+2N^2+213N+30\right)$	$\frac{1}{8}N\left(107N^3+2N^2+57N-30\right)$
			$\frac{1}{2}N\left(12N^3+13N^2+46N+19\right)$	$\frac{1}{2}N\left(12N^3+13N^2+8N+1\right)$

Table 4.C.3: Number of flavorful real and imaginary parameters in SMEFT at dimension-six with N flavors. For each type of operator, the first line (in white) counts the number of physical parameters, while the second one (highlighted in gray) counts those which are also primary.

θ-parameter of QCD, associated to the following topological term,

$$\mathcal{L}_{\rm QCD} \supset -\theta_{QCD}\frac{g_s^2}{16\pi^2}\,{\rm Tr}\Big(G\tilde{G}\Big)\ . \tag{4.F.1}$$

θ_{QCD} has the following flavor charges (and no lepton-type charge),

	$SU(3)_{Q_L}$	$U(1)_{Q_L}$	$SU(3)_{u_R}$	$U(1)_{u_R}$	$SU(3)_{d_R}$	$U(1)_{d_R}$
$e^{i\theta_{QCD}}$	**1**	6	**1**	-3	**1**	-3

Wilson coefficient	Number of phases	Minimal set
$C_e \equiv \begin{cases} C_{eH} \\ C_{eW} \\ C_{eB} \end{cases}$	3	$\left\{ L_0\left(C_e Y_e^\dagger\right) \quad L_1\left(C_e Y_e^\dagger\right) \quad L_2\left(C_e Y_e^\dagger\right) \right\}$
$C_u \equiv \begin{cases} C_{uH} \\ C_{uG} \\ C_{uW} \\ C_{uB} \end{cases}$	9	$\begin{Bmatrix} L_{0000}\left(C_u Y_u^\dagger\right) & L_{1000}\left(C_u Y_u^\dagger\right) & L_{0100}\left(C_u Y_u^\dagger\right) \\ L_{1100}\left(C_u Y_u^\dagger\right) & L_{0110}\left(C_u Y_u^\dagger\right) & L_{2200}\left(C_u Y_u^\dagger\right) \\ L_{0220}\left(C_u Y_u^\dagger\right) & L_{1220}\left(C_u Y_u^\dagger\right) & L_{0122}\left(C_u Y_u^\dagger\right) \end{Bmatrix}$
$C_d \equiv \begin{cases} C_{dH} \\ C_{dG} \\ C_{dW} \\ C_{dB} \end{cases}$		Same with $C_u Y_u^\dagger \to C_d Y_d^\dagger$
C_{Hud}		Same with $C_u Y_u^\dagger \to Y_u C_{Hud} Y_d^\dagger$
$C_{HL}^{(1,3)}, C_{He}$	0	$\emptyset$
$C_{HQ}^{(1,3)}$	3	$\left\{ L_{1100}\left(C_{HQ}^{(1,3)}\right) \quad L_{2200}\left(C_{HQ}^{(1,3)}\right) \quad L_{1122}\left(C_{HQ}^{(1,3)}\right) \right\}$
C_{Hu}		Same with $C_{HQ}^{(1,3)} \to Y_u C_{Hu} Y_u^\dagger$
C_{Hd}		Same with $C_{HQ}^{(1,3)} \to Y_d C_{Hd} Y_d^\dagger$

Table 4.D.1: Minimal sets of CP-odd flavor invariants for all SMEFT dimension-six Wilson coefficients associated to operators bilinear in fermion fields. We recall that $X_u \equiv Y_u Y_u^\dagger$, and similarly for down quarks or electrons. We also recall the definition in Eq. (4.4.1). We also defined for the leptons

$$L_a(\tilde{C}) \equiv \mathrm{Im}\,\mathrm{Tr}\Big(X_e^a \tilde{C}\Big) \ , \ \text{with } a = 1, 2 \ .$$

These charges allow us to build the usual flavor-invariant, physical $\bar{\theta}$-angle, defined as follows,

$$e^{-i\theta_{QCD}} \det Y_u \det Y_d = |\det\left(Y_u Y_d\right)| e^{-i\left[\theta_{QCD} - \arg\det(Y_u Y_d)\right]} = |\det\left(Y_u Y_d\right)| e^{-i\bar{\theta}} \ . \tag{4.F.2}$$

As θ_{QCD} provides a dimension-four flavor-charged quantity, one can wonder whether its presence makes new SMEFT coefficients primary, which would mean that new invariants featuring explicitly θ_{QCD} should be included in the minimal sets. The answer is however negative: the secondary sources of CPV from the dimension-six Wilson coefficients are all charged under unbroken vector-like flavor symmetries of the dimension-four lagrangian, under which θ_{QCD} is neutral. Indeed, as the anomalous angle of a vector-like gauge theory, it only shifts under chiral transformations.

Nevertheless, some SMEFT coefficients can be arranged with θ_{QCD} to form flavor-invariants (albeit redundant in terms of primary parameter counting at dimension-six), which may yield a more natural description of some non-perturbative contributions of the strong interactions

Wilson coefficient	Number of phases	Minimal set
C_{LL}, C_{ee}	0	$\emptyset$
C_{Le}	3	$\left\{ B_0^0\left(C_{LL\tilde{e}\tilde{e}}\right) \quad B_0^1\left(C_{LL\tilde{e}\tilde{e}}\right) \quad B_0^2\left(C_{LL\tilde{e}\tilde{e}}\right) \right\}$
C_{Qe}	9	$\left\{ \begin{array}{ccc} A_0^{1100}\left(C_{QQee}\right) & A_1^{1100}\left(C_{QQee}\right) & A_2^{1100}\left(C_{QQee}\right) \\ A_0^{2200}\left(C_{QQee}\right) & A_1^{2200}\left(C_{QQee}\right) & A_2^{2200}\left(C_{QQee}\right) \\ A_0^{1122}\left(C_{QQee}\right) & A_1^{1122}\left(C_{QQee}\right) & A_2^{1122}\left(C_{QQee}\right) \end{array} \right\}$
C_{ed}		Same with $C_{QQee} \to C_{ee\tilde{d}\tilde{d}}$ (exchanging upper with lower indices and with $Y_e \leftrightarrow Y_e^\dagger$)
C_{eu}		Same with $C_{QQee} \to C_{ee\tilde{u}\tilde{u}}$ (exchanging upper with lower indices and with $Y_e \leftrightarrow Y_e^\dagger$)
$C_{LQ}^{(1,3)}$		$\left\{ \begin{array}{ccc} A_{1100}^0\left(C_{LQ}^{(1,3)}\right) & A_{1100}^1\left(C_{LQ}^{(1,3)}\right) & A_{1100}^2\left(C_{LQ}^{(1,3)}\right) \\ A_{2200}^0\left(C_{LQ}^{(1,3)}\right) & A_{2200}^1\left(C_{LQ}^{(1,3)}\right) & A_{2200}^2\left(C_{LQ}^{(1,3)}\right) \\ A_{1122}^0\left(C_{LQ}^{(1,3)}\right) & A_{1122}^1\left(C_{LQ}^{(1,3)}\right) & A_{1122}^2\left(C_{LQ}^{(1,3)}\right) \end{array} \right\}$
C_{Ld}		Same with $C_{LQ}^{(1,3)} \to C_{LL\tilde{d}\tilde{d}}$
C_{Lu}		Same with $C_{LQ}^{(1,3)} \to C_{LL\tilde{u}\tilde{u}}$
$C_{LeQu}^{(1,3)}$	27	$\left\{ \begin{array}{ccc} A_{0000}^0\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0000}^1\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0000}^2\left(C_{L\tilde{e}Q\tilde{u}}\right) \\ A_{1000}^0\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{1000}^1\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{1000}^2\left(C_{L\tilde{e}Q\tilde{u}}\right) \\ A_{0100}^0\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0100}^1\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0100}^2\left(C_{L\tilde{e}Q\tilde{u}}\right) \\ A_{1100}^0\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{1100}^1\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{1100}^2\left(C_{L\tilde{e}Q\tilde{u}}\right) \\ A_{0110}^0\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0110}^1\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0110}^2\left(C_{L\tilde{e}Q\tilde{u}}\right) \\ A_{2200}^0\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{2200}^1\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{2200}^2\left(C_{L\tilde{e}Q\tilde{u}}\right) \\ A_{0220}^0\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0220}^1\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0220}^2\left(C_{L\tilde{e}Q\tilde{u}}\right) \\ A_{1220}^0\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{1220}^1\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{1220}^2\left(C_{L\tilde{e}Q\tilde{u}}\right) \\ A_{0122}^0\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0122}^1\left(C_{L\tilde{e}Q\tilde{u}}\right) & A_{0122}^2\left(C_{L\tilde{e}Q\tilde{u}}\right) \end{array} \right\}$
C_{LedQ}		Same with $C_{L\tilde{e}Q\tilde{u}} \to C_{L\tilde{e}\tilde{d}Q}$ and $A_{bcde}^a \to A_{edcb}^a$

Table 4.E.1: Minimal sets of CP-odd flavor invariants for all the SMEFT dimension-six Wilson coefficients associated to operators quartic in fermion fields (continued in Tables 4.E.2, 4.E.3). We recall that $X_u \equiv Y_u Y_u^\dagger$, and similarly for down quarks or electrons. We use the generalized traces introduced in Eq. (4.4.8), as well as the compact notations in Eqs. (4.4.9)-(4.4.10). We also defined for the leptons $A_b^a(C) \equiv \mathrm{Tr}_A\left(X_e^a, X_e^b, C\right)$, $B_b^a(C) \equiv \mathrm{Tr}_B\left(X_e^a, X_e^b, C\right)$ with $a, b = 1, 2$, $A_{bcde}^f(C) \equiv \mathrm{Tr}_A\left(X_e^f, X_u^b X_d^c X_u^d X_d^e, C\right)$, $A_f^{abcd}(C) \equiv \mathrm{Tr}_A\left(X_u^a X_d^b X_u^c X_d^d, \left(Y_e^\dagger Y_e\right)^f, C\right)$ and $B_{bcde}^f(C) \equiv \mathrm{Tr}_B\left(X_e^f, X_u^b X_d^c X_u^d X_d^e, C\right)$

to CP-odd observables.[23] Those invariants would not have the single trace structure which we used to build our sets of invariants, since δ_m^n is $U(3)^5$-invariant, while θ_{QCD} is charged under some abelian parts of the flavor group. Therefore, it will rather offset the abelian charges of

[23] In the perturbative phase of QCD, the magnitude of such invariants is expected to be suppressed by an additional non-perturbative factor $e^{-8\pi^2/g_s^2}$. For low-energy observables, such as the EDMs of the neutron [303] and of the electron [304, 305], no further suppression would be needed.

Wilson coefficient	Number of phases	Minimal set
$C_{QQ}^{(1,3)}$	18	$\left\{\begin{array}{ccc} A_{1100}^{0000}\,(C_{QQQQ}) & A_{1100}^{1000}\,(C_{QQQQ}) & A_{1100}^{0100}\,(C_{QQQQ}) \\ A_{2200}^{0000}\,(C_{QQQQ}) & A_{1100}^{1100}\,(C_{QQQQ}) & A_{2200}^{1000}\,(C_{QQQQ}) \\ A_{2200}^{0100}\,(C_{QQQQ}) & A_{1122}^{0000}\,(C_{QQQQ}) & A_{2200}^{1100}\,(C_{QQQQ}) \\ A_{2100}^{1200}\,(C_{QQQQ}) & A_{1122}^{1000}\,(C_{QQQQ}) & A_{1122}^{0100}\,(C_{QQQQ}) \\ A_{1122}^{1100}\,(C_{QQQQ}) & A_{2200}^{2200}\,(C_{QQQQ}) & B_{1100}^{0000}\,(C_{QQQQ}) \\ B_{2200}^{0000}\,(C_{QQQQ}) & B_{1122}^{0000}\,(C_{QQQQ}) & A_{1122}^{2200}\,(C_{QQQQ}) \end{array}\right\}$
C_{uu}	18	$\left\{\begin{array}{ccc} A_{1100}^{0000}\,(C_{uu\tilde{u}\tilde{u}}) & A_{1100}^{1000}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) & A_{1100}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) \\ A_{2200}^{0000}\,(C_{uu\tilde{u}\tilde{u}}) & A_{1100}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) & A_{1100}^{0200}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) \\ A_{2200}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) & A_{1122}^{0000}\,(C_{uu\tilde{u}\tilde{u}}) & A_{2200}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) \\ A_{1122}^{1000}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) & A_{1122}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) & A_{0122}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) \\ A_{2200}^{1200}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) & B_{1100}^{0000}\,(C_{u\tilde{u}\tilde{u}u}) & B_{1100}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) \\ B_{2100}^{0200}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) & A_{1122}^{1200}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) & B_{1200}^{1000}\,(C_{\tilde{u}\tilde{u}\tilde{u}\tilde{u}}) \end{array}\right\}$
C_{dd}	18	$\left\{\begin{array}{ccc} A_{1100}^{0000}\,(C_{dd\tilde{d}\tilde{d}}) & A_{1100}^{1000}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & A_{2200}^{0000}\,(C_{dd\tilde{d}\tilde{d}}) \\ A_{2000}^{1100}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & A_{1100}^{0100}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & A_{1100}^{1100}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) \\ A_{2200}^{1000}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & A_{1122}^{0000}\,(C_{dd\tilde{d}\tilde{d}}) & A_{2200}^{1100}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) \\ A_{1122}^{1000}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & A_{1220}^{1100}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & A_{2110}^{1200}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) \\ A_{0122}^{2100}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & A_{1220}^{2200}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & B_{1100}^{0000}\,(C_{dd\tilde{d}\tilde{d}}) \\ B_{2100}^{0100}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & B_{1100}^{1000}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) & B_{2000}^{1200}\,(C_{\tilde{d}\tilde{d}\tilde{d}\tilde{d}}) \end{array}\right\}$
$C_{Qu}^{(1,8)}$	36	$\left\{\begin{array}{ccc} A_{0000}^{1100}\,(C_{QQuu}) & A_{1100}^{0000}\,(C_{QQ\tilde{u}\tilde{u}}) & A_{1100}^{1000}\,(C_{QQ\tilde{u}\tilde{u}}) \\ A_{0100}^{1100}\,(C_{QQ\tilde{u}\tilde{u}}) & A_{1100}^{1100}\,(C_{QQ\tilde{u}\tilde{u}}) & A_{0110}^{1100}\,(C_{QQ\tilde{u}\tilde{u}}) \\ A_{1000}^{1200}\,(C_{QQ\tilde{u}\tilde{u}}) & A_{0000}^{2200}\,(C_{QQuu}) & A_{2200}^{1100}\,(C_{QQ\tilde{u}\tilde{u}}) \\ A_{0220}^{1100}\,(C_{QQ\tilde{u}\tilde{u}}) & A_{0110}^{2200}\,(C_{QQ\tilde{u}\tilde{u}}) & A_{1122}^{1100}\,(C_{QQ\tilde{u}\tilde{u}}) \\ A_{1220}^{1200}\,(C_{QQ\tilde{u}\tilde{u}}) & A_{1122}^{2200}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{0100}^{0000}\,(C_{QQ\tilde{u}\tilde{u}}) \\ B_{1000}^{0000}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{0110}^{0000}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{0220}^{0000}\,(C_{QQ\tilde{u}\tilde{u}}) \\ B_{1100}^{0000}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{0221}^{0000}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{1000}^{0100}\,(C_{QQ\tilde{u}\tilde{u}}) \\ B_{1100}^{0100}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{2200}^{0100}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{2110}^{0100}\,(C_{QQ\tilde{u}\tilde{u}}) \\ B_{2000}^{0200}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{2100}^{0200}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{2110}^{0200}\,(C_{QQ\tilde{u}\tilde{u}}) \\ B_{0110}^{1000}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{0220}^{1000}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{0221}^{1000}\,(C_{QQ\tilde{u}\tilde{u}}) \\ B_{1100}^{1100}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{2200}^{1100}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{2100}^{1200}\,(C_{QQ\tilde{u}\tilde{u}}) \\ B_{2210}^{1200}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{1200}^{2100}\,(C_{QQ\tilde{u}\tilde{u}}) & B_{0221}^{0110}\,(C_{QQ\tilde{u}\tilde{u}}) \end{array}\right\}$
$C_{Qd}^{(1,8)}$	36	$\left\{\begin{array}{ccc} A_{0000}^{1100}\,(C_{QQdd}) & A_{1100}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) & A_{1100}^{1000}\,(C_{QQ\tilde{d}\tilde{d}}) \\ A_{1000}^{1100}\,(C_{QQ\tilde{d}\tilde{d}}) & A_{0000}^{2200}\,(C_{QQdd}) & A_{1100}^{0100}\,(C_{QQ\tilde{d}\tilde{d}}) \\ A_{2200}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) & A_{1100}^{1100}\,(C_{QQ\tilde{d}\tilde{d}}) & A_{2100}^{1100}\,(C_{QQ\tilde{d}\tilde{d}}) \\ A_{0000}^{1122}\,(C_{QQdd}) & A_{1122}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) & A_{2200}^{1100}\,(C_{QQ\tilde{d}\tilde{d}}) \\ A_{0220}^{1100}\,(C_{QQ\tilde{d}\tilde{d}}) & A_{1122}^{1000}\,(C_{QQ\tilde{d}\tilde{d}}) & A_{1122}^{1100}\,(C_{QQ\tilde{d}\tilde{d}}) \\ A_{0122}^{2100}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{0100}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{1000}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) \\ B_{0110}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{0220}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{1100}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) \\ B_{0221}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{2200}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{2210}^{0000}\,(C_{QQ\tilde{d}\tilde{d}}) \\ B_{1000}^{0100}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{0120}^{0100}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{1100}^{0100}\,(C_{QQ\tilde{d}\tilde{d}}) \\ B_{2210}^{0100}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{0110}^{1000}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{0220}^{1000}\,(C_{QQ\tilde{d}\tilde{d}}) \\ B_{0221}^{1000}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{1200}^{1000}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{2200}^{1100}\,(C_{QQ\tilde{d}\tilde{d}}) \\ B_{2210}^{1100}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{2100}^{1200}\,(C_{QQ\tilde{d}\tilde{d}}) & B_{2211}^{2100}\,(C_{QQ\tilde{d}\tilde{d}}) \end{array}\right\}$

Table 4.E.2: Continuation of Table 4.E.1

determinant-like $SU(3)^5$-invariants. For instance, for the operator C_{QuQd}, we can form

$$\mathrm{Im}\left(e^{-i\theta_{QCD}}\varepsilon^{ABC}\varepsilon^{abc}\varepsilon^{DEF}\varepsilon^{def}Y_{u,Aa}Y_{u,Bb}C_{QuQd,CcDd}Y_{d,Ee}Y_{d,Ff}\right) = \\ = \big|_{\text{up basis}} 4y_b y_s y_t y_c \mathrm{Im}C_{QuQd,1111} + \dots\,.$$

Wilson coefficient	Number of phases	Minimal set
$C_{ud}^{(1,8)}$	36	$\left\{\begin{array}{lll} A_{0000}^{1100}\,(C_{\tilde{u}\tilde{u}dd}) & A_{1100}^{0000}\,(C_{uu\tilde{d}\tilde{d}}) & A_{1100}^{1000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ A_{1000}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & A_{0000}^{2200}\,(C_{\tilde{u}\tilde{u}dd}) & A_{1100}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ A_{2200}^{0000}\,(C_{uu\tilde{d}\tilde{d}}) & A_{1100}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & A_{0110}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ A_{2200}^{1000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & A_{2100}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & A_{0000}^{1122}\,(C_{\tilde{u}\tilde{u}dd}) \\ A_{2200}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & A_{1122}^{0000}\,(C_{uu\tilde{d}\tilde{d}}) & A_{2200}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ A_{1122}^{1000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & A_{1122}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & A_{1122}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ B_{0100}^{0000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{1000}^{0000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{0110}^{0000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ B_{1100}^{0000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{0221}^{0000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{2200}^{0000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ B_{1000}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{0110}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{2110}^{0100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ B_{2000}^{0200}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{2110}^{0200}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{0110}^{1000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ B_{0221}^{1000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{1200}^{1000}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{2200}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \\ B_{2211}^{1100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{2100}^{1200}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) & B_{1200}^{2100}\,(C_{\tilde{u}\tilde{u}\tilde{d}\tilde{d}}) \end{array}\right\}$
$C_{QuQd}^{(1,8)}$	81	$\left\{\begin{array}{lll} A_{0000}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1000}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0000}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{1000}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0100}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0000}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{1100}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0110}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1000}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{0100}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0000}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0000}^{0110}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{1100}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0110}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1000}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{0100}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1100}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0110}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{0100}^{0110}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{2200}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0220}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{2000}^{0200}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1100}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0110}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{0200}^{2000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0100}^{2100}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1100}^{0110}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{0110}^{0110}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1000}^{0210}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1220}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{2000}^{1200}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0122}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1220}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{0122}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{2200}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0220}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{2100}^{1200}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1200}^{2100}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0210}^{2100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{0110}^{2200}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{2200}^{0110}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0220}^{0110}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{2000}^{0112}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1220}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0112}^{2100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{1220}^{1200}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{2200}^{2200}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1122}^{0110}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ A_{2100}^{0122}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{0220}^{0220}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0000}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{0100}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{1000}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{1100}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{2200}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0110}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0122}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{0220}^{0000}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0000}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{1000}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{1100}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{2100}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0120}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{1220}^{0100}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{1120}^{0200}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0000}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{0100}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{1200}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0110}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{0122}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0210}^{1000}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0000}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{1100}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{2200}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0110}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{0220}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{1122}^{1100}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{2100}^{1200}\,(C_{Q\tilde{u}Q\tilde{d}}) \\ B_{0122}^{2100}\,(C_{Q\tilde{u}Q\tilde{d}}) & B_{0000}^{2200}\,(C_{Q\tilde{u}Q\tilde{d}}) & A_{1122}^{2200}\,(C_{Q\tilde{u}Q\tilde{d}}) \end{array}\right\}$

Table 4.E.3: Continuation of Tables 4.E.1 and 4.E.2

CHAPTER 5

Conclusion

Because it's there

English mountaineer George Mallory
when asked why he wanted
to climb Mount Everest [306]

What physicists have learned in the centuries-old history of the discipline is that dealing with the understanding of Nature implies the awareness that Nature may not have any specific interest in being studied and understood. The last 10 years have made this observation particularly apparent, as the lack of new clear experimental signatures at colliders have brought the physics community to rethink a lot of its beliefs, and to try and challenge what was taken for granted. A useful guide, as always, is represented by our collective past experience, and the possibility to take a step back and gain some larger perspective. Indeed, if we have managed to gather so much knowledge about the physical world, it is only because of a long path of trial and error, made possible by employing each time different angles to approach the same problems, until one of the perspectives would turn out to be the right key to break the code. It is easy, when looking back, to fall in the trap of flattening this process of natural selection and only single out the approaches that turned out to be the most suitable ones. Such narrow perspective is wrong twice. First, because it does not make justice to the struggle our discipline had to undergo to make each step in the direction of progress, and secondly, because we risk falling in the misconception that the apparent stalemate we find ourselves in at the moment is an exception in the history of physics, when it is rather much closer to being the rule. Endowed with this realization, we should resolve to approach the issue from various angles with an open mind.

In this sense, the paradigm of Effective Field Theories offers, to a Particle Physicist, a compelling vantage point of observation. Clearly, while on one hand its rather model independent nature represents an indisputable advantage, the proliferation of free coefficients may be

disheartening to the curious scientist attempting to approach the subject for the first time. Because of this very reason, it is crucial, on the theoretical side, to try and characterize to the best of our capabilities the distinct features of this framework, as to extract as much information as possible from them. This is the vision we tried to imbue this work with, and the philosophy that stands behind it.

First of all, we dedicated Chapters 1 and 2 to the description of the tools of EFTs, specifically to the characterization of how an Effective Theory is put together and what it is composed of, what its connections to specific UV completions look like, and how we can use it to make predictions. Particular attention has been given to reviewing how to build the so called Hilbert Series, which turns out to be a rather handy instrument for the unambiguous construction of specific EFTs.

Then, we illustrated in Chapter 3 how assumptions of locality and causality of the UV theory imply that some of the EFT coefficients are forced to lie within a subset of the full parameter space. While these constraints exhibit a rather straightforward nature in the simplest cases, their structure becomes gradually more complicated as the number of degrees of freedom is increased. A clear example is represented by the rich flavor structure of the SMEFT, that implies quite convoluted bounds on its dimension-eight coefficients. We studied the interplay between such flavor structure and the bounds when a specific assumption, Minimal Flavor Violation, is made on the coefficients of operators containing four fermionic fields. Moreover, we played with the possibility that this relationship could be exploited to, in a sense, turn the bounds around and gain knowledge on the dimension-4, renormalizable coefficients.

In Chapter 4 we turned our focus on the impact of the EFT approach on the breaking of the CP symmetry. As we reviewed, CP is broken in a very peculiar manner in the SM, in a way that can be captured by a single quantity, the Jarlskog invariant, which does not depend on the specific basis we pick for the quark generations. This picture needs to be complemented when enlarging the SM to its EFT counterpart, and we addressed this issue at the level of dimension-six operators, as those are the ones we expect to have the most relevant impact on observables. We showed how the breaking of CP symmetry is here, too, most suitably described through flavor-reparametrization invariant quantities. This method, combined with the correct use of EFT power-counting, allowed us to identify a large fraction of the CP-violating quantities appearing at dimension-six as subleading. This is a clear exemplification of what we mean by exploiting the properties of EFTs to squeeze as much information about them as possible.

More work in the directions explored here is definitely needed. For what concerns the most punctual issues, we saw how the connection between the Hilbert Series and the Flavor invariants fell just short of providing us with a well rounded characterization of the physical quantities we were interested in. This is certainly a hint that the technique need to be further refined to be more flexible and adaptable to a wider range of problems. Moreover, although we showed an example of the interplay between positivity bounds and symmetry, it seems clear that this is just the start of a quest to fully understand their deep relationship and the role played here e.g. by spurions and symmetry breaking. In a broader perspective, we feel that there is still a

lot of room for the strengthening of our theoretical tools, and that a deeper understanding of the EFT construction and its peculiar structures is crucial to build the correct framework to accommodate any hint of New Physics that Nature decides to give us.

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