Compactifications of IIA Supergravity on $SU(2)$-Structure Manifolds

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Bastiaan Spanjaard
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Gutachter der Dissertation:	Prof. Dr. J. Louis Prof. Dr. H. Samtleben
Gutachter der Disputation:	Prof. Dr. J. Louis Prof. Dr. J. Teschner
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Vorsitzender des Prüfungsausschusses:	Prof. Dr. J. Bartels
Vorsitzender des Promotionsausschusses:	Prof. Dr. J. Bartels
Dekan der Fakultät für Mathematik, Informatik und Naturwissenschaften:	Prof. Dr. A. Frühwald

Abstract

In this thesis, we study compactifications of type IIA supergravity on six-dimensional manifolds with an $SU(2)$-structure. A general study of six-dimensional manifolds with $SU(2)$-structure shows that IIA supergravity compactified on such a manifold should yield a four-dimensional gauged $\mathcal{N} = 4$ supergravity. We explicitly derive the bosonic spectrum, gauge transformations and action for IIA supergravity compactified on two different manifolds with $SU(2)$-structure, one of which also has an $H_{10}^{(3)}$-flux, and confirm that the resulting four-dimensional theories are indeed $\mathcal{N} = 4$ gauged supergravities. In the second chapter, we study an explicit construction of a set of $SU(2)$-structure manifolds. This construction involves a Scherk-Schwarz duality twist reduction of the half-maximal six-dimensional supergravity obtained by compactifying IIA supergravity on a $K3$. This reduction results in a gauged $\mathcal{N} = 4$ four-dimensional supergravity, where the gaugings can be divided into three classes of parameters. We relate two of the classes to parameters we found before, and argue that the third class of parameters could be interpreted as a mirror flux.

Zusammenfassung

In dieser Dissertation untersuchen wir Kompaktifizierungen von Typ IIA Supergravitation auf sechsdimensionalen Mannigfaltigkeiten mit einer $SU(2)$-Struktur. Allgemeine Untersuchungen von sechsdimensionalen Mannigfaltigkeiten mit einer $SU(2)$-Struktur zeigen, dass die Kompaktifizierung von IIA Supergravitation auf solchen Mannigfaltigkeiten eine vierdimensionale geeichte Supergravitation ergeben sollte. Wir berechnen das bosonische Spektrum, die Eichtransformationen und die Wirkung von IIA Supergravitation kompaktifiziert auf zwei Mannigfaltigkeiten mit $SU(2)$-Struktur, eine davon mit einem $H_{10}^{(3)}$-Flux, und bestätigen, dass die resultierenden vierdimensionalen Theorien tatsächlich $\mathcal{N} = 4$ geeichte Supergravitationstheorien sind. Im zweiten Kapitel untersuchen wir eine explizite Konstruktion von einer Menge von $SU(2)$-Struktur Mannigfaltigkeiten. Für diese Konstruktion benutzen wir eine Scherk-Schwarz Dualitätstwist-Reduktion der halbmaximalen sechsdimensionalen Supergravitation, die man aus der Kompaktifizierung von IIA Supergravitation auf $K3$ erhält. Diese Scherk-Schwarz Reduktion ergibt eine geeichte $\mathcal{N} = 4$ vierdimensionale Supergravitation, in der die Eichungen in drei Parameterklassen aufgeteilt werden können. Wir setzen zwei dieser Klassen mit zuvor gefundenen Parametern in Verbindung, und behaupten, dass die dritte Klasse als Mirrorflux interpretiert werden könnte.

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Contents

Chapter 1

Introduction

1.1 String Theory and Compactifications

To describe the fundamental interactions in nature, two different theories are needed: the Standard Model and General Relativity. The Standard Model describes the world at small length scales and very accurately predicts how particles such as quarks, electrons and neutrinos interact. According to the Standard Model, these interactions are due to three different forces: the strong force, the weak force and the electromagnetic force. These forces act by the exchange of a particle related to the force, such as the photon for electromagnetism. At larger length scales we know, however, that we need a fourth force to describe nature, namely gravity. Gravity is described by the theory of General Relativity, in which objects move in a space time curved by the stress-energy tensor of matter and radiation. The predictions of this theory are also highly accurate.

Unfortunately, these theories do not seem to be compatible. At small length scales, General Relativity breaks down and constructing a renormalizable quantum field theory that treats gravity on the same footing as the other three forces has not yet been done, despite numerous tries. However, it is expected that there is one theory that describes nature, a theory that unifies the Standard Model and General Relativity. Not only would this be aesthetically pleasing, there are also instances where a quantum theory of gravity is needed, such as the early universe and black holes.

String theory ([1, 2]) has emerged as a leading candidate to solve this problem. The fundamental objects in string theory are tiny, vibrating strings that can be both closed and open. Each of their vibrational modes represents a particle. Strings can interact by joining or splitting, which in turn causes interactions between the different vibrational modes of the strings, meaning that string theory can describe particle interaction. Amongst the possible particles are scalars, fermions, gauge bosons and also a spin-2 field that is a candidate for the graviton, the particle associated with gravity. This indicates that string theory could contain a quantum theory of gravity.

There are five consistent superstring theories that live in ten dimensions. All these theories are supersymmetric, meaning that there is a symmetry that interchanges the fermions and the bosons of the theory. These five theories are type I string theory, consisting of unoriented open and closed strings with a gauge group $SO(32)$, type IIA and IIB string theory, made of open and closed strings, and two heterotic string theories that only have closed strings, one with gauge group $SO(32)$, and one with gauge group $E_8 \times E_8$.

Since supersymmetry interchanges bosons and fermions, the supersymmetry parameter itself must be a spinor. The number $\mathcal{N}$ of supersymmetry parameters, together with the dimension of the representation in which the parameters live, determines the amount of conserved charges associated with the supersymmetry. These charges are called the supercharges. For example, type IIA and IIB string theory are $\mathcal{N} = 2$ theories, with the parameters in 16 real-dimensional Majorana-Weyl representations, meaning that the theories have 32 supercharges. Similarly, type I and the heterotic string theories are ten-dimensional $\mathcal{N} = 1$ theories with 16 supercharges. In four dimensions, we will mainly be concerned with $\mathcal{N} = 4$ theories, these are half-maximal theories with 16 supercharges.

The calculations we will present here are all done in the supergravity limit of the string theories. When the string length goes to zero, string theory can be approximated by an effective field theory that describes the massless states of the string theory. This field theory inherits the supersymmetry from the string theory, and the graviton state of the string means that the metric is a field in the field theory as well. Therefore, the field theory thus obtained is a supersymmetric theory of gravity. In this thesis, we will use both the ten-dimensional $\mathcal{N} = 2$ type IIA supergravity that is related to type IIA string theory, and lower-dimensional supergravities.

Since the observable world is four-dimensional, it is necessary to have a way to extract four-dimensional physics from this ten-dimensional theory; this is called dimensional reduction. First devised by Kaluza [3] and Klein [4] as a means to unify gravity and electromagnetism, dimensional reduction requires us to make an Ansatz about the form of space-time. The specific Ansatz we will use here is that space-time is a product of four-dimensional observable space and a compact, internal manifold; therefore this sort of dimensional reduction is alternatively called compactification. The higher-dimensional fields are then expanded in a basis of eigenfunctions[1] of the Laplacian on the internal space, and the coefficients of this expansion are the four-dimensional fields. Entering the field expansions into the action, one can integrate over the internal manifold and obtain a lower-dimensional theory.

One of the aims of string theory is to show that it reduces to a four-dimensional theory that is a supersymmetric extension of the Standard Model. Amongst its features, such a theory should have a gauge group that includes the Standard Model gauge group. The theory should also be $\mathcal{N} = 1$ supersymmetric, where the supersymmetry would be broken in some way.

One important early attempt to obtain the Standard Model from string theory was the compactification of heterotic string theory on a Calabi-Yau manifold [5]. A Calabi-Yau manifold is a manifold with one covariantly constant spinor. The supersymmetry parameter of the string theory must be expanded in this one internal spinor, resulting in one space-time supersymmetry parameter, or an $\mathcal{N} = 1$ theory. Furthermore, a subgroup of the gauge group of the heterotic string theory is preserved under the compactification, leading to gauge groups large enough to accomodate some viable candidates for gauge

[1] We use the term function loosely here; what we expand in depends on the field we are expanding.

groups that include the Standard Model gauge group.

Since then, many different ways of compactifying supergravities to obtain four-dimensional theories have been developed. Two methods to obtain lower-dimensional gauged supergravities from possibly ungauged higher-dimensional supergravity are compactifications on manifolds with reduced structure and compactifications with fluxes. As the Standard Model is a gauged theory, it is clearly in our best interests to understand how to obtain different gauge groups from dimensional reductions. Furthermore, in gauged supergravities, fields also acquire mass terms. Through these mass terms, it could be possible to tune the effective field content to one that we want to obtain.

In this thesis, we will focus on compactifications that yield four-dimensional $\mathcal{N} = 4$ supergravities. Because of the high degree of supersymmetry, these theories are not usually thought to be feasible candidates for phenomenology. However, the high degree of supersymmetry also makes the theories very controlled; we will see later on that a four-dimensional gauged $\mathcal{N} = 4$ supergravity can be determined by very few parameters.

1.2 Compactifications on Manifolds with Fluxes and Reduced Structure

Two related configurations for dimensional reductions that have received much attention recently are reductions on manifolds with fluxes and on manifolds with reduced structure. Dimensional reductions with fluxes were mentioned already in [6], and have become a viable option in phenomenology (see for instance [7] and [8]). For an overview of compactifications with fluxes on reduced structure manifolds, see [9] and references therein.

The structure group of a Euclidian n-dimensional manifold is $SO(n)$, this is the group in which the transition functions of the tangent bundle to the manifold take their values. If the transition functions of the tangent space of a manifold take values in a subgroup of $SO(n)$, we say that that manifold has reduced structure. An example of this is a six-dimensional Calabi-Yau manifold, whose structure group is $SU(3)$.

The structure group also determines the transition functions of the spinors on the manifold. Specifying to six dimensions, the spinors generically transform under $SU(4) \cong SO(6)$. This means that the vector space of spinors is a complex four-dimensional space, and the four spinors that span this space transform into one another when going from patch to patch on the manifold. However, if the manifold has a reduced structure, one or more spinors are singlets under the transition functions. Therefore, these spinors are globally well-defined on the manifold.

If one or more globally well-defined spinors exist on the manifold, they can be used to construct certain structures on that manifold. For example, if the manifold has $SU(3)$-structure (see, for instance, [10]), there is one globally well-defined spinor, and that spinor can be used to construct an almost complex structure and a two-form. If, moreover, the spinor were to be covariantly constant, both the almost complex structure and the two-form would be closed, making the manifold complex and Kähler, making it a Calabi-Yau manifold. If, on the other hand, the manifold has $SU(2)$-structure (see, for instance, [11], [12], [13] and [10]), there are two globally well-defined spinors. Both of these can be used to define an $SU(3)$-structure with an almost complex structure and a two-form, and in this case the product of the two two-forms determines an almost product structure: locally, the manifold looks like the product of a two and a four-dimensional manifold.

Strictly speaking, a configuration with $H_{10}^{(3)}$-flux is a configuration in which the integral of the field strength $H_{10}^{(3)}$ over a non-trivial cycle in the internal space is non-zero. After a dimensional reduction, the value of this integral becomes a mass parameter in the four-dimensional action. In this way, turning on fluxes can be used as a way to obtain gauged four-dimensional theories.

At a first glance, there does not seem to be much correspondence between fluxes and manifolds with reduced structure. However, it is known that for a compactification of IIA supergravity on a Calabi-Yau manifold Y, there exists another Calabi-Yau manifold $\tilde{Y}$, such that the compactification of IIB supergravity on $\tilde{Y}$ yields the samelower-dimensional theory as the compactification of IIA on Y. This correspondence is called mirror symmetry [14]

One can study what happens to fluxes under mirror symmetry: if IIA on a Calabi-Yau Y is dual to IIB on a Calabi-Yau $\tilde{Y}$, what is the mirror of the IIA theory if we turn on fluxes on Y? It has been shown that fluxes on some of the fields in IIA can be mirrored by fluxes on the fields in IIB [15], another flux, namely the $H_{10}^{(3)}$-flux, can not be mirrored by a flux in IIB. Instead, the mirror theory is obtained by compactifying IIB on a manifold $\tilde{Y}$ that is no longer Calabi-Yau but has reduced structure [16]. A similar phenomenon occurs in toroidal compactifications [17], [18]. It seems to be a good approach to look at reduced structure and fluxes as two sides of the same coin.

1.3 Outline of the Thesis

In this thesis, our aim is find out what four-dimensional supergravities we can obtain by dimensionally reducing IIA supergravity on a manifold with a reduced structure, namely $SU(2)$-structure, with $H_{10}^{(3)}$-flux turned on. We concentrate on the bosonic part of the theory. One specific example of a manifold with $SU(2)$-structure is $K3 \times T^2$, and compactification of IIA on a $K3 \times T^2$ gives a four-dimensional supergravity theory with sixteen supercharges, or $\mathcal{N} = 4$ supergravity. As we will see, compactifying IIA supergravity on different $SU(2)$-structure manifolds, with and without $H_{10}^{(3)}$-flux, always gives a four-dimensional gauged $\mathcal{N} = 4$ supergravity.

We also want to see whether we can make an explicit construction of $SU(2)$-structure manifolds. We start from a Scherk-Schwarz duality twist of the six-dimensional supergravity one obtains by compactifying IIA supergravity on a $K3$. This yields a gauged $\mathcal{N} = 4$ four-dimensional supergravity. We find that the gauging can be contributed to three classes of parameters, and show that we have already encountered two of these parameters, while the third seems to be related to mirror symmetry.

The outline of this thesis is as follows: in Chapter 2, we reduce IIA supergravity on different $SU(2)$-structure manifolds with and without $H_{10}^{(3)}$-flux, and show that they yield $\mathcal{N} = 4$ supergravity. We first discuss IIA supergravity, presenting the spectrum, the action and its symmetries. We then describe manifolds with $SU(2)$-structure and $H_{10}^{(3)}$-flux; we define the different structures on the $SU(2)$-structure manifold in terms of the spinors and show that a rotation of the spinors rotates the complex and Kähler structures into one another. In section 2.3, IIA supergravity is compactified on a $K3 \times T^2$, since the compactification of IIA on $K3 \times T^2$ and on different $SU(2)$-structure follows roughly the same steps. In section 2.4, we compactify IIA supergravity on a manifold with $SU(2)$-structure , Y_1, with $H_{10}^{(3)}$-flux. In order to show that this and the following compactification do indeed result in $\mathcal{N} = 4$ supergravities, we describe how these theories can be classified, following the formalism of [19], and we show that the compactifications we have done up to that point give $\mathcal{N} = 4$ supergravities. Finally, in section 2.6, we compactify IIA supergravity on a more complicated $SU(2)$-structure manifold, Y_2, and show that the resulting theory is also an $\mathcal{N} = 4$ supergravity.

In Chapter 3, we discuss an explicit construction of a set of $SU(2)$-structure manifolds. This construction is done by performing a Scherk-Schwarz duality twist of the six-dimensional half-maximal supergravity obtained from the reduction of IIA supergravity on a $K3$. We will first review the compactification of IIA supergravity on a $K3$ and then perform a Scherk-Schwarz reduction of the six-dimensional theory to four dimensions, showing that the resulting theory is a gauged $\mathcal{N} = 4$ supergravity, with gauging depending on the Scherk-Schwarz twist employed. We will then show that the gaugings can be divided into three classes of parameters, and interpret the first two classes as equivalent to a class of parameters in the $SU(2)$-structure manifolds we considered before, and $H_{10}^{(3)}$-fluxes. We will finally argue that the last class of parameters can be interpreted as another $H_{10}^{(3)}$-flux, but this time applied to the mirror $K3$.

Chapter 2

Gauged Supergravities from IIA Supergravity on $SU(2)$-Structure Manifolds

Let us take a general look at the procedure of compactifying a ten-dimensional theory down to four dimensions. More information on this can be found in [20] and references therein. As in this whole thesis, we focus on bosonic fields and assume that the fermionic fields and action can be determined by supersymmetry.

We assume that ten-dimensional spacetime $M^{1,9}$ has the following product form:

$$M^{1,9} = M^{1,3} \times Y, \tag{2.1}$$

with $M^{1,3}$ a four-dimensional space and Y a compact, orientable, six-dimensional manifold. The coordinate on $M^{1,9}$ is X, the coordinate on $M^{1,3}$ is x, and the coordinate on Y is y. The split (2.1) means that the Lorentz group is reduced as well:

$$SO(1,9) \to SO(1,3) \times SO(6). \tag{2.2}$$

This means, for example, that a ten-dimensional scalar field Φ_{10} can be expanded in a set of functions on Y:

$$\Phi_{10}(X) = \sum_{n=1}^{\infty} \Phi_n(x) f^n(y), \tag{2.3}$$

the coefficients Φ_n being the resulting four-dimensional fields.

The spectrum can be determined by taking a closer look at the functions f^n. We will now also assume that Y is Kähler, and that the metric on $M^{1,9}$ is block-diagonal. The equation of motion for Φ_{10} is

$$\Delta_{10}\Phi_{10} - m^2\Phi_{10} = 0. \tag{2.4}$$

Since the metric is block-diagonal, $\Delta_{10} = \Delta_6 + \Delta_4$; therefore, using eq. (2.3) here tells us that

$$(m^2 - m_n^2) f^n = \Delta_6 f^n, \tag{2.5}$$

for $m_n^2\Phi_n = \Delta_4\Phi_n$. In other words, the mass of the four-dimensional fields depends on the mass of the ten-dimensional field and the eigenvalues of the Laplace operator Δ_6. We will restrict ourselves to the massless case, that is, to the case where $m = 0$ and $\Delta_6 f^n = 0$. This means that f^n has to be a constant, so the ten-dimensional scalar field $\Phi_{10}(X)$ gives rise to one four-dimensional scalar field $\Phi(x)$.

Generally speaking, bosonic tensor fields can always be expanded in eigenvalues of the six-dimensional Laplace operator. If Y is Kähler, the eigenfunctions with eigenvalue zero are in a one-to-one correspondence with the cohomology of Y. This means that we can decompose all bosonic fields in a basis of the cohomology of Y.

If Y is not Kähler, the above argument no longer holds. In particular, in sections 2.4 and 2.6 we will want to expand the fields in forms, including the Kähler form, that are no longer closed. However, it can be argued [16] that the resulting masses will be much smaller than the masses coming from expanding the fields in eigenfunctions of the six-dimensional Laplace operator with a non-zero eigenvalue.

Once we have established the decomposition of the fields, this allows us to determine the spectrum and the action of the four-dimensional theory. The spectrum is given by the coefficients of the expansions of the ten-dimensional fields. To find the four-dimensional action, we insert all the field expansions in the ten-dimensional action. This action then splits into a four-dimensional and six-dimensional integral. The six-dimensional integral is given by the intersection numbers of Y, and this leaves us with a four-dimensional action.

In the following sections, we will see how all of this is done for IIA supergravity on a manifold with $SU(2)$-structure. We will start by giving the spectrum and action of IIA supergravity. After that, we will describe what an $SU(2)$-structure manifold is and see that $K3 \times T^2$ is a particular example of such a manifold. Then we will perform the compactification of IIA on a $K3 \times T^2$. In the next section, we will compactify IIA on a simple manifold with $SU(2)$-structure that is a generalization of $K3 \times T^2$, with an added $H_{10}^{(3)}$-flux. We will then discuss four-dimensional gauged supergravities, before turning to IIA compactified on a more complex manifold with $SU(2)$-structure.

2.1 IIA Supergravity

Type IIA supergravity is a ten-dimensional, $\mathcal{N} = 2$, non-chiral supergravity theory. A supergravity theory is a supersymmetric theory that includes gravity; the ten-dimensional metric $g_{10\,MN}$ is a bosonic field that enters the action. The fact that $\mathcal{N} = 2$ means that the supersymmetry transformations of the fields are generated by two independent parameters $\epsilon^{1,2}$. In ten dimensions, this means that the theory has 32 conserved supercharges. The two supersymmetry parameters could have either opposite chirality, meaning

$$\Gamma_{10}\epsilon^1 = \pm\epsilon^1 \tag{2.6}$$

$$\Gamma_{10}\epsilon^2 = \mp\epsilon^2, \tag{2.7}$$

or the same chirality, meaning

$$\Gamma_{10}\epsilon^{1,2} = \pm\epsilon^{1,2}. \tag{2.8}$$

IIA supergravity is non-chiral, which means that the supersymmetry parameters have opposite chiralities. Type IIB supergravity, which will not be used in this thesis, is a ten-dimensional, $\mathcal{N} = 2$, chiral supergravity theory with both supersymmetry parameters having the same chirality.

The bosonic spectrum of IIA supergravity contains the graviton $g_{10\,MN}$, the dilaton ϕ_{10} and the two-form field $B_{10}^{(2)}$. These fields come from the Neveu-Schwarz-Neveu-Schwarz sector of the string and this sector is therefore often called the *NS sector*. The other two bosonic fields in IIA supergravity are the one-form $A_{10}^{(1)}$ and the three-form $C_{10}^{(3)}$. These come from the Ramond-Ramond sector of the string and this sector is called the *Ramond sector*.

The action is manifestly invariant under a coordinate transformation

$$X^M \to X^M + \xi^M, \tag{2.9}$$

for infinitesimal ξ^M. Since coordinate transformations can be written as transformations of the fields, this is equivalent to saying that the action is invariant under these field transformations. Under eq. (2.9), a tensor field $T^{N_1,\ldots,N_q}_{M_1,\ldots,M_p}$ transforms as

$$\begin{aligned} \delta_\xi T^{N_1,\ldots,N_q}_{M_1,\ldots,M_p} = -\xi^R \partial_R T^{N_1,\ldots,N_q}_{M_1,\ldots,M_p} - \sum_{i=1}^{p} \partial_{M_i} \xi^R T^{N_1,\ldots,N_q}_{M_1,\ldots,M_{i-1},R,M_{i+1},\ldots,M_p} \\ + \sum_{i=1}^{q} \partial_R \xi_{N_i} T^{N_1,\ldots,N_{i-1},R,N_{i+1},\ldots,N_q}_{M_1,\ldots,M_p}. \end{aligned} \tag{2.10}$$

The form fields also transform as gauge fields. More precisely, they have the following gauge transformations:

$$\delta_0 A_{10}^{(1)} = d\Lambda_{10}, \tag{2.11}$$

$$\delta_1 B_{10}^{(2)} = d\Lambda_{10}^{(1)}, \tag{2.12}$$

$$\delta_2 C_{10}^{(3)} = d\Lambda_{10}^{(2)} + \Lambda_{10}\, dB_{10}^{(2)}. \tag{2.13}$$

This means that whereas the field strengths of $A_{10}^{(1)}$ and $B_{10}^{(2)}$ are simply $F_{10}^{(2)} := d_{10}\, A_{10}^{(1)}$ and $H_{10}^{(3)} := d_{10}\, B_{10}^{(2)}$, the exterior derivative of $F_{10}^{(4)} := d_{10}\, C_{10}^{(3)}$ does not transform covariantly. Instead, it can be easily seen that the field strength

$$\tilde{F}_{10}^{(4)} := d_{10}\, C_{10}^{(3)} - A_{10}^{(1)} \wedge H_{10}^{(3)} \tag{2.14}$$

is left invariant by the transformations in eq. (2.13).

The bosonic action for these fields is made out of three parts,

$$S_{IIA} = S_{NS} + S_R + S_{CS}, \tag{2.15}$$

where

$$S_{NS} = \int e^{-2\phi_{10}} \left\{ d^{10}x\sqrt{-g_{10}} \Big(R_{10} + 4\partial_M \phi_{10} \partial^M \phi_{10} \Big) + \frac{1}{2} H_{10}^{(3)} \wedge *H_{10}^{(3)} \right\} \tag{2.16}$$

$$S_R = \frac{1}{2} \int \left\{ F_{10}^{(2)} \wedge *F_{10}^{(2)} + \tilde{F}_{10}^{(4)} \wedge *\tilde{F}_{10}^{(4)} \right\} \tag{2.17}$$

$$S_{CS} = -\frac{1}{2} \int B_{10}^{(2)} \wedge F_{10}^{(4)} \wedge F_{10}^{(4)} . \tag{2.18}$$

It can be checked that this action is invariant under all symmetry transformations described above.

2.2 Structures and Spinors

Let us now explain what we mean when we say that the internal manifold Y is a six-dimensional manifold with $SU(2)$-structure. Such manifolds have been discussed, for example, in refs. [10, 12, 13] and in this section we summarize some of their results.

2.2.1 Topological Requirements for Supersymmetry Breaking

In general n-dimensional manifolds with G-structure are defined to have a reduced structure group $G \subset SO(n)$.[1] For the case at hand the structure group $SO(6)$ is reduced to $SU(2)$. This in turn implies that two nowhere vanishing spinors η^1, η^2 can be globally defined on Y. These are precisely the two singlets in the decomposition of the four-dimensional spinor representation of $SO(6)$ under $SU(2)$

$$SO(6) \cong SU(4) \to SU(2) : \qquad \mathbf{4} \to \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \ . \tag{2.19}$$

The ten-dimensional supersymmetry parameters are Majorana-Weyl and they reside in the **16** of $SO(1,9)$. For backgrounds of the form (2.1) they decompose as

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \ , \tag{2.20}$$

where the $\mathbf{2}$ and $\bar{\mathbf{2}}$ correspond to Weyl spinors of the four-dimensional Lorentz group $SO(1,3)$. Keeping only the two singlets of (2.19) leaves eight out of the original 16 supercharges in $M^{1,3}$ intact. For type IIA where one starts with 32 supercharges in ten dimensions, one ends up with 16 supercharges in $M^{1,3}$ or in other words $\mathcal{N} = 4$ supersymmetry. More precisely we have

$$\begin{aligned} \epsilon^1 &= \xi^1_{+i} \otimes \eta^i_+ + \xi^1_{-i} \otimes \eta^i_- \ , \quad i = 1,2 \\ \epsilon^2 &= \xi^2_{+i} \otimes \eta^i_- + \xi^2_{-i} \otimes \eta^i_+ \ , \end{aligned} \tag{2.21}$$

where $\epsilon^{1,2}$ are the two supersymmetry parameters in $d = 10$, $\xi_i^{1,2}$ are the four supersymmetry parameters of $\mathcal{N} = 4$ in $d = 4$ and by abuse of notation the subscript $\pm$ indicates both the four- and six-dimensional chiralities.

[1] The structure group is the group the transition functions of the tangent bundle TY take values in.

2.2.2 Spinors on an $SU(2)$-Structure Manifold

Instead of characterizing manifolds with G-structure by the existence of globally defined spinors they can equivalently be described by invariant tensors constructed out of the spinors. We will be using the spinor conventions laid out in Appendix G.

For manifolds with $SU(2)$-structure one can define a pair of 2-forms J^i and a pair of 3-forms Ω^i via

$$J^i_{mn} = i\eta^{i\dagger}_- \gamma_{mn} \eta^i_- \ , \qquad \Omega^i_{mnp} = i\eta^{i\dagger}_- \gamma_{mnp} \eta^i_+ \ , \qquad m,n = 1,\dots,6 \ . \tag{2.22}$$

By raising an index with the metric one obtains two almost complex structures

$$I^i{}_m{}^n := J^i_{mp} g^{pn} \ , \qquad (I^i)^2 = -\mathbf{1} \ , \tag{2.23}$$

which generically are not integrable since the Nijenhuis-tensor is not necessarily vanishing. With respect to I^i the two forms J^i are $(1,1)$-forms while the Ω^i are $(3,0)$ forms.

If the manifold has an $SU(2)$-structure the two almost complex structure commute $[I_1, I_2] = 0$ and define an almost product structure Π via

$$\Pi_m{}^n := I^1{}_m{}^p I^2{}_p{}^n \ , \quad \text{with} \quad \Pi^2 = \mathbf{1} \ . \tag{2.24}$$

One can check that Π has four negative and two positive eigenvalues which in turn implies that locally the tangent space splits into a four-dimensional and a two-dimensional component and the metric can be choosen block-diagonal [10, 12]. In other words locally the six-manifolds Y is a product of the form $Y \simeq Y^{(4)} \times Y^{(2)}$ where $Y^{(4)}$ is a four-manifold while $Y^{(2)}$ is a two-dimensional manifold. Furthermore since the spinors are never parallel also a globally defined complex one-form

$$\sigma_m := \sigma^1_m - i\sigma^2_m := \eta^{2\dagger}_+ \gamma_m \eta^1_- \ , \tag{2.25}$$

exists.

With this information at hand the four tensors J^i, Ω^i can be expressed in terms of the one forms σ^i, a $(1,1)$-form j and a $(2,0)$-form ω via [10, 12]

$$J^{1,2} = j \pm \sigma^1 \wedge \sigma^2 \ , \qquad \Omega^{1,2} = \omega \wedge (\sigma^1 \pm i\sigma^2) \ , \tag{2.26}$$

or equivalently [13]

$$j = \frac{1}{2}(J^1 - J^2) \ , \quad \omega_{mn} = i\eta^{1\dagger}_- \gamma_{mn} \eta^2_- \ . \tag{2.27}$$

As shown in refs. [10, 12] σ^i can be viewed as one forms on the two-dimensional component $Y^{(2)}$ while j and ω define an $SU(2)$ structure on the four-dimensional component $Y^{(4)}$.

There is an arbitrariness in our choice of spinors. Any linear combination of the two globally well-defined spinors η^1, η^2, as long as it leaves the lengths invariant, would yield two-forms $j, \mathrm{Re}\,\omega, \mathrm{Im}\,\omega$ and a one-form σ. Let us consider a complex 2×2-matrix g acting on $\eta_- := (\eta^1_-, \eta^2_-)$ such that

$$\eta^\dagger_- \eta_- \to \eta^\dagger_- g^\dagger g \eta_- = \eta^\dagger_- \eta_- . \tag{2.28}$$

This means $g \in SU(2)$, so we can write it as

$$g = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \tag{2.29}$$

with $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$. While the negative chirality spinors transform as

$$\eta_- \to g\eta_-, \tag{2.30}$$
$$\eta_-^\dagger \to \eta_-^\dagger g^\dagger, \tag{2.31}$$

the positive chirality-spinors are defined as

$$\eta_+^\dagger = \eta_-^T C. \tag{2.32}$$

Therefore, they transform as

$$\eta_+ \to g^* \eta_+, \tag{2.33}$$
$$\eta_+^\dagger \to \eta_+^\dagger g^{\mathrm{T}}. \tag{2.34}$$

This $SU(2)$-rotation of the spinors corresponds to an $SO(3)$-rotation of the three two-forms $j, \mathrm{Re}\,\omega, \mathrm{Im}\,\omega$. Let $G(g)$ be the action of g on any spinor bilinear given by the $SU(2)$-rotation g of the spinors, written as a matrix multiplying the vector

$$\begin{pmatrix} j \\ \mathrm{Re}\,\omega \\ \mathrm{Im}\,\omega \end{pmatrix}. \tag{2.35}$$

We want to consider the action of g on this vector. In terms of spinors, we have

$$j_{mn} = \frac{i}{2}\left(\eta_-^{1\dagger}\gamma_{mn}\eta_-^1 - \eta_-^{2\dagger}\gamma_{mn}\eta_-^2\right), \tag{2.36}$$
$$\mathrm{Re}\,\omega = \frac{i}{2}\left(\eta_-^{1\dagger}\gamma_{mn}\eta_-^2 + \eta_-^{2\dagger}\gamma_{mn}\eta_-^1\right), \tag{2.37}$$
$$\mathrm{Im}\,\omega = \frac{1}{2}\left(\eta_-^{1\dagger}\gamma_{mn}\eta_-^2 - \eta_-^{2\dagger}\gamma_{mn}\eta_-^1\right), \tag{2.38}$$

and writing $a = t + ix$, $b = y + iz$ we find

$$G(g) = \begin{pmatrix} t^2 + x^2 - y^2 - z^2 & 2ty + 2xz & 2xy - 2tz \\ 2xz - 2ty & t^2 - x^2 - y^2 + z^2 & 2tx + 2yz \\ 2xy + 2tz & 2yz - 2tx & t^2 - x^2 + y^2 - z^2 \end{pmatrix}. \tag{2.39}$$

For $g \in SU(2)$ we have $t^2 + x^2 + y^2 + z^2 = 1$, and using this we can calculate that $GG^T = 1$. This means $G \in O(3)$, and since $\det G = 1$ for $t = 1, x = y = z = 0$, we conclude $G \in SO(3)$.

On the other hand, the one-form σ is left invariant under this rotation. To show this we need that

$$\sigma_m := \eta_-^{2\dagger}\gamma_m\eta_+^1 = -\eta_-^{1\dagger}\gamma_m\eta_+^2. \tag{2.40}$$

This can be shown as follows:

$$\begin{aligned}\eta_-^{2\dagger}\gamma_m\eta_+^1 &= (\eta_+^2)^T C\gamma_m\eta_+^1 = -(\eta_+^2)^T\gamma_m^T C\eta_+^1 \\ &= -(\eta_+^2)^T\gamma_m^T(\eta_-^{1\dagger})^T = -\eta_-^{1\dagger}\gamma_m\eta_+^2.\end{aligned} \tag{2.41}$$

We calculate

$$G(g)v_m = (-b\eta_-^{1\dagger} + a\eta_-^{2\dagger})\gamma_m(a^*\eta_+^1 + b^*\eta_+^2) = (|a|^2 + |b|^2)v_m = v_m. \tag{2.42}$$

We recognize these transformations from $K3 \times T^2$. $K3$ has a triplet of complex structures, while T^2 has one. The rotation of the spinors exactly duplicates this for the case of an $SU(2)$-structure manifold: the three complex structures on $Y^{(4)}$, $j, \mathrm{Re}\,\omega$, and $\mathrm{Im}\,\omega$ rotate into each other, while the complex structure on $Y^{(2)}$, σ, is left invariant.

2.3 IIA on $K3 \times T^2$

We will start the compactifications by giving an overview of the compactification of IIA supergravity on $K3 \times T^2$. The computations of IIA supergravity compactified on manifolds with $SU(2)$-structure will be based on this, as the internal manifolds are inspired by $K3 \times T^2$. We start by looking at the internal manifold to determine what forms to expand the ten-dimensional fields in. Having done that, we move on to calculate the spectrum and the action.

2.3.1 $K3$ and T^2

Here, we will introduce the forms and geometric objects needed to compactify on a $K3 \times T^2$. The concepts used here are explained in Appendix A. Since $K3 \times T^2$ is a product manifold, one can look at $K3$ and T^2 separately. Let us start with T^2.

T^2 is a two-torus; we will denote coordinates by y^i for $i = 1, 2$. It has two one-forms, $\sigma^i = dy^i$, and one two-form, $\sigma^i \wedge \sigma^j = \epsilon^{ij} dy^1 \wedge dy^2$. The one integral we need to know is

$$\int_{T^2} \sigma^i \wedge \sigma^j = \epsilon^{ij}. \tag{2.43}$$

The geometry of T^2 is determined by its metric g_{ij}. From this metric, the volume of the T^2, $e^{-2\eta}$ can be derived as $e^{-2\eta} := \det g_{ij}$.

$K3$, with local coordinates denoted by z^a for $a = 1, \ldots, 4$, is slightly more complicated; it is the only non-trivially Calabi-Yau manifold of four real dimensions. This means that it is a Ricci-flat Kähler manifold with vanishing first Chern class. From this, one can infer (see, for instance, [21]) that the Hodge diamond is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{2.44}$$

The 22 2-forms are denoted as Ω^A $(A = 1, \dots, 22)$ and they determine the following intersection metric:

$$\eta^{AB} := \int_{K3} \Omega^A \wedge \Omega^B. \tag{2.45}$$

Note that η^{AB} is symmetric. The signature of this metric is $(3, 19)$, which corresponds to the fact that the two-forms can be split into three anti-selfdual ones (the Kähler form, and the $(2,0)$ and $(0,2)$-form), and nineteen selfdual ones.

Since the Hodge-dual of a two-form on a four-dimensional manifold is again a two-form, $*\Omega^A$ is a linear combination of two-forms:

$$*\Omega^A =: H^A{}_B \Omega^B. \tag{2.46}$$

Since $**\Omega^A = \Omega^A$, $H^A{}_B H^B{}_C = \delta^A_C$. Furthermore, $\Omega^A \wedge *\Omega^B = \Omega^B \wedge *\Omega^A$, so $\eta^{AB} H^C{}_B = \eta^{CB} H^A{}_B$. From this, it follows that

$$H^A{}_B \eta^{BC} H^D{}_C = \eta^{AD}, \tag{2.47}$$

or, in other words, $H^A{}_B \in SO(3,19)$. In fact, $H^A{}_B$ is an element of the 57-dimensional *coset space* [22]

$$\frac{SO(3,19)}{SO(3) \times SO(19)}. \tag{2.48}$$

Since the metric is used in the definition of the Hodge star operator $*$, $H^A{}_B$ is determined by the metric g_{ab} on the $K3$. In fact, $H^A{}_B$ contains 57 out of the 58 parameters we need to specify the $K3$ metric [21]. The one metric parameter that is not caught in $H^A{}_B$ is the volume of the $K3$. Analogous to the T^2, the volume $e^{-2\rho}$ is defined as $e^{-2\rho} := \det g_{ab}$.

2.3.2 The Spectrum

We are now in the position to determine the bosonic spectrum of IIA supergravity compactified on $K3 \times T^2$ by expanding the bosonic spectrum of IIA supergravity in the cohomology of the internal manifold. The four-dimensional fields that arise have transformations that come from the ten-dimensional field transformations. We will discuss these and redefine the four-dimensional fields accordingly. Finally, we will specify the field dualizations we need to perform to make the global $SO(6,22)$-symmetry of the theory manifest.

The four-dimensional dilaton is defined such that it contains the ten-dimensional dilaton and the volume of the $K3$:

$$\phi := \phi_{10} + \frac{1}{2}\rho. \tag{2.49}$$

The ten-dimensional metric gives rise to the four-dimensional metric $g_{\mu\nu}$, two vector-fields $G^{(1)i}$ and metric moduli coming from g_{ij} and g_{ab}:

$$\begin{aligned} ds^2 =& g_{\mu\nu}(x) dx^\mu dx^\nu + g_{ab}(x,z) dz^a dz^b \\ &+ g_{ij}(x) \left(\sigma^i - G^{(1)i}_\mu(x) dx^\mu\right) \left(\sigma^j - G^{(1)j}_\nu(x) dx^\nu\right). \end{aligned} \tag{2.50}$$

The existence of the two vectorfields $G^{(1)i}$ is directly related to the fact that a T^2 has two one-forms. On the other hand, there are no one-forms on the $K3$ so $g_{10\,\mu a} = 0$.

The combination

$$\nu^i := \sigma^i - G^{(1)i} \tag{2.51}$$

is invariant under the remnants of the ten-dimensional coordinate transformations. These remnants are generated by a vector $\xi^i(x)s_i$ with the vectors s_i dual to the one-forms σ^i on T^2. Using eq. (2.10), we can calculate the transformation of g_{ij} under ξ^i:

$$\delta_\xi g_{10\,ij} = -\xi^k\partial_k g_{ij} - \partial_i\xi^k g_{kj} - \partial_j\xi^k g_{ik}. \tag{2.52}$$

Since both g_{ij} and ξ^i only depend on x, we conclude that

$$\delta_\xi g_{ij} = 0. \tag{2.53}$$

This, and the calculation

$$\delta_\xi g_{10\,\mu i} = -\partial_\mu\xi^j g_{10\,ji} = -g_{ij}\partial_\mu\xi^j, \tag{2.54}$$

tells us that

$$\delta_\xi G^{(1)i} = d\xi^i. \tag{2.55}$$

This means $G^{(1)i}$ is a gauge field for the parameter ξ^i. Other fields coming from $g_{10\,MN}$ do not transform under ξ. The one-form $\sigma^i = dy^i$ does transform under $y^i \to y^i + \xi^i$, of course, and the transformations of σ^i and $-G^{(1)i}$ exactly cancel each other out. Therefore, we expand the fields in terms of the invariant combination

$$\nu^i := \sigma^i - G^{(1)i}. \tag{2.56}$$

Since the metric is block-diagonal in terms of ν^i, the ten-dimensional Hodge-star splits when written in this basis. That means that apart from the integral in eq. (2.43) we can also calculate the integral

$$\int_{T^2} \nu^i \wedge *\nu^j = e^{-\eta} g^{ij}. \tag{2.57}$$

We will need both these integrals to calculate the four-dimensional action.

The form fields can be expanded in terms of the fields on $K3 \times T^2$ as follows:

$$A_{10}^{(1)} = A^{(1)}(x) + a_i(x)\nu^i \tag{2.58}$$

$$B_{10}^{(2)} = B^{(2)}(x) + B_i^{(1)}(x) \wedge \nu^i + \frac{1}{2} b_{ij}(x)\, \nu^i \wedge \nu^j + b_A(x)\, \Omega^A \tag{2.59}$$

$$\begin{aligned} C_{10}^{(3)} = {} & \hat{C}^{(3)}(x) + \hat{C}_i^{(2)}(x) \wedge \nu^i + \frac{1}{2}\hat{C}_{ij}^{(1)}(x)\, \nu^i \wedge \nu^j \\ & + \hat{C}_A^{(1)}(x) \wedge \Omega^A + \hat{c}_{iA}(x)\, \nu^i \wedge \Omega^A\ . \end{aligned} \tag{2.60}$$

This means that the ten-dimensional form fields give the following four-dimensional fields: one three-form, three two-forms, twenty-four one-forms and sixty-nine scalars.

The four-dimensional fields inherit the gauge behavior of the ten-dimensional fields. This is especially interesting for the fields coming from $C_{10}^{(3)}$, as it has a non-trivial transformation under the ten-dimensional gauge transformations. $\hat{C}^{(1)}_A$, for example, transforms like

$$\delta_\Lambda \hat{C}^{(1)}_A = d\Lambda_A + \Lambda_{10} db_A. \tag{2.61}$$

We find it convenient to perform the following field redefinitions:

$$\begin{aligned} c_{iA} &:= \hat{c}_{iA} + a_i b_A, \\ C^{(1)A} &:= \eta^{AB}\left(\hat{C}^{(1)}_B + A^{(1)} b_B\right), \\ C^{(1)}_{ij} &:= \hat{C}^{(1)}_{ij} + A^{(1)} b_{ij}, \\ C^{(2)}_i &:= \hat{C}^{(2)}_i + A^{(1)} \wedge B^{(1)}_i, \\ C^{(3)} &:= \hat{C}^{(3)} + A^{(1)} \wedge B^{(2)}. \end{aligned} \tag{2.62}$$

Note that we have used this redefinition to also raise the index on $C^{(1)A}$. This is only done for notational purposes and has no physical significance. $C^{(1)A}$ is a $U(1)$ gauge boson with parameter λ^A defined by

$$\lambda^A := \eta^{AB}\left(\Lambda_B + \Lambda b_B\right). \tag{2.63}$$

To compare this to the formulation of $\mathcal{N} = 4$ supergravity in the literature, we have to dualize some of the fields. This will make the global $SO(6, 22)$-symmetry of the theory manifest. We have written down the procedure of dualizing fields in Appendix B. In particular, to write the action in terms of the fields found in the literature we do the following: we integrate out the three-form $C^{(3)}$, since it does not represent any degrees of freedom in four dimensions, and dualize the two antisymmetric tensor fields $C^{(2)}_i$ to two scalars γ_i. We then dualize the one-form field $C^{(1)}_{ij}$ to a one-form field $\tilde{C}^{(1)}$, and the two-form field $B^{(2)}$ to a scalar $\beta_{ij} = -\beta_{ji}$. Note that since $i, j \in \{1, 2\}$, β_{ij} only represents one independent scalar field. Finally, we dualize the one-form fields $B^{(1)}_i$ to the one-form fields $\tilde{B}^{(1)\iota}$ ($\iota = 1, 2$). In this final frame, the bosonic spectrum consists of one graviton, 28 vector fields and 134 scalar fields. In section 2.5 we shall see this is the spectrum for $\mathcal{N} = 4$ supergravity with 22 vector multiplets.

2.3.3 The Reduction

It is now possible to perform the actual dimensional reduction of the Lagrangian (see [23] and references therein), dualize the fields as explained above, and end up with a four-dimensional theory that has a manifest $SO(6, 22)$ global symmetry. In this section, we will mainly present the results of the reduction of the action. The reader interested in the calculations behind it is referred to Appendix C.

We start by entering the expansions (2.58), (2.59) and (2.60) into the supergravity action (2.15) and make the field redefinitions as in section 2.3.2. This yields kinetic terms for all the fields, and a topological term. We then perform the dualizations discussed above.

The 22 field strengths in the resulting action are given by

$$\begin{aligned}
\mathcal{F}^{(2)i+} &:= dG^{(1)i}, \\
\mathcal{F}^{(2)\iota+} &:= d\tilde{B}^{(1)\iota}, \\
\mathcal{F}^{(2)5+} &:= dA^{(1)}, \\
\mathcal{F}^{(2)6+} &:= d\tilde{C}^{(1)}, \\
\mathcal{F}^{(2)A+} &:= dC^{(1)A}.
\end{aligned} \tag{2.64}$$

The $+$ index on the field strengths is added for the comparison to $\mathcal{N}=4$ supergravity and holds no significance at this moment. For these gauge bosons and field strengths, we now introduce an $SO(6,22)$-index M that runs over i, ι, 5, 6, and over A. Using this, we put all field strengths in an $SO(6,22)$-vector of field strengths, $\mathcal{F}^{(2)M+}$, defined as

$$\mathcal{F}^{(2)M+} := (\mathcal{F}^{(2)i+}, \mathcal{F}^{(2)\iota+}, \mathcal{F}^{(2)5+}, \mathcal{F}^{(2)6+}, \mathcal{F}^{(2)A+}) \tag{2.65}$$

We also define a 28×28 matrix M_{MN} that contains 132 out of the 134 scalars; its definition is in Appendix F. From the definition given there, it can be calculated that $M_{MN} \in SO(6,22)$. In fact, M_{MN} spans the coset space [22]

$$\frac{SO(6,22)}{SO(6) \times SO(22)}. \tag{2.66}$$

The two scalar fields that are not in M_{MN} make up the complex scalar field

$$\tau := -\frac{1}{4}\epsilon^{ij} b_{ij} + \frac{i}{2} e^{-\eta}, \tag{2.67}$$

which spans the coset space [22]

$$\frac{SL(2)}{SO(2)}. \tag{2.68}$$

Using the definitions of $\mathcal{F}^{(2)M+}$, M_{MN} and τ, the action of the field strengths can then be written as

$$S_{fs} = \int \left\{ \mathrm{Im}(\tau) M_{MN} \mathcal{F}^{(2)M+} \wedge *\mathcal{F}^{(2)N+} + \mathrm{Re}(\tau) L_{MN} \mathcal{F}^{(2)M+} \wedge \mathcal{F}^{(2)N+} \right\}, \tag{2.69}$$

where L_{MN} is the metric on $SO(6,22)$:

$$L_{MN} = \begin{pmatrix} 0 & \delta_{i\iota} & 0 & 0 & 0 \\ \delta_{i\iota} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_{AB} \end{pmatrix}. \tag{2.70}$$

This action has a manifest global $SO(6,22)$-symmetry: let $T^M{}_N \in SO(6,22)$, then the action is invariant under the transformation

$$\begin{aligned}
(LML)^{MN} &\to T^M{}_P (LML)^{PQ} (T^T)_Q{}^N, \\
\mathcal{F}^{(2)M+} &\to T^M{}_R \mathcal{F}^{(2)R+},
\end{aligned} \tag{2.71}$$

the other fields being invariant. The transformation of $(M^{-1}) = LML$ means that M transforms as

$$M_{MN} \to (T^{-1})^O{}_M (T^{-1})^P{}_N M_{OP}, \tag{2.72}$$

and we see that eq. (2.69) is invariant under the transformations.

To bring the scalar term in a $SO(6, 22)$-invariant form, we need to perform the Weyl rescaling $g_{\mu\nu} \to e^{2\phi+\eta} g_{\mu\nu}$. While this operation leaves S_{fs} invariant, it changes the scalar term. After a calculation, the main steps of which can be found in Appendix C, the scalar term becomes:

$$S_{sc} = \int d^4x \sqrt{-g} \left[R - \frac{1}{2\mathrm{Im}^2(\tau)} \partial_\mu \tau \partial^\mu \tau^* + \frac{1}{8} \partial_\mu M_{MN} \partial^\mu (LML)^{MN} \right]. \tag{2.73}$$

This term is again invariant under the global symmetry given in eq. (2.71).

The action resulting from the compactification of IIA supergravity on a $K3 \times T^2$ is therefore given by

$$S_{sc} + S_{fs}. \tag{2.74}$$

In section 2.5 we will compare this to the literature to see that this is an $\mathcal{N} = 4$ supergravity action with 22 vector multiplets.

2.4 IIA on Y_1 with $H_{10}^{(3)}$-Flux

After compactifying on $K3 \times T^2$, we will now perform the compactification on a more general manifold with $SU(2)$ structure, that we call Y_1. We will also add a flux to the $H_{10}^{(3)}$-field. The forms we expand the fields in will no longer be closed. The masses of the fields will therefore be nonzero, controlled by the parameter that determines the derivatives of the forms and the $H_{10}^{(3)}$-flux. These same parameters will also determine the gauging of the theory, as we will see.

2.4.1 The Internal Manifold with $H_{10}^{(3)}$-Flux

In section 2.2, we discussed some of the properties of a manifold with $SU(2)$ structure. One of these was the existence of a two one-forms σ^i, $i = 1, 2$, and another was the existence of a real two-form j and a complex two-form ω. This is reminiscent of $K3 \times T^2$, where the two-form j is the Kähler form of $K3$ and the complex two-form ω determines the complex structure of $K3$, spliting into a $(2, 0)$ and a $(0, 2)$-form. However, in comparison with $K3 \times T^2$ we now make two generalizations: we set the number of two-forms Ω^A to n, meaning $A = 1, \dots, n$, and we allow for non-closed two-forms. In particular, we want to examine what happens for

$$\begin{aligned} d\sigma^i &= 0, \\ d\Omega^A &= D^A_{iB} \sigma^i \wedge \Omega^B. \end{aligned} \tag{2.75}$$

Since we want the forms to be generated by σ^i and Ω^A, the most general three-form we can have is a linear combination of $\sigma^i \wedge \Omega^A$'s.

Consistency of form derivatives means that we have to make sure that $d^2 = 0$ still holds, and that Stokes' theorem is still obeyed. These two conditions will translate to two conditions on the parameters D_{iA}^{B}. First of all, we can calculate that

$$\begin{aligned} d^2\Omega^A &= d\left(D_{iB}^{A}\sigma^i \wedge \Omega^B\right) \\ &= -D_{iB}^{A} D_{jC}^{B} \sigma^i \wedge \sigma^j \wedge \Omega^C \\ &= -\frac{1}{2}\left(D_{iB}^{A} D_{jC}^{B} - D_{jB}^{A} D_{iC}\right)\sigma^i \wedge \sigma^j \wedge \Omega^C. \end{aligned} \tag{2.76}$$

This is zero if and only if

$$D_{iB}^{C} D_{jA}^{B} = D_{jB}^{C} D_{iA}^{B}. \tag{2.77}$$

Furthermore, Stokes' theorem, in its general form, says that

$$\int_Y d\omega_{d-1} = \int_{\partial Y} \omega_{d-1}, \tag{2.78}$$

for Y a d-dimensional manifold with boundary ∂Y, and ω_{d-1} a $(d-1)$-form. Since Y_1 is compact it has no boundary, so we need to assure that

$$\int_{Y_1} d\left(\sigma^i \wedge \Omega^A \wedge \Omega^B\right) = 0 \tag{2.79}$$

The derivative gives

$$d\left(\sigma^i \wedge \Omega^A \wedge \Omega^B\right) = -\sigma^i \wedge \sigma^j \wedge \left(D_{jC}^{A}\Omega^C \wedge \Omega^B + D_{jC}^{B}\Omega^A \wedge \Omega^C\right). \tag{2.80}$$

Performing the integration gives the constraint

$$\eta^{AC} D_{iC}^{B} = -\eta^{BC} D_{iC}^{A}. \tag{2.81}$$

The integrals on Y_1 can be defined analogous to those on $K3 \times T^2$. The intersection metric η^{AB} is now defined by

$$\eta^{AB}\epsilon^{ij} := \int_{Y_1} \sigma^i \wedge \sigma^j \wedge \Omega^A \wedge \Omega^B, \tag{2.82}$$

with η^{AB} still symmetric and now a metric with signature $(3, n)$.

Since Y_1 possesses an almost product structure, it locally looks like a product of a four and a two-dimensional manifold. This, again, is reminiscent of $K3 \times T^2$, where the product structure is global. However, an almost product structure already allows us to split the metric locally into g_{ij} and g_{ab}, so we can still define the scalars

$$e^{-2\rho} := \det g_{ab} \tag{2.83}$$

$$e^{-2\eta} := \det g_{ij}. \tag{2.84}$$

Finally, the definition of ${H^A}_B$ in the $K3 \times T^2$-case can be generalised to

$$*\Omega^B =: e^{-\eta} {H^B}_C \sigma^1 \wedge \sigma^2 \wedge \Omega^C. \tag{2.85}$$

Again, this matrix contains all the information about the metric apart from $e^{-2\rho}$.

In this compactification, we also want to consider an $H_{10}^{(3)}$-flux. This means that we assign a non-zero value to the integral of $H_{10}^{(3)}$ over a non-trivial three-cycle:

$$\int_{[\sigma^i \wedge \Omega^A]} H_{10}^{(3)} = k_{iA} \tag{2.86}$$

for $[\sigma^i \wedge \Omega^A]$ the Poincaré dual of the three-form $\sigma^i \wedge \Omega^A$. Locally, we can add the flux by adding a term

$$\frac{1}{2}\omega_{ab} dy^a \wedge dy^b \tag{2.87}$$

obeying

$$\partial_i \omega_{ab} dy^i \wedge dy^a \wedge dy^b = k_{iA} dy^i \wedge \Omega^A_{ab} dy^a \wedge dy^b \tag{2.88}$$

to $B_{10}^{(2)}$.

There is a constraint on this flux: we require the Bianchi identity for $H_{10}^{(3)}$ to hold. In other words, we require that

$$dH_{10}^{(3)} = 0. \tag{2.89}$$

Since Ω^A is non-closed, this does not automatically hold. Instead, we have to impose

$$k_{iA} D^A_{jB} = k_{jA} D^A_{iB}. \tag{2.90}$$

2.4.2 The Spectrum

Let us now turn our attention to the spectrum of the four-dimensional theory. In comparison to section 2.3.2, three things have changed: the number of two-forms Ω^A has changed from 22 to n, those same two-forms are no longer closed, and we have added an $H_{10}^{(3)}$-flux. We will see that the first change leads to a change in the number of lower-dimensional fields, as could be expected. The second and third changes lead to changes in the transformation behaviour.

However, some things stay the same. These include the four-dimensional dilaton

$$\phi := \phi_{10} + \frac{1}{2}\rho. \tag{2.91}$$

and the expansion of the metric

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + g_{ab}(x,y,z) dz^a dz^b + g_{ij}(x) \nu^i \nu^j. \tag{2.92}$$

Here, ν^i is again defined as

$$\nu^i := \sigma^i - G_\mu^{(1)i}(x) dx^\mu. \tag{2.93}$$

Since the vector fields $G^{(1)i}$ transform like

$$\delta_\xi G^{(1)i} = d\xi^i, \tag{2.94}$$

ν^i is invariant under ξ, as explained in section 2.3.2

The only fields from the ten-dimensional metric that are different are the ${H^A}_B$, since A now no longer runs to 22, but to n. We assume the matrix ${H^A}_B$ is still in a coset space, although this time in the $(3n-9)$-dimensional generalization of (2.48),

$$\frac{SO(3, n-3)}{SO(3) \times SO(n-3)}. \tag{2.95}$$

It can also still be thought of as parametrizing all but one of the metric moduli of the four-dimensional part of Y_1.

To the expansion of the form field $B_{10}^{(2)}$ in eq. (2.59) we add the local term (2.87), as discussed above. The expansion of the fields $A_{10}^{(1)}$ and $C_{10}^{(3)}$ remain formally the same as in eqs. (2.58) and (2.60). However, since the number of two-forms Ω^A has now become n, the number of four-dimensional fields has changed, compared to the compactification on $K3 \times T^2$. We still have one three-form and three two-forms, but we now have $n+4$ one-form fields coming from the ten-dimensional formfields. And we have a total of $2n+3$ scalars.

By the same reasoning as before, the transformation of the fields coming from $C_{10}^{(3)}$ we discussed in section 2.3.2 are still there, necessitating the same field redefinitions (2.62). But there are new field transformations related to the fact that the Ω^A are no longer closed.

Some of these transformations come from coordinate transformations in the torus direction:

$$y^i \to y^i + \xi^i(x). \tag{2.96}$$

The transformation of ten-dimensional fields under this is given in eq. (2.10). As shown in, for example, [24], for m-forms $\omega^{(m)}$, this transformation is

$$\delta_\xi \omega^{(m)} = \mathcal{L}_{-\xi}\omega^{(m)} := -\left(d \circ \iota_\xi + \iota_\xi \circ d\right)\omega^{(m)}, \tag{2.97}$$

where $\iota_\xi \omega^{(m)}$ is defined as

$$\iota_\xi \omega^{(m)} := \frac{1}{(m-1)!}\xi^\nu \omega^{(m)}_{\nu\mu_1\cdots\mu_m} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_m}. \tag{2.98}$$

For $k_{iA} = 0$, the expansion of the $B_{10}^{(2)}$-field contains a piece $b_A \Omega^A$. Under a coordinate transformation (2.96), then, this transforms as

$$\delta_\xi b_A \Omega^A = -\xi^i D_{iA}^B b_B \Omega^A. \tag{2.99}$$

This means that the action is invariant under the transformation

$$\delta_\xi b_A = -\xi^i D_{iA}^B b_B. \tag{2.100}$$

When $k_{iA} \neq 0$, this becomes

$$\delta_\xi b_A = -\xi^i \Big(k_{iA} + D_{iA}^B b_B\Big) \tag{2.101}$$

by a similar calculation. With the same reasoning we also find that c_{iA} and $C^{(1)A}$ have a similar transformation with respect to ξ:

$$\delta_\xi c_{iA} = -\xi^k \Big(k_{kA} a_i + D^B_{jA} c_{iB} \Big) \tag{2.102}$$

$$\delta_\xi C^{(1)A} = \xi^j \Big(D^A_{jB} C^{(1)B} - \eta^{AB} k_{jB} A^{(1)} \Big). \tag{2.103}$$

The transformation under Λ also yields a new transformation of the four-dimensional fields. Nothing changes for the fields coming from $A^{(1)}_{10}$ and $B^{(2)}_{10}$, but in the expansion of $\Lambda^{(2)}_{10}$,

$$\Lambda^{(2)}_{10} = \Lambda^{(2)} + \Lambda^{(1)}_i \wedge \nu^i + \frac{1}{2}\Lambda_{ij}\nu^i \wedge \nu^j + \Lambda_A \Omega^A, \tag{2.104}$$

we see Ω^A, meaning there will be an additional term to $d\Lambda^{(2)}_{10}$. This term, $\Lambda_B D^B_{iA}\sigma^i \wedge \Omega^A$, generates the transformation

$$\lambda^B D^C_{iA} \eta_{BC}. \tag{2.105}$$

With the redefinitions (2.62), c_{iA} also transforms under the gauge transformation of $A^{(1)}$, so we find

$$\delta_\lambda c_{iA} = \Lambda k_{iA} + \lambda^B D^C_{iA} \eta_{BC}. \tag{2.106}$$

Because of the redefinitions (2.62), this generates the transformation

$$\delta_\lambda C^{(1)A} = d\lambda^A + \Lambda \eta^{AB} k_{kB} G^{(1)k} - \lambda^B D^A_{kB} G^{(1)k}. \tag{2.107}$$

To obtain the action in the correct, manifestly globally symmetric form, we need to perform the same dualizations as we did in section 2.3.2. We can read off in appendix B that the transformations of the dual fields are given by:

$$\delta \tilde{B}^{(1)\iota} = d\tilde{B}^{(1)\iota} - \lambda^A \eta_{AB} \delta^{i\iota} D^B_{iC} C^{(1)C} \tag{2.108}$$

$$\delta \tilde{C}^{(1)} = d\tilde{\lambda} \tag{2.109}$$

$$\delta \gamma_i = 0 \tag{2.110}$$

$$\delta \beta_{ij} = -\frac{1}{2}\lambda^A \left(c_{iB} D^B_{jA} - c_{jB} D^B_{iA} \right) \tag{2.111}$$

2.4.3 The Reduction

We will proceed analogous to section 2.3, but in contrast to the reduction there, we will now find modified field strengths, covariant scalar derivatives and a potential. In this section, we will mainly present the results of the reduction of the action. The reader interested in the calculations behind it is referred to Appendix D, where we will also show the gauge invariance of the action.

As before, we enter the expansions of the $A^{(1)}_{10}$, $B^{(2)}_{10}$ and $C^{(3)}_{10}$-field into the supergravity action. We then perform the procedure discussed at the end of section 2.3.2, replacing the three-form field with its equation of motion and dualizing the two-forms $C^{(2)}_i$ to the scalars γ_i, the one-form $C^{(1)}_{ij}$ to the one-form $\tilde{C}^{(1)}$, the two-form $B^{(2)}$ to the scalar β_{ij} and the one-forms $B^{(1)}_i$ to the one-forms $\tilde{B}^{(1)\iota}$.

The $6+n$ field strengths in the resulting action are given by

$$\begin{aligned}
\mathcal{F}^{(2)i+} &:= dG^{(1)i}, \\
\mathcal{F}^{(2)\iota+} &:= d\tilde{B}^{(1)\iota} - \frac{1}{2}\eta_{BC}\delta^{i\iota}D^{C}_{iA}C^{(1)A}\wedge C^{(1)B} + A^{(1)}\wedge\delta^{i\iota}k_{iA}C^{(1)A}, \\
\mathcal{F}^{(2)5+} &:= dA^{(1)}, \\
\mathcal{F}^{(2)6+} &:= d\tilde{C}^{(1)} + G^{(1)k}\wedge k_{kA}C^{(1)A}, \\
\mathcal{F}^{(2)A+} &:= dC^{(1)A} - G^{(1)k}\wedge D^{A}_{kB}C^{(1)B} + G^{(1)k}\wedge k_{kB}\eta^{AB}A^{(1)}.
\end{aligned} \tag{2.112}$$

with $i,j,k=1,2$, $\iota=1,2$ and $A,B,C=1,\dots,n$.

Again in close analogy with the reduction on $K3\times T^2$, we now introduce an $SO(6,n)$-index M that runs over i, ι, 5, 6, and over A. Using this, we put all field strengths in an $SO(6,n)$-vector of field strengths, $\mathcal{F}^{(2)M+}$, defined as

$$\mathcal{F}^{(2)M+} := (\mathcal{F}^{(2)i+},\mathcal{F}^{(2)\iota+},\mathcal{F}^{(2)5+},\mathcal{F}^{(2)6+},\mathcal{F}^{(2)A+}) \tag{2.113}$$

We also define an $SO(6,n)$ matrix M_{MN} that contains $6n$ out of the $6n+2$ scalars; its definition is in Appendix F. M_{MN} spans the coset space

$$\frac{SO(6,n)}{SO(6)\times SO(n)}. \tag{2.114}$$

The two scalar fields that are not in M_{MN} make up the complex scalar field

$$\tau := -\frac{1}{4}\epsilon^{ij}b_{ij} + \frac{i}{2}e^{-\eta}, \tag{2.115}$$

with $e^{-\eta}=\sqrt{\det g_{ij}}$, which spans the coset space

$$\frac{SL(2)}{SO(2)}. \tag{2.116}$$

Using the definitions of $\mathcal{F}^{(2)M+}$, M_{MN} and τ, the action of the field strengths can then be written as

$$S_{fs} = \int\left\{\mathrm{Im}(\tau)M_{MN}\mathcal{F}^{(2)M+}\wedge *\mathcal{F}^{(2)N+} + \mathrm{Re}(\tau)L_{MN}\mathcal{F}^{(2)M+}\wedge\mathcal{F}^{(2)N+}\right\}, \tag{2.117}$$

where L_{MN} is the metric on $SO(6,n;\mathbb{R})$:

$$L_{MN} = \begin{pmatrix} 0 & \delta_{i\iota} & 0 & 0 & 0 \\ \delta_{i\iota} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_{AB} \end{pmatrix}. \tag{2.118}$$

As in section 2.3, we now perform a the Weyl rescaling $g_{\mu\nu}\to e^{2\phi+\eta}g_{\mu\nu}$. While leaving S_{fs} invariant, changes the potential and scalar term. The scalar term becomes (see Appendix D):

$$S_{sc} = \int d^4x\sqrt{-g}\left(R - \frac{1}{2\mathrm{Im}^2(\tau)}\partial_\mu\tau\partial^\mu\tau^* + \frac{1}{8}\mathcal{D}_\mu M_{MN}\mathcal{D}^\mu(LML)^{MN}\right). \tag{2.119}$$

The covariant derivatives of the scalars are given in Appendix D.

The potential can be written in terms of the scalars τ and M_{MN}, and after the Weyl rescaling it becomes:

$$
\begin{aligned}
S_{pot} =& \frac{1}{8}\int \sqrt{-g}\Bigg\{\delta^{i\iota}\delta^{j\iota'}\mathrm{Im}(\tau)^{-1}\Big\{D^B_{iA}D^A_{jB}M_{\iota\iota'} \\
&- D^C_{iE}D^D_{jF}\eta^{AE}\eta^{BF}\Big(M_{\iota\iota'}M_{AB}M_{CD} - 2M_{\iota B}M_{\iota' A}M_{CD}\Big) \\
&+ k_{iA}\eta^{BD}D^C_{jD}\Big(M_{\iota\iota'}M_{6B}M_{AC} + M_{\iota C}M_{6\iota'}M_{AB} + M_{\iota B}M_{6C}M_{A\iota'}\Big) \\
&- k_{iC}k_{jD}\eta^{AC}\eta^{BD}\Big(M_{\iota\iota'}M_{66}M_{AB} + 2M_{\iota B}M_{6\iota'}M_{6A}\Big)\Big\}\Bigg\}
\end{aligned}
\tag{2.120}
$$

Our results can be summarized in the action

$$
S_{sc} + S_{fs} + S_{pot}. \tag{2.121}
$$

For $D^B_{iA} = 0$, $k_{iA} = 0$ and $n = 22$, this reduces to the action of IIA supergravity compactified on $K3 \times T^2$, with field strengths and covariant derivatives given by ordinary exterior derivatives, and without a potential, that can be found in section 2.3.3. At the end of the next section, after discussing the general framework of four-dimensional $\mathcal{N} = 4$ supergravities, we will show the theory we obtained in this section is a gauged $\mathcal{N} = 4$ supergravity when $D^B_{iA} \neq 0$ or $k_{iA} \neq 0$. In Appendix D, we show the gauge invariance of the action (2.121).

2.5 Four-Dimensional Gauged $\mathcal{N} = 4$ Supergravities

Here, we will give an overview of four-dimensional gauged $\mathcal{N} = 4$ supergravities, as described in [19]. The authors of this paper use the formalism of the embedding tensor as developed in [25]-[30]. The embedding tensor embeds the gauge group of a theory in the global symmetry of that theory. This formalism is developed to allow us to write down all possible gaugings of supergravities. In [19] it is used to write down the most general gauged four-dimensional $\mathcal{N} = 4$ supergravities.

We will go on to show that the four-dimensional theory obtained in section 2.4 is in fact an example of such a supergravity. This was already expected from the theory in section 2.2.

2.5.1 Formulation

We will give a short summary of [19], in which the authors describe the general form of four-dimensional $\mathcal{N} = 4$ gauged supergravity. In section 2.5.2 we will show this includes the results in sections 2.3.3 and 2.4.3. Since any four-dimensional $\mathcal{N} = 4$ supergravity can be uniquely characterized by its field content and its gauging, we will start by describing the fields, then the possible gaugings, and finally write down the action.

To write down the most general gauge-invariant action, not only need the fields that carry degrees of freedom are needed, but also their duals. The bosonic fields that carry degrees of freedom are the graviton, the electric vectors and the scalars. The duals of the electric vectors are the magnetic vectors, and the two-form are the duals of the scalars. These dual fields have gauge transformations as well, magnetic vectors are used to give the correct transformation to covariant derivatives, while the two-forms are used for the field strengths. These magnetic vectors and two-forms do not have kinetic terms in the action and the equations of motion show that they are dual to the electric vectors and the scalars respectively. For every gauging, it is possible to select a symplectic frame, i.e. to specify which vectors are called electric and magnetic, such that the action for this frame does not contain magnetic vectors or two-forms.

An $\mathcal{N} = 4$, four-dimensional supergravity theory consists of one gravity multiplet coupled to an arbitrary number, n, of vector multiplets. The theory has a global on-shell symmetry

$$SL(2,\mathbb{R}) \times SO(6,n;\mathbb{R}), \tag{2.122}$$

and the scalars in the theory fall into two coset spaces

$$\frac{SL(2,\mathbb{R})}{SO(2,\mathbb{R})} \times \frac{SO(6,n;\mathbb{R})}{SO(6,\mathbb{R}) \times SO(n,\mathbb{R})}. \tag{2.123}$$

The coset space $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ is parametrized by the complex scalar τ, or equivalently by an $SL(2,\mathbb{R})$-matrix $M_{\alpha\beta}$ ($\alpha, \beta = +, -$). The definition of $M_{\alpha\beta}$ and its inverse $M^{\alpha\beta}$ in terms of τ is

$$\begin{aligned} M_{\alpha\beta} &:= \frac{1}{\mathrm{Im}(\tau)} \begin{pmatrix} |\tau|^2 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & 1 \end{pmatrix} \\ M^{\alpha\beta} &:= \frac{1}{\mathrm{Im}(\tau)} \begin{pmatrix} 1 & -\mathrm{Re}(\tau) \\ -\mathrm{Re}(\tau) & |\tau|^2 \end{pmatrix}. \end{aligned} \tag{2.124}$$

The coset space

$$\frac{SO(6,n;\mathbb{R})}{SO(6,\mathbb{R}) \times SO(n,\mathbb{R})} \tag{2.125}$$

is parametrized by the matrix $M_{MN} \in SO(6,22;\mathbb{R})$ ($M, N = 1, \ldots, n+6$). It contains $6n$ scalars.

The theory becomes gauged if a part of the global symmetry becomes local. How the resulting gauge group is embedded in the symmetry group is encoded in the embedding tensor ${\Theta_{M\alpha}}^a$. Here $\alpha = +, -$, $M = 1, \ldots, 6+n$ and a is a general index for all the fields. In terms of this embedding tensor, a covariant derivative can be written as

$$D_\mu = \partial_\mu - A_\mu^{M\alpha} {\Theta_{M\alpha}}^a t_a, \tag{2.126}$$

where t_a is the generator of the global symmetry (2.122).

The embedding tensor ${\Theta_{M\alpha}}^a$ can be described by two tensors, $f_{\alpha MNP} = f_{\alpha[MNP]}$ and $\xi_{\alpha M}$. To do this, note that the generators of the global symmetry group are split into $t_{\alpha\beta} = t_{(\alpha\beta)}$ for $SL(2,\mathbb{R})$ and $t_{MN} = t_{[MN]}$ for $SO(6,n;\mathbb{R})$. Therefore, ${\Theta_{M\alpha}}^a$ can be split into ${\Theta_{M\alpha}}^{\beta\gamma}$ and ${\Theta_{M\alpha}}^{NP}$, and as explained in [19], these can be further decomposed into irreducible representations $\xi_{\alpha M}$ and $f_{\alpha[MNP]}$:

$${\Theta_{M\alpha}}^{\beta\gamma} = \xi_{\delta M} \epsilon^{\delta(\beta} \delta_\alpha^{\gamma)}, \quad {\Theta_{M\alpha}}^{NP} = {f_{\alpha M}}^{NP} + \delta_M^{[N} \xi_\alpha^{P]}. \tag{2.127}$$

The $SO(6,n;\mathbb{R})$-metric L_{MN} is used to raise and lower indices.

The most general formulation of a gauged $\mathcal{N}=4$ four-dimensional supergravity, as found in [19], has ξ_{+M}, ξ_{-M}, f_{+MNP} and f_{-MNP} all nonzero. However, the compactifications in this paper will both have $\xi_{-M}=f_{-MNP}=0$, and we will reduce the formulas of [19] accordingly in order to increase readability. The only place where we will use the index $\alpha=+,-$ is in eq. (2.130), since it is an intrinsic part of that equation.

The theory described only exhibits $\mathcal{N}=4$ supersymmetry if the embedding tensor obeys certain constraints. These constraints can be written out in terms of the tensors $\xi_{\alpha M}$ and $f_{\alpha MNP}$. For $\xi_{-M}=f_{-MNP}=0$, the only constraint that is not trivially satisfied is

$$3f_{+R[MN}f_{+PQ]}{}^{R}+2\xi_{+[M}f_{+NPQ]}=0. \tag{2.128}$$

Together, the spectrum and the embedding tensor fully determine the action. It has a kinetic part, a potential part, and a further topological part. The kinetic part is

$$S_{kin}=\int\Bigg\{d^4x\sqrt{-g}\left(R-\frac{1}{2\mathrm{Im}^2(\tau)}\mathcal{D}_\mu\tau\mathcal{D}^\mu\tau^*+\frac{1}{8}\mathcal{D}_\mu M_{MN}\mathcal{D}^\mu(LML)^{MN}\right)$$
$$+\mathrm{Im}(\tau)M_{MN}\mathcal{F}^{(2)M+}\wedge*\mathcal{F}^{(2)N+}+\mathrm{Re}(\tau)L_{MN}\mathcal{F}^{(2)M+}\wedge\mathcal{F}^{(2)N+}\Bigg\}. \tag{2.129}$$

Incidentally, this kinetic part will look the same for all possible gaugings. Of course, the covariant derivatives and field strengths will not, as can be seen from their definitions:

$$\mathcal{D}M_{\alpha\beta}=dM_{\alpha\beta}+A^{M\gamma}\xi_{(\alpha M}M_{\beta)\gamma}-A^{M\delta}\xi_{\epsilon M}\epsilon_{\delta(\alpha}\epsilon^{\epsilon\gamma}M_{\beta)\gamma} \tag{2.130}$$
$$\mathcal{D}M_{MN}=dM_{MN}+2A^{P\alpha}\Theta_{P\alpha(M}{}^{Q}M_{N)Q}\,, \tag{2.131}$$
$$\mathcal{F}^{(2)M+}=dA^{M+}-\frac{1}{2}\hat{f}_{+NP}{}^{M}A^{N+}\wedge A^{P+}+\frac{1}{2}\xi_{+}{}^{M}B^{(2)++}, \tag{2.132}$$

where $\hat{f}_{+MNP}$ is defined as

$$\hat{f}_{+MNP}:=f_{+MNP}-\xi_{+[M}L_{P]N}-\frac{3}{2}\xi_{+N}L_{MP}. \tag{2.133}$$

The magnetic field strengths do not enter in the action, but can still be written down:

$$\mathcal{F}^{(2)M-}=dA^{M-}-\frac{1}{2}\hat{f}_{+NP}{}^{M}A^{N+}\wedge A^{P-}-\frac{1}{2}\Theta_{+}{}^{M}{}_{NP}B^{(2)NP} \tag{2.134}$$
$$+\frac{1}{2}\xi_{+}{}^{M}B^{(2)+-}.$$

We will encounter these in our compactifications. The potential term is

$$S_{pot}=-\frac{1}{8}\int\Bigg\{f_{+}{}^{MNP}f_{+}{}^{QRS}M^{++}\Big[\frac{1}{3}M_{MQ}M_{NR}M_{PS}$$
$$+\left(\frac{2}{3}L_{MQ}-M_{MQ}\right)L_{NR}L_{PS}\Big]$$
$$+3\xi_{+}{}^{M}\xi_{+}{}^{N}M^{++}M_{MN}\Bigg\}. \tag{2.135}$$

Finally, the remaining topological term is

$$S_{top} = -\int \Big\{ \xi_{+M} L_{NP} A^{M-} \wedge A^{N+} \wedge dA^{P+}$$
$$- \frac{1}{4} \hat{f}_{+MNR} \hat{f}_{+PQ}{}^{R} A^{M+} \wedge A^{N+} \wedge A^{P+} \wedge A^{Q-}$$
$$- \xi_{+M} B^{++} \wedge \Big(dA^{M-} - \frac{1}{2} \hat{f}_{+QR}{}^{M} A^{Q+} \wedge A^{R-} \Big) \Big\}. \tag{2.136}$$

2.5.2 Comparison

In this section we will compare the compactification of IIA on $K3 \times T^2$ (see section 2.3) and on Y_1 with $H_{10}^{(3)}$-flux (see section 2.4) with the above description of four-dimensional $\mathcal{N} = 4$ supergravity to show that both compactifications result in $\mathcal{N} = 4$ supergravities. We will compare the bosonic spectrum, the form of the action and the gauging of the action, and show how the constraints on the embedding tensor are obeyed.

Compactifying IIA on a $K3 \times T^2$ gives an ungauged theory, meaning we only have to compare the bosonic spectrum and the form of the action. The bosonic fields coming from the reduction are a graviton, 28 vectors

$$A^{M+} := (G^{(1)i}, \tilde{B}^{(1)\iota}, A^{(1)}, \tilde{C}^{(1)}, C^{(1)A}), \tag{2.137}$$

a complex scalar τ and a matrix of scalars M_{MN} that contains 132 scalars. Together, the scalars span the produc of coset spaces

$$\frac{SL(2,\mathbb{R})}{SO(2,\mathbb{R})} \times \frac{SO(6,22;\mathbb{R})}{SO(6,\mathbb{R}) \times SO(22,\mathbb{R})}. \tag{2.138}$$

The action after this dimensional reduction is given in eq. (2.74). Since the spectrum and action agree with those of an ungauged $\mathcal{N} = 4$ supergravity with 22 vector multiplets, we conclude that this compactification gives an ungauged $\mathcal{N} = 4$ supergravity with 22 vector multiplets.

For the $SU(2)$-structure manifold Y_1 with $H_{10}^{(3)}$-flux, the bosonic spectra coincide as well, but this time for a general n vector multiplets. In this case, the spectrum contains a graviton, the scalars in the product of coset spaces

$$\frac{SL(2,\mathbb{R})}{SO(2,\mathbb{R})} \times \frac{SO(6,n;\mathbb{R})}{SO(6,\mathbb{R}) \times SO(n,\mathbb{R})}, \tag{2.139}$$

and the $6 + n$ vectors

$$A^{M+} := (G^{(1)i}, \tilde{B}^{(1)\iota}, A^{(1)}, \tilde{C}^{(1)}, C^{(1)A}). \tag{2.140}$$

The gauging of IIA on Y_1 with $H_{10}^{(3)}$-flux agrees with $\mathcal{N} = 4$ supergravity for an embedding tensor that can be determined from the field strengths in eq. (2.112). Since there are no two-forms in the field strengths, we conclude that

$$\xi_{+M} = 0, \tag{2.141}$$

meaning that

$$\hat{f}_{+MNP} = f_{+MNP}. \tag{2.142}$$

The field strengths then agree with the $\mathcal{N} = 4$ field strengths in eq. (2.132) for the choice

$$f_{+iAB} = -\eta_{AC} D^C_{iB}, \quad f_{+i5A} = -k_{iA}, \tag{2.143}$$

all components with different indices being zero. The scalar derivatives from $\mathcal{N} = 4$ supergravity, given in eqs. (2.130) and (2.131), agree with those of IIA on Y_1 with $H^{(3)}_{10}$-flux given in eq. (F.44) for this embedding tensor.

The embedding tensor defined by (2.143) is antisymmetric and satisfies the consistency constraint (2.128). Antisymmetry holds because, according to eq. (2.81),

$$\eta_{AC} D^C_{iB} = -\eta_{BC} D^C_{iA}. \tag{2.144}$$

Since $\xi_{+M} = 0$, the consistency constraint (2.128) reduces to

$$f_{+R[MN} f_{+PQ]}{}^R = 0. \tag{2.145}$$

The summation can only be over $R = C$. The constraint takes two different forms, one when one of (M, N, P, Q) is 5, and one when none of (M, N, P, Q) is 5. The first form reads as follows:

$$f_{+C5i} f_{+jB}{}^C - f_{+C5j} f_{+iB}{}^C = 0, \tag{2.146}$$

or

$$k_{iC} D^C_{jB} - k_{jC} D^C_{iB} = 0. \tag{2.147}$$

This is true because of eq. (2.90). The second form is

$$f_{+CiA} f_{+jB}{}^C - f_{+CjA} f_{+iB}{}^C = 0, \tag{2.148}$$

or

$$\eta_{CD} D^D_{iA} D^C_{jB} - \eta_{CD} D^D_{jA} D^C_{iB} = 0. \tag{2.149}$$

We can show this holds using eq. (2.90).

Finally, we will show that the action of IIA on Y_1 with $H^{(3)}_{10}$-flux is the same as the $\mathcal{N} = 4$ supergravity action. We ahve already seen that the field strengths and covariant derivatives agree. Between the action of IIA on Y_1 with $H^{(3)}_{10}$-flux (2.121) and the action of $\mathcal{N} = 4$ supergravity, then, we only need to compare the topological term and the potential. We see that there is no topological term in the Y_1-action, apart from the one in S_{fs}. And in the $\mathcal{N} = 4$ action (2.136), the only remaining topological term that can be nonzero, has prefactor

$$\hat{f}_{+R[MN} \hat{f}_{+P]Q}{}^R. \tag{2.150}$$

There is antisymmetrization over M, N and P but not over Q, since M, N and P are indices for electric vectors, but Q for a magnetic vector. The sum can only be over $R = C$, and choosing $Q = i$ or $Q = A$ gives two different topological terms. However, using (2.81) and (2.90), we see that all prefactors are zero. To compare the potential, we note that for an embedding tensor given by eq. (2.143), only the first and third term of the $\mathcal{N} = 4$

potential (2.135) are non-zero. Writing out the indices shows us that this equals the potential (2.120) from the compactification. We conclude that IIA supergravity on Y_1 with H-flux gives a four-dimensional, $\mathcal{N}=4$ gauged supergravity with

$$f_{+iAB} = -\eta_{AC} D^C_{iB}, \quad f_{+i5A} = -k_{iA}. \tag{2.151}$$

as only nonzero components of the embedding tensor.

2.6 IIA on Y_2

In this section, we will compactify IIA supergravity on Y_2, a more complex manifold with $SU(2)$-structure. After some calculations, it can again be cast in the $SO(6,n)$-symmetric form and we conclude from section 2.5 that this is, as expected, again an $\mathcal{N}=4$ gauged supergravity.

2.6.1 The Internal Manifold

On a generic manifold with $SU(2)$-structure inspired by $K3\times T^2$, the one- and two-forms are not necessarily closed. Instead we can have

$$\begin{aligned} d\sigma^i &= \frac{1}{2} D^i_{jk}\sigma^j \wedge \sigma^k + D^i_A \Omega^A \ , \\ d\Omega^A &= D^A_{iB}\sigma^i \wedge \Omega^B. \end{aligned} \tag{2.152}$$

Here, $i=1,2$ and $A=1,\ldots,n$ as before. On Y_2, we will restrict ourselves to $D^i_A = 0$.

We now proceed analogous to section 2.4.1, and define the intersection metric η^{AB} as

$$\eta^{AB}\epsilon^{ij} := \int_{Y_2} \sigma^i \wedge \sigma^j \wedge \Omega^A \wedge \Omega^B. \tag{2.153}$$

Furthermore, requiring $d^2=0$ now implies

$$D^C_{iB} D^B_{jA} - D^C_{jB} D^B_{iA} = D^k_{ij} D^C_{kA} \ , \tag{2.154}$$

writing out the indices i, j explicitly gives

$$D^k_{il} D^l_{jm} - D^k_{jl} D^l_{im} = D^k_{ij} D^l_{lm}, \tag{2.155}$$

while Stokes' theorem yields the constraint

$$-\eta^{AB} D^k_{ik} = \eta^{AC} D^B_{iC} + \eta^{BC} D^A_{iC}. \tag{2.156}$$

The scalar fields are defined in the same way as before:

$$e^{-2\rho} := \det g_{ab} \tag{2.157}$$

$$e^{-2\eta} := \det g_{ij} \tag{2.158}$$

$$*\Omega^B =: e^{-\eta} H^B{}_C \sigma^1 \wedge \sigma^2 \wedge \Omega^C. \tag{2.159}$$

2.6.2 The Spectrum

This time the internal manifold not only has non-closed two-forms we expand in, also the one-forms we expand in are nonclosed. The expansion of the ten-dimensional fields, however, looks the same as in eqs. (2.49), (2.50) and (2.58)-(2.60). The only difference is that we again set $A, B = 1, \dots, n$ for n any integer. We make the same redefinitions for the fields coming from $C_{10}^{(3)}$ as before, given in eq. (2.62).

But the gauge behavior of the fields has changed. New transformations come from coordinate transformations in the torus direction and the ten-dimensional gauge transformations. Recalling that the transformation of a form under a torus coordinate transformation

$$y^i \to y^i + \xi^i \tag{2.160}$$

is determined by the Lie derivative $\mathcal{L}_{-\xi}$ acting on that form, we calculate

$$\mathcal{L}_{-\xi}\sigma^i = -\iota_\xi \circ d\sigma^i - d \circ \iota_\xi \sigma^i = -\iota_\xi \frac{1}{2} D^i_{jk}\sigma^j \wedge \sigma^k = -\xi^j D^i_{jk}\sigma^k. \tag{2.161}$$

This means that a coordinate transformation acting on the metric can be rewritten as an active transformation for g_{ij} as

$$\delta_\xi g_{ij} = -\xi^k \left(D^l_{ki} g_{lj} + D^l_{kj} g_{il} \right), \tag{2.162}$$

together with an active transformation for $G^{(1)i}$ as

$$\delta_\xi G^{(1)i} = d\xi^i + \xi^k D^i_{kl} G^{(1)l}. \tag{2.163}$$

Fields in the expansions of the $A_{10}^{(1)}$, $B_{10}^{(2)}$ and $C_{10}^{(3)}$-fields with lower indices i transform in the same way, for example

$$\delta_\xi a_i = -\xi^k D^l_{ki} a_l. \tag{2.164}$$

The transformation of all fields under ξ^i is given in appendix F.

Similar to the Y_1-case, the Λ-transformations of the form fields also generate new transformations for the four-dimensional fields. The reason for this is that the ten-dimensional Λ-parameters have σ^i's in their expansion, so transformations $d\Lambda_{10}^{(1)}$ and $d\Lambda_{10}^{(2)}$ receive extra contributions. For example, $\delta_\Lambda B_{10}^{(2)} = d\Lambda_{10}^{(1)}$, and

$$\Lambda_{10}^{(1)} = \Lambda^{(1)} + \Lambda_i \nu^i. \tag{2.165}$$

Written in terms of the expanded fields, this means that

$$\delta_\Lambda b_{ij} = \Lambda_k D^k_{ij}, \tag{2.166}$$

while

$$\delta_\Lambda B_i^{(1)} = d\Lambda_i + \Lambda_k G^{(1)l} D^k_{il}. \tag{2.167}$$

Again, the transformation of all fields under the Λ-parameters is given in appendix F.

The transformations of the dual fields is a different matter. From appendix B, it follows that their transformations are given by:

$$\delta\tilde{B}^{(1)\iota} = d\tilde{\lambda}^{\iota} + \xi^k D^l_{kl}\tilde{B}^{(1)\iota} - \lambda^A\eta_{AB}\delta^{i\iota}D^B_{iC}C^{(1)C} + \lambda\delta^{i\iota}D^l_{il}\tilde{C}^{(1)} \tag{2.168}$$

$$\delta\tilde{C}^{(1)} = d\tilde{\lambda} + \xi^k D^l_{kl}\tilde{C}^{(1)} - \tilde{\lambda}D^l_{kl}G^{(1)k} \tag{2.169}$$

$$\delta\gamma_i = -\xi^k(D^l_{ki}\gamma_l - D^l_{kl}\gamma_i) - \tilde{\lambda}D^k_{ik} \tag{2.170}$$

$$\delta\beta_{ij} = -\xi^k(D^l_{ki}\beta_{lj} + D^l_{kj}\beta_{il} - D^l_{kl}\beta_{ij}) - \frac{1}{2}\lambda^A\left(c_{iB}D^B_{jA} - c_{jB}D^B_{iA}\right) \tag{2.171}$$

2.6.3 The Reduction

This will proceed analogously to section 2.3; however, we will now find modified field strengths, covariant scalar derivatives, a potential and a topological term. The gauge invariance of the action is shown and the action is a four-dimensional gauged $\mathcal{N} = 4$ supergravity. Again, we will focus on the results and important steps here, a fuller account of the calculations can be found in Appendix E.

Just as in section 2.4.3, we start by inserting the expansions of the $A^{(1)}_{10}$, $B^{(2)}_{10}$ and $C^{(3)}_{10}$-fields into the action (2.15), and make the field redefinitions as given in eq. (2.62). We replace the three-form field $C^{(3)}$ with its equations of motion and dualize the fields $C^{(2)}_i$ and $C^{(1)}_{ij}$. For a nonzero D^k_{ij}, a linear combination of $C^{(2)}_i$'s becomes massive by eating $C^{(1)}_{ij}$:

$$\mathcal{F}^{(2)-}_{ij} := dC^{(1)}_{ij} + G^{(1)k}\wedge\left(D^l_{ki}C^{(1)}_{lj} + D^l_{kj}C^{(1)}_{il}\right) + D^k_{ij}C^{(2)}_k. \tag{2.172}$$

Although that makes the procedure of dualizing $C^{(2)}_i$ and $C^{(1)}_{ij}$ different than in section 2.4.3, the outcome is comparable: two scalars γ_i and a vector $\tilde{C}^{(1)}$ (for the detailed calculation, see Appendix B). Finally, we dualize $B^{(2)}$ to β_{ij}. In section 2.4.3, we finished by dualizing $B^{(1)}_i$ to $\tilde{B}^{(1)\iota}$, but this is not possible here. A linear combination of $B^{(1)}_i$'s becomes massive by eating b_{ij}, as we can see from its covariant derivative

$$\mathcal{D}b_{ij} := db_{ij} + G^{(1)k}D^l_{kl}b_{ij} - D^k_{ij}B^{(1)}_k. \tag{2.173}$$

Dualizing $B^{(1)}_i$ would therefore replace b_{ij} by a dual two-form as well in the same way as dualizing $C^{(2)}_i$ automatically dualizes $C^{(1)}_{ij}$. Since we want to write the action in terms of scalars and vectors, we do not dualize $B^{(1)}_i$. For the degrees of freedom, this does not matter: we still have $6n+2$ scalars, $6+n$ vectors and one graviton in the bosonic spectrum, to construct one $\mathcal{N} = 4$ gravity multiplet and n $\mathcal{N} = 4$ vector multiplets.

However, we do want to show that the action we obtain with this compactification, S_{Y_2}, is a gauged $\mathcal{N} = 4$ supergravity, by comparing it to the results of [19]. We will do this in a different way than in section 2.4: we are going to determine the embedding tensor from the information we have from the compactification. We can then enter the embedding tensor into the formulation of [19] to see what the action should look like in the electric frame. We will call this action S_{el} and we want to show that it is the same as S_{Y_2}. In the electrical frame action, we will find a two-form Lagrange multiplier that is obviously not present in the spectrum of IIA compactified on Y_2. Integrating out this two-form, we will recover S_{Y_2}, thus showing that IIA supergravity compactified on Y_2 gives a gauged $\mathcal{N} = 4$ supergravity. Note that since the two-form is a Lagrange multiplier, it does not change the degrees of freedom; both S_{Y_2} and S_{el} describe $6n + 2$ scalars, n vectors and one graviton. As we saw in section 2.5, that is the spectrum of $\mathcal{N} = 4$ supergravity.

We can find the embedding tensor by looking at the covariant derivatives and field strengths in S_{Y_2}. By comparing the field strengths

$$\mathcal{F}^{(2)i+} := dG^{(1)i} - \frac{1}{2}G^{(1)k} \wedge G^{(1)l} D^i_{kl}, \tag{2.174}$$

$$\mathcal{F}^{(2)5+} := dA^{(1)}, \tag{2.175}$$

$$\mathcal{F}^{(2)6+} := d\tilde{C}^{(1)} - G^{(1)k} \wedge D^l_{kl}\tilde{C}^{(1)}, \tag{2.176}$$

$$\mathcal{F}^{(2)A+} := dC^{(1)A} - G^{(1)k} \wedge \Big(D^A_{kB}C^{(1)B} + D^l_{kl}C^{(1)A}\Big), \tag{2.177}$$

to eq. (2.132) we see that $\xi_{+M} = 0$ for $M \neq i$. We can also read off most of the $\hat{f}_{+MNP}$-matrices. To determine f from them, we need to know ξ_{+i}.

To find ξ_{+i} we look at the kinetic term for $\tau := -\frac{1}{4}\epsilon^{ij}b_{ij} + \frac{i}{2}e^{-\eta}$. We perform a Weyl rescaling $g_{\mu\nu} \to e^{2\phi+\eta}g_{\mu\nu}$ to find it is given by

$$-\frac{1}{2\mathrm{Im}^2(\tau)}\mathcal{D}_\mu\tau\mathcal{D}^\mu\tau^* = \frac{1}{4}\mathcal{D}_\mu M_{\alpha\beta}\mathcal{D}^\mu M^{\alpha\beta}, \tag{2.178}$$

where, according to eq. (2.124), $M_{\alpha\beta}$ is defined as

$$M_{\alpha\beta} := 2e^\eta \begin{pmatrix} \frac{1}{16}\left[(\epsilon^{ij}b_{ij})^2 + 4e^{-2\eta}\right] & -\frac{1}{4}\epsilon^{ij}b_{ij} \\ -\frac{1}{4}\epsilon^{ij}b_{ij} & 1 \end{pmatrix}. \tag{2.179}$$

From the covariant derivative of b_{ij} in eq. (2.173) and that of $e^{-\eta}$,

$$\mathcal{D}e^{-\eta} := de^{-\eta} - G^{(1)k}D^l_{kl}e^{-\eta}, \tag{2.180}$$

we conclude that

$$\mathcal{D}M_{--} := dM_{--} - G^{(1)k}D^l_{kl}M_{--}, \tag{2.181}$$

$$\mathcal{D}M_{+-} := dM_{+-} + \frac{1}{4}\epsilon^{ij}B^{(1)}_k D^k_{ij}M_{--}, \tag{2.182}$$

$$\mathcal{D}M_{++} := dM_{++} + \frac{1}{2}\epsilon^{ij}B^{(1)}_k D^k_{ij}M_{+-} + G^{(1)k}D^l_{kl}M_{++}. \tag{2.183}$$

Comparing this to eq. (2.130) tells us that $\xi_{+i} = D^k_{ik}$.

We now use the definition of $\hat{f}_{+MNP}$ in eq. (2.133) to read off the components of the embedding tensor in terms of the matrices D^B_{iA} and D^k_{ij}:

$$
\begin{aligned}
\xi_{+i} &= D^k_{ik}, \\
f_{+ij\iota} &= \frac{1}{2} D^k_{ij}\delta_{k\iota}, \\
f_{+i56} &= \frac{1}{2} D^k_{ik}, \\
f_{+iAB} &= -\eta_{AC} D^C_{iB} - \frac{1}{2}\eta_{AB} D^k_{ik}.
\end{aligned}
\tag{2.184}
$$

Some details on this can be found in Appendix E. There, we also show that $f_{+iAB} = -f_{+iBA}$ and that the embedding tensor satisfies the consistency constraint (2.128).

The scalar part of the action S_{Y_2} can be used immediately to check this embedding tensor. The compactification gives us the following kinetic term for the scalars (see Appendix E for details):

$$
S_{sc} = \int d^4x\sqrt{-g}\left[R - \frac{1}{2\mathrm{Im}^2(\tau)}\partial_\mu\tau\partial^\mu\tau^* + \frac{1}{8}\mathcal{D}_\mu M_{MN}\mathcal{D}^\mu(LML)^{MN}\right]. \tag{2.185}
$$

The covariant derivatives of the scalars M_{MN} are in agreement with [19] for the embedding tensor (2.184). We also obtain a potential

$$
\begin{aligned}
S_{pot} = -\int d^4x\sqrt{-g}\Bigg\{ & \frac{e^{2\phi+\eta+\rho}}{2}\eta^{AC}H^B{}_C g^{ij} b_D b_E D^D_{iA} D^E_{jB} \\
& + \frac{e^{4\phi+\eta-\rho}}{4} g^{ik} g^{jl} a_m a_n D^m_{ij} D^n_{kl} \\
& + 2e^{4\phi+3\eta+\rho}\eta^{AB}\eta^{CD} b_A b_C \epsilon^{ij}\epsilon^{kl}\left(D^E_{iA}(c_{jE} - a_j b_E) + \frac{1}{2}D^m_{ij}(c_{mA} - a_m b_A)\right)\times \\
& \left(D^F_{kB}(c_{lF} - a_l b_F) + \frac{1}{2}D^n_{kl}(c_{nB} - a_n b_B)\right) \\
& + \frac{e^{4\phi+\eta}}{4}\eta^{AC}H^B{}_C g^{ik} g^{jl}\left(D^D_{iA}c_{jD} - D^D_{jA}c_{iD} + D^m_{ij}(c_{mA} - a_m b_A)\right)\times \\
& \left(D^E_{kB}c_{lE} - D^E_{lB}c_{kE} + D^n_{kl}(c_{nB} - a_n b_B)\right)\Bigg\}.
\end{aligned}
\tag{2.186}
$$

We can rewrite this potential in terms of scalars in M_{MN}, and using the definitions in eq. (2.184), rewrite it as

$$
\begin{aligned}
S_{pot} = -\frac{1}{8}\int \Big\{ & f_+{}^{MNP} f_+{}^{QRS} M^{++}\Big(\frac{1}{3}M_{MQ}M_{NR}M_{PS} - M_{MQ}L_{NR}L_{PS}\Big) \\
& + 3\xi_+{}^M \xi_+{}^N M^{++} M_{MN}.\Big\}
\end{aligned}
\tag{2.187}
$$

With embedding tensor (2.184), these are the only terms in the potential (2.135). We conclude that the scalar section agrees with that of an $\mathcal{N} = 4$ gauged supergravity.

As for the field strengths, the action of all of them except for those of $B_i^{(1)}$ agrees with S_{el} as well and is given by (we will use the index $\hat{M}$ to mean that the index $M = \iota$ is skipped):

$$S_{\mathcal{F}^{(2)\hat{M}+}} = \int \Big\{ \mathrm{Im}(\tau) \left(M_{\hat{M}\hat{N}} - e^{-2\phi} g_{kl} M^k{}_{\hat{M}} M^k{}_{\hat{N}} \right) \mathcal{F}^{(2)\hat{M}+} \wedge *\mathcal{F}^{(2)\hat{N}+} + \mathrm{Re}(\tau) L_{\hat{M}\hat{N}} \mathcal{F}^{(2)\hat{M}+} \wedge \mathcal{F}^{(2)\hat{N}+} \Big\}. \tag{2.188}$$

In the action resulting from compactifying IIA on Y_2 we furthermore find the action for $B_i^{(1)}$, the one-form coming from $B_{10}^{(2)}$. This part can not be found in S_{el}, and it is given by

$$\begin{aligned} S_{B_i^{(1)}} = \int \Big\{ & \frac{e^{-2\phi-\eta}}{2} g^{ij} \left(\mathcal{F}_i^{(2)-} - \mathcal{F}^{(2)k+} b_{ik} \right) \wedge * \left(\mathcal{F}_j^{(2)-} - \mathcal{F}^{(2)l-} b_{jl} \right) \\ & + \epsilon^{ij} \left(\mathcal{F}_i^{(2)-} - \mathcal{F}^{(2)k-} b_{ik} \right) \wedge \Big(\mathcal{F}^{(2)A+} c_{jA} - a_j \mathcal{F}^{(2)6+} - \gamma_j \mathcal{F}^{(2)5+} \\ & + \mathcal{F}^{(2)k+} (\beta_{jk} - a_j \gamma_k + \frac{1}{2} \eta^{AB} c_{jA} c_{kB}) \Big) \\ & + \frac{1}{2} \epsilon^{ij} \Big(\mathcal{F}_i^{(2)-} \wedge \eta_{AC} D_{iB}^C C^{(1)A} \wedge C^{(1)B} - 2 B_i^{(1)} \wedge \mathcal{F}^{(2)5+} \wedge \tilde{C}^{(1)} D_{jk}^k \\ & + B_i^{(1)} \wedge \eta_{AB} \mathcal{F}^{(2)A+} \wedge C^{(1)B} D_{jk}^k \Big) \Big\}, \end{aligned} \tag{2.189}$$

with the field strength $\mathcal{F}_i^{(2)-}$ defined as

$$\mathcal{F}_i^{(2)-} := dB_i^{(1)} - G^{(1)k} \wedge D_{ki}^l B_l^{(1)}. \tag{2.190}$$

The electric frame dual of $B_i^{(1)}$ is given by $\tilde{B}^{(1)\iota}$. Using the formulation of [19], we are now going to determine the action for $\tilde{B}^{(1)\iota}$ in S_{el}, and then show that this is dual to $S_{B_i^{(1)}}$. According to eq. (2.132) and (2.184), the field strength $\mathcal{F}^{(2)\iota+}$ of $\tilde{B}^{(1)i}$ is given by:

$$\begin{aligned} \mathcal{F}^{(2)\iota+} := d\tilde{B}^{(1)\iota} + \frac{1}{2} \delta^{i\iota} \Big(& D_{ik}^k G^{(1)m} \wedge \tilde{B}^{(1)\iota'} \delta_{m\iota'} + D_{ik}^k \tilde{C}^{(1)} \wedge A^{(1)} \\ & + \eta_{AC} D_{iB}^C C^{(1)A} \wedge C^{(1)B} \Big) + \frac{\delta_i^\iota \epsilon^{ij}}{2} D_{jk}^k B^{(2)++}. \end{aligned} \tag{2.191}$$

We notice the occurrence of a two-form Lagrange multiplier $B^{(2)++}$. Using $\mathcal{F}^{(2)\iota+}$, the action for $\tilde{B}^{(1)\iota}$ in S_{el} is:

$$S_{\tilde{B}^{(1)i},\mathcal{N}=4} = \int \Bigg\{ \frac{e^{-\eta} M_{\iota\iota'}}{2} \left(\mathcal{F}^{(2)\iota +} + e^{-2\phi}\delta^{i\iota} g_{ik}\delta^{k\iota''} M_{\iota''\hat{M}}\mathcal{F}^{(2)\hat{M}+}\right) \wedge \\ \wedge * \left(\mathcal{F}^{(2)\iota' +} + e^{-2\phi}\delta^{j\iota'} g_{jl}\delta^{l\iota'''} M_{\iota'''\hat{N}}\mathcal{F}^{(2)\hat{N}+}\right) \\ + \frac{\epsilon^{ij}}{2} D^m_{im} B^{(2)++} \wedge \mathcal{F}^{(2)-}_j - 2\mathrm{Re}(\tau)\delta_{i\iota}\mathcal{F}^{(2)\iota+} \wedge \mathcal{D}G^{(1)i} + \mathcal{L}_{top} \Bigg\}, \tag{2.192}$$

where we have used the shorthand

$$\mathcal{L}_{top} = \epsilon^{ij} B^{(1)}_i \wedge \left(\frac{1}{2}\eta_{AB}\mathcal{F}^{(2)A+} \wedge C^{(1)B} D^k_{jk} - \mathcal{F}^{(2)5+} \wedge \tilde{C}^{(1)} D^k_{jk}\right) \\ + \epsilon^{ij}\left(\delta_{i\iota} d\tilde{B}^{(1)\iota} + \frac{1}{2} D^l_{il}\delta_{k\iota} G^{(1)k} \wedge \tilde{B}^{(1)\iota} + \frac{1}{2} D^k_{ik}\tilde{C}^{(1)} \wedge A^{(1)}\right) \wedge \mathcal{F}^{(2)-}_j \tag{2.193}$$

From this we see that $B^{(1)}_i$ is indeed present as a magnetic vector in the formulation of [19]. We can now integrate out the two-form field $B^{(2)++}$ in eq. (2.192) to obtain eq. (2.189).

We conclude that the field strength term in our theory is given by

$$S_{fs} = \int \left\{ \mathrm{Im}(\tau) M_{MN}\mathcal{F}^{(2)M+} \wedge *\mathcal{F}^{(2)N+} + \mathrm{Re}(\tau) L_{MN}\mathcal{F}^{(2)M+} \wedge \mathcal{F}^{(2)N+} \right\}, \tag{2.194}$$

whereas a remaining topological term is given by

$$S_{top} = \int \Bigg\{ \frac{\epsilon^{ij}}{2} D^m_{im} B^{(2)++} \wedge \mathcal{F}^{(2)-}_j \\ + \epsilon^{ij} B^{(1)}_i \wedge \left(\frac{1}{2}\eta_{AB}\mathcal{F}^{(2)A+} \wedge C^{(1)B} D^k_{jk} - \mathcal{F}^{(2)5+} \wedge \tilde{C}^{(1)} D^k_{jk}\right) \\ + \epsilon^{ij}\left(\delta_{i\iota} d\tilde{B}^{(1)\iota} + \frac{1}{2} D^l_{il}\delta_{k\iota} G^{(1)k} \wedge \tilde{B}^{(1)\iota} + \frac{1}{2} D^k_{ik}\tilde{C}^{(1)} \wedge A^{(1)}\right) \wedge \mathcal{F}^{(2)-}_j \Bigg\}. \tag{2.195}$$

The resulting action,

$$S_{sc} + S_{fs} + S_{pot} + S_{top}, \tag{2.196}$$

reduces for $D^k_{ij} = 0$ to the action we found in section 2.4.3, and is gauge-invariant, as we show in Appendix E. It is in agreement with [19] and we conclude that compactifying IIA supergravity on a manifold with $SU(2)$-structure does indeed give $\mathcal{N} = 4$ four-dimensional gauged supergravities. To our knowledge, it is the first such theory obtained from dimensional reduction in which $f_{+MNP} \neq 0$ and $\xi_{+M} \neq 0$ simultaneously.

Chapter 3

Explicit Construction of $SU(2)$-Structure Manifolds

In this chapter, we will explicitly construct a set of $SU(2)$-structure manifolds and compactify IIA supergravity on manifolds in this set to obtain gauged four-dimensional $\mathcal{N} = 4$ supergravities. In contrast to Chapter 2, the manifolds constructed in this chapter will be based on T^2 and $K3$, setting $n = 22$ instead of allowing for all values of n. For this one value of n, we find more gaugings than we did in section 2.4. More specifically, we will show that the gaugings we find here can be split into three classes, the first equivalent to allowing the two-forms to be non-closed, determined by D_{iA}^{B}, the second equivalent to adding an $H_{10}^{(3)}$-flux. The third class of gaugings we find here was not discovered in Chapter 2. We will offer an interpretation for this class.

The construction we will use is based on a Scherk-Schwarz duality twist reduction [31] of a half-maximal supergravity in six dimensions. A Scherk-Schwarz duality twist reduction is a dimensional reduction that can be used to construct gauged supergravities from higher-dimensional supergravities. The internal manifold is a torus, and in the reduction ansatz, the fields depend on the torus coordinate in such a way that when we follow a path around the torus, the fields come back to themselves up to a symmetry transformation. The symmetry that the fields are twisted by then determines the gauge symmetry of the dimensionally-reduced theory. In section 3.1, we will perform a Scherk-Schwarz duality twist reduction of a six-dimensional half-maximal supergravity to four dimensions.

The half-maximal supergravity in question is obtained by reducing IIA supergravity on a $K3$, and it has a global $SO(4, 20)$-symmetry. It can also be shown that the string theory behind it has a smaller symmetry: a global $SO(4, 20; \mathbb{Z})$-symmetry [32]. This smaller subgroup is the group we use for the Scherk-Schwarz duality twist.

The Scherk-Schwarz duality twist reduction using the discrete symmetry group yields a four-dimensional $\mathcal{N} = 4$ gauged supergravity with 22 vector multiplets where the gauging can be split into three classes. As mentioned above, one of those classes can be interpreted as the parameter controlling the fact that the two-forms are nonclosed, or equivalently, controlling the twist of the $K3$ over the T^2, and the second class can be interpreted as the parameter that determines the $H_{10}^{(3)}$-flux of the ten-dimensional theory. We will argue that the third class could be a mirror flux, an $H_{10}^{(3)}$-flux applied to the mirror of the $K3$ in the internal space.

The context of this research is that of the duality ([33], [34], [35], [36]) between heterotic string theory on T^4 and IIA string theory on $K3$. Just as we give an interpretation to the different classes of gaugings on the IIA side here, we interpret the different classes of gaugings on the heterotic side in [37] This interpretation includes results found in [38] and [31], as well as a new class of parameters interpreted by our collaborator R. Reid-Edwards.

We will start by reviewing the compactification of IIA supergravity on $K3$, and then we will perform a Scherk-Schwarz duality twist reduction on the resulting six-dimensional action. We will then show that the most general twist is made up of three different classes of parameters, and we will relate two of these classes to the gaugings we have described in section 2.4. We finally offer an interpretation of the remaining class of parameters.

3.1 Scherk-Schwarz Duality Twist Reduction to Four Dimensions

To compactify IIA supergravity on a $K3$, a reduction procedure similar to the one outlined in section 2.3 is performed. The field expansions given in eqs. (2.49), (2.50) and (2.58)-(2.60) still hold, albeit without the T^2-expansions. These fields are entered into the action (2.15) and the action is then integrated over $K3$. Finally, the three-form $\hat{C}^{(3)}$ is dualized to a vector. For more detail on this procedure, see [39].

The resulting action is best written in terms of an $SO(4,20)$-matrix that we call $M_{6\,IJ}$, given by

$$M_{6\,IJ} := \begin{pmatrix} e^{-\rho} + \eta^{AC} H^B{}_C b_A b_B + e^\rho e^2 & e^\rho e & -H^C{}_B b_C - e^\rho b_B e \\ e^\rho e & e^\rho & -e^\rho b_B \\ -H^C{}_A b_C - e^\rho b_A e & -e^\rho b_A & \eta_{AC} H^C{}_B + e^\rho b_A b_B \end{pmatrix} \tag{3.1}$$

with $e := \frac{1}{2}\eta^{AB} b_A b_B$. It obeys

$$M_{6\,IK} L^{6\,KL} M_{6\,LJ} = L_{6\,IJ}, \quad M_{6\,IJ} = M_{6\,JI}, \tag{3.2}$$

with L_6 given by

$$L_{6\,IJ} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \eta_{AB} \end{pmatrix}. \tag{3.3}$$

Counting the scalar fields tells us that $M_{6\,IJ}$ contains 80 scalar degrees of freedom. It can be shown that ([40],[32])

$$M_{6\,IJ} \in \frac{SO(4,20)}{SO(4) \times SO(20)}. \tag{3.4}$$

In the theory are also a metric, a dilaton ϕ_6 and a two-form field $B_6^{(2)}$, and 24 gauge fields. Of these gauge fields, one comes from the ten-dimensional gauge field, 22 from the expansion of the three-form field in the two-forms of $K3$, and the last one is the dual of the three-form field in six dimensions. The six-dimensional supergravity action (this can be found, for example, in [39]) is given by

$$S_6 = \int \Big\{ d^6\sqrt{-g_6}e^{-2\phi_6}\Big(R_6 + 4\partial_M\phi_6\partial^M\phi_6 + \frac{1}{8}\partial_M M_{6\,IJ}\partial^M(L_6M_6L_6)^{IJ}\Big) \\ + \frac{e^{-2\phi_6}}{2}\mathcal{H}_6^{(3)}\wedge *\mathcal{H}_6^{(3)} + \frac{1}{2}M_{6\,IJ}\mathcal{F}_6^{(2)I}\wedge *\mathcal{F}_6^{(2)J} - \frac{1}{2}L_{6\,IJ}B_6\wedge\mathcal{F}_6^{(2)I}\wedge\mathcal{F}_6^{(2)J}\Big\}. \tag{3.5}$$

The field strengths are

$$\mathcal{H}_6^{(3)} = dB_6^{(2)}, \tag{3.6}$$

$$\mathcal{F}_6^{(2)I} = dA_6^{(1)I}. \tag{3.7}$$

This action is invariant under an $SO(4,20;\mathbb{R})$ global symmetry:

$$M_6 \to (\Omega^{-1})^T M_6\Omega^{-1}, \quad A_6^{(1)} \to \Omega A_6^{(1)}, \tag{3.8}$$

where

$$\Omega L_6\Omega^T = L_6. \tag{3.9}$$

In [32], it was shown that the discrete subgroup $SO(4,20;\mathbb{Z}) \subset SO(4,20;\mathbb{R})$ is the symmetry group of the underlying string theory.

We will now perform a Scherk-Schwarz duality twist reduction of the six-dimensional $\mathcal{N}=2$ supergravity we discussed in the last section, following the procedure outlined in [31]. We will reduce on a T^2 with coordinates y^i such that $y^i \sim y^i+1$ and twist the fields over this space. That means that the fields acquire a monodromy, depending on the twist, upon circumnavigating the T^2. The reduction is well-defined when this monodromy is an element of the global symmetry group; in that case, we are identifying two symmetric field configurations. The global symmetry group of the supergravity action is $SO(4,20;\mathbb{R})$, however, with the application to the IIA/heterotic duality in mind, we constrain ourselves to monodromies lying in the symmetry group of the string theory, $SO(4,20;\mathbb{Z})$.

Let us now concretely say what we mean when we twist fields by $SO(4,20)$. The six-dimensional metric, dilaton and the two-form field do not transform under $SO(4,20)$; their reduction Ansätze are

$$ds_6^2 = ds_4^2 + g_{ij}\nu^i\otimes\nu^j, \tag{3.10}$$

$$\phi_6 = \phi, \tag{3.11}$$

$$B_6^{(2)} = B^{(2)} + B_i^{(1)}\wedge\nu^i + \frac{1}{2}b_{ij}\nu^i\wedge\nu^j. \tag{3.12}$$

The scalars $M_{6\,IJ}$ and one-forms $A_6^{(1)I}$ do transform under $SO(4,20)$, their reduction Ansätze are

$$A_6^{(1)I} = (e^{yt})^I{}_J\left(A^{(1)J} + a_j^J\nu^j\right), \tag{3.13}$$

$$M_{6\,IJ} = (e^{-yt^T})_I{}^K\hat{M}_{KL}(e^{-yt})^L{}_J. \tag{3.14}$$

Here, we have denoted the 24×24-scalar matrix $\hat{M}_{KL}$ to avoid confusion with the 28×28 scalar matrix M_{MN} that we will use later on. The twist matrix $(e^{ty})^J{}_I$ is defined as

$$(e^{ty})^J{}_I := 1 + y^i t_{iI}{}^J + \frac{1}{2} y^i y^j t_{iI}{}^K t_{jK}{}^J + \cdots . \tag{3.15}$$

The matrices $t_{iI}{}^J$ are chosen such that

$$(e^{yt})^J{}_I \in SO(4,20) \tag{3.16}$$

for all $y^i \in T^2$. This requirement is tantamount to requiring

$$L_{IK} t_{iJ}{}^K = -L_{JK} t_{iI}{}^K. \tag{3.17}$$

Requiring that $d^2 = 0$ gives us the requirement

$$t_{iI}{}^K t_{jK}{}^J = t_{jI}{}^K t_{iK}{}^J. \tag{3.18}$$

We already said that the monodromy fields acquire after going around the T^2 must be in the global symmetry group of the theory. This means that

$$(e^{t_i})^J{}_I \in SO(4,20;\mathbb{Z}), \quad i = 1,2. \tag{3.19}$$

Finally, when we enter the expansions (3.10)-(3.14) into the action, we find that the y-dependence drops out, so we can integrate over T^2.

We can see that the four-dimensional theory is gauged by looking at the field strengths and covariant derivatives. Let us take the six-dimensional field strength $\mathcal{F}_6^{(2)I}$ as an example. We enter the expansion (3.13) into $\mathcal{F}_6^{(2)I} = dA_6^{(1)I}$, and since

$$d(e^{ty})^J{}_I = (e^{ty})^K{}_I \Big(t_{iK}{}^J \nu^i + t_{iK}{}^J G^{(1)i} \Big), \tag{3.20}$$

we find

$$\begin{aligned} \mathcal{F}_6^{(2)I} = (e^{ty})^I{}_J \bigg(\Big(& \mathcal{F}^{(2)J} - \mathcal{F}^{(2)k} a_k^J \Big) \\ & + \mathcal{D}a_i^J \nu^i + \frac{1}{2} \Big(t_{iK}{}^J a_j^K - t_{jK}{}^J a_i^K \Big) \nu^i \wedge \nu^j \bigg). \end{aligned} \tag{3.21}$$

The field strength and covariant derivative

$$\mathcal{F}^{(2)I} = dA^{(1)I} + G^{(1)k} \wedge t_{kK}{}^I A^{(1)K}, \tag{3.22}$$

$$\mathcal{D}a_i^I = da_i^I - A^{(1)K} t_{iK}{}^I + G^{(1)k} t_{kK}{}^I a_i^K \tag{3.23}$$

agree with the transformation laws of the fields that can be obtained in the same we way obtained them in section 2.4.2. The full spectrum, its transformation and its field strengths and covariant derivatives can be found in Appendix F.

From the reduction, we want to obtain a four-dimensional theory that is cast in the same form as the $\mathcal{N} = 4$ gauged supergravities described in section 2.5, to allow for comparison. To do this, we first enter the Ansätze (3.10)-(3.14) into the action (3.5), and then follow a procedure very similar to the second half of the procedure of the compactification of IIA on Y_1 (see section 2.4 and Appendix D for calculations): we dualize the two-form B to a scalar β_{ij} and the one-forms B_i to one-forms $\tilde{B}^\iota$ (see Appendix B for an explanation on how to dualize fields). We end by performing a Weyl rescaling $g_{\mu\nu} \to e^{2\phi+\eta} g_{\mu\nu}$. The final action we find is

$$S = \int \Big\{ d^4x\sqrt{-g}\Big(R - \frac{1}{2\mathrm{Im}^2(\tau)}\partial_\mu\tau\partial^\mu\tau^* + \frac{1}{8}\mathcal{D}_\mu M_{MN}\mathcal{D}^\mu(LML)^{MN} + V\Big) + \mathrm{Im}(\tau)M_{MN}\mathcal{F}^{(2)M+}\wedge *\mathcal{F}^{(2)N+} + \mathrm{Re}(\tau)L_{MN}\mathcal{F}^{(2)M+}\wedge\mathcal{F}^{(2)N+}\Big\}. \quad (3.24)$$

Here, we have used the definitions

$$\tau := -\frac{1}{4}\epsilon^{ij}b_{ij} + \frac{i}{2}e^{-\eta}, \quad (3.25)$$

$$V := \frac{e^\eta}{4}t^{\iota IK}t^{\iota' JL}M_{\iota\iota'}L_{IJ}L_{KL} + \frac{e^\eta}{2}t^{\iota IK}t^{\iota' JL}M_{\iota J}M_{\iota' I}M_{KL} - \frac{e^\eta}{4}t^{\iota IK}t^{\iota' JL}M_{\iota\iota'}M_{IJ}M_{KL}. \quad (3.26)$$

while the scalar matrix M_{MN} is in $SO(6,22)$ and is given in Appendix F.

It is easy to see that this action is a four-dimensional $\mathcal{N}=4$ gauged supergravity. The spectrum and the form of the action correspond to a $\mathcal{N}=4$ gauged supergravity with 22 vector multiplets as described in section 2.5. Furthermore, from the field strengths

$$F^i = dG^{(1)i}, \quad (3.27)$$

$$F^\iota = d\tilde{B}^\iota - \frac{1}{2}L_{JK}\delta^{i\iota}t_{iI}{}^K A^I\wedge A^J, \quad (3.28)$$

$$F^I = dA^I + G^{(1)k}\wedge t_{kK}{}^I A^K \quad (3.29)$$

we can infer that the embedding tensor is given by

$$f_{+iIJ} = L_{6\,IK}t_{iJ}{}^K, \quad (3.30)$$

all other components being zero. This embedding tensor obeys the constraint given in eq. (2.128) because of eq. (3.18), while eq. (3.17) tells us that the embedding tensor is antisymmetric in its indices. The form of the covariant derivatives and of the potential confirms that we are looking at a four-dimensional $\mathcal{N}=4$ gauged supergravity.

The action is invariant under the gauge transformations given in Appendix F. The argument for this is akin to the one given in Appendix D: we use the transformation rules in Appendix F, together with the constraints (3.17) and (3.18) to show that $\delta V = 0$.

Let us compare the embedding tensor (3.30) to the one we found in our compactifications on Y_1 in eq. (2.151). First of all, it should be noted that on Y_1, we allowed for any integer n, whereas here we are restricting to $n = 22$. In this sense, the manifolds discussed here are more restricted than Y_1. However, for $n = 22$, the embedding tensor (3.30) possibly contains more parameters than the one given in eq. (2.151), since I, J runs over $1, \ldots, 24$, whereas A, B in eq. (2.151) runs over $1, \ldots, 22$. In the next section, we will take a closer look at the monodromies allowed by the Scherk-Schwarz procedure, and relate them to the ones in eq. (2.151).

3.2 Interpreting the Different Monodromies

We are now going to give a ten-dimensional interpretation of the different classes of monodromies in the $SO(4,20;\mathbb{Z})$-twist. Let us first describe the generators of these classes.

We want to find the generators of $SO(4,20)$-matrices written as $(e^{yt})^J{}_I$, with the requirement that $(e^t)^J{}_I \in SO(4,20;\mathbb{Z})$. As described before, this requirement is due to the fact that for $y^1 = 1$ or $y^2 = 1$, the fields twisted by $(e^{yt})^J{}_I$ should be related to the untwisted fields by a symmetry of the theory. Since the symmetry group of the string theory is $SO(4,20;\mathbb{Z})$, we require $(e^t)^J{}_I \in SO(4,20;\mathbb{Z})$.

This knowledge enables us to find the generators of the twist, together with constraints on these generators. According to Aspinwall and Morrison [32], twist matrices in $SO(4,20;\mathbb{Z})$ are generated, in the $K3$-case, by $SO(3,19;\mathbb{Z})$, $\mathbb{Z}^{3,19}$ and a $\mathbb{Z}_2$. We can see this as follows. The fact that

$$(e^{yt})^J{}_I \in SO(4,20) \qquad \forall y \in T^2 \tag{3.31}$$

imposes the condition that $f + L^T f^T L = 0$, which is solved by

$$t_{iI}{}^J = \begin{pmatrix} a_i & 0 & \eta^{BC} g_{iC} \\ 0 & -a_i & \eta^{BC} h_{iC} \\ h_{iA} & g_{iA} & E^B_{iA} \end{pmatrix}, \tag{3.32}$$

with arbitrary a_i, g_{iA} and h_{iA}, while

$$\eta^{AC} E^B_{iC} + \eta^{BC} E^A_{iC} = 0. \tag{3.33}$$

However, requiring that $(e^t)^J{}_I \in SO(4,20;\mathbb{Z})$ excludes a_i, since that would mean that $e^{a_i} \in \mathbb{Z}$ and $e^{-a_i} \in \mathbb{Z}$, which is only possible for $a_i = 0$. Finally, we can calculate that

$$L^T e^{yg} L = e^{-yh}. \tag{3.34}$$

Here,

$$g_{iI}{}^J := \begin{pmatrix} 0 & 0 & \eta^{BC} g_{iC} \\ 0 & 0 & 0 \\ 0 & g_{iA} & 0 \end{pmatrix} \tag{3.35}$$

and

$$h_{iI}{}^J := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \eta^{BC} h_{iC} \\ h_{iA} & 0 & 0 \end{pmatrix}. \tag{3.36}$$

We conclude that the $SO(4,20;\mathbb{Z})$-twists are generated by $SO(3,19;\mathbb{Z})$ mapping to E^B_{iA}, $\mathbb{Z}^{3,19}$ mapping to h_{iA}, and a $\mathbb{Z}_2$ in the form of L that interchanges h_{iA} and g_{iA}.

We will use this decomposition to interpret the Scherk-Schwarz twist on the IIA side. We will find that turning on E^B_{iA} corresponds to the $SU(2)$-structure manifold Y_1 with $n = 22$, while turning on g_{iA} corresponds to an $H^{(3)}_{10}$-flux. Finally, we will discuss how to interpret h_{iA} and what all of this tells us about $SU(2)$-structure manifolds.

3.2.1 $K3$ Fibered over T^2

We will first show that compactifying IIA on Y_1 with 22 two-forms is equivalent to a twist by a $SO(3,19;\mathbb{Z})$-monodromy. This monodromy is implemented by a twist matrix

$$t_{iI}{}^{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E^B_{iA} \end{pmatrix} \tag{3.37}$$

where $(e^{E_i})^B{}_A \in SO(3,19;\mathbb{Z})$ for $i = 1, 2$.

Both reductions give a four-dimensional, $\mathcal{N} = 4$ supergravity with 22 multiplets. For IIA on Y_1, we have seen this in section 2.4. For a Scherk-Schwarz twist compactification of $d = 6, \mathcal{N} = 2$ supergravity we have seen this in section 3.1. To compare the two theories further, we will split the $SO(4,20)$-index M in the latter case exactly as we did in the former case: we split up the index I into $5, 6$ and A. Eq. (3.22) then tells us that the field strengths exactly equal those in the compactification of IIA supergravity on Y_1 for the choice $E^B_{iA} = D^B_{iA}$. We conclude that the compactification of IIA supergravity on Y_1 with $n = 22$ is equivalent to the compactification of IIA supergravity on $K3$, further compactified by an $SO(3,19;\mathbb{Z})$ Scherk-Schwarz reduction over T^2.

In fact, it is possible to interpret Y_1 for $n = 22$ as $K3$ twisted over T^2 by an $SO(3,19;\mathbb{Z})$-monodromy. Let us re-examine the Ansätze (3.10)-(3.14) for $t_{iI}{}^J$ given by (3.37). In these, all the six-dimensional fields coming from the expansion of the ten-dimensional fields in the $K3$ 2-forms Ω^A are twisted by the $SO(3,19;\mathbb{Z})$-monodromy. Equivalently, we can twist the 2-forms Ω^A by setting

$$\Omega^A \to \Omega'^A = (e^{yt})^A{}_B \Omega^B. \tag{3.38}$$

These twisted two-forms obey

$$d\Omega'^A = E^B_{iA}\sigma^i \wedge \Omega'^B, \tag{3.39}$$

exactly as in eq. (2.75). So Y_1 with twenty-two two-forms is a $K3$ twisted over T^2 by an $SO(3,19;\mathbb{Z})$-monodromy.

3.2.2 Including $H^{(3)}_{10}$-Flux

We will now argue that turning on an $H^{(3)}_{10}$-flux on Y_1 with $n = 22$, as described in section 2.4, is equivalent to turning on g_{iA} and E^B_{iA} in eq. (3.32). In this case, the monodromy is given by

$$t_{iI}{}^{J} = \begin{pmatrix} 0 & 0 & \eta^{BC} g_{iC} \\ 0 & 0 & 0 \\ 0 & g_{iA} & E^B_{iA} \end{pmatrix}. \tag{3.40}$$

We already know, from sections 2.4 and 3.1, that both these compactifications yield an $\mathcal{N} = 4$ gauged supergravity with 22 vector multiplets. To show that the lower-dimensional actions are the same, we need only look at the gauging.

We can determine the gauging of both theories from the field strengths, since the only non-zero component of the embedding tensor is f_{+MNP}. The field strengths of IIA compactified on Y_1 with $H_{10}^{(3)}$-flux are given in eq. (2.112). To compare these with the field strengths of the Scherk-Schwarz twist, we once again split the index I in 5,6 and A; eq. (3.22) then shows us that the field strengths in the Scherk-Schwarz reduction are the same as those in the reduction on Y_1 with $H_{10}^{(3)}$-flux, in the case that $n = 22$.

That turning on an $H_{10}^{(3)}$-flux corresponds to performing a duality twist reduction with $(e^{yg})^J{}_I$ as twist matrix could have been expected. Turning on an $H_{10}^{(3)}$-flux is equivalent to requiring that, for $[\omega]$ the cycle dual to the form ω,

$$\int_{[\sigma^i \wedge \Omega^A]} H_{10}^{(3)} = k_{iA}. \tag{3.41}$$

To show that a duality twist by $(e^{yg})_I{}^J$ exhibits the same behavior, we must show that

$$\int_{[\sigma^i]} d_6 b_A = k_{iA}. \tag{3.42}$$

Since the twisted matrix

$$(e^{-yg^T})_I{}^K \hat{M}_{KL} (e^{-yg})^L{}_J \tag{3.43}$$

equals the original matrix $\hat{M}_{IJ}$ with the subsitution

$$b_A \to b_A + y^i g_{iA}, \tag{3.44}$$

we see that

$$d_6 b_A = db_A + \sigma^i k_{iA}, \tag{3.45}$$

such that eq. (3.42) holds.

3.2.3 Mirrorfold

We would now like to offer an interpretation for the last generator of the $SO(4, 20; \mathbb{Z})$-twists, h_{iA}. According to eq. (3.34), h_{iA} can be interpreted as an $H_{10}^{(3)}$-flux, applied to a $\mathbb{Z}_2$-conjugate theory. According to Aspinwall, the final $\mathbb{Z}_2$ corresponds to a mirror map, an $\mathcal{N} = 4$ equivalent of mirror symmetry. This means that h_{iA} would correspond to performing a mirror symmetry, applying an $H_{10}^{(3)}$-flux, and then performing a mirror symmetry again.

We have not been able to offer any concrete proof for this. For compactifications to $\mathcal{N} = 2$ supergravities, mirror symmetry interchanges complex and complexified Kähler moduli. The complex moduli are the moduli that determine the complex structure of the internal manifold, whereas the complexified Kähler moduli determine the Kähler structure and the B-field on the internal manifold. For a $K3$, the matrix $H^A{}_B$ contains both complex and Kähler moduli, and it is not a priori clear how one could interchange them without knowing exactly how $H^A{}_B$ depends on these moduli. Walton ([41]), for exampls, suggests that the metric moduli can be written roughly as

$$\delta g_{a\bar{b}} = v_A \Omega^A_{a\bar{b}}, \tag{3.46}$$

$$\delta g_{ab} = z_A (\epsilon_{ac} \Omega^{A\,c}{}_b + \epsilon_{bc} \Omega^{A\,c}{}_a), \tag{3.47}$$

$$\delta g_{\bar{a}\bar{b}} = (\delta g_{ab})^*, \tag{3.48}$$

with v_A being real and z_A complex. This could be used to write ${H^A}_B$ in terms of the Kähler moduli v_A and the complex moduli z_A. Mirror symmetry should then exchange

$$v_A + ib_A \leftrightarrow z_A. \tag{3.49}$$

This could give a strong hint of mirror symmetry and warrants, in our opinion, further research.

Chapter 4

Summary, Conclusion and Outlook

In this thesis, we have discussed the compactification of IIA supergravity on manifolds with $SU(2)$-structure. These manifolds can be seen as generalizations of $K3 \times T^2$, and we have seen how compactifying IIA supergravity on $SU(2)$-structure manifolds generalizes the known results for IIA supergravity on $K3 \times T^2$. The two results we have generalized are the compactification of IIA supergravity on $K3 \times T^2$, and the duality between IIA on $K3 \times T^2$ and heterotic on T^6.

We have seen that these compactifications produce different gauged $\mathcal{N} = 4$ supergravities. Compactifying IIA on Y_1 with $H_{10}^{(3)}$-flux, we found a gauged supergravity, with the gauging determined by the parameters used to specify the $SU(2)$-structure, namely D_{iA}^B, and the $H_{10}^{(3)}$-flux, namely k_{iA}. Requiring $d^2 = 0$ and Stokes' theorem to still hold gave us the constraints on the parameters that make the theory invariant under gauged transformations. After the dualization of fields we could compare the spectrum and the action to the literature [19], and we have seen that this is an $\mathcal{N} = 4$ gauged supergravity with N vector multiplets, where n is the number of two-forms of the internal manifold. In the language of [19], one of the tensors that make up the embedding tensor is non-zero, namely f_{+MNP}.

Compactifying IIA supergravity on Y_2, we have again found a gauged supergravity. The gauging in this case is determined by the two tensors D_{ij}^k and D_{iA}^B that specify the $SU(2)$-structure. Once more, requiring $d^2 = 0$ and Stokes' theorem provides the constraints on D_{ij}^k and D_{iA}^B that make the theory gauge invariant. In this case, not all the dualizations of fields that we did for IIA on Y_1 are possible. Instead, we identify the embedding tensor and then dualize a field in the action of [19] to obtain the same action we got from the compactification. The reason that we had to follow this procedure is that two of the four tensors that make up the embedding tensor, f_{+MNP} and ξ_{+M}, are now turned on. As far as we are aware, this is the first compactification to a gauged supergravity such that these two tensors are turned on similarly.

In both the compactifications of IIA supergravity on Y_1 and Y_2, we have not been able to calculate the reduction of the Ricci scalar completely. Based on analogy with the compactification on $K3 \times T^2$, we have assumed that terms in the Ricci scalar for which all indices lie in the four-dimensional component of Y_1 or Y_2 vanish or cancel each other out. We have furthermore assumed that the fields $e^{-\rho}$ and $H^A{}_B$, that come from the metric on the four-dimensional component, transform in such a way that the theory is gauge-invariant. Since the transformations of all the other fields could be calculated directly from the reduction, this assumption told us the exact transformation of $e^{-\rho}$ and $H^A{}_B$. The problem underlying this issue is that it does not seem clear how, precisely, the metric of $K3$, g_{ab}, is related to the moduli fields $H^A{}_B$. This prohibits us from calculating exactly how the kinetic and potential terms for $H^A{}_B$ arise from the Ricci scalar when we compactify on Y_1 and Y_2. Further study into this matter will be necessary to be able to explicitly calculate the reduction of the Ricci scalar on manifolds with $SU(2)$-structure like the ones we have studied here.

In Chapter 3, we have provided an explicit construction of a set of $SU(2)$-structure manifolds. Our starting point here was the six-dimensional supergravity obtained from compactifying IIA on a $K3$. We have shown that performing a Scherk-Schwarz duality twist reduction of this theory gives an $\mathcal{N} =$ gauged four-dimensional supergravity. We have shown that every twist can be characterised by three components. We have identified one of these components, E^B_{iA}, with the D^B_{iA}-parameters of Y_1 when Y_1 has 22 two-forms. A second component, g_{iA}, has been identified with the $H^{(3)}_{10}$-flux parameter k_{iA}. We have conjectured that the third component can be interpreted as $H^{(3)}_{10}$-flux applied to a mirror $K3$.

It would now be interesting to see what supergravity results from the compactification of IIA supergravity on more general manifolds with $SU(2)$-structure. The most general prescription for the derivatives of the forms is

$$d\sigma^i = \frac{1}{2} D^i_{kl} \sigma^k \wedge \sigma^l + D^i_A \Omega^A, \tag{4.1}$$

$$d\Omega^A = D^B_{iA} \sigma^i \wedge \Omega^A. \tag{4.2}$$

In this thesis, we have kept D^i_A zero, but it could be turned on. On this manifold with $SU(2)$-structure, Y_3, we could then apply $H^{(3)}_{10}$-fluxes and also give fluxes to the Ramond-Ramond field. Finally, we could see how the mirror flux we conjectured in section 3.2.3 fits in all of this: is this an artifact of the specific compactification we studied in Chapter 3, or can this be applied to other manifolds with $SU(2)$-structure as well?

One way to provide evidence that the $\mathbb{Z}_2$-transformation discussed in section 3.2.3 is really reminiscent of mirror symmetry is to look at the complex, Kähler, and B-field moduli of the internal manifold. Mirror symmetry is known to interchange the complex moduli with the *complexified Kähler moduli*, complex moduli fields made out of Kähler and B-field moduli. However, for a $K3$ and the extensions of $K3$ we have studied here, both the complex and the Kähler moduli reside in the same matrix $H^A{}_B$. Since we have been unable to find what $H^A{}_B$ looks like in terms of the individual moduli, we have not been able to show that the complex and complexified Kähler moduli are interchanged under the $\mathbb{Z}_2$-transformation.

The explicit constructions of $SU(2)$-structure manifolds in Chapter 3 were inspired by the duality between IIA string theory on $K3$ and heterotic string theory on T^4. It is possible to split the gaugings obtained from the Scherk-Schwarz duality twist in such a way that they can be interpreted in a heterotic supergravity framework as well. We hope to further report on this issue in [37].

Appendix A

Differential Forms

Let M be a d-dimensional manifold with local coordinates x^i, $i = 1, \ldots, d$. An *m-form* $\omega^{(m)}$ is defined as

$$\omega^{(m)} := \frac{1}{m!}\omega^{(m)}_{i_1 \ldots i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m}. \tag{A.1}$$

Given an m-form $\omega^{(m)}$ and an n-form $\chi^{(n)}$, their wedge product is defined as

$$\omega \wedge \chi := \frac{1}{m!n!}\omega^{(m)}_{i_1 \ldots i_m} \chi^{(n)}_{j_1 \ldots j_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_n}. \tag{A.2}$$

The exterior derivative d acts on $\omega^{(m)}$ as

$$d\omega^{(m)} = \frac{1}{m!}\partial_j \omega^{(m)}_{i_1 \ldots i_m} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_m} \tag{A.3}$$

From the definition, it is clear that $d^2\omega = 0$. Furthermore, Stokes' theorem says that for N an r-dimensional submanifold of M, and $\omega^{(r-1)}$ an $r-1$-form,

$$\int_N d\omega^{(r-1)} = \int_{\partial N} \omega^{(r-1)}. \tag{A.4}$$

The Hodge star operator $*$ takes an m-form to a $(d-m)$-form:

$$*\omega^{(m)} := \frac{1}{\sqrt{|g|}} \frac{1}{m!(d-m)!} \omega^{(m)}_{i_1 \ldots i_m} \epsilon^{i_1 \ldots i_m}{}_{i_{m+1} \ldots i_d} dx^{i_{m+1}} \wedge \cdots \wedge dx^{i_d}. \tag{A.5}$$

Here, $\epsilon^{i_1 \ldots i_d}$ is the fully antisymmetric tensor that obeys $\epsilon^{1 \ldots d} = 1$. Indices are lowered by the metric g_{ij} with determinant g such that

$$\epsilon^{i_1 \ldots i_p i_{p+1} \ldots i_d} \epsilon_{j_1 \ldots j_p i_{p+1} i_d} = g(d-p)! \delta^{i_1 \ldots i_p}_{j_1 \ldots j_p}, \tag{A.6}$$

where $\delta^{i_1 \ldots i_p}_{j_1 \ldots j_p}$ is an antisymmetrized product of delta-functions obeying

$$\delta^{i_1 \ldots i_p}_{j_1 \ldots j_p} \omega^{(p)}_{i_1 \ldots i_p} = p! \omega^{(p)}_{j_1 \ldots j_p}. \tag{A.7}$$

With these definitions, it can be shown that

$$\omega^{(m)} \wedge *\chi^{(m)} = -\frac{1}{m!}\omega^{(m)i_1\dots i_m}\chi^{(m)}_{i_1\dots i_m}\sqrt{|g|}dx^1\wedge\cdots\wedge dx^d \tag{A.8}$$

on a Lorentzian manifold. On a Euclidean manifold, the minus sign would be absent. Also,

$$**\,\omega^{(m)} = -(-1)^{m(d-m)}\omega^{(m)} \tag{A.9}$$

holds for a Lorentzian manifold, while there would, again, be one minus sign less on a Euclidean manifold. Finally, if the metric is block-diagonal and splits into D and $(d-D)$-dimensional blocks, the Hodge star splits into two parts:

$$*_d(F^{(n)}\wedge\theta^{(m)}) = (-1)^{mn} *_{d-D} F^{(n)} \wedge *_D\theta^{(m)} \tag{A.10}$$

for $F^{(n)}$ an n-form defined on the $(d-D)$-dimensional part, and $\theta^{(m)}$ an m-form defined on the D-dimensional part.

The mth *de Rham cohomology* group $H^m(M)$ is defined as the group of closed forms quotiented out by the exact forms

$$H^m(M) := \frac{Z^m(M)}{d\Omega^{(m-1)}}, \tag{A.11}$$

with

$$Z^m(M) := \left\{\omega^{(m)}|d\omega^{(m)}=0\right\}. \tag{A.12}$$

The dimension of the mth cohomology group of M is called the mth *Betti number* $b^m(M)$ of M. Poincaré duality implies that, in fact, there is a bijection

$$H^r(M) \cong H^{d-m}(M), \tag{A.13}$$

so $b^m(M) = b^{d-m}(M)$.

On a complex or almost complex manifold, an m-form $\omega^{(m)}$ can locally be written in complex coordinates z^i:

$$\omega^{(m)} = \frac{1}{r!s!}\omega^{(m)}_{i_1\dots i_p j_1\dots j_q} dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q} \tag{A.14}$$

for some integer $p,q\geq 0$ such that $m=p+q$. Such a form is then called a (p,q)-*form*, and the vector space of such forms is denoted $\Omega^{p,q}(M)$. This means that

$$\Omega^m(M) = \bigoplus_{p+q=m}\Omega^{p,q}(M). \tag{A.15}$$

Therefore, every m-form in the cohomology group $H^m(M)$ can be written as a sum of (p,q)-forms, with $p+q=m$. The *Hodge numbers* $b^{p,q}$ indicate the number of linearly independent (p,q)-forms we have in $H^m(M)$. All these numbers can be written down in the *Hodge diamond*, which looks like

$$\begin{array}{ccccccccc} & & & & b^{0,0} & & & & \\ & & & b^{1,0} & & b^{0,1} & & & \\ & & & & \cdots & & & & \\ b^{m,0} & b^{m-1,1} & & & \cdots & & & b^{1,m-1} & b^{0,m} \\ & & & & \cdots & & & & \\ & & & b^{m,m-1} & & b^{m-1,m} & & & \\ & & & & b^{m,m} & & & & \end{array} \tag{A.16}$$

Appendix B

Bosonic Fields and their Duals

Using Hodge duality, it is possible to rewrite an action $S[A^{(p)}, dA^{(p)}]$ for a p-form field $A^{(p)}$ as an action for a dual q-form field $D^{(q)}$. We will first describe how to do this for massless fields, as described in, for example, [42] and then move on to massive fields, as described in, for example, [43].

For this procedure, we will often make use of the Euler-Lagrange equation. Since we are working with form fields, we want to find an expression of the Euler-Lagrange equation that applies to form fields. The action is left invariant under a transformation $A^{(p)} \to A^{(p)} + \delta A^{(p)}$, which means that

$$\begin{aligned} 0 = \delta S &= \int \left\{ \delta A^{(p)} \wedge \frac{\delta \mathcal{L}}{\delta A^{(p)}} + \delta(dA^{(p)}) \wedge \frac{\delta \mathcal{L}}{\delta dA^{(p)}} \right\} \\ &= \int \left\{ \delta A^{(p)} \wedge \frac{\delta \mathcal{L}}{\delta A^{(p)}} + d\left(\delta(A^{(p)}) \wedge \frac{\delta \mathcal{L}}{\delta dA^{(p)}} \right) - (-1)^p \delta A^{(p)} \wedge d\frac{\delta \mathcal{L}}{\delta dA^{(p)}} \right\}. \end{aligned} \tag{B.1}$$

Our convention for functional form derivatives is that

$$\frac{\delta}{\delta A^{(p)}} \left(B^{(q)} \wedge A^{(p)} \wedge C^{(r)} \right) = (-1)^{pq} B^{(q)} \wedge C^{(r)}. \tag{B.2}$$

The total derivative term is zero since we assume the variation of $A^{(p)}$ goes to zero at infinity. Therefore we conclude that

$$\frac{\delta \mathcal{L}}{\delta A^{(p)}} - (-1)^p d\frac{\delta \mathcal{L}}{\delta dA^{(p)}} = 0. \tag{B.3}$$

B.1 Massless

In d dimensions, a massless p-form field has $\binom{d-2}{p}$ degrees of freedom. For $d = 4$, this means that a two-form field and a scalar have the same number of degrees of freedom. We could describe those degrees of freedom by a two-form or by a scalar, we are free to choose which. For example, if we had a two-form $B^{(2)}$ with kinetic term

$$\frac{1}{2} dB^{(2)} \wedge *dB^{(2)}, \tag{B.4}$$

we could also interpret this as the kinetic form of a scalar field ϕ, called the *dual field* of $B^{(2)}$ with $*dB^{(2)}$ (a one-form) as its derivative.

By the same reasoning, a one-form $A^{(1)}$ with kinetic term

$$\frac{1}{2}dA^{(1)} \wedge *dA^{(1)} \tag{B.5}$$

can equivalently be described by a dual one-form field with $*dA^{(1)}$ as its fieldstrength. In the following two sections we will show what the action looks like in terms of these dual fields.

B.1.1 One-Form

Consider the action

$$S_{C^{(1)}} = \int \left\{ \frac{g}{2}(\mathcal{F}^{(2)} - J^{(2)}) \wedge *(\mathcal{F}^{(2)} - J^{(2)}) - \mathcal{F}^{(2)} \wedge K^{(2)} \right\}, \tag{B.6}$$

with $\mathcal{F}^{(2)} = dC^{(1)}$. Now treat $\mathcal{F}^{(2)}$ as an independent field and write a new action $S'_{C^{(1)}}$ with a Lagrange-multiplier $D^{(1)}$ that enforces the Bianchi-identity $d\mathcal{F}^{(2)} = 0$:

$$\begin{aligned} S'_{C^{(1)}} = \int \Bigg\{ & \frac{g}{2}(\mathcal{F}^{(2)} - J^{(2)}) \wedge *(\mathcal{F}^{(2)} - J^{(2)}) \\ & - \mathcal{F}^{(2)} \wedge K^{(2)} - D^{(1)} \wedge d\mathcal{F}^{(2)} \Bigg\}. \end{aligned} \tag{B.7}$$

Now we can determine the equation of motion of $\mathcal{F}^{(2)}$ and use that to eliminate it from the action. The result is

$$\begin{aligned} S'_{C^{(1)}} = \int \Bigg\{ & \frac{1}{2g}(dD^{(1)} + K^{(2)}) \wedge *(dD^{(1)} + K^{(2)}) \\ & - (dD^{(1)} + K^{(2)}) \wedge J^{(2)} \Bigg\}. \end{aligned} \tag{B.8}$$

Let us now see how to determine the transformation behavior of the dual field $D^{(1)}$. First of all, it is clear that both (B.7) and (B.8) are invariant under

$$\delta D^{(1)} = d\lambda. \tag{B.9}$$

Furthermore, since in equation (B.6), $\mathcal{F}^{(2)} = dC^{(1)}$, the topological term is invariant under a transformation

$$\delta K^{(2)} = dT^{(1)}. \tag{B.10}$$

The form of $T^{(1)}$ depends on the fields that $K^{(2)}$ consists of. However, in equation (B.7), $\mathcal{F}^{(2)}$ is no longer $dC^{(1)}$, and therefore the term $\mathcal{F}^{(2)} \wedge K^{(2)}$ is not invariant under the transformation (B.10) anymore. This means that the Lagrange multiplier $D^{(1)}$ must transform as

$$\delta D^{(1)} = -T^{(1)}. \tag{B.11}$$

This means that the term $(dD^{(1)} + K^{(2)})$ which appears in the final action (B.8) is invariant.

B.1.2 Two-Form

Consider the action

$$S_{B^{(2)}} = \int \left\{ \frac{a}{2} \left(\mathcal{H}^{(3)} - J^{(3)} \right) \wedge * \left(\mathcal{H}^{(3)} - J^{(3)} \right) + \frac{1}{2} \epsilon^{ij} \mathcal{H}^{(3)} \wedge A^{(1)}_{ij} \right\}. \tag{B.12}$$

Here, $\mathcal{H}^{(3)} := dB^{(2)}$. We want to write the action in terms of the dual scalar $\beta_{ij} = -\beta_{ij}$ for $i, j \in \{1, 2\}$, since this is the dualization we perform in the main text. Note that β_{ij} is only one independent scalar field. We first write down the equivalent action

$$S'_{\mathcal{H}^{(3)}} = \int \left\{ \frac{a}{2} \left(\mathcal{H}^{(3)} - J^{(3)} \right) \wedge * \left(\mathcal{H}^{(3)} - J^{(3)} \right) + \frac{1}{2} \epsilon^{ij} \mathcal{H}^{(3)} \wedge A^{(1)}_{ij} - \frac{1}{2} \epsilon^{ij} \beta_{ij} d\mathcal{H}^{(3)} \right\}. \tag{B.13}$$

Again, $\mathcal{H}^{(3)}$ is now an arbitrary field and $S_{B^{(2)}}$ can be regained by integrating out β_{ij}. But by determining the equation of motion for $\mathcal{H}^{(3)}$, and using that to eliminate $\mathcal{H}^{(3)}$ from the action, we get the dual action

$$S_{\beta_{ij}} = \int \left\{ \frac{g}{4a} g^{ik} g^{jl} \left(d\beta_{ij} - A^{(1)}_{ij} \right) \wedge * \left(d\beta_{kl} - A^{(1)}_{kl} \right) + \frac{1}{2} \epsilon^{ij} \left(d\beta_{ij} - A^{(1)}_{ij} \right) \wedge J^{(3)} \right\}. \tag{B.14}$$

Here, $g := \det(g_{ij})$ and we have used

$$\epsilon^{ij} \epsilon^{kl} \beta_{ij} \gamma_{kl} = 2g g^{ik} g^{jl} \beta_{ij} \gamma_{kl} \tag{B.15}$$

for arbitrary antisymmetric β_{ij} and γ_{ij}.

Again, depending on the transformations of $A^{(1)}_{ij}$, β_{ij} may acquire non-trivial transformations

B.2 Massive

A massive p-form field in d dimensions has $\binom{d-1}{p}$ degrees of freedom. Taking $d = 4$ again, this means that while a massless one-form is dual to another massless one-form, and a massless two-form to a massless scalar, a massive one-form is dual to a massive two-form. Furthermore, if, for example, a two-form field becomes massive through a Stueckelberg mechanism, we can interpret that in terms of massless fields: a massless two-form field, having one degree of freedom, becomes massive by eating a massless one-form field. In that process it acquires two more degrees of freedom, so it has the three degrees of freedom of a massive two-form. We can dualize this to a vector with two degrees of freedom, that becomes massive by eating a scalar, giving it one more degree of freedom. We want to work out the details for this interpretation.

To illustrate the main concepts, we will first describe the easiest case, of one two-form field $B^{(2)}$ that eats one one-form $C^{(1)}$. The action here is

$$S_{B^{(2)},C^{(1)}} = \int \left\{ \frac{1}{2}\mathcal{H}^{(3)} \wedge *\mathcal{H}^{(3)} + \frac{1}{2}\mathcal{F}^{(2)} \wedge *\mathcal{F}^{(2)} \right\}, \tag{B.16}$$

with

$$\mathcal{H}^{(3)} := dB^{(2)}, \tag{B.17}$$

$$\mathcal{F}^{(2)} := dC^{(1)} + mB^{(2)}. \tag{B.18}$$

We can write this action as

$$S_{\mathcal{H}^{(3)},B^{(2)},C^{(1)}} = \int \left\{ -\frac{1}{2}\mathcal{H}^{(3)} \wedge *\mathcal{H}^{(3)} + dB^{(2)} \wedge *\mathcal{H}^{(3)} + \frac{1}{2}\mathcal{F}^{(2)} \wedge *\mathcal{F}^{(2)} \right\}, \tag{B.19}$$

which reduces to the former action if we invoke the equation of motion for $\mathcal{H}^{(3)}$.

Now the Hodge dual of $\mathcal{H}^{(3)}$ is a one-form, and we want to describe the action in terms of this one-form. In the massless case, this one-form can be written as the derivative of a scalar, in the massive case we know there must also be a one-form that cannot be written as a derivative. Since everything must reduce to the massless case for $m = 0$, we write

$$*\mathcal{H}^{(3)} =: d\phi + m\tilde{C}^{(1)}. \tag{B.20}$$

From this equation, we can immediately determine the transformation behavior of the dual fields, as they must transform in such a way that the right hand side and the left hand side have the same transformation.

The equation of motion for $B^{(2)}$ gives us the equations

$$\mathcal{F}^{(2)} = - * d\tilde{C}^{(1)} \tag{B.21}$$

$$*\mathcal{F}^{(2)} = d\tilde{C}^{(1)}, \tag{B.22}$$

and plugging this into the action, together with the expression for $*\mathcal{H}^{(3)}$, gives us the action of the dualized fields ϕ and $\tilde{C}^{(1)}$:

$$S_{\phi,\tilde{C}^{(1)}} = \int \left\{ \frac{1}{2}(d\phi + m\tilde{C}^{(1)}) \wedge *(d\phi + m\tilde{C}^{(1)}) + \frac{1}{2}d\tilde{C}^{(1)} \wedge *d\tilde{C}^{(1)} \right\}. \tag{B.23}$$

Note that the degrees of freedom are left unchanged: we started with a massless one and two-form, corresponding to three degrees of freedom, or equivalently a two-form that became massive through eating a one-form, also corresponding to three degrees of freedom. We end with a massless scalar and one-form, corresponding to three degrees of freedom, or equivalently a massive one-form that has eaten a scalar, also corresponding to three degrees of freedom.

We will now describe the dualization of massive two-forms $C_i^{(2)}$ that eat a one-form $C_{ij}^{(1)}$ $(i,j \in \{1,2\})$, which we will use in Appendix E. We start from the action

$$\begin{aligned} S_{C_i^{(2)},C_{ij}^{(1)}} = \int \Bigg\{ & \frac{hg^{ij}}{2} \left(\mathcal{H}_i^{(3)} - \mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)} + J_i^{(3)}\right) \wedge \\ & * \left(\mathcal{H}_j^{(3)} - \mathcal{F}^{(2)l+} \wedge C_{jl}^{(1)} + J_j^{(3)}\right) \\ & + \frac{hg^{ik}g^{jl}}{4} \left(\mathcal{F}_{ij}^{(2)-} + J_{ij}^{(2)}\right) \wedge * \left(\mathcal{F}_{kl}^{(2)-} + J_{kl}^{(2)}\right) \\ & + \epsilon^{ij} \left(\mathcal{H}_i^{(3)} - \mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)}\right) \wedge K_j^{(1)} + \epsilon^{ij} \mathcal{F}_{ij}^{(2)-} \wedge K^{(2)} \Bigg\}, \end{aligned} \tag{B.24}$$

with

$$\mathcal{H}_i^{(3)} := dC_i^{(2)} - G^{(1)l} \wedge D_{li}^k C_k^{(2)}, \tag{B.25}$$

$$\mathcal{F}_{ij}^{(2)-} := dC_{ij}^{(1)} - G^{(1)l} \wedge \left(D_{li}^k C_{kj}^{(1)} + D_{lj}^k C_{ik}^{(1)}\right) + C_k^{(2)} D_{ij}^k. \tag{B.26}$$

In principle, $h, J_i^{(3)}, J_{ij}^{(2)}, K_i^{(1)}$ and $K^{(2)}$ are combinations of fields in the theory that do not depend on $C_i^{(2)}$ and $C_{ij}^{(1)}$, with as only restriction that $S_{C_i^{(2)},C_{ij}^{(1)}}$ is gauge-invariant. In the calculation of Appendix E, they are given by

$$h := e^{-\eta-\rho}, \tag{B.27}$$

$$J_i^{(3)} := a_i \left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+} \wedge B_k^{(1)}\right) - \mathcal{F}^{(2)5+} \wedge B_i^{(1)}, \tag{B.28}$$

$$J_{ij}^{(2)} := -\mathcal{F}^{(2)5+} \wedge b_{ij} - \Big(a_i\big(\mathcal{F}_j^{(2)-} + \mathcal{F}^{(2)k+} b_{jk}\big) - a_j\big(\mathcal{F}_i^{(2)-} + \mathcal{F}^{(2)l+} b_{il}\big)\Big), \tag{B.29}$$

$$K_i^{(1)} := \eta^{AB} b_A \Big(\mathcal{D}c_{jB} - a_j \mathcal{D}b_B - \mathcal{D}a_j b_B\Big), \tag{B.30}$$

$$K^{(2)} := -\Big(b_A \mathcal{F}^{(2)A+} - \frac{1}{2}\eta^{AB} b_A b_B \mathcal{F}^{(2)5+} + \eta^{AB} b_A \mathcal{F}^{(2)k+}(c_{kB} - a_k b_B)\Big). \tag{B.31}$$

The action $S_{C_i^{(2)},C_{ij}^{(1)}}$ is then equal to

$$\begin{aligned} \int \Bigg\{ & - \frac{hg^{ij}}{2} \left(\mathcal{H}_i^{(3)} + \mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)} + J_i^{(3)}\right) \wedge * \left(\mathcal{H}_j^{(3)} + \mathcal{F}^{(2)l+} \wedge C_{jl}^{(1)} + J_j^{(3)}\right) \\ & + hg^{ij} \left(dC_i^{(2)} - G^{(1)o} \wedge D_{oi}^k C_k^{(2)} + \mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)} + J_i^{(3)}\right) \wedge \\ & * \left(\mathcal{H}_j^{(3)} + \mathcal{F}^{(2)l+} \wedge C_{jl}^{(1)} + J_j^{(3)}\right) \\ & + \frac{hg^{ik}g^{jl}}{4} \left(\mathcal{F}_{ij}^{(2)-} + J_{ij}^{(2)}\right) \wedge * \left(\mathcal{F}_{kl}^{(2)-} + J_{kl}^{(2)}\right) \\ & + \epsilon^{ij} \left(dC_i^{(2)} - G^{(1)l} \wedge D_{li}^k C_k^{(2)} + \mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)}\right) \wedge K_j^{(1)} + \epsilon^{ij} \mathcal{F}_{ij}^{(2)-} \wedge K^{(2)} \Bigg\}, \end{aligned} \tag{B.32}$$

which we can show by invoking the equations of motion for $\mathcal{H}_i^{(3)}$. Again, we let the Hodge dual of the two-form fields be determined by a scalar and a one-form field, although the equation is slightly more complex now:

$$hg^{ij} * \left(\mathcal{H}_j^{(3)} + J_j^{(3)}\right) + \epsilon^{ij} K_j^{(1)} = \epsilon^{ij} \left(d\phi_j + D_{jk}^k(\tilde{C}^{(1)} + G^{(1)l}\phi_l)\right). \tag{B.33}$$

Again, this equation tells us the transformation laws for the dual fields:

$$\delta\tilde{C}^{(1)} = d\tilde{\lambda} - \xi^k D_{kl}^l \tilde{C}^{(1)} + \tilde{\lambda} D_{kl}^l G^{(1)k}, \tag{B.34}$$

$$\delta\phi_i = -\xi^l D_{ik}^k \phi_l - \tilde{\lambda} D_{ik}^k. \tag{B.35}$$

The equations of motion for the two-form now give us the equations

$$* \left(\mathcal{F}_{ij}^{(2)-} + J_{ij}^{(2)}\right) = -\frac{\epsilon^{kl} g_{ik} g_{jl}}{h} \left(\mathcal{F}^{(2)6+} + \mathcal{F}^{(2)k+}\phi_k + 2K^{(2)}\right), \tag{B.36}$$

$$\mathcal{F}_{ij}^{(2)-} + J_{ij}^{(2)} = \frac{\epsilon^{kl} g_{ik} g_{jl}}{h} * \left(\mathcal{F}^{(2)6+} + \mathcal{F}^{(2)k+}\phi_k + 2K^{(2)}\right), \tag{B.37}$$

with

$$\mathcal{F}^{(2)6+} := d\tilde{C}^{(1)} + G^{(1)k} \wedge D_{kl}^l \tilde{C}^{(1)}. \tag{B.38}$$

Entering these into the action gives us the result

$$\begin{aligned} S_{\tilde{C}^{(1)},\phi_i,C_{ij}^{(1)}} = \int \Bigg\{ & \frac{\det(g_{ij}) g^{ij}}{2h} \left(\mathcal{D}\phi_i - K_i^{(1)}\right) \wedge * \left(\mathcal{D}\phi_j - K_j^{(1)}\right) \\ & + \frac{\det(g_{ij})}{2h} \left(\mathcal{F}^{(2)6+} + \mathcal{F}^{(2)k+}\phi_k + 2K^{(2)}\right) \wedge \\ & * \left(\mathcal{F}^{(2)6+} + \mathcal{F}^{(2)k+}\phi_k + 2K^{(2)}\right) \\ & + \epsilon^{ij} J_i^{(3)} \wedge \left(\mathcal{D}\phi_i - K_i^{(1)}\right) \\ & - \frac{1}{2}\epsilon^{ij} J_{ij}^{(2)} \wedge \left(\mathcal{F}^{(2)6+} + \mathcal{F}^{(2)k+}\phi_k + 2K^{(2)}\right) \Bigg\}. \end{aligned} \tag{B.39}$$

Here,

$$\mathcal{D}\phi_i := d\phi_i + D_{ik}^k \tilde{C}^{(1)} + G^{(1)l} D_{ik}^k \phi_l. \tag{B.40}$$

Appendix C

Calculations for the Compactification on $K3 \times T^2$

This appendix contains calculations for the reduction presented in section 2.3.3. We will start by giving the four-dimensional action obtained from the dimensional reduction, and then proceed to dualize fields until we can rewrite the action in a manifestly $SL(2,\mathbb{R}) \times SO(6,22;\mathbb{R})$-covariant way.

The Kaluza-Klein action is the action we get by entering the expansions (2.58), (2.59) and (2.60) into the supergravity action (2.15), performing the integration over the internal manifold and making the field redefinitions as in section (2.3.2). Like its ten-dimensional predecessor, we split the resulting action into a Neveu-Schwarz, Ramond-Ramond and Chern-Simons part.

Let us treat the reduction of the Ricci scalar and the kinetic term for the dilaton separately. Since $K3$ is Ricci-flat, the reduction of the ten-dimensional Ricci scalar will not yield any terms that depend only on the $K3$-coordinates. With this knowledge, one can calculate that the Ricci scalar and the kinetic term for the dilaton reduce to (relevant formulas for this calculation can be found in [1] and references therein)

$$\begin{aligned} \int d^4x\sqrt{-g}e^{-2\phi-\eta}\Big\{ & R - \nabla_\mu\big(g^{ij}\partial^\mu g_{ij}\big) - \nabla_\mu\big(g^{ab}\partial^\mu g_{ab}\big) + 4\partial_\mu\phi_{10}\partial^\mu\phi_{10} \\ & + \frac{1}{4}\partial_\mu g_{ij}\partial^\mu g^{ij} + \frac{1}{4}\partial_\mu g_{ab}\partial^\mu g^{ab} - \frac{1}{4} g_{ij}\mathcal{F}^{(2)i+}_{\mu\nu}\mathcal{F}^{(2)j+\,\mu\nu} \\ & - \frac{1}{4}g^{ij}g^{kl}\partial_\mu g_{ij}\partial^\mu g_{kl} - \frac{1}{4}g^{ab}g^{cd}\partial_\mu g_{ab}\partial^\mu g_{cd} - \frac{1}{2}g^{ij}g^{ab}\partial_\mu g_{ij}\partial^\mu g_{ab}\Big\}. \end{aligned} \tag{C.1}$$

Here,

$$\nabla_\mu V^\mu := \partial_\mu V^\mu + \Gamma^\mu_{\mu\nu}V^\nu. \tag{C.2}$$

Using

$$\partial_\mu\big(\sqrt{-g}V^\mu\big) = \sqrt{-g}\nabla_\mu V^\mu \tag{C.3}$$

yields (this result is shown in, for example, [31] and [44])

$$\int d^4x\sqrt{-g}e^{-2\phi-\eta}\Big\{R+4\partial_\mu(\phi+\frac{1}{2}\eta)\partial^\mu(\phi+\frac{1}{2}\eta)-\frac{1}{4}g_{ij}\mathcal{F}^{(2)i+}_{\mu\nu}\mathcal{F}^{(2)j+\,\mu\nu} + \frac{1}{4}\partial_\mu g_{ij}\partial^\mu g^{ij}+\frac{1}{4}\partial_\mu g_{ab}\partial^\mu g^{ab}\Big\}. \tag{C.4}$$

For a $K3$, it is known that (see for instance [39]) that

$$\frac{1}{4}\partial_\mu g_{ab}\partial^\mu g^{ab}=\frac{1}{4}\partial_\mu e^\rho\partial^\mu e^{-\rho}+\frac{1}{8}\partial_\mu H^A{}_B\partial^\mu H^B{}_A. \tag{C.5}$$

This means that the Ricci scalar and the kinetic term for the dilaton together reduce to

$$\int d^4x\sqrt{-g}e^{-2\phi-\eta}\Big\{R+4\partial_\mu(\phi+\frac{1}{2}\eta)\partial^\mu(\phi+\frac{1}{2}\eta)-\frac{1}{4}g_{ij}\mathcal{F}^{(2)i+}_{\mu\nu}\mathcal{F}^{(2)j+\,\mu\nu} + \frac{1}{4}\partial_\mu g_{ij}\partial^\mu g^{ij}+\frac{1}{4}\partial_\mu e^\rho\partial^\mu e^{-\rho}+\frac{1}{8}\partial_\mu H^A{}_B\partial^\mu H^B{}_A\Big\}. \tag{C.6}$$

We then see that the Neveu-Schwarz action, obtained by reducing the ten-dimensional Neveu-Schwarz action, is

$$\begin{aligned} S_{NS}=\int e^{-2\phi-\eta}\Big\{&d^4x\sqrt{-g}\Big[R+4\partial_\mu(\phi+\frac{1}{2}\eta)\partial^\mu(\phi+\frac{1}{2}\eta)\\ &+\frac{1}{4}(\partial_\mu e^\rho\partial^\mu e^{-\rho}+\partial_\mu g_{ij}\partial^\mu g^{ij})+\frac{1}{8}\partial_\mu H^A{}_B\partial^\mu H^B{}_A\Big]\\ &+\frac{1}{2}\Big[e^\rho\eta^{AC}H^B{}_C\,(db_A\wedge *db_B)+\frac{1}{2}g^{ik}g^{jl}db_{ij}\wedge *db_{kl}\\ &+g_{ij}\mathcal{F}^{(2)i+}\wedge *\mathcal{F}^{(2)j+}\\ &+g^{ij}\left(\mathcal{F}^{(2)-}_i+\mathcal{F}^{(2)k+}b_{ik}\right)\wedge *\left(\mathcal{F}^{(2)-}_j+\mathcal{F}^{(2)l+}b_{jl}\right)\\ &+\left(\mathcal{H}^{(3)}+\mathcal{F}^{(2)k+}\wedge B^{(1)}_k\right)\wedge *\left(\mathcal{H}^{(3)}+\mathcal{F}^{(2)l+}\wedge B^{(1)}_l\right)\Big]\Big\}. \end{aligned} \tag{C.7}$$

Fieldstrengths here are

$$\mathcal{F}^{(2)-}_i := dB^{(1)}_i, \tag{C.8}$$

$$\mathcal{F}^{(2)i+} := dG^{(1)i}, \tag{C.9}$$

$$\mathcal{H}^{(3)} := dB^{(2)}. \tag{C.10}$$

The ten-dimensional Ramond-Ramond section gives the action

$$
\begin{aligned}
S_{RR} =& \frac{1}{2}\int e^{-\eta}\Bigg\{ e^{-\rho}\Bigg[g^{ij} da_i \wedge * da_j \\
&+ \left(\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)k+} a_k\right) \wedge * \left(\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)l+} a_l\right) \\
&+ \tilde{F}^{(4)} \wedge * \tilde{F}^{(4)} + g^{ij} \tilde{F}^{(3)}_i \wedge * \tilde{F}^{(3)}_j + \frac{1}{2} g^{ik} g^{jl} \tilde{F}^{(2)}_{ij} \wedge * \tilde{F}^{(2)}_{kl} \Bigg] \\
&+ \eta^{AC} H^B{}_C \Bigg[\left(\eta_{AD}\mathcal{F}^{(2)D+} - \mathcal{F}^{(2)5+} b_A - \mathcal{F}^{(2)k+}(c_{kA} - a_k b_A)\right) \wedge \\
&* \left(\eta_{BE}\mathcal{F}^{(2)E+} - \mathcal{F}^{(2)5+} b_B - \mathcal{F}^{(2)l+}(c_{lB} - a_l b_B)\right) \\
&+ g^{ij}(dc_{iA} - da_i b_A) \wedge *(dc_{jB} - da_j b_B)\Bigg]\Bigg\},
\end{aligned}
\tag{C.11}
$$

with the definitions

$$
\tilde{F}^{(4)} := dC^{(3)} - \mathcal{F}^{(2)5+} \wedge B^{(2)} - \mathcal{F}^{(2)k+} \wedge C^{(2)}_k, \tag{C.12}
$$

$$
\begin{aligned}
\tilde{F}^{(3)}_i :=& \mathcal{H}^{(3)}_i - \mathcal{F}^{(2)5+} \wedge B^{(1)}_i - \mathcal{F}^{(2)k+} \wedge C^{(1)}_{ik} \\
&+ a_i \left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+} \wedge B^{(1)}_k\right),
\end{aligned}
\tag{C.13}
$$

$$
\begin{aligned}
\tilde{F}^{(2)}_{ij} :=& \mathcal{F}^{(2)-}_{ij} - \mathcal{F}^{(2)5+} b_{ij} \\
&- \left[a_i(\mathcal{F}^{(2)-}_j + \mathcal{F}^{(2)k+} b_{jk}) - a_j(\mathcal{F}^{(2)-}_i + \mathcal{F}^{(2)l+} b_{il})\right].
\end{aligned}
\tag{C.14}
$$

New fieldstrengths here are

$$
\mathcal{F}^{(2)5+} := dA^{(1)}, \tag{C.15}
$$

$$
\mathcal{H}^{(3)}_i := dC^{(2)}_i, \tag{C.16}
$$

$$
\mathcal{F}^{(2)-}_{ij} := dC^{(1)}_{ij}, \tag{C.17}
$$

$$
\mathcal{F}^{(2)A+} := dC^{(1)A}. \tag{C.18}
$$

Finally, the Chern-Simons term gives the topological action

$$
\begin{aligned}
S_{CS} =&\frac{1}{2}\epsilon^{ij}\eta^{AB}\int\Bigg\{-\frac{1}{2}b_{ij}\eta_{AC}\eta_{BD}\mathcal{F}^{(2)C+}\wedge\mathcal{F}^{(2)D+} \\
&+\left(\mathcal{F}_i^{(2)-}+\mathcal{F}^{(2)k+}b_{ik}\right)\wedge\left(2\eta_{AC}\mathcal{F}^{(2)C+}-\mathcal{F}^{(2)l+}(c_{lA}-a_lb_A)\right)\times \\
&(c_{jB}-a_jb_B) \\
&-\left(\mathcal{H}^{(3)}+\mathcal{F}^{(2)k+}\wedge B_k^{(1)}\right)\wedge\left(dc_{jB}-a_jdb_B-da_jb_B\right)\left(c_{iA}-a_ib_A\right) \\
&-\left(\mathcal{F}_{ij}^{(2)-}-\mathcal{F}^{(2)5+}b_{ij}\right)\wedge \\
&\left(\eta_{BC}b_A\mathcal{F}^{(2)C+}-b_A\mathcal{F}^{(2)k+}(c_{kB}-a_kb_B)-\frac{1}{2}b_Ab_A\mathcal{F}^{(2)5+}\right) \\
&+2\left(\mathcal{H}_i^{(3)}-\mathcal{F}^{(2)5+}\wedge B_i^{(1)}-\mathcal{F}^{(2)k+}\wedge C_{ik}^{(1)}\right)\wedge b_Ad\Big(c_{jB}-a_jb_B\Big)\Bigg\}.
\end{aligned}
\tag{C.19}
$$

As explained in section 2.3.2, we are now going to replace $C^{(3)}$ with its equations of motion, and dualize $C_i^{(2)}$, $C_{ij}^{(1)}$, $B^{(2)}$ and $B_i^{(1)}$, using the procedure explained in Appendix B. In fact, the equation of motion for $C^{(3)}$ just says that $\tilde{F}^{(4)}=0$, so we can remove it from the Lagrangian. The action for $C_i^{(2)}$ is

$$
\begin{aligned}
S_{C_i^{(2)}} = \int\Bigg\{&\frac{1}{2}e^{-\eta-\rho}g^{ij}\tilde{F}_i^{(3)}\wedge *\tilde{F}_j^{(3)} \\
&+\epsilon^{ij}\eta^{AB}(\mathcal{H}_i^{(3)}-\mathcal{F}^{(2)5+}\wedge B_i^{(1)}-\mathcal{F}^{(2)k+}\wedge C_{ik}^{(1)}) \\
&\wedge b_Ad(c_{jB}-a_jb_B)\Bigg\},
\end{aligned}
\tag{C.20}
$$

and dualizing $C_i^{(2)}$ to scalar fields γ_i gives

$$
\begin{aligned}
\int\Bigg\{&\frac{e^{\rho-\eta}}{2}g^{ij}\left(d\gamma_i-\eta^{AB}b_Adc_{iB}+\frac{1}{2}\eta^{AB}b_Ab_Bda_i\right)\wedge \\
&*\left(d\gamma_j-\eta^{CD}b_Cdc_{jD}+\frac{1}{2}\eta^{CD}b_Cb_Dda_j\right) \\
&+\epsilon^{ij}a_i\left(\mathcal{H}^{(3)}+\mathcal{F}^{(2)k+}\wedge B_k^{(1)}\right)\wedge \\
&\left(d\gamma_j-\eta^{AB}b_Adc_{jB}+\frac{1}{2}\eta^{AB}b_Ab_Bda_j\right) \\
&-\epsilon^{ij}\left(\mathcal{F}^{(2)k+}\wedge C_{ik}^{(1)}+\mathcal{F}^{(2)5+}\wedge B_i^{(1)}\right)\wedge d\left(\gamma_k-\frac{1}{2}\eta^{AB}b_Ab_Ba_j\right)\Bigg\}.
\end{aligned}
\tag{C.21}
$$

The action for $C_{ij}^{(1)}$ is

$$S_{C_{ij}^{(1)}} = \int \Bigg\{ \frac{e^{-\eta-\rho}}{4} g^{ik} g^{jl} \tilde{F}_{ij}^{(2)} \wedge *\tilde{F}_{kl}^{(2)} + \frac{1}{2}\epsilon^{ij} \left(\mathcal{F}_{ij}^{(2)-} - \mathcal{F}^{(2)5+} b_{ij}\right) \wedge \Bigg[- b_A \mathcal{F}^{(2)A+} + \frac{1}{2}\eta^{AB} b_A b_B \mathcal{F}^{(2)5+} - \mathcal{F}^{(2)k+} \left(\gamma_k - \eta^{AB} b_A c_{kB} + \frac{1}{2}\eta^{AB} b_A b_B a_k\right)\Bigg]\Bigg\}, \tag{C.22}$$

and this is equivalent to a dual field $\tilde{C}^{(1)}$ described by the action

$$\int \Bigg\{ \frac{e^{\rho-\eta}}{2} \Bigg(\mathcal{F}^{(2)6+} - b_A \mathcal{F}^{(2)A+} + \frac{1}{2}\eta^{AB} b_A b_B \mathcal{F}^{(2)5+} - \mathcal{F}^{(2)k+}\Big(\gamma_k - \eta^{AB} b_A c_{kB} + \frac{1}{2}\eta^{AB} b_A b_B a_k\Big)\Bigg) \wedge * \Bigg(\mathcal{F}^{(2)6+} - b_C \mathcal{F}^{(2)C+} + \frac{1}{2}\eta^{CD} b_C b_D \mathcal{F}^{(2)5+} - \mathcal{F}^{(2)k+}\Big(\gamma_k - \eta^{CD} b_C c_{kD} + \frac{1}{2}\eta^{CD} b_C b_D a_k\Big)\Bigg) + \frac{1}{2}\epsilon^{ij} b_{ij} \mathcal{F}^{(2)5+} \wedge \mathcal{F}^{(2)6+} + \frac{1}{2}\epsilon^{ij} \Big(a_i(\mathcal{F}_j^{(2)-} + \mathcal{F}^{(2)k+} b_{jk}) - a_j(\mathcal{F}_i^{(2)-} + \mathcal{F}^{(2)k+} b_{ik})\Big) \wedge \Bigg(\mathcal{F}^{(2)6+} - b_A \mathcal{F}^{(2)A+} + \frac{1}{2}\eta^{AB} b_A b_B \mathcal{F}^{(2)5+} - \mathcal{F}^{(2)k+}\Big(\gamma_k - \eta^{AB} b_A c_{kB} + \frac{1}{2}\eta^{AB} b_A b_B a_k\Big)\Bigg)\Bigg\}. \tag{C.23}$$

The field strength $\mathcal{F}^{(2)6+}$ of $\tilde{C}^{(1)}$ is

$$\mathcal{F}^{(2)6+} := d\tilde{C}^{(1)}. \tag{C.24}$$

The action of $B^{(2)}$ is

$$S_{B^{(2)}} = \int \Bigg\{ \frac{e^{-2\phi-\eta}}{2} \left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+} \wedge B_k^{(1)}\right) \wedge * \left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)l+} \wedge B_l^{(1)}\right) + \epsilon^{ij} \mathcal{H}^{(3)} \wedge \left(a_i d\gamma_j - \frac{1}{2}\eta^{AB} c_{iA} dc_{jB}\right)\Bigg\}. \tag{C.25}$$

By defining

$$A_{ij}^{(1)} := a_i d\gamma_j - a_j d\gamma_i - \frac{1}{2}\eta^{AB}\left(c_{iA} dc_{jB} - c_{jA} dc_{iB}\right), \tag{C.26}$$

we can rewrite the topological term as $\frac{1}{2}\epsilon^{ij}\mathcal{H}^{(3)} \wedge A^{(1)}_{ij}$. The action written in terms of the dual field $\beta_{ij} = -\beta_{ji}$ becomes

$$\begin{aligned}\int \Bigg\{ & \frac{e^{2\phi-\eta}}{4} g^{ik} g^{jl} \left(d\beta_{ij} - A^{(1)}_{ij} \right) \wedge * \left(d\beta_{kl} - A^{(1)}_{kl} \right) \\ & - \frac{1}{2}\epsilon^{ij} \left(d\beta_{ij} - A^{(1)}_{ij} \right) \wedge \mathcal{F}^{(2)k+} \wedge B^{(1)}_k \Bigg\}.\end{aligned} \tag{C.27}$$

The last fields we dualize are the $B^{(1)}_i$. The action is

$$\begin{aligned}\int \Bigg\{ & \frac{e^{-2\phi-\eta}}{2} g^{ij} \left(\mathcal{F}^{(2)-}_i + \mathcal{F}^{(2)k+} b_{ik} \right) \wedge * \left(\mathcal{F}^{(2)-}_j + \mathcal{F}^{(2)l+} b_{jl} \right) \\ & - \epsilon^{ij} \left(\mathcal{F}^{(2)-}_i + \mathcal{F}^{(2)k+} b_{ik} \right) \wedge \\ & \Big(a_j \mathcal{F}^{(2)6+} - c_{jA} \mathcal{F}^{(2)A+} + \gamma_j \mathcal{F}^{(2)5+} \\ & - \mathcal{F}^{(2)l+} (a_j \gamma_l - \frac{1}{2}\eta^{AB} c_{jA} c_{lB}) \Big) \\ & + \epsilon^{ij} \mathcal{F}^{(2)-}_i \wedge \mathcal{F}^{(2)k+} \beta_{kj} \Bigg\}\end{aligned} \tag{C.28}$$

This action, written in terms of the dual of $B^{(1)}_i$, $\tilde{B}^{(1)\iota}$, is

$$\begin{aligned}\int \Bigg\{ & \frac{e^{2\phi-\eta}}{2} g^{ij} \delta_{i\iota} \delta_{j\iota'} \left(\mathcal{F}^{(2)\iota+} - L^{(2)\iota} \right) \wedge * \left(\mathcal{F}^{(2)\iota+} - L^{(2)\iota'} \right) \\ & - \frac{1}{2}\epsilon^{ij} b_{ij} \delta_{k\iota} \mathcal{F}^{(2)k+} \wedge \mathcal{F}^{(2)\iota+} \Bigg\}.\end{aligned} \tag{C.29}$$

We have used the shorthand

$$\begin{aligned}L^{(2)\iota} := \delta^{i\iota} \Bigg\{ & a_i \mathcal{F}^{(2)6+} - c_{iA} \mathcal{F}^{(2)A+} + \gamma_i \mathcal{F}^{(2)5+} \\ & - \mathcal{F}^{(2)k+} (a_i \gamma_k - \frac{1}{2}\eta^{AB} c_{iA} c_{kB} - \beta_{ik}) \Bigg\}.\end{aligned} \tag{C.30}$$

The field strength $\mathcal{F}^{(2)\iota+}$ of $\tilde{B}^{(1)\iota}$ is

$$\mathcal{F}^{(2)\iota+} := d\tilde{B}^{(1)\iota}. \tag{C.31}$$

We can now simplify the action for the field strengths of the vectors. The kinetic term for the field strengths is

$$\int \Bigg\{ \frac{e^{-2\phi-\eta}}{2} g_{ij} \mathcal{F}^{(2)i+} \wedge * \mathcal{F}^{(2)j+} \tag{C.32}$$

$$+ \frac{e^{2\phi-\eta}}{2} g^{ij} \delta_{i\iota} \delta_{j\iota'} \left(\mathcal{F}^{(2)\iota+} - L^{(2)\iota} \right) \wedge * \left(\mathcal{F}^{(2)\iota+} - L^{(2)\iota'} \right)$$

$$+ \frac{e^{-\eta-\rho}}{2} \left(\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)k+} a_k \right) \wedge * \left(\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)l+} a_l \right)$$

$$+ \frac{e^{\rho-\eta}}{2} \Bigg(\mathcal{F}^{(2)6+} - b_A \mathcal{F}^{(2)A+} + \frac{1}{2} \eta^{AB} b_A b_B \mathcal{F}^{(2)5+}$$

$$- \mathcal{F}^{(2)k+} \left(\gamma_k - \eta^{AB} b_A c_{kB} + \frac{1}{2} \eta^{AB} b_A b_B a_k \right) \Bigg) \wedge$$

$$* \Bigg(\mathcal{F}^{(2)6+} - b_C \mathcal{F}^{(2)C+} + \frac{1}{2} \eta^{CD} b_C b_D \mathcal{F}^{(2)5+}$$

$$- \mathcal{F}^{(2)l+} \left(\gamma_l - \eta^{CD} b_C c_{lD} + \frac{1}{2} \eta^{CD} b_C b_D a_l \right) \Bigg)$$

$$+ \frac{e^{-\eta}}{2} \eta^{AC} H^B{}_C \left(\eta_{AD} \mathcal{F}^{(2)D+} - \mathcal{F}^{(2)5+} b_A - \mathcal{F}^{(2)k+} (c_{kA} - a_k b_A) \right) \wedge$$

$$* \left(\eta_{BE} \mathcal{F}^{(2)E+} - \mathcal{F}^{(2)5+} b_B - \mathcal{F}^{(2)l+} (c_{lB} - a_l b_B) \right) \Bigg\},$$

and the topological term for the field strengths is

$$\int \Bigg\{ - \frac{1}{2} \epsilon^{ij} b_{ij} \delta_{k\iota} \mathcal{F}^{(2)k+} \wedge \mathcal{F}^{(2)\iota+} + \frac{1}{2} \epsilon^{ij} b_{ij} \mathcal{F}^{(2)5+} \wedge \mathcal{F}^{(2)6+}$$

$$- \frac{1}{4} \epsilon^{ij} b_{ij} \eta_{AB} \mathcal{F}^{(2)A+} \wedge \mathcal{F}^{(2)B+} \Bigg\}. \tag{C.33}$$

We now introduce an $SO(6,22)$-index M that runs over i, ι, 5, 6, and A. Using this, we put all field strengths in an $SO(6,22)$-vector of fieldstrengths, $\mathcal{F}^{(2)M+}$, defined as

$$\mathcal{F}^{(2)M+} := (\mathcal{F}^{(2)i+}, \mathcal{F}^{(2)\iota+}, \mathcal{F}^{(2)5+}, \mathcal{F}^{(2)6+}, \mathcal{F}^{(2)A+}) \tag{C.34}$$

We also use the $SO(6,22)$-index M for the $SO(6,22;\mathbb{R})$ matrix M_{MN} that contains 132 out of the 134 scalars; its definition is in Appendix F. Finally, we define the complex scalar τ as

$$\tau := -\frac{1}{4} \epsilon^{ij} b_{ij} + i \frac{e^{-\eta}}{2} \tag{C.35}$$

With these definitions, we can rewrite the action for the field strengths as

$$S_{fs} = \int \Bigg\{ \mathrm{Im}(\tau) M_{MN} \mathcal{F}^{(2)M+} \wedge * \mathcal{F}^{(2)N+} \tag{C.36}$$

$$+ \mathrm{Re}(\tau) L_{MN} \mathcal{F}^{(2)M+} \wedge \mathcal{F}^{(2)N+} \Bigg\},$$

with the $SO(6,22)$-metric

$$L_{MN} = \begin{pmatrix} 0 & \delta_{i\iota} & 0 & 0 & 0 \\ \delta_{i\iota} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_{AB} \end{pmatrix}. \tag{C.37}$$

We will now simplify the kinetic term for the scalars. After a Weyl rescaling $g_{\mu\nu} \to e^{2\phi+\eta} g_{\mu\nu}$, it becomes

$$\begin{aligned} S_{sc} = \int d^4x \sqrt{-g} \Bigg\{ & R - 2\partial_\mu(\phi + \frac{1}{2}\eta)\partial^\mu(\phi + \frac{1}{2}\eta) \\ & + \frac{1}{4}(\partial_\mu e^\rho \partial^\mu e^{-\rho} + \partial_\mu g_{ij}\partial^\mu g^{ij}) + \frac{1}{8}\partial_\mu H^A{}_B \partial^\mu H^B{}_A \\ & - \frac{1}{4} g^{ik} g^{jl} \partial_\mu b_{ij} \partial^\mu b_{kl} - \frac{e^\rho}{2}\eta^{AC} H^B{}_C \partial_\mu b_A \partial^\mu b_B \\ & - \frac{e^{2\phi-\rho}}{2} g^{ij} \partial_\mu a_i \partial^\mu a_j \\ & - \frac{e^{2\phi}}{2}\eta^{AC} H^B{}_C g^{ij} (\partial_\mu c_{iA} - \partial_\mu a_i b_A)(\partial^\mu c_{jB} - \partial^\mu a_j b_B) \\ & - \frac{e^{2\phi+\rho}}{2} g^{ij} \left(\partial_\mu \gamma_i - \eta^{AB} b_A \partial_\mu c_{iB} + \frac{1}{2}\eta^{AB} b_A b_B \partial_\mu a_i\right) \times \\ & \left(\partial^\mu \gamma_j - \eta^{CD} b_C \partial^\mu c_{jD} + \frac{1}{2}\eta^{CD} b_C b_D \partial^\mu a_j\right) \\ & - \frac{e^{4\phi}}{4} g^{ik} g^{jl} \left(\partial_\mu \beta_{ij} - A^{(1)}_{\mu ij}\right)\left(\partial^\mu \beta_{kl} - A^{(1)\mu}_{kl}\right) \Bigg\} \end{aligned} \tag{C.38}$$

Using the fact that $g^{ij}\partial_\mu g_{ij} = -2\partial_\mu \eta$, we find that

$$\begin{aligned} & -2\partial_\mu(\phi + \frac{1}{2}\eta)\partial^\mu(\phi + \frac{1}{2}\eta) - \frac{1}{4} g^{ik} g^{jl} \partial_\mu b_{ij} \partial^\mu b_{kl} \\ & = -2\partial_\mu\phi\partial^\mu\phi - 2\partial_\mu\phi\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{e^{2\eta}}{8}\epsilon^{ij}\epsilon^{kl}\partial_\mu b_{ij}\partial^\mu b_{kl} \\ & = -2\partial_\mu\phi\partial^\mu\phi + g^{ij}\partial_\mu\phi\partial^\mu g_{ij} - \frac{1}{2\mathrm{Im}^2(\tau)}\partial_\mu\tau\partial^\mu\tau^* \end{aligned} \tag{C.39}$$

$M_{MN} \in SO(6,22;\mathbb{R})$, so its inverse, $(M^{-1})^{MN}$, is given by

$$(M^{-1})^{MN} = L^{MO} M_{OP} L^{PN}. \tag{C.40}$$

Since the index $M = 1, \ldots, 28$ splits into i, ι, 5, 6, and A, we calculate

$$\begin{aligned}
&\frac{g^{\mu\nu}}{8}\partial_\mu M_{MN}\partial_\nu (LML)^{MN} = \\
&\frac{1}{8}g^{\mu\nu}\Bigg\{2\delta^{i\iota}\delta^{j\iota'}\partial_\mu M_{ij}\partial^\mu M_{\iota\iota'} + 2\delta^{i\iota}\delta^{j\iota'}\partial_\mu M_{i\iota'}\partial^\mu M_{j\iota} \\
&+ 4\delta^{i\iota}\partial_\mu M_{\iota 5}\partial^\mu M_{i6} + 4\delta^{i\iota}\partial_\mu M_{\iota 6}\partial^\mu M_{i5} - 4\delta^{i\iota}\eta^{IJ}\partial_\mu M_{\iota I}\partial^\mu M_{iJ} \\
&+ 2\partial_\mu M_{55}\partial^\mu M_{66} + 2\partial_\mu M_{56}\partial^\mu M_{56} \\
&- 4\eta^{IJ}\partial_\mu M_{5I}\partial^\mu M_{6J} + \eta^{AC}\eta^{BD}\partial_\mu M_{AB}\partial^\mu M_{CD}\Bigg\}.
\end{aligned} \tag{C.41}$$

Using eqs. (C.41) and (F.97), calculating the scalar term is a straightforward procedure, resulting in

$$S_{sc} = \int d^4x\sqrt{-g}\Bigg\{R - \frac{1}{2\mathrm{Im}^2(\tau)}\partial_\mu\tau\partial^\mu\tau^* + \frac{1}{8}\partial_\mu M_{MN}\partial^\mu (LML)^{MN}\Bigg\}. \tag{C.42}$$

Appendix D

Calculations for the Compactification on Y_1 with $H_{10}^{(3)}$-flux

This appendix contains calculations for the reduction presented in section 2.4, of IIA supergravity on Y_1 with an $H_{10}^{(3)}$-flux. We will start by giving the four-dimensional action obtained from the dimensional reduction, and then proceed to dualize fields until we can rewrite the action in a manifestly $SL(2,\mathbb{R}) \times SO(6,n;\mathbb{R})$-covariant way. Finally, we show that it is invariant under gauge transformations.

We enter the expansions (2.58), (2.59) and (2.60), together with the expression for $H_{10}^{(3)}$ as given in (2.86) into the supergravity action (2.15), perform the integration over the internal manifold and make the field redefinitions in eq. (2.62).

Again, we start by looking at the reduction of the action of the Ricci scalar and the kinetic term for the dilaton. In this reduction, we will make three assumptions that are, as yet, unproven. The first assumption is that, in analogy to the compactification on $K3 \times T^2$, the reduction of the Ricci scalar will not yield any terms that depend only on the $Y_1^{(4)}$-coordinate. With this assumption in place, the reduction of the Ricci scalar and the kinetic term for the dilaton gives

$$\begin{aligned}\int d^4x\sqrt{-g}e^{-2\phi-\eta}\Big\{ & R - \nabla_\mu\big(g^{ij}\partial^\mu g_{ij}\big) - \nabla_\mu\big(g^{ab}\partial^\mu g_{ab}\big) + 4\partial_\mu\phi_{10}\partial^\mu\phi_{10} \\ & - \partial_i\big(g^{ab}\partial_i g_{ab} - g^{ab}G_\mu^{(1)i}\mathcal{D}^\mu g_{ab}\big) \\ & + \frac{1}{4}\partial_\mu g_{ij}\partial^\mu g^{ij} + \frac{1}{4}\mathcal{D}_\mu g_{ab}\mathcal{D}^\mu g^{ab} + \frac{1}{4}\partial_i g_{ab}\partial^i g^{ab} \\ & - \frac{1}{4}g^{ij}g^{kl}\partial_\mu g_{ij}\partial^\mu g_{kl} - \frac{1}{4}g^{ab}g^{cd}\mathcal{D}_\mu g_{ab}\mathcal{D}^\mu g_{cd} - \frac{1}{2}g^{ij}g^{ab}\partial_\mu g_{ij}\mathcal{D}^\mu g_{ab} \\ & - \frac{1}{4}g^{ab}g^{cd}\partial_i g_{ab}\partial^i g_{cd} - \frac{1}{4}g_{ij}\mathcal{F}_{\mu\nu}^{(2)i+}\mathcal{F}^{(2)j+\mu\nu}\Big\}\end{aligned} \quad \text{(D.1)}$$

with

$$\mathcal{D}_\mu g_{ab} = \partial_\mu g_{ab} + G_\mu^{(1)i}\partial_i g_{ab}. \tag{D.2}$$

From the transformation of other fields, we know that the fields coming from g_{ab}, $H^A{}_B$ and $e^{-\rho}$, transform in such a way that

$$\mathcal{D}_\mu H^A{}_B := \partial_\mu H^A{}_B + G_\mu^{(1)k} D^C_{kB} H^A{}_C - G_\mu^{(1)k} D^A_{kC} H^C{}_B, \tag{D.3}$$

$$\mathcal{D}_\mu e^{-\rho} := \partial_\mu e^{-\rho}. \tag{D.4}$$

We have not actually shown that this follows from the reduction of the Ricci scalar; that this does follow is our second assumption. Since the term

$$g^{ab}\mathcal{D}_\mu g_{ab} \tag{D.5}$$

gives the covariant derivative of the volume $e^{-\rho}$, and we know from (D.4) that the volume should not transform under gauge transformations, we are going to make our final assumption:

$$g^{ab}\partial^i g_{ab} = 0. \tag{D.6}$$

Using this, the second line in eq. (D.1) vanishes, and we can calculate that the reduction gives

$$\int d^4x\sqrt{-g}e^{-2\phi-\eta}\Big\{R + 4\partial_\mu\big(\phi + \frac{1}{2}\eta\big)\partial^\mu\big(\phi+\frac{1}{2}\eta\big) - \frac{1}{4}g_{ij}\mathcal{F}^{(2)i+}_{\mu\nu}\mathcal{F}^{(2)j+\,\mu\nu} + \frac{1}{4}\partial_\mu g_{ij}\partial^\mu g^{ij} + \frac{1}{4}\mathcal{D}_\mu g_{ab}\mathcal{D}^\mu g^{ab} + \frac{1}{4}g^{ij}\partial_i g_{ab}\partial_j g^{ab}\Big\}. \tag{D.7}$$

The second assumption tells us that

$$\frac{1}{4}\mathcal{D}_\mu g_{ab}\mathcal{D}^\mu g^{ab} = \frac{1}{4}\partial_\mu e^\rho \partial^\mu e^{-\rho} + \frac{1}{8}\mathcal{D}_\mu H^A{}_B \mathcal{D}^\mu H^B{}_A. \tag{D.8}$$

To calculate the potential term $\frac{1}{4}g^{ij}\partial_i g_{ab}\partial_j g^{ab}$, we look at the term in $\mathcal{D}_\mu g_{ab}\mathcal{D}^\mu g^{ab}$ that is quadratic in $G_\mu^{(1)i}$. That term is $\partial_i g_{ab}\partial_j g^{ab}$. From eq. (D.3) we can then infer that

$$\frac{1}{4}g^{ij}\partial_i g_{ab}\partial_j g^{ab} = \frac{1}{8}g^{ij}\Big(D^C_{iA}H_{CB} + D^C_{iB}H_{AC}\Big)\Big(D^A_{jD}H^{DB} + D^B_{jD}H^{AD}\Big) \tag{D.9}$$

for

$$H^{AB} := \eta^{AC}H^B{}_C, \quad H_{AB} := \eta_{AC}H^C{}_B. \tag{D.10}$$

The Neveu-Schwarz action, obtained by reducing the ten-dimensional Neveu-Schwarz action, is then

$$
\begin{aligned}
S_{NS} = \int e^{-2\phi-\eta}\Bigg\{ & d^4x\sqrt{-g}\Big[R + 4\partial_\mu(\phi+\frac{1}{2}\eta)\partial^\mu(\phi+\frac{1}{2}\eta) \\
& + \frac{1}{4}\Big(\partial_\mu e^{\rho}\partial^\mu e^{-\rho} + \partial_\mu g_{ij}\partial^\mu g^{ij}\Big) + \frac{1}{8}\mathcal{D}_\mu H^A{}_B \mathcal{D}^\mu H^B{}_A \\
& - \frac{1}{8}g^{ij}\Big(D^C_{iA}H_{CB} + D^C_{iB}H_{AC}\Big)\Big(D^A_{jD}H^{DB} + D^B_{jD}H^{AD}\Big)\Big] \\
& + \frac{1}{2}\Big[e^{\rho}H^{AB}\Big(\mathcal{D}b_A \wedge *\mathcal{D}b_B + g^{ij}(b_D D^D_{iA} + k_{iA}) \wedge *(b_E D^E_{jB} + k_{jB})\Big) \\
& + \frac{1}{2}g^{ik}g^{jl}db_{ij}\wedge *db_{kl} + g_{ij}\mathcal{F}^{(2)i+}\wedge *\mathcal{F}^{(2)j+} \\
& + g^{ij}\left(\mathcal{F}^{(2)-}_i + \mathcal{F}^{(2)k+}b_{ik}\right)\wedge *\left(\mathcal{F}^{(2)-}_j + \mathcal{F}^{(2)l+}b_{jl}\right) \\
& + \left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+}\wedge B^{(1)}_k\right)\wedge *\left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)l+}\wedge B^{(1)}_l\right)\Big]\Bigg\}.
\end{aligned}
\tag{D.11}
$$

Covariant derivatives and fieldstrengths here are

$$\mathcal{D}b_A := db_A + G^{(1)k}(k_{kA} + D^C_{kA}b_C), \tag{D.12}$$

$$\mathcal{F}^{(2)-}_i := dB^{(1)}_i, \tag{D.13}$$

$$\mathcal{F}^{(2)i+} := dG^{(1)i}, \tag{D.14}$$

$$\mathcal{H}^{(3)} := dB^{(2)}. \tag{D.15}$$

The ten-dimensional Ramond-Ramond section gives the action

$$
\begin{aligned}
S_{RR} = & \frac{1}{2}\int e^{-\eta}\Bigg\{e^{-\rho}\Big[\Big(\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)k+}a_k\Big)\wedge *\Big(\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)l+}a_l\Big) \\
& + g^{ij}da_i\wedge *da_j + \tilde{F}^{(4)}\wedge *\tilde{F}^{(4)} + g^{ij}\tilde{F}^{(3)}_i\wedge *\tilde{F}^{(3)}_j + \frac{1}{2}g^{ik}g^{jl}\tilde{F}^{(2)}_{ij}\wedge *\tilde{F}^{(2)}_{kl}\Big] \\
& + H^{AB}\Big[\Big(\eta_{AD}\mathcal{F}^{(2)D+} - \mathcal{F}^{(2)5+}b_A - \mathcal{F}^{(2)k+}(c_{kA} - a_k b_A)\Big)\wedge \\
& * \Big(\eta_{BE}\mathcal{F}^{(2)E+} - \mathcal{F}^{(2)5+}b_B - \mathcal{F}^{(2)l+}(c_{lB} - a_l b_B)\Big) \\
& + g^{ij}\Big(\mathcal{D}c_{iA} - da_i b_A\Big)\wedge *\Big(\mathcal{D}c_{jB} - da_j b_B\Big) + \frac{1}{2}g^{ik}g^{jl}\tilde{F}_{ijA}\wedge *\tilde{F}_{klB}\Big]\Bigg\},
\end{aligned}
\tag{D.16}
$$

with the definitions

$$\tilde{F}^{(4)} := dC^{(3)} - \mathcal{F}^{(2)5+} \wedge B^{(2)} - \mathcal{F}^{(2)k+} \wedge C_k^{(2)}, \tag{D.17}$$

$$\tilde{F}_i^{(3)} := \mathcal{H}_i^{(3)} - \mathcal{F}^{(2)5+} \wedge B_i^{(1)} - \mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)} + a_i \left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+} \wedge B_k^{(1)} \right), \tag{D.18}$$

$$\tilde{F}_{ij}^{(2)} := \mathcal{F}_{ij}^{(2)-} - \mathcal{F}^{(2)5+} \wedge b_{ij} - \left(a_i (\mathcal{F}_j^{(2)-} + \mathcal{F}^{(2)k+} b_{jk}) - a_j (\mathcal{F}_i^{(2)-} + \mathcal{F}^{(2)l+} b_{il}) \right), \tag{D.19}$$

$$\tilde{F}_{ijA} := D_{iA}^D c_{jD} - D_{jA}^D c_{iD} - (a_i k_{jA} - a_j k_{iA}). \tag{D.20}$$

New covariant derivatives and fieldstrengths are

$$\mathcal{F}^{(2)5+} := dA^{(1)}, \tag{D.21}$$

$$\mathcal{H}_i^{(3)} := dC_i^{(2)}, \tag{D.22}$$

$$\mathcal{F}_{ij}^{(2)-} := dC_{ij}^{(1)}, \tag{D.23}$$

$$\mathcal{F}^{(2)A+} := dC^{(1)A} - G^{(1)k} \wedge D_{kB}^A C^{(1)B} + G^{(1)k} k_{kB} \eta^{AB} A^{(1)}, \tag{D.24}$$

$$\mathcal{D} c_{iA} := dc_{iA} - C^{(1)B} \eta_{BC} D_{iA}^C + G^{(1)k} (D_{kA}^C c_{iC} + k_{kB} a_i) - A^{(1)} k_{iA}. \tag{D.25}$$

Finally, the Chern-Simons term gives the topological action

$$\begin{aligned} S_{CS} =& \frac{1}{2} \epsilon^{ij} \eta^{AB} \int \Bigg\{ - \frac{1}{2} b_{ij} \Big(\eta_{AC} \eta_{BD} \mathcal{F}^{(2)C+} \wedge \mathcal{F}^{(2)D+} \\ & - 2\eta_{AC} C^{(1)C} \wedge d\big(G^{(1)k} \wedge k_{kB} A^{(1)}\big) \Big) \\ & + \left(\mathcal{F}_i^{(2)-} + \mathcal{F}^{(2)k+} b_{ik} \right) \wedge \Big(\big(2\eta_{AC} \mathcal{F}^{(2)C+} - \mathcal{F}^{(2)l+} (c_{lA} - a_l b_A) \big) (c_{jB} - a_j b_B) \\ & + 2A^{(1)} k_{jA} \eta_{BC} C^{(1)C} - D_{jB} \eta_{AD} C^{(1)D} \wedge \eta_{CE} C^{(1)E} \Big) \\ & - \left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+} \wedge B_k^{(1)} \right) \wedge \left(c_{iA} - a_i b_A \right) \times \\ & \left(\mathcal{D} c_{jB} - a_j \mathcal{D} b_B - d a_j b_B - D_{jB}^C \eta_{CD} C^{(1)D} - A^{(1)} k_{jB} \right) \\ & - \left(\mathcal{F}_{ij}^{(2)-} - \mathcal{F}^{(2)5+} b_{ij} \right) \wedge \Big(\eta_{BC} b_A \mathcal{F}^{(2)C+} - \frac{1}{2} b_A b_A \mathcal{F}^{(2)5+} \\ & - b_A \mathcal{F}^{(2)k+} (c_{kB} - a_k b_B) - G^{(1)k} \wedge k_{kA} \eta_{BC} C^{(1)C} \Big) \\ & + 2 \Big(\mathcal{H}_i^{(3)} - \mathcal{F}^{(2)5+} \wedge B_i^{(1)} - \mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)} \Big) \wedge \\ & \Big(b_A \mathcal{D} c_{jB} - b_A a_j \mathcal{D} b_B - d a_j b_A b_B - k_{jA} A^{(1)} b_B - G^{(1)k} k_{kA} (c_{jB} - a_j b_B) \Big) \\ & + 2 \left(dC^{(3)} - \mathcal{F}^{(2)k+} \wedge B^{(2)} + \mathcal{F}^{(2)k+} \wedge C_k^{(2)} \right) \left(k_{iA} (c_{jB} - a_j b_B) + D_{iA}^C b_C c_{jB} \right) \Bigg\}. \end{aligned} \tag{D.26}$$

As explained in section 2.4, we are now going to replace $C^{(3)}$ with its equations of motion, and dualize $C_i^{(2)}$, $C_{ij}^{(1)}$, $B^{(2)}$ and $B_i^{(1)}$, following the procedure outlined in Appendix B. We start by using $C^{(3)}$'s equations of motion to calculate

$$\begin{aligned}S_{C^{(3)}} &= \int \left\{ \frac{e^{-\eta-\rho}}{2} \tilde{F}^{(4)} \wedge *\tilde{F}^{(4)} + \tilde{F}^{(4)} \epsilon^{ij} \eta^{AB} \big(k_{iA}(c_{jB} - a_j b_B) + D_{iA}^C b_C c_{jB} \big) \right\} \quad \text{(D.27)} \\ &= \int \left\{ \frac{e^{\eta+\rho}}{2} \epsilon^{ij} \epsilon^{kl} \eta^{AB} \eta^{CD} \Big(k_{iA}(c_{jB} - a_j b_B) + D_{iA}^E b_E c_{jB} \Big) \wedge \right. \\ &\quad \left. * \Big(k_{kC}(c_{lD} - a_l b_D) + D_{kC}^F b_F c_{lD} \Big) \right\}.\end{aligned}$$

The action for $C_i^{(2)}$ is

$$\begin{aligned}S_{C_i^{(2)}} = \int \Bigg\{ & \frac{e^{-\eta-\rho}}{2} g^{ij} \tilde{F}_i^{(3)} \wedge *\tilde{F}_j^{(3)} \qquad \text{(D.28)} \\ & + \epsilon^{ij} \eta^{AB} \Big(\mathcal{H}_i^{(3)} - \mathcal{F}^{(2)5+} \wedge B_i^{(1)} - \mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)} \Big) \wedge \\ & \Big(b_A \mathcal{D} c_{jB} - a_j b_A \mathcal{D} b_B - da_j b_A b_B - k_{jA} A^{(1)} b_B - G^{(1)k} k_{kA} (c_{jB} - a_j b_B) \Big) \Bigg\},\end{aligned}$$

and dualizing $C_i^{(2)}$ to scalar fields γ_i gives

$$\begin{aligned}\int \Bigg\{ & \frac{e^{\rho-\eta}}{2} g^{ij} \Big(\mathcal{D}\gamma_i - \eta^{AB} b_A \mathcal{D} c_{iB} + \frac{1}{2} \eta^{AB} b_A b_B da_i \Big) \wedge \qquad \text{(D.29)} \\ & * \Big(\mathcal{D}\gamma_j - \eta^{CD} b_C \mathcal{D} c_{jD} + \frac{1}{2} \eta^{CD} b_C b_D da_j \Big) \\ & + \epsilon^{ij} a_i \Big(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+} \wedge B_k^{(1)} \Big) \wedge \Big(\mathcal{D}\gamma_j - \eta^{AB} b_A \mathcal{D} c_{jB} + \frac{1}{2} \eta^{AB} b_A b_B da_j \Big) \\ & - \epsilon^{ij} \Big(\mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)} + \mathcal{F}^{(2)5+} \wedge B_i^{(1)} \Big) \wedge d\Big(\gamma_k - \frac{1}{2} \eta^{AB} b_A b_B a_j \Big) \Bigg\}.\end{aligned}$$

The covariant derivative of γ_i is

$$\mathcal{D}\gamma_i := d\gamma_i - C^{(1)A} k_{iA} + G^{(1)k} \eta^{AB} k_{kA} c_{iB}. \qquad \text{(D.30)}$$

The action for $C_{ij}^{(1)}$ is

$$\begin{aligned}S_{C_{ij}^{(1)}} = \int \Bigg\{ & \frac{e^{-\eta-\rho}}{4} g^{ik} g^{jl} \tilde{F}_{ij}^{(2)} \wedge *\tilde{F}_{kl}^{(2)} \qquad \text{(D.31)} \\ & + \frac{1}{2} \epsilon^{ij} \Big(\mathcal{F}_{ij}^{(2)-} - \mathcal{F}^{(2)5+} b_{ij} \Big) \wedge \Big(- b_A \mathcal{F}^{(2)A+} + \frac{1}{2} \eta^{AB} b_A b_B \mathcal{F}^{(2)5+} \\ & - \mathcal{F}^{(2)k+} (\gamma_k - \eta^{AB} b_A c_{kB} + \frac{1}{2} \eta^{AB} b_A b_B a_k) + G^{(1)k} \wedge k_{kA} C^{(1)A} \Big) \Bigg\},\end{aligned}$$

and this is equivalent to a dual field $\tilde{C}^{(1)}$ described by the action

$$\begin{aligned}\int \Bigg\{ &\frac{e^{\rho-\eta}}{2}\Big(\mathcal{F}^{(2)6+} - b_A\mathcal{F}^{(2)A+} + \frac{1}{2}\eta^{AB}b_Ab_B\mathcal{F}^{(2)5+} \\ &- \mathcal{F}^{(2)k+}(\gamma_k - \eta^{AB}b_Ac_{kB} + \frac{1}{2}\eta^{AB}b_Ab_Ba_k)\Big)\wedge \\ &* \Big(\mathcal{F}^{(2)6+} - b_C\mathcal{F}^{(2)C+} + \frac{1}{2}\eta^{CD}b_Cb_D\mathcal{F}^{(2)5+} \\ &- \mathcal{F}^{(2)k+}(\gamma_k - \eta^{CD}b_Cc_{kD} + \frac{1}{2}\eta^{CD}b_Cb_Da_k)\Big) \\ &+ \frac{1}{2}\epsilon^{ij}b_{ij}\mathcal{F}^{(2)5+}\wedge d\tilde{C}^{(1)} \\ &+ \frac{1}{2}\epsilon^{ij}\Big(a_i\big(\mathcal{F}_j^{(2)-} + \mathcal{F}^{(2)k+}b_{jk}\big) - a_j\big(\mathcal{F}_i^{(2)-} + \mathcal{F}^{(2)k+}b_{ik}\big)\Big)\wedge \\ &\Big(\mathcal{F}^{(2)6+} - b_A\mathcal{F}^{(2)A+} + \frac{1}{2}\eta^{AB}b_Ab_B\mathcal{F}^{(2)5+} \\ &- \mathcal{F}^{(2)k+}(\gamma_k - \eta^{AB}b_Ac_{kB} + \frac{1}{2}\eta^{AB}b_Ab_Ba_k)\Big)\Bigg\}. \end{aligned} \tag{D.32}$$

The field strength $\mathcal{F}^{(2)6+}$ of $\tilde{C}^{(1)}$ is

$$\mathcal{F}^{(2)6+} := d\tilde{C}^{(1)} + G^{(1)k}\wedge k_{kA}C^{(1)A}. \tag{D.33}$$

The action of $B^{(2)}$ is

$$\begin{aligned} S_{B^{(2)}} = \int \Bigg\{ &\frac{e^{-2\phi-\eta}}{2}\Big(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+}\wedge B_k^{(1)}\Big)\wedge * \Big(\mathcal{H}^{(3)} + \mathcal{F}^{(2)l+}\wedge B_l^{(1)}\Big) \\ &+ \epsilon^{ij}\Big(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+}\wedge B_k^{(1)}\Big)\wedge \\ &\Big(a_i\mathcal{D}\gamma_j - \frac{1}{2}\eta^{AB}c_{iA}\big(\mathcal{D}c_{jB} - \eta_{CD}C^{(1)D}D^C_{jB} - A^{(1)}k_{jB}\big)\Big)\Bigg\}. \end{aligned} \tag{D.34}$$

By defining

$$A_{ij}^{(1)} := a_i\mathcal{D}\gamma_j - a_j\mathcal{D}\gamma_i - \frac{1}{2}\eta^{AB}\left(c_{iA}\mathcal{D}c_{jB} - c_{jA}\mathcal{D}c_{iB}\right), \tag{D.35}$$

we can rewrite the topological term as

$$\frac{1}{2}\epsilon^{ij}\mathcal{H}^{(3)}\wedge\Big(A_{ij}^{(1)} + \eta^{AB}G^{(1)k}k_{kA}c_{iB}a_j + \eta^{AB}A^{(1)}c_{iA}k_{jB} - C^{(1)B}c_{iA}D^A_{jB}\Big). \tag{D.36}$$

The action written in terms of the dual field $\beta_{ij} = -\beta_{ji}$ becomes

$$\begin{aligned}\int \Bigg\{ &\frac{e^{2\phi-\eta}}{4}g^{ik}g^{jl}\left(\mathcal{D}\beta_{ij} - A_{ij}^{(1)}\right)\wedge * \left(\mathcal{D}\beta_{kl} - A_{kl}^{(1)}\right) \\ &- \frac{1}{2}\epsilon^{ij}\left(\mathcal{D}\beta_{ij} - A_{ij}^{(1)}\right)\wedge\mathcal{F}^{(2)k+}\wedge B_k^{(1)}\Bigg\}, \end{aligned} \tag{D.37}$$

(D.38)

with the covariant derivative

$$\mathcal{D}\beta_{ij} :=d\beta_{ij} + \frac{1}{2}G^{(1)k}k_{kA}\eta^{AB}(a_i c_{jB} a_j c_{iB}) + \frac{1}{2}A^{(1)}\eta^{AB}(k_{iA}c_{jB} - k_{jA}c_{iB}) \\ + \frac{1}{2}C^{(1)A}(c_{iB}D^B_{jA} - c_{jB}D^B_{iA}). \tag{D.39}$$

The last fields we dualize are the $B_i^{(1)}$. The action is

$$\int \Bigg\{ \frac{e^{-2\phi-\eta}}{2} g^{ij} \left(\mathcal{F}_i^{(2)-} + \mathcal{F}^{(2)k+}b_{ik}\right) \wedge * \left(\mathcal{F}_j^{(2)-} + \mathcal{F}^{(2)l+}b_{jl}\right) \\ - \epsilon^{ij}\left(\mathcal{F}_i^{(2)-} + \mathcal{F}^{(2)k+}b_{ik}\right) \wedge \\ \Big(a_j\mathcal{F}^{(2)6+} - c_{jA}\mathcal{F}^{(2)A+} + \gamma_j\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)l+}(a_j\gamma_l - \frac{1}{2}\eta^{AB}c_{jA}c_{lB}) \\ + A^{(1)} \wedge k_{jA}C^{(1)A} - \frac{1}{2}\eta_{BC}D^C_{jA}C^{(1)A}\wedge C^{(1)B}\Big)\Bigg\} \tag{D.40}$$

This action, written in terms of the dual of $B_i^{(1)}$, $\tilde{B}^{(1)\iota}$, is

$$\int \Bigg\{ \frac{e^{2\phi-\eta}}{2} g^{ij}\delta_{i\iota}\delta_{j\iota'} \left(\mathcal{F}^{(2)\iota+} - L^{(2)\iota}\right) \wedge * \left(\mathcal{F}^{(2)\iota+} - L^{(2)\iota'}\right) \\ - \frac{1}{2}\epsilon^{ij}b_{ij}\delta_{k\iota}\mathcal{F}^{(2)k+}\wedge\mathcal{F}^{(2)\iota+}\Bigg\}, \tag{D.41}$$

where we have used the shorthand

$$L^{(2)\iota} := \delta^{i\iota}\Big\{a_i\mathcal{F}^{(2)6+} - c_{iA}\mathcal{F}^{(2)A+} + \gamma_i\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)k+}(a_i\gamma_k - \frac{1}{2}\eta^{AB}c_{iA}c_{kB} - \beta_{ik})\Big\}. \tag{D.42}$$

The field strength $\mathcal{F}^{(2)\iota+}$ of $\tilde{B}^{(1)\iota}$ is

$$\mathcal{F}^{(2)\iota+} := d\tilde{B}^{(1)\iota} + A^{(1)}\wedge k_{iA}C^{(1)A} - \frac{1}{2}\eta_{BC}\delta^{i\iota}D^C_{iA}C^{(1)A}\wedge C^{(1)B}. \tag{D.43}$$

We can now simplify the action for the field strengths of the vectors. The kinetic term for the field strengths is

$$
\begin{aligned}
\int \Bigg\{ & \frac{e^{-2\phi-\eta}}{2} g_{ij} \mathcal{F}^{(2)i+} \wedge *\mathcal{F}^{(2)j+} \\
& + \frac{e^{2\phi-\eta}}{2} g^{ij} \delta_{i\iota} \delta_{j\iota'} \left(\mathcal{F}^{(2)\iota+} - L^{(2)\iota} \right) \wedge * \left(\mathcal{F}^{(2)\iota+} - L^{(2)\iota'} \right) \\
& + \frac{e^{-\eta-\rho}}{2} \left(\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)k+} a_k \right) \wedge * \left(\mathcal{F}^{(2)5+} - \mathcal{F}^{(2)l+} a_l \right) \\
& + \frac{e^{\rho-\eta}}{2} \Bigg(\mathcal{F}^{(2)6+} - b_A \mathcal{F}^{(2)A+} + \frac{1}{2} \eta^{AB} b_A b_B \mathcal{F}^{(2)5+} \\
& - \mathcal{F}^{(2)k+} \left(\gamma_k - \eta^{AB} b_A c_{kB} + \frac{1}{2} \eta^{AB} b_A b_B a_k \right) \Bigg) \wedge \\
& * \Bigg(\mathcal{F}^{(2)6+} - b_C \mathcal{F}^{(2)C+} + \frac{1}{2} \eta^{CD} b_C b_D \mathcal{F}^{(2)5+} \\
& - \mathcal{F}^{(2)l+} \left(\gamma_l - \eta^{CD} b_C c_{lD} + \frac{1}{2} \eta^{CD} b_C b_D a_l \right) \Bigg) \\
& + \frac{e^{-\eta}}{2} H^{AB} \left(\eta_{AD} \mathcal{F}^{(2)D+} - \mathcal{F}^{(2)5+} b_A - \mathcal{F}^{(2)k+} (c_{kA} - a_k b_A) \right) \wedge \\
& * \left(\eta_{BE} \mathcal{F}^{(2)E+} - \mathcal{F}^{(2)5+} b_B - \mathcal{F}^{(2)l+} (c_{lB} - a_l b_B) \right) \Bigg\},
\end{aligned} \tag{D.44}
$$

and the topological term for the field strengths is

$$
\begin{aligned}
\int \Bigg\{ & -\frac{1}{2} \epsilon^{ij} b_{ij} \delta_{k\iota} \mathcal{F}^{(2)k+} \wedge \mathcal{F}^{(2)\iota+} + \frac{1}{2} \epsilon^{ij} b_{ij} \mathcal{F}^{(2)5+} \wedge \mathcal{F}^{(2)6+} \\
& - \frac{1}{4} \epsilon^{ij} b_{ij} \eta_{AB} \mathcal{F}^{(2)A+} \wedge \mathcal{F}^{(2)B+} \Bigg\}.
\end{aligned} \tag{D.45}
$$

We now introduce an $SO(6,n)$-index M that runs over i, ι, 5, 6, and A. Using this, we put all field strengths in an $SO(6,n)$-vector of fieldstrengths, $\mathcal{F}^{(2)M+}$, defined as

$$
\mathcal{F}^{(2)M+} := (\mathcal{F}^{(2)i+}, \mathcal{F}^{(2)\iota+}, \mathcal{F}^{(2)5+}, \mathcal{F}^{(2)6+}, \mathcal{F}^{(2)A+}) \tag{D.46}
$$

We also use the $SO(6,n)$-index M for the $SO(6,n;\mathbb{R})$ matrix M_{MN} that contains $6n$ out of the $6n+2$ scalars; its definition is in Appendix F. Finally, we define the complex scalar τ as

$$
\tau := -\frac{1}{4} \epsilon^{ij} b_{ij} + i \frac{e^{-\eta}}{2} \tag{D.47}
$$

With all these definitions, we can rewrite the action for the field strengths as

$$
S_{fs} = \int \Bigg\{ \mathrm{Im}(\tau) M_{MN} \mathcal{F}^{(2)M+} \wedge *\mathcal{F}^{(2)N+} + \mathrm{Re}(\tau) L_{MN} \mathcal{F}^{(2)M+} \wedge \mathcal{F}^{(2)N+} \Bigg\}, \tag{D.48}
$$

with the $SO(6,n;\mathbb{R})$-metric

$$L_{MN}=\begin{pmatrix} 0 & \delta_{i\iota} & 0 & 0 & 0 \\ \delta_{i\iota} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_{AB} \end{pmatrix}. \tag{D.49}$$

We will now simplify the kinetic term for the scalars. After a Weyl rescaling $g_{\mu\nu} \to e^{2\phi+\eta}g_{\mu\nu}$, it has become

$$\begin{aligned} S_{sc} = \int d^4x\sqrt{-g}\Big\{ & R - 2\partial_\mu(\phi+\frac{1}{2}\eta)\partial^\mu(\phi+\frac{1}{2}\eta) \\ & + \frac{1}{4}\big(\partial_\mu e^\rho \partial^\mu e^{-\rho} + \partial_\mu g_{ij}\partial^\mu g^{ij}\big) + \frac{1}{8}\mathcal{D}_\mu H^A{}_B \mathcal{D}^\mu H^B{}_A \\ & - \frac{1}{4}g^{ik}g^{jl}\partial_\mu b_{ij}\partial^\mu b_{kl} - \frac{e^\rho}{2}H^{AB}\mathcal{D}_\mu b_A \mathcal{D}^\mu b_B - \frac{e^{2\phi-\rho}}{2}g^{ij}\partial_\mu a_i \partial^\mu a_j \\ & - \frac{e^{2\phi}}{2}H^{AB}g^{ij}\big(\mathcal{D}_\mu c_{iA} - \partial_\mu a_i b_A\big)\big(\mathcal{D}^\mu c_{jB} - \partial^\mu a_j b_B\big) \\ & - \frac{e^{2\phi+\rho}}{2}g^{ij}\Big(\mathcal{D}_\mu \gamma_i - \eta^{AB}b_A \mathcal{D}_\mu c_{iB} + \frac{1}{2}\eta^{AB}b_A b_B \partial_\mu a_i\Big)\times \\ & \Big(\mathcal{D}^\mu \gamma_j - \eta^{CD}b_C \mathcal{D}^\mu c_{jD} + \frac{1}{2}\eta^{CD}b_C b_D \partial^\mu a_j\Big) \\ & - \frac{e^{4\phi}}{4}g^{ik}g^{jl}\Big(\mathcal{D}_\mu \beta_{ij} - A^{(1)}_{\mu ij}\Big)\Big(\mathcal{D}^\mu \beta_{kl} - A^{(1)\mu}_{kl}\Big)\Big\} \end{aligned} \tag{D.50}$$

As in Appendix C, we find that

$$\begin{aligned} & -2\partial_\mu(\phi+\frac{1}{2}\eta)\partial^\mu(\phi+\frac{1}{2}\eta) - \frac{1}{4}g^{ik}g^{jl}\partial_\mu b_{ij}\partial^\mu b_{kl} \\ & = -2\partial_\mu\phi\partial^\mu\phi - 2\partial_\mu\phi\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{e^{2\eta}}{8}\epsilon^{ij}\epsilon^{kl}\partial_\mu b_{ij}\partial^\mu b_{kl} \\ & = -2\partial_\mu\phi\partial^\mu\phi + g^{ij}\partial_\mu\phi\partial^\mu g_{ij} - \frac{1}{2\mathrm{Im}^2(\tau)}\partial_\mu\tau\partial^\mu\tau^* \end{aligned} \tag{D.51}$$

The inverse of M_{MN} is again given by

$$(M^{-1})^{MN} = L^{MO}M_{OP}L^{PN}. \tag{D.52}$$

Since the index $M = 1, \ldots, 6+n$ splits into i, ι, 5, 6, and A, we calculate

$$\frac{g^{\mu\nu}}{8}\mathcal{D}_\mu M_{MN}\mathcal{D}_\nu(LML)^{MN} =$$
$$\frac{1}{8}g^{\mu\nu}\Bigg\{2\delta^{i\iota}\delta^{j\iota'}\mathcal{D}_\mu M_{ij}\mathcal{D}^\mu M_{\iota\iota'} + 2\delta^{i\iota}\delta^{j\iota'}\mathcal{D}_\mu M_{i\iota'}\mathcal{D}^\mu M_{j\iota} + 4\delta^{i\iota}\mathcal{D}_\mu M_{\iota 5}\mathcal{D}^\mu M_{i6}$$
$$+ 4\delta^{i\iota}\mathcal{D}_\mu M_{\iota 6}\mathcal{D}^\mu M_{i5} - 4\delta^{i\iota}\eta^{IJ}\mathcal{D}_\mu M_{\iota I}\mathcal{D}^\mu M_{iJ} + 2\mathcal{D}_\mu M_{55}\mathcal{D}^\mu M_{66}$$
$$+ 2\mathcal{D}_\mu M_{56}\mathcal{D}^\mu M_{56} - 4\eta^{IJ}\mathcal{D}_\mu M_{5I}\mathcal{D}^\mu M_{6J} + \eta^{AC}\eta^{BD}\mathcal{D}_\mu M_{AB}\mathcal{D}^\mu M_{CD}\Bigg\}. \qquad \text{(D.53)}$$

The covariant derivatives are given in Appendix F. Using eqs. (D.53) and (F.44), calculating the scalar term is a straightforward procedure, resulting in

$$S_{sc} = \int d^4x\sqrt{-g}\Bigg\{R - \frac{1}{2\text{Im}^2(\tau)}\partial_\mu\tau\partial^\mu\tau^* + \frac{1}{8}\mathcal{D}_\mu M_{MN}\mathcal{D}^\mu(LML)^{MN}\Bigg\}. \qquad \text{(D.54)}$$

The potential as obtained from the reduction and dualization is

$$S_{pot} = -\int\sqrt{-g}\Bigg\{\frac{e^{2\phi+\eta}}{8}g^{ij}\Big(D^C_{iA}H_{CB} + D^C_{iB}H_{AC}\Big)\Big(D^A_{jD}H^{DB} + D^B_{jD}H^{AD}\Big) \qquad \text{(D.55)}$$
$$+ \frac{e^{2\phi+\eta+\rho}}{2}H^{AB}g^{ij}(b_D D^D_{iA} + k_{iA})(b_E D^E_{jB} + k_{jB})$$
$$+ \frac{e^{4\phi+3\eta+\rho}}{2}\epsilon^{ij}\epsilon^{kl}\eta^{AB}\eta^{CD}\Big(k_{iA}(c_{jB} - a_j b_B) + D^E_{iA}b_E c_{jB}\Big)\times$$
$$\Big(k_{kC}(c_{lD} - a_l b_D) + D^F_{kC}b_F c_{lD}\Big)$$
$$+ \frac{e^{4\phi+\eta}}{4}\eta^{AC}H^B{}_C g^{ik}g^{jl}\tilde{F}_{ijA}\tilde{F}_{klB}\Bigg\}.$$

We can rewrite this, using the scalar matrix M_{MN}, as

$$S_{pot} = \frac{1}{8}\int\sqrt{-g}\Bigg\{\delta^{i\iota}\delta^{j\iota'}\text{Im}(\tau)^{-1}\Bigg\{D^B_{iA}D^A_{jB}M_{\iota\iota'} \qquad \text{(D.56)}$$
$$- D^C_{iE}D^D_{jF}\eta^{AE}\eta^{BF}\Big(M_{\iota\iota'}M_{AB}M_{CD} - 2M_{\iota B}M_{\iota' A}M_{CD}\Big)$$
$$+ k_{iA}\eta^{BD}D^C_{jD}\Big(M_{\iota\iota'}M_{6B}M_{AC} + M_{\iota C}M_{6\iota'}M_{AB} + M_{\iota B}M_{6C}M_{A\iota'}\Big)$$
$$- k_{iC}k_{jD}\eta^{AC}\eta^{BD}\Big(M_{\iota\iota'}M_{66}M_{AB} + 2M_{\iota B}M_{6\iota'}M_{6A}\Big)\Bigg\}\Bigg\}.$$

It remains to be shown that the action is invariant under the various gauge transformations corresponding to the gauge fields. The parameters belonging to the gauge fields are ξ^i for $G^{(1)i}$, λ^ι for $\tilde{B}^{(1)\iota}$, Λ for $A^{(1)}$, $\tilde{\lambda}$ for $\tilde{C}^{(1)}$, and λ^A for $C^{(1)A}$. How the fields transform under these transformations can be found in Appendix (F), taking into account that $D_{ij}^k = 0$ for now, and that $B^{(2)++}$ is not present in this case. We immediately see that R and τ are invariant under gauge transformations.

The transformations of $\mathcal{F}^{(2)M+}$, M_{MN} and $\mathcal{D}M_{MN}$, as shown in Appendix F, can all be written as infinitesimal $SO(6,n)$-rotations generated by the gauge parameters. Since the action for the scalars, S_{kin}, and the action for the field strengths, S_{fs}, are both $SO(6,n)$-scalars, they are gauge invariant.

Finally, using eqs. (2.77), (2.81), (2.90) and the formulation of the potential as written in eq. (D.55), it is easily seen that the potential is invariant under gauge transformations as well.

Appendix E

Calculations for the Compactification on Y_2

This Appendix will contain calculations for the reduction presented in section 2.6. We will start exactly as we did in Appendix D, by presenting the Kaluza-Klein action and the dualization of several fields. The last step towards the manifestly $SL(2,\mathbb{R})\times SO(6,n;\mathbb{R})$-covariant way of writing the action is taken in detail in the main text so we do not present it here. We will, however, show how the potential and topological term can be written in the form as found in [19] and described in section 2.5. Finally, we show the gauge invariance of the action.

Once again, let us first say something about the reduction of the action of the Ricci scalar and the kinetic term for the dilaton. We are going to make the assumptions similar to the ones we made in Appendix D. We assume that the reduction of the Ricci scalar will not yield any terms that depend on the $Y_2^{(4)}$-coordinate alone, and that

$$\partial_i\big(g^{ab}\partial^i g_{ab}\big) = 0. \tag{E.1}$$

The reason for the second assumption can, again, be found in the covariant derivative of the volume modulus $e^{-\rho}$. We can calculate the covariant derivative

$$\mathcal{D}_\mu g_{ij} := \partial_\mu g_{ij} + G_\mu^{(1)k}\left(D^l_{kj}g_{il} + D^l_{ki}g_{lj}\right), \tag{E.2}$$

directly from the ten-dimensional Ricci scalar. Together with the transformation of the other fields in the theory, it follows that

$$\mathcal{D}_\mu e^{-\rho} := \partial_\mu e^{-\rho} - G_\mu^{(1)k}D^l_{kl}e^{-\rho}, \tag{E.3}$$

$$\mathcal{D}_\mu H^A{}_B := \partial_\mu H^A{}_B + G_\mu^{(1)k}\left(D^C_{kB}H^A{}_C - D^A_{kC}H^C{}_B\right). \tag{E.4}$$

In particular, we see that the term linear in $G^{(1)i}$ in the covariant derivative of $e^{-\rho}$ does only depend on the spacetime-coordinate x. Up to a factor, that term is given by $g^{ab}\partial^i g_{ab}$. Therefore we assume that

$$\partial_i\big(g^{ab}\partial^i g_{ab}\big) = 0. \tag{E.5}$$

With these two assumptions in place, we find that the reduction of the Ricci scalar gives

$$\int d^4x\sqrt{-g}e^{-2\phi-\eta}\Big\{R - \nabla_\mu\big(g^{ij}\partial^\mu g_{ij}\big) - \nabla_\mu\big(g^{ab}\partial^\mu g_{ab}\big) \tag{E.6}$$
$$+\frac{1}{4}\mathcal{D}_\mu g_{ij}\mathcal{D}^\mu g^{ij} + \frac{1}{4}\mathcal{D}_\mu g_{ab}\mathcal{D}^\mu g^{ab} + \frac{1}{4}\partial_i g_{ab}\partial^i g^{ab}$$
$$-\frac{1}{4}g^{ij}g^{kl}\mathcal{D}_\mu g_{ij}\mathcal{D}^\mu g_{kl} - \frac{1}{4}g^{ab}g^{cd}\mathcal{D}_\mu g_{ab}\mathcal{D}^\mu g_{cd} - \frac{1}{2}g^{ij}g^{ab}\mathcal{D}_\mu g_{ij}\mathcal{D}^\mu g_{ab}$$
$$-\frac{1}{4}g^{ab}g^{cd}\partial_i g_{ab}\partial^i g_{cd} - g^{ij}D^k_{ik}D^l_{jl} - \frac{1}{4}g_{ij}\mathcal{F}^{(2)i+}_{\mu\nu}\mathcal{F}^{(2)j+\,\mu\nu}\Big\}.$$

Here,

$$\mathcal{D}_\mu g_{ab} := \partial_\mu g_{ab} + G^{(1)i}_\mu \partial_i g_{ab}. \tag{E.7}$$

Again assuming that the reduction of the Ricci scalar does indeed give the kinetic terms for $e^{-\rho}$ and $H^A{}_B$, we can calculate

$$\frac{1}{4}\mathcal{D}_\mu g_{ab}\mathcal{D}^\mu g^{ab} = \frac{1}{4}\mathcal{D}_\mu e^\rho \mathcal{D}^\mu e^{-\rho} + \frac{1}{8}\mathcal{D}_\mu H^A{}_B\mathcal{D}^\mu H^B{}_A. \tag{E.8}$$

and

$$\frac{1}{4}g^{ij}\partial_i g_{ab}\partial_j g^{ab} - \frac{1}{4}g^{ab}g^{cd}\partial_i g_{ab}\partial^i g_{cd}$$
$$= \frac{1}{8}g^{ij}\Big(D^C_{iA}H_{CB} + D^C_{iB}H_{AC}\Big)\Big(D^A_{jD}H^{DB} + D^B_{jD}H^{AD}\Big) \tag{E.9}$$

for

$$H^{AB} := \eta^{AC}H^B{}_C, \quad H_{AB} := \eta_{AC}H^C{}_B. \tag{E.10}$$

The Neveu-Schwarz action is

$$S_{NS} = \int e^{-2\phi-\eta}\Big\{d^4x\sqrt{-g}\Big[R + 4\partial_\mu(\phi + \frac{1}{2}\eta)\partial^\mu(\phi+\frac{1}{2}\eta) \tag{E.11}$$
$$+\frac{1}{4}\Big(\mathcal{D}_\mu e^\rho\mathcal{D}^\mu e^{-\rho} + \mathcal{D}_\mu g_{ij}\mathcal{D}^\mu g^{ij}\Big) + \frac{1}{8}\mathcal{D}_\mu H^A{}_B\mathcal{D}^\mu H^B{}_A$$
$$-\frac{1}{8}g^{ij}\Big(D^C_{iA}H_{CB} + D^C_{iB}H_{AC}\Big)\Big(D^A_{jD}H^{DB} + D^B_{jD}H^{AD}\Big)\Big]$$
$$-g^{ij}D^k_{ik}D^l_{jl}$$
$$+\frac{1}{2}\Big[e^\rho H^{AB}\Big(\mathcal{D}b_A\wedge *\mathcal{D}b_B + g^{ij}(b_D D^D_{iA})\wedge *(b_E D^E_{jB})\Big)$$
$$+\frac{1}{2}g^{ik}g^{jl}\mathcal{D}b_{ij}\wedge *\mathcal{D}b_{kl} + g_{ij}\mathcal{F}^{(2)i+}\wedge *\mathcal{F}^{(2)j+}$$
$$+g^{ij}\Big(\mathcal{F}^{(2)-}_i + \mathcal{F}^{(2)k+}b_{ik}\Big)\wedge *\Big(\mathcal{F}^{(2)-}_j + \mathcal{F}^{(2)l+}b_{jl}\Big)$$
$$+\Big(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+}\wedge B^{(1)}_k\Big)\wedge *\Big(\mathcal{H}^{(3)} + \mathcal{F}^{(2)l+}\wedge B^{(1)}_l\Big)\Big]\Big\},$$

Here, we have the following covariant derivatives and fieldstrengths:

$$\mathcal{D}b_A := db_A + G^{(1)k} D^C_{kA} b_C, \tag{E.12}$$

$$\mathcal{D}b_{ij} := db_{ij} - B^{(1)}_k D^k_{ij} + G^{(1)k}\left(D^l_{kj} b_{il} + D^l_{ki} b_{lj}\right), \tag{E.13}$$

$$\mathcal{F}^{(2)-}_i := dB^{(1)}_i + G^{(1)k} \wedge D^l_{ki} B^{(1)}_l, \tag{E.14}$$

$$\mathcal{F}^{(2)i+} := dG^{(1)i} - \frac{1}{2} G^{(1)k} \wedge G^{(1)l} D^i_{kl}, \tag{E.15}$$

$$\mathcal{H}^{(3)} := dB^{(2)}. \tag{E.16}$$

and we have again defined

$$H^{AB} := \eta^{AC} H^B{}_C, \quad H_{AB} := \eta_{AC} H^C{}_B. \tag{E.17}$$

The Ramond-Ramond action is given by

$$\begin{aligned} S_{RR} =& \frac{1}{2}\int e^{-\eta}\Bigg\{ e^{-\rho}\Bigg[\left(\mathcal{F}^{(2)5+} + \mathcal{F}^{(2)k+} a_k\right) \wedge * \left(\mathcal{F}^{(2)5+} + \mathcal{F}^{(2)l+} a_l\right) \\ &+ g^{ij} \mathcal{D}a_i \wedge * \mathcal{D}a_j + \frac{1}{2} g^{ik} g^{jl} (a_m D^m_{ij}) \wedge *(a_n D^n_{kl}) \\ &+ \tilde{F}^{(4)} \wedge * \tilde{F}^{(4)} + g^{ij} \tilde{F}^{(3)}_i \wedge * \tilde{F}^{(3)}_j + \frac{1}{2} g^{ik} g^{jl} \tilde{F}^{(2)}_{ij} \wedge * \tilde{F}^{(2)}_{kl} \Bigg] \\ &+ H^{AB}\Bigg[\left(\eta_{AD} \mathcal{F}^{(2)D+} - \mathcal{F}^{(2)5+} b_A - \mathcal{F}^{(2)k+}(c_{kA} - a_k b_A)\right) \wedge \\ &* \left(\eta_{BE} \mathcal{F}^{(2)E+} - \mathcal{F}^{(2)5+} b_B - \mathcal{F}^{(2)l+}(c_{lB} - a_l b_B)\right) \\ &+ g^{ij}\left(\mathcal{D}c_{iA} - \mathcal{D}a_i b_A\right) \wedge * \left(\mathcal{D}c_{jB} - \mathcal{D}a_j b_B\right) + \frac{1}{2} g^{ik} g^{jl} \tilde{F}_{ijA} \wedge * \tilde{F}_{klA} \Bigg]\Bigg\}, \end{aligned} \tag{E.18}$$

with the shorthands

$$\tilde{F}^{(4)} := dC^{(3)} - \mathcal{F}^{(2)5+} \wedge B^{(2)} - \mathcal{F}^{(2)k+} \wedge C^{(2)}_k, \tag{E.19}$$

$$\tilde{F}^{(3)}_i := \mathcal{H}^{(3)}_i - \mathcal{F}^{(2)5+} \wedge B^{(1)}_i - \mathcal{F}^{(2)k+} \wedge C^{(1)}_{ik} + a_i\left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+} \wedge B^{(1)}_k\right), \tag{E.20}$$

$$\tilde{F}^{(2)}_{ij} := \mathcal{F}^{(2)-}_{ij} - \mathcal{F}^{(2)5+} b_{ij} - \left(a_i\big(\mathcal{F}^{(2)-}_j + \mathcal{F}^{(2)k+} b_{jk}\big) - a_j\big(\mathcal{F}^{(2)-}_i + \mathcal{F}^{(2)l+} b_{il}\big)\right), \tag{E.21}$$

$$\tilde{F}_{ijA} := D^k_{ij}\big(c_{kA} - a_k b_A\big) + D^C_{iA} c_{jC} - D^C_{jA} c_{iC}. \tag{E.22}$$

The covariant derivatives and fieldstrengths here are

$$\mathcal{D}a_i := da_i + G^{(1)k} D^l_{ki} a_l, \tag{E.23}$$

$$\mathcal{F}^{(2)5+} := dA^{(1)}, \tag{E.24}$$

$$\mathcal{H}^{(3)}_i := dC^{(2)}_i + G^{(1)k} \wedge D^l_{ki} C^{(2)}_l, \tag{E.25}$$

$$\mathcal{F}^{(2)-}_{ij} := dC^{(1)}_{ij} + C^{(2)}_k D^k_{ij} + G^{(1)k}\Big(D^l_{kj} C^{(1)}_{il} + D^l_{ki} C^{(1)}_{lj}\Big), \tag{E.26}$$

$$\mathcal{F}^{(2)A+} := dC^{(1)A} - G^{(1)k} \wedge \Big(D^A_{kB} C^{(1)B} + D^l_{kl} C^{(1)A}\Big), \tag{E.27}$$

$$\mathcal{D}c_{iA} := dc_{iA} - \eta_{BC} D^B_{iA} C^{(1)C} + G^{(1)k}\left(D^C_{kA} c_{iC} + D^l_{ki} c_{lA}\right). \tag{E.28}$$

The Chern-Simons term looks very much like before, the main difference is in the definition of the covariant derivatives and fieldstrengths.

$$\begin{aligned}
S_{CS} =& \frac{1}{2}\epsilon^{ij}\eta^{AB}\int\Bigg\{-\frac{1}{2}b_{ij}\eta_{AC}\mathcal{F}^{(2)C+}\wedge\eta_{BD}\mathcal{F}^{(2)+}_{D} \\
&+\Big(\mathcal{F}^{(2)-}_i + \mathcal{F}^{(2)k+}b_{ik}\Big)\wedge\Big(2\eta_{AC}\mathcal{F}^{(2)C+} - \mathcal{F}^{(2)l+}(c_{lA} - a_l b_A)\Big)\Big(c_{jB} - a_j b_B\Big) \\
&- 2B^{(1)}_i \eta_{AD}\eta_{CE}\mathcal{F}^{(2)D+}\wedge C^{(1)E} D^C_{jB} \\
&- \Big(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+}\wedge B^{(1)}_k\Big)\wedge\Big(\mathcal{D}c_{jB} - a_j\mathcal{D}b_B - \mathcal{D}a_j b_B - \eta_{CD}C^{(1)D}D^C_{jB}\Big)\times \\
&\Big(c_{iA} - a_i b_A\Big) - \Big(\mathcal{F}^{(2)-}_{ij} - \mathcal{F}^{(2)5+}b_{ij}\Big)\wedge \\
&\Big(\eta_{BC}b_A\mathcal{F}^{(2)C+} - b_A\mathcal{F}^{(2)k+}(c_{kB} - a_k b_B) - \frac{1}{2}b_A b_B\mathcal{F}^{(2)5+}\Big) \\
&+ 2\Big(\mathcal{H}^{(3)}_i - \mathcal{F}^{(2)5+}\wedge B^{(1)}_i - \mathcal{F}^{(2)k+}\wedge C^{(1)}_{ik}\Big)\wedge \\
&b_A\Big(\mathcal{D}c_{jB} - a_j\mathcal{D}b_B - \mathcal{D}a_j b_B - \eta_{CD}C^{(1)D}D^C_{jB}\Big) \\
&- \Big(dC^{(3)} - \mathcal{F}^{(2)5+}\wedge B^{(2)} - \mathcal{F}^{(2)k+}\wedge C^{(2)}_k\Big)\times \\
&b_A\Big(D^k_{ij}c_{kB} + D^C_{iB}c_{jC} - D^C_{jB}c_{iC}\Big)\Bigg\}.
\end{aligned} \tag{E.29}$$

We are now going to replace $C^{(3)}$ with its equations of motion, and dualize $C^{(2)}_i$, $C^{(1)}_{ij}$ and $B^{(2)}$. We start by using $C^{(3)}$'s equations of motion to rewrite

$$S_{C^{(3)}} = \frac{1}{2}\int\Bigg\{e^{-\eta-\rho}\tilde{F}^{(4)}\wedge *\tilde{F}^{(4)} - \epsilon^{ij}\eta^{AB}b_A\tilde{F}^{(4)}\Big(D^m_{ij}c_{mB} + D^C_{iB}c_{jC} - D^C_{jB}c_{iC}\Big)\Bigg\} \tag{E.30}$$

to

$$\frac{1}{2}\int\Bigg\{\frac{e^{\eta+\rho}}{4}\epsilon^{ij}\epsilon^{kl}\eta^{AB}\eta^{CD}b_A b_C\Big(D^m_{ij}c_{mB} + 2D^E_{iB}c_{jE}\Big)\wedge *\Big(D^n_{kl}c_{nD} + 2D^E_{kD}c_{jE}\Big)\Bigg\}. \tag{E.31}$$

The next step is to dualize the two-forms $C_i^{(2)}$, as we did in the last section as well. As we can see from the definition of $\mathcal{F}_{ij}^{(2)-}$ in eq. (E.26), however, a linear combination of the $C_i^{(2)}$'s is massive through a Stueckelberg mechanism. As detailed in Appendix B, we will dualize the two-forms $C_i^{(2)}$ and the one-form $C_{ij}^{(1)}$ into two scalars γ_i, a linear combination of which is eaten by a one-form $\tilde{C}^{(1)}$. The action for $C_i^{(2)}$ and $C_{ij}^{(1)}$ is

$$\begin{aligned} S_{C_i^{(2)},C_{ij}^{(1)}} = \int \Bigg\{ & \frac{e^{-\eta-\rho}g^{ij}}{2}\tilde{F}_i^{(3)} \wedge *\tilde{F}_j^{(3)} + \frac{e^{-\eta-\rho}g^{ik}g^{jl}}{4}\tilde{F}_{ij}^{(2)} \wedge *\tilde{F}_{kl}^{(2)} \\ & + \epsilon^{ij}\eta^{AB}\Big(\mathcal{H}_i^{(3)} - \mathcal{F}^{(2)k+} \wedge C_{ik}^{(1)}\Big) \wedge b_A\Big(\mathcal{D}c_{jB} - a_j\mathcal{D}b_B - \mathcal{D}a_j b_B\Big) \\ & - \frac{1}{2}\epsilon^{ij}\eta^{AB}\mathcal{F}_{ij}^{(2)-} \wedge \\ & \Big(b_A\eta_{BC}\mathcal{F}^{(2)C+} + b_A\mathcal{F}^{(2)k+}(c_{kB} - a_k b_B) - \frac{1}{2}b_A b_B\mathcal{F}^{(2)5+}\Big)\Bigg\}, \end{aligned} \tag{E.32}$$

Referring to Appendix B, we see that the dualized action is

$$\begin{aligned} \int \Bigg\{ & \frac{e^{\rho-\eta}g^{ij}}{2}\left(\mathcal{D}\gamma_i - \eta^{AB}b_A\mathcal{D}c_{iB} + \frac{1}{2}\eta^{AB}b_A b_B\mathcal{D}a_i\right) \wedge \\ & *\left(\mathcal{D}\gamma_j - \eta^{CD}b_C\mathcal{D}c_{jD} + \frac{1}{2}\eta^{CD}b_C b_D\mathcal{D}a_j\right) \\ & + \frac{e^{\rho-\eta}}{2}\Big(\mathcal{F}^{(2)6+} - b_A\mathcal{F}^{(2)A+} + \frac{1}{2}\eta^{AB}b_A b_B\mathcal{F}^{(2)5+} - \\ & \mathcal{F}^{(2)k+}\big(\gamma_k - \eta^{AB}b_A c_{iB} + \frac{1}{2}\eta^{AB}b_A b_B a_k\big)\Big) \wedge \\ & *\Big(\mathcal{F}^{(2)6+} - b_C\mathcal{F}^{(2)C+} + \frac{1}{2}\eta^{CD}b_C b_D\mathcal{F}^{(2)5+} \\ & - \mathcal{F}^{(2)k+}\big(\gamma_k - \eta^{CD}b_C c_{iD} + \frac{1}{2}\eta^{CD}b_C b_D a_k\big)\Big) \\ & + \frac{1}{2}\left(\mathcal{F}^{(2)5+}b_{ij} + a_i\left(\mathcal{F}_j^{(2)-} + \mathcal{F}^{(2)k+}b_{jk}\right) - a_j\left(\mathcal{F}_i^{(2)-} + \mathcal{F}^{(2)k+}b_{ik}\right)\right) \wedge \\ & \Big(\mathcal{F}^{(2)6+} - b_A\mathcal{F}^{(2)A+} + \frac{1}{2}\eta^{AB}b_A b_B\mathcal{F}^{(2)5+} \\ & - \mathcal{F}^{(2)k+}\big(\gamma_k - \eta^{AB}b_A c_{iB} + \frac{1}{2}\eta^{AB}b_A b_B a_k\big)\Big) \\ & + \epsilon^{ij}\Big(a_i\big(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+} \wedge B_k^{(1)}\big) - \mathcal{F}^{(2)5+} \wedge B_i^{(1)}\Big) \wedge \\ & \Big(\mathcal{D}\gamma_j - \eta^{AB}b_A\mathcal{D}c_{jB} + \frac{1}{2}\eta^{AB}b_A b_B\mathcal{D}a_j\Big)\Bigg\}. \end{aligned} \tag{E.33}$$

The field strength of the new vector field $\tilde{C}^{(1)}$ is

$$\mathcal{F}^{(2)6+} := d\tilde{C}^{(1)} - G^{(1)k} \wedge D_{kl}^l\tilde{C}^{(1)}, \tag{E.34}$$

while the covariant derivative of γ_i,

$$\mathcal{D}\gamma_i := d\gamma_i + D^k_{ik}\tilde{C}^{(1)} - G^{(1)l} D^k_{ik}\gamma_l, \tag{E.35}$$

shows that one of the two scalars γ_i is eaten by $\tilde{C}^{(1)}$, making $\tilde{C}^{(1)}$ massive through the Stueckelberg mechanism. Incidentally, the field γ_i is related to the field ϕ_i from Appendix B by $\gamma_i := \phi_i + \frac{1}{2}\eta^{AB} b_A b_B a_i$.

The action of $B^{(2)}$ is

$$\begin{aligned} S_{B^{(2)}} = \int \Bigg\{ & \frac{e^{-2\phi-\eta}}{2} \left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)k+} \wedge B^{(1)}_k\right) \wedge * \left(\mathcal{H}^{(3)} + \mathcal{F}^{(2)l+} \wedge B^{(1)}_l\right) \\ & + \epsilon^{ij}\mathcal{H}^{(3)} \wedge \left(a_i \mathcal{D}\gamma_j - \frac{1}{2}\eta^{AB} c_{iA}(\mathcal{D}c_{jB} - C^{(1)C}\eta_{CD} D^D_{jB}\right) \Bigg\}. \end{aligned} \tag{E.36}$$

By defining

$$A^{(1)}_{ij} := a_i d\gamma_j - a_j d\gamma_i - \frac{1}{2}\eta^{AB}\left(c_{iA}\mathcal{D}c_{jB} - c_{jA}\mathcal{D}c_{iB}\right), \tag{E.37}$$

we can rewrite the topological term as

$$\frac{1}{2}\epsilon^{ij}\mathcal{H}^{(3)} \wedge \left(A^{(1)}_{ij} + \frac{1}{2}\eta^{AB}\eta_{CD} C^{(1)D}(c_{iA}D^C_{jB} - c_{jA}D^C_{iB})\right). \tag{E.38}$$

The action written in terms of the dual field β_{ij} becomes

$$\begin{aligned} \int \Bigg\{ & \frac{e^{2\phi-\eta}}{4} g^{ik} g^{jl} \left(\mathcal{D}\beta_{ij} - A^{(1)}_{ij}\right) \wedge * \left(\mathcal{D}\beta_{kl} - A^{(1)}_{kl}\right) \\ & + \frac{1}{2}\epsilon^{ij} \left(\mathcal{D}\beta_{ij} - A^{(1)}_{ij}\right) \wedge \mathcal{F}^{(2)k+} \wedge B^{(1)}_k \Bigg\}, \end{aligned} \tag{E.39}$$

with the covariant derivative

$$\mathcal{D}\beta_{ij} := d\beta_{ij} + \frac{1}{2} C^{(1)A}\Big(\left(c_{iB}D^B_{jA} + c_{iA}D^k_{jk}\right) - \left(c_{jB}D^B_{iA} + c_{jA}D^k_{ik}\right)\Big). \tag{E.40}$$

Let us now take a look at the kinetic term for the scalars. We perform a Weyl rescaling

$g_{\mu\nu} \to e^{2\phi+\eta} g_{\mu\nu}$, and after that the term is

$$\begin{aligned}
S_{sc} = \int d^4x\sqrt{-g}\Bigg\{ & R - 2\partial_\mu(\phi + \frac{1}{2}\eta)\partial^\mu(\phi + \frac{1}{2}\eta) \\
& + \frac{1}{4}(\mathcal{D}_\mu e^\rho \mathcal{D}^\mu e^{-\rho} + \mathcal{D}_\mu g_{ij}\mathcal{D}^\mu g^{ij}) + \frac{1}{8}\mathcal{D}_\mu H^A{}_B \mathcal{D}^\mu H^B{}_A \\
& - \frac{1}{4} g^{ik} g^{jl} \mathcal{D}_\mu b_{ij} \mathcal{D}^\mu b_{kl} - \frac{e^\rho}{2}\eta^{AC} H^B{}_C \mathcal{D}_\mu b_A \mathcal{D}^\mu b_B - \frac{e^{2\phi-\rho}}{2} g^{ij}\mathcal{D}_\mu a_i \mathcal{D}^\mu a_j \\
& - \frac{e^{2\phi}}{2}\eta^{AC} H^B{}_C g^{ij}(\mathcal{D}_\mu c_{iA} - \mathcal{D}_\mu a_i b_A)(\mathcal{D}^\mu c_{jB} - \mathcal{D}^\mu a_j b_B) \\
& - \frac{e^{2\phi+\rho}}{2} g^{ij}\left(\mathcal{D}_\mu \gamma_i - \eta^{AB} b_A \mathcal{D}_\mu c_{iB} + \frac{1}{2}\eta^{AB} b_A b_B \mathcal{D}_\mu a_i\right)\times \\
& \left(\mathcal{D}^\mu \gamma_j - \eta^{CD} b_C \mathcal{D}^\mu c_{jD} + \frac{1}{2}\eta^{CD} b_C b_D \mathcal{D}^\mu a_j\right) \\
& - \frac{e^{4\phi}}{4} g^{ik} g^{jl}\left(\mathcal{D}_\mu \beta_{ij} - A^{(1)}_{\mu ij}\right)\left(\mathcal{D}^\mu \beta_{kl} - A^{(1)\mu}_{kl}\right)\Bigg\}
\end{aligned} \tag{E.41}$$

Using the fact that $g^{ij}\mathcal{D}_\mu g_{ij} = 2e^\eta \mathcal{D}_\mu e^{-\eta}$, with

$$\mathcal{D}e^{-\eta} := de^{-\eta} - G^{(1)k} D^l_{kl} e^{-\eta}, \tag{E.42}$$

we find that

$$\begin{aligned}
& - 2\partial_\mu(\phi + \frac{1}{2}\eta)\partial^\mu(\phi + \frac{1}{2}\eta) - \frac{1}{4} g^{ik} g^{jl}\mathcal{D}_\mu b_{ij}\mathcal{D}^\mu b_{kl} \\
= & \frac{1}{2}\mathcal{D}_\mu e^{2\phi}\mathcal{D}^\mu e^{-2\phi} + \frac{1}{2}\mathcal{D}_\mu e^\eta \mathcal{D}^\mu e^\eta \\
= & + \frac{1}{2} e^{2\phi-\eta}\mathcal{D}_\mu e^{-2\phi}\mathcal{D}^\mu e^\eta + \frac{1}{2} e^{-2\phi+\eta}\mathcal{D}_\mu e^{2\phi}\mathcal{D}^\mu e^{-\eta} - \frac{e^{2\eta}}{8}\epsilon^{ij}\epsilon^{kl}\mathcal{D}_\mu b_{ij}\mathcal{D}^\mu b_{kl} \\
= & \frac{1}{2}\mathcal{D}_\mu e^{2\phi}\mathcal{D}^\mu e^{-2\phi} - \frac{1}{2\mathrm{Im}^2(\tau)}\mathcal{D}_\mu \tau \mathcal{D}^\mu \tau^* \\
& . + \frac{1}{4}\left(g^{ij} e^{-2\phi}\mathcal{D}_\mu e^{2\phi}\mathcal{D}^\mu g_{ij} + g_{ij} e^{2\phi}\mathcal{D}_\mu e^{-2\phi}\mathcal{D}^\mu g^{ij}\right)
\end{aligned} \tag{E.43}$$

Here,

$$\tau := -\frac{1}{4}\epsilon^{ij} b_{ij} + i\frac{e^{-\eta}}{2}, \tag{E.44}$$

and the covariant derivative of τ,

$$\mathcal{D}\tau := d\tau + G^{(1)k} D^l_{kl}\tau + \frac{1}{4} B^{(1)}_k \epsilon^{ij} D^k_{ij}, \tag{E.45}$$

follows from eqs. (E.13) and (E.42). The covariant derivatives of M_{MN} are defined in eq. (F.97). Using the expansion of the kinetic scalar term, (D.53), we can calculate that the kinetic term for the scalars is given by

$$S_{sc} = \int d^4x\sqrt{-g}\left\{ R - \frac{1}{2\mathrm{Im}^2(\tau)}\mathcal{D}_\mu \tau \mathcal{D}^\mu \tau^* + \frac{1}{8}\mathrm{Tr}\mathcal{D}_\mu M \mathcal{D}^\mu (LML)\right\}. \tag{E.46}$$

The information in $\mathcal{D}\tau$, together with the field strengths $\mathcal{F}^{(2)k+}$ (E.15), $\mathcal{F}^{(2)5+}$ (E.24), $\mathcal{F}^{(2)6+}$ (E.34) and $\mathcal{F}^{(2)A+}$ (E.27), allows us to read off the non-zero components of the embedding tensor. We can use the definition of $M_{\alpha\beta}$,

$$M_{\alpha\beta} := \frac{1}{\mathrm{Im}(\tau)} \begin{pmatrix} |\tau|^2 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & 1 \end{pmatrix}, \tag{E.47}$$

to rewrite $\mathcal{D}\tau$ to

$$\mathcal{D}M_{--} := dM_{--} - G^{(1)k} D^l_{kl} M_{--}, \tag{E.48}$$

$$\mathcal{D}M_{+-} := dM_{-+} + B^{(1)}_k (\frac{1}{4}\epsilon^{ij} D^k_{ij}) M_{--}, \tag{E.49}$$

$$\mathcal{D}M_{++} := dM_{++} + G^{(1)k} D^l_{kl} M_{++} + B^{(1)}_k (\frac{1}{2}\epsilon^{ij} D^k_{ij}) M_{+-}. \tag{E.50}$$

A comparison of this with eq. (2.130) tells us that

$$\xi_{+i} = D^k_{ik}, \tag{E.51}$$

while all other ξ_{+M}'s and all ξ_{-M}'s are zero. We also see that the vector A^{-i} is given by

$$A^{-i} = -\frac{1}{2}\epsilon^{ij} B^{(1)}_j. \tag{E.52}$$

Knowing this, let us take a look at the field strengths. For $\mathcal{F}^{(2)i+}$, we have

$$\mathcal{F}^{(2)i+} := dG^{(1)i} - \frac{1}{2} D^i_{kl} G^{(1)k} \wedge G^{(1)l}. \tag{E.53}$$

This has to be equal to eq. (2.132),

$$\mathcal{F}^{(2)i+} = dG^{(1)i} - \frac{1}{2}\hat{f}_{+NP}{}^{i} A^N \wedge A^P, \tag{E.54}$$

and using the definition of $\hat{f}_{+MNP}$ in eq. (2.133),

$$\hat{f}_{+MNP} := f_{+MNP} - \xi_{+[M} L_{P]N} - \frac{3}{2}\xi_{+N} L_{MP}, \tag{E.55}$$

we find that $f_{+ij\iota} = \frac{1}{2} D^k_{ij} \delta_{k\iota}$. We can find the other f's in the same way; the nonzero components of the embedding tensor are

$$\begin{aligned} \xi_{+i} &= D^k_{ik}, \\ f_{+ij\iota} &= \frac{1}{2} D^k_{ij} \delta_{k\iota}, \\ f_{+i56} &= \frac{1}{2} D^k_{ik}, \\ f_{+iAB} &= -\eta_{AC} D^C_{iB} - \frac{1}{2}\eta_{AB} D^k_{ik}. \end{aligned} \tag{E.56}$$

These components have to satisfy the constraint equation

$$3f_{+R[MN}f_{+PQ]}{}^{R} + 2\xi_{[M}f_{+NPQ]} = 0, \tag{E.57}$$

and the f_{+MNP} have to be antisymmetric in all indices. This last requirement is only non-trival for f_{+iAB}, and it is satisfied because of eq. (2.156). The constraint equation gives us two constraints, one for $(M, N, P, Q) = (5, 6, i, j)$ (or any permutation thereof):

$$3f_{+R[56}f_{+ij]}{}^{R} + 2\xi_{[5}f_{+6ij]} = 0, \tag{E.58}$$

and one for $(M, N, P, Q) = (i, C, j, D)$ (or any permutation thereof):

$$3f_{+R[iC}f_{+jD]}{}^{R} + 2\xi_{[i}f_{+CjD]} = 0. \tag{E.59}$$

Entering eq. (E.56) into (E.58), we see that both terms are independently zero. The same procedure for eq. (E.59) gives us, using eqs. (2.154), (2.155) and (2.156),

$$2\xi_{[i}f_{+CjD]} = \frac{1}{2}D^{k}_{ij}\eta_{CA}D^{A}_{kD}, \tag{E.60}$$

and

$$\begin{aligned} 3f_{+R[56}f_{+ij]}{}^{R} &= \delta^{\iota k}f_{+ij\iota}f_{+kCD} - \eta^{EF}\left(f_{+iDE}f_{+jCF} + f_{+iCE}f_{jFD}\right) \\ &= \frac{1}{2}D^{k}_{ij}\eta_{CA}D^{A}_{kD} - D^{k}_{ij}\eta_{CA}D^{A}_{kD}. \end{aligned} \tag{E.61}$$

So the embedding tensor given by eq. (E.56) satisfies the constraint equation.

After the Weyl rescaling $g_{\mu\nu} \to e^{2\phi+\eta}$ we did, the potential takes the form

$$\begin{aligned} S_{pot} = -\int d^4x\sqrt{-g}\Bigg\{ &\frac{e^{2\phi+\eta+\rho}}{2}H^{AB}g^{ij}b_D b_E D^{D}_{iA}D^{E}_{jB} \\ &+ \frac{e^{2\phi+\eta}}{8}g^{ij}\Big(D^{C}_{iA}H_{CB} + D^{C}_{iB}H_{AC}\Big)\Big(D^{A}_{jD}H^{DB} + D^{B}_{jD}H^{AD}\Big)\Big] \\ &+ \frac{e^{2\phi+\eta}}{4}g^{mn}\Big(D^{k}_{mk}D^{l}_{nl} + \big(D^{k}_{mi}g_{kj} + D^{k}_{mj}g_{ik}\big)\big(D^{i}_{nl}g^{lj} + D^{j}_{nl}g^{il}\big)\Big) \\ &+ \frac{e^{4\phi+\eta-\rho}}{4}g^{ik}g^{jl}a_m a_n D^{m}_{ij}D^{n}_{kl} \\ &+ 2e^{4\phi+3\eta+\rho}\epsilon^{ij}\epsilon^{kl}\eta^{AC}\eta^{BD}b_C b_D\left(D^{E}_{iA}(c_{jE} - a_j b_E) + \frac{1}{2}D^{m}_{ij}(c_{mA} - a_m b_A)\right)\times \\ &\left(D^{F}_{kB}(c_{lF} - a_l b_F) + \frac{1}{2}D^{n}_{kl}(c_{nB} - a_n b_B)\right) \\ &+ \frac{e^{4\phi+\eta}}{4}H^{AB}g^{ik}g^{jl}\Big(D^{D}_{iA}c_{jD} - D^{D}_{jA}c_{iD} + D^{m}_{ij}\big(c_{mA} - a_m b_A\big)\Big)\times \\ &\Big(D^{E}_{kB}c_{lE} - D^{E}_{lB}c_{kE} + D^{n}_{kl}\big(c_{nB} - a_n b_B\big)\Big)\Bigg\}. \end{aligned} \tag{E.62}$$

We can now use the definition of the scalar matrix M_{MN} in Appendix F and the embedding tensor (E.56) to show that this equals

$$S_{pot} = -\frac{1}{8}\int \Big\{ f_+{}^{MNP} f_+{}^{QRS} M^{++}\Big(\frac{1}{3}M_{MQ}M_{NR}M_{PS} - M_{MQ}L_{NR}L_{PS}\Big) \tag{E.63}$$

$$+ 3\xi_+{}^M \xi_+{}^N M^{++} M_{MN}.\Big\} \tag{E.64}$$

This reproduces the potential given in [19].

The remaining topological term, according to [19], is

$$\begin{aligned} S_{top} = -\Bigg\{ & \xi_{+M} L_{NP} A^{M-} \wedge A^{N+} \wedge dA^{P+} \\ & - \frac{1}{4}\hat{f}_{+MNR}\hat{f}_{+PQ}{}^{R} A^{M+} \wedge A^{N+} \wedge A^{P+} \wedge A^{Q-} \\ & - \xi_{+M} B^{++} \wedge \left(dA^{M-} - \frac{1}{2}\hat{f}_{+QR}{}^{M} A^{Q+} \wedge A^{R-} \right) \Bigg\}. \end{aligned} \tag{E.65}$$

Writing out the indices tells us that for the embedding tensor (E.56),

$$\begin{aligned} & \frac{1}{4}\hat{f}_{+MNR}\hat{f}_{+PQ}{}^{R} A^{M+} \wedge A^{N+} \wedge A^{P+} \wedge A^{Q-} = \\ & - \frac{1}{2}\epsilon^{ij}\delta_{k\iota} G^{(1)k} \wedge D^{l}_{il}\tilde{B}^{(1)\iota} \wedge G^{(1)p} \wedge D^{o}_{pj} B^{(1)}_{p} \\ & + \frac{1}{2}\epsilon^{ij} D^{m}_{im}\tilde{C}^{(1)} \wedge A^{(1)} \wedge G^{(1)k} \wedge D^{l}_{kj} B^{(1)}_{l} \\ & + \frac{1}{2}\epsilon^{ij} B^{(1)}_{i} \wedge G^{(1)l} \wedge D^{B}_{lA}\eta_{BC} C^{(1)C} \wedge C^{(1)A} D^{k}_{jk}. \end{aligned} \tag{E.66}$$

Writing out the other two terms as well, we find that

$$\begin{aligned} S_{top} = \int \Bigg\{ & \frac{\epsilon^{ij}}{2} D^{m}_{im} B^{(2)++} \wedge \mathcal{F}^{(2)-}_{j} \\ & + \epsilon^{ij} B^{(1)}_{i} \wedge \left(\frac{1}{2}\eta_{AB}\mathcal{F}^{(2)A+} \wedge C^{(1)B} D^{k}_{jk} - \mathcal{F}^{(2)5+} \wedge \tilde{C}^{(1)} D^{k}_{jk} \right) \\ & + \epsilon^{ij} \left(\delta_{i\iota} d\tilde{B}^{(1)\iota} - \frac{1}{2} D^{l}_{il}\delta_{k\iota} G^{(1)k} \wedge \tilde{B}^{(1)\iota} + \frac{1}{2} D^{k}_{ik}\tilde{C}^{(1)} \wedge A^{(1)} \right) \wedge \mathcal{F}^{(2)-}_{j} \Bigg\}. \end{aligned} \tag{E.67}$$

It remains to be shown that the action is invariant under the various gauge transformations corresponding to the gauge fields. The parameters belonging to the gauge fields are ξ^i for $G^{(1)i}$, λ^ι for $\tilde{B}^{(1)\iota}$, Λ for $A^{(1)}$, $\tilde{\lambda}$ for $\tilde{C}^{(1)}$, and λ^A for $C^{(1)A}$. Furthermore, we also have the transformations with respect to λ_i (for which the magnetic field $B^{(1)}_i$ is the gauge field) and Ξ^{++} (the gauge transformation parameter for $B^{(2)++}$) into account. How the fields transform under these can be found in Appendix F. We immediately see that R is invariant under gauge transformations.

The scalar τ transforms as

$$\delta\tau = -\xi^k D^l_{kl}\tau - \frac{1}{4}\epsilon^{ij}\lambda_k D^k_{ij}. \tag{E.68}$$

From the definition of the fieldstrength $\mathcal{D}\tau$, we can see that $\delta_{\lambda_i}\mathcal{D}\tau = 0$ and

$$\delta_{\xi^i}\mathcal{D}\tau = -\xi^j D^l_{kl}\mathcal{D}\tau. \tag{E.69}$$

Since

$$\delta_{\xi^i}\mathrm{Im}(\tau) = -\xi^k D^l_{kl}\mathrm{Im}(\tau), \tag{E.70}$$

the inverse of $\mathrm{Im}(\tau)$ will transform exactly the same, except with a minus sign:

$$\delta_{\xi^i}\frac{1}{\mathrm{Im}(\tau)} = +\xi^k D^l_{kl}\frac{1}{\mathrm{Im}(\tau)}. \tag{E.71}$$

Therefore, the kinetic τ-term is invariant under gauge transformations.

Let us now look at the kinetic term for M_{MN}, and the action for the field strengths. First of all note that, because of its definition (2.191), $\mathcal{F}^{(2)\iota+}$ is invariant under gauge transformations with parameter Ξ^{++}. No other fields transform under Ξ^{++}, so the action is invariant under Ξ^{++}. Furthermore, for transformations under ξ^i, λ^ι, Λ, $\tilde{\lambda}$, and λ^A, the situation is the same as in Appendix 2.4: a transformation with respect to these parameters performes an $SO(6,n)$-rotation on $\mathcal{F}^{(2)M+}$, M_{MN} and $\mathcal{D}M_{MN}$. Again, the action is an $SO(6,n)$-scalar and therefore invariant. However, S_{fs} is not invariant under λ_i. $\mathrm{Re}(\tau)$ transforms under it, so we find

$$\delta_{\lambda_i}\Big(\mathrm{Re}(\tau)L_{MN}\mathcal{F}^{(2)M+}\wedge\mathcal{F}^{(2)N+}\Big) = -\frac{1}{4}\lambda_k\epsilon^{ij}D^k_{ij}L_{MN}\mathcal{F}^{(2)M+}\wedge\mathcal{F}^{(2)N+}. \tag{E.72}$$

This term, however, cancels out against the transformation of S_{top}. Rewritten as

$$S_{top} = \int\epsilon^{ij}\bigg\{ -\delta_{i\iota}\mathcal{F}^{(2)\iota+}\wedge\mathcal{F}^{(2)-}_j - B^{(1)}_i\wedge dC^{(1)A}\wedge C^{(1)B}\eta_{BC}D^C_{jA} \tag{E.73}$$
$$- B^{(1)}_i\wedge\mathcal{F}^{(2)5+}\wedge\tilde{C}^{(1)}D^k_{jk}\bigg\},$$

it is quite easily shown that it is invariant under all gauge transformations, except for δ_{λ_i}. Under λ_i, it transforms as

$$\delta_{\lambda_i}S_{top} = \frac{1}{4}\lambda_k\epsilon^{ij}D^k_{ij}L_{MN}\mathcal{F}^{(2)M+}\wedge\mathcal{F}^{(2)N+}.$$

This exactly cancels out the non-zero gauge-transformation of the topological field-strength term.

Finally, using the scalar transformations as given in eq. (F.96), together with the constraints (2.154), (2.155) and (2.156), it can be shown that the potential, written down as in eq. (E.62), is gauge invariant.

Appendix F

The Spectrum and its Transformations

This appendix contains the spectra, the transformation rules of all fields, the definition of the scalar matrices, the covariant derivatives and the fieldstrengths for the different compactifications: we start with the compactifications of IIA supergravity on $SU(2)$-structure manifolds, giving the definition of the scalar τ and the scalar matrix M_{MN}. Then we will give the transformation behavior, field strengths and scalar derivatives for Y_1 with $H_{10}^{(3)}$-flux in a first subsection, and Y_2 in a second. In the second section we will start by giving the scalar τ and the scalar matrix M_{MN} for the Scherk-Schwarz duality twist, followed by the transformation behavior, field strengths and scalar derivatives for that case.

F.1 The spectrum of IIA on $SU(2)$-structure manifolds

Before turning to the specific case of either Y_1 with $H_{10}^{(3)}$-flux, or Y_2, we present the general definitions of the complex scalar τ and the scalar matrix M_{MN} in terms of the fields obtained from the dimensional reduction.

The scalars $e^{-\eta}$ and b_{ij} are put into one complex scalar τ:

$$\tau := -\frac{1}{4}\epsilon^{ij}b_{ij} + \frac{i}{2}e^{-\eta}, \tag{F.1}$$

Equivalently, we can use the scalar matrix $M_{\alpha\beta}$ for $\alpha, \beta \in \{+, -\}$; this contains the same information as τ. It is defined as

$$M_{\alpha\beta} := \frac{1}{\mathrm{Im}(\tau)} \begin{pmatrix} |\tau|^2 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & 1 \end{pmatrix}. \tag{F.2}$$

The scalar matrix M_{MN} and its inverse $(LML)^{MN}$ are given in eqs. (F.3) and (F.4).

$$
M_{MN} = \left(\begin{array}{c|c|c|c|c}
\begin{array}{l} e^{-2\phi}g_{ij} + e^{-\rho}a_i a_j + \\ \eta^{AC}H^{B}{}_{C}a_{iA}a_{jB} + \\ e^{\rho}b_i b_j + e^{2\phi}g^{kl}c_{ki}c_{lj} \end{array} & e^{2\phi}\delta_{j\iota}g^{jk}c_{ki} & \begin{array}{l} -e^{-\rho}a_i + \\ \eta^{AC}H^{B}{}_{C}a_{iA}b_B - \\ e^{\rho}eb_i - \\ e^{2\phi}g^{kl}\gamma_k c_{li} \end{array} & \begin{array}{l} -e^{\rho}b_i - \\ e^{2\phi}g^{kl}a_k c_{li} \end{array} & \begin{array}{l} e^{\rho}b_i b_B - H^{C}{}_{B}a_{iC} + \\ e^{2\phi}g^{kl}c_{kB}c_{li} \end{array} \\ \hline
e^{2\phi}\delta_{i\iota}g^{ik}c_{kj} & e^{2\phi}\delta_{i\iota}\delta_{j\iota'}g^{ij} & -e^{2\phi}\delta_{i\iota}g^{ik}\gamma_k & -e^{2\phi}\delta_{i\iota}g^{ik}a_k & e^{2\phi}\delta_{i\iota}g^{ik}c_{kB} \\ \hline
\begin{array}{l} -e^{-\rho}a_j + \\ \eta^{AC}H^{B}{}_{C}a_{jA}b_B - \\ e^{\rho}eb_j - e^{2\phi}g^{kl}\gamma_k c_{lj} \end{array} & -e^{2\phi}\delta_{j\iota}g^{jk}\gamma_k & \begin{array}{l} e^{-\rho} + \\ \eta^{AC}H^{B}{}_{C}b_A b_B + \\ e^{\rho}e^2 + e^{2\phi}g^{kl}\gamma_k\gamma_l \end{array} & e^{\rho}e + e^{2\phi}g^{kl}\gamma_k a_l & \begin{array}{l} -H^{C}{}_{B}b_C - e^{\rho}b_B e - \\ e^{2\phi}g^{kl}c_{kB}\gamma_l \end{array} \\ \hline
-e^{\rho}b_j - e^{2\phi}g^{kl}a_k c_{lj} & -e^{2\phi}\delta_{j\iota}g^{jk}a_k & e^{\rho}e + e^{2\phi}g^{kl}\gamma_k a_l & e^{\rho} + e^{2\phi}g^{kl}a_k a_l & -e^{\rho}b_B - e^{2\phi}g^{kl}a_k c_{lB} \\ \hline
\begin{array}{l} e^{\rho}b_j b_A - H^{C}{}_{A}a_{jC} + \\ e^{2\phi}g^{kl}c_{kA}c_{lj} \end{array} & e^{2\phi}\delta_{j\iota}g^{jk}c_{kB} & \begin{array}{l} -H^{C}{}_{A}b_C - \\ e^{\rho}b_A e - \\ e^{2\phi}g^{kl}c_{kA}\gamma_l \end{array} & \begin{array}{l} -e^{\rho}b_A - \\ e^{2\phi}g^{kl}a_k c_{lA} \end{array} & \begin{array}{l} \eta_{AC}H^{C}{}_{B} + e^{\rho}b_A b_B + \\ e^{2\phi}g^{kl}c_{kA}c_{lB} \end{array}
\end{array} \right) \quad \text{(F.3)}
$$

$$
(LML)^{MN} = \left(\begin{array}{c|c|c|c|c}
e^{2\phi}g^{ij} & e^{2\phi}g^{ik}\delta_{j\iota}c_{kj} & e^{2\phi}g^{ik}a_k & e^{2\phi}g^{ik}\gamma_k & e^{2\phi}g^{ik}\eta^{BC}c_{kC} \\ \hline
e^{2\phi}g^{jk}\delta_{i\iota}c_{ki} & \begin{array}{l} \delta^{i\iota}\delta^{j\iota'}\big(e^{-2\phi}g_{ij} + \\ e^{-\rho}a_i a_j + \\ \eta^{AC}H^{B}{}_{C}a_{iA}a_{jB} + \\ e^{\rho}b_i b_j + \\ e^{2\phi}g^{kl}c_{ki}c_{lj}\big) \end{array} & \begin{array}{l} \delta^{i\iota}\big(e^{\rho}b_i + \\ e^{2\phi}g^{kl}a_k c_{li}\big) \end{array} & \begin{array}{l} \delta^{i\iota}\big(e^{-\rho}a_i - \\ \eta^{AC}H^{B}{}_{C}a_{iA}b_B + \\ e^{\rho}eb_i + \\ e^{2\phi}g^{kl}\gamma_k c_{li}\big) \end{array} & \begin{array}{l} \delta^{i\iota}\big(- \\ \eta^{BC}H^{D}{}_{C}a_{iD} + \\ \eta^{BC}e^{\rho}b_i b_C + \\ \eta^{BC}e^{2\phi}g^{kl}c_{kC}c_{li}\big) \end{array} \\ \hline
e^{2\phi}g^{jk}a_k & \begin{array}{l} \delta^{j\iota}\big(e^{\rho}b_j + \\ e^{2\phi}g^{kl}a_k c_{lj}\big) \end{array} & e^{\rho} + e^{2\phi}g^{kl}a_k a_l & e^{\rho}e + e^{2\phi}g^{kl}\gamma_k a_l & \begin{array}{l} e^{\rho}\eta^{BC}b_C + \\ e^{2\phi}g^{kl}a_k\eta^{BC}c_{lC} \end{array} \\ \hline
e^{2\phi}g^{jk}\gamma_k & \begin{array}{l} \delta^{j\iota}\big(e^{-\rho}a_j - \\ \eta^{AC}H^{B}{}_{C}a_{jA}b_B + \\ e^{\rho}eb_j + \\ e^{2\phi}g^{kl}\gamma_k c_{lj}\big) \end{array} & e^{\rho}e + e^{2\phi}g^{kl}\gamma_k a_l & \begin{array}{l} e^{-\rho} + \\ \eta^{AC}H^{B}{}_{C}b_A b_B + \\ e^{\rho}e^2 + e^{2\phi}g^{kl}\gamma_k\gamma_l \end{array} & \begin{array}{l} \eta^{BC}H^{D}{}_{C}b_D + \\ e^{\rho}\eta^{BC}b_C e + \\ e^{2\phi}g^{kl}\eta^{BC}c_{kC}\gamma_l \end{array} \\ \hline
e^{2\phi}g^{jk}\eta^{AC}c_{kC} & \begin{array}{l} \delta^{j\iota}\big(- \\ \eta^{AC}H^{D}{}_{C}a_{jD} + \\ e^{\rho}b_j\eta^{AC}b_C + \\ e^{2\phi}g^{kl}\eta^{AC}c_{kC}c_{lj}\big) \end{array} & \begin{array}{l} e^{\rho}\eta^{AC}b_C + \\ e^{2\phi}g^{kl}a_k\eta^{AC}c_{lC} \end{array} & \begin{array}{l} \eta^{AC}H^{D}{}_{C}b_D + \\ e^{\rho}\eta^{AC}b_C e + \\ e^{2\phi}g^{kl}\eta^{AC}c_{kC}\gamma_l \end{array} & \begin{array}{l} \eta^{AC}H^{B}{}_{C} + \\ e^{\rho}\eta^{AC}\eta^{BD}b_C b_D + \\ e^{2\phi}g^{kl}\eta^{AC}\eta^{BD}c_{kC}c_{lD} \end{array}
\end{array} \right) \quad \text{(F.4)}
$$

We have used the following shorthands:

$$a_{iA} := c_{iA} - a_i b_A, \tag{F.5}$$

$$b_i := \gamma_i - \eta^{AB} b_A c_{iB} + e a_i, \tag{F.6}$$

$$c_{ij} := a_i \gamma_j - \frac{1}{2}\eta^{AB} c_{iA} c_{jB} - \beta_{ij}, \tag{F.7}$$

$$e := \frac{\eta^{AB} b_A b_B}{2}. \tag{F.8}$$

F.1.1 The Spectrum of IIA on Y_1 with $H_{10}^{(3)}$-Flux

We now give the transformation rules, field strengths and covariant derivatives for the compactification of IIA supergravity on Y_1 with an $H_{10}^{(3)}$-flux. We show only the fields in terms of which the supergravity is written, although we discuss the scalars both in- and outside the scalar matrix. Note that the transformations we show are valid on Y_1 with $H_{10}^{(3)}$-flux, the transformations on Y_1 are obtained by setting $Dk_{iA} = 0$ and those on $K3 \times T^2$ by setting $k_{iA} = D_{iA}^B = 0$.

The gauge bosons in the final electric frame transform as follows:

$$\delta G^{(1)i} = d\xi^i, \tag{F.9}$$

$$\delta \tilde{B}^{(1)\iota} = d\tilde{\lambda}^\iota - \lambda^A \delta^{i\iota} D_{iA}^B \eta_{BC} C^{(1)C} \tag{F.10}$$

$$- \Lambda \delta^{i\iota} k_{iA} C^{(1)A} + \lambda^A \delta^{i\iota} k_{iA} A^{(1)}, \tag{F.11}$$

$$\delta A^{(1)} = d\Lambda, \tag{F.12}$$

$$\delta \tilde{C}^{(1)} = d\tilde{\lambda} + \lambda^A k_{kA} G^{(1)k} - \xi^k k_{kA} C^{(1)A}, \tag{F.13}$$

$$\delta C^{(1)A} = d\lambda^A + \xi^k \Big(D_{kB}^A C^{(1)B} - \eta^{AB} k_{kB} A^{(1)} \Big) \tag{F.14}$$

$$- G^{(1)k} \Big(D_{kB}^A \lambda^B - \eta^{AB} k_{kB} \Lambda \Big).$$

The two-form field strengths for the final frame bosons are:

$$\mathcal{F}^{(2)i+} := dG^{(1)i}, \tag{F.15}$$

$$\mathcal{F}^{(2)\iota+} := d\tilde{B}^{(1)\iota} + A^{(1)} \wedge \delta^{i\iota} k_{iA} C^{(1)A} + \frac{1}{2}\delta^{i\iota} \eta_{AC} D_{iB}^C C^{(1)A} \wedge C^{(1)B}, \tag{F.16}$$

$$\mathcal{F}^{(2)5+} := dA^{(1)}, \tag{F.17}$$

$$\mathcal{F}^{(2)6+} := d\tilde{C}^{(1)} + G^{(1)k} \wedge k_{kA} C^{(1)A}, \tag{F.18}$$

$$\mathcal{F}^{(2)A+} := dC^{(1)A} - G^{(1)k} \wedge D_{kB}^A C^{(1)B} + G^{(1)k} \wedge k_{kB} \eta^{AB} A^{(1)}. \tag{F.19}$$

These fieldstrengths transform covariantly, meaning

$$\delta\mathcal{F}^{(2)i+} = 0, \tag{F.20}$$

$$\delta\mathcal{F}^{(2)\iota+} = -\lambda^A\delta^{i\iota}D^B_{iA}\eta_{BC}\mathcal{F}^{(2)C+} \tag{F.21}$$

$$-\Lambda\delta^{i\iota}k_{iA}\mathcal{F}^{(2)A+} + \lambda^A\delta^{i\iota}k_{iA}\mathcal{F}^{(2)5+}, \tag{F.22}$$

$$\delta\mathcal{F}^{(2)5+} = 0, \tag{F.23}$$

$$\delta\mathcal{F}^{(2)6+} = \lambda^A k_{kA}\mathcal{F}^{(2)k+} - \xi^k k_{kA}\mathcal{F}^{(2)A+}, \tag{F.24}$$

$$\delta\mathcal{F}^{(2)A+} = \xi^k\Big(D^A_{kB}\mathcal{F}^{(2)B+} - \eta^{AB}k_{kB}\mathcal{F}^{(2)5+}\Big) - \mathcal{F}^{(2)k+}\Big(D^A_{kB}\lambda^B - \eta^{AB}k_{kB}\Lambda\Big). \tag{F.25}$$

We now turn to the transformation of the scalars. The dilaton does not transform:

$$\delta e^{-2\phi} = 0, \tag{F.26}$$

the only scalars from the metric sector that have a non-zero transformation are $H^A{}_B$:

$$\delta e^{-\eta} = 0, \tag{F.27}$$

$$\delta e^{-\rho} = 0, \tag{F.28}$$

$$\delta g_{ij} = 0, \tag{F.29}$$

$$\delta H^A{}_B = \xi^k\Big(D^C_{kB}H^A{}_C - D^A_{kC}H^C{}_B\Big). \tag{F.30}$$

The scalars from the form fields $A^{(1)}_{10}$, $B^{(2)}_{10}$ and $C^{(3)}_{10}$ transform like

$$\delta a_i = 0, \tag{F.31}$$

$$\delta b_{ij} = 0, \tag{F.32}$$

$$\delta b_A = -\xi^k\Big(k_{kA} + D^C_{kA}b_C\Big), \tag{F.33}$$

$$\delta c_{iA} = -\xi^k\Big(k_{kA}a_i + D^C_{kA}c_{iC}\Big) + \Lambda k_{iA} + \lambda^B D^C_{iA}\eta_{BC}, \tag{F.34}$$

whereas the fields that are the duals of two-form fields transform like

$$\delta\gamma_i = -\xi^k k_{kA}\eta^{AB}c_{iB} + \lambda^A k_{iA}, \tag{F.35}$$

$$\delta\beta_{ij} = -\frac{1}{2}\eta^{AB}\Big(\xi^k k_{kA}\big(a_i c_{jB} - a_j c_{iB}\big) + \Lambda\big(k_{iA}c_{jB} - k_{jA}c_{iB}\big)\Big) - \frac{1}{2}\lambda^A\big(c_{iB}D^B_{jA} - c_{jB}D^B_{iA}\big). \tag{F.36}$$

This means we get one non-trivial scalar derivatives from the metric sector,

$$\mathcal{D}H^A{}_B := dH^A{}_B + G^{(1)k}\Big(D^C_{kB}H^A{}_C - D^A_{kC}H^C{}_B\Big), \tag{F.37}$$

the non-trivial scalar derivatives from the ten-dimensional form fields are

$$\mathcal{D}b_A := db_A + G^{(1)k}\Big(k_{kA} + D^C_{kA}b_C\Big), \tag{F.38}$$

$$\mathcal{D}c_{iA} := dc_{iA} + G^{(1)k}\Big(k_{kA}a_i + D^C_{kA}c_{iC}\Big) - A^{(1)}k_{iA} - D^B_{iA}\eta_{BC}C^{(1)C}, \tag{F.39}$$

and the scalars dual to the two-form fields have the covariant derivative

$$\mathcal{D}\gamma_i := d\gamma_i + G^{(1)k}\eta^{AB}k_{kA}c_{iB} - C^{(1)A}k_{iA}, \tag{F.40}$$

$$\begin{aligned}\mathcal{D}\beta_{ij} := d\beta_{ij} + \frac{1}{2}G^{(1)k}\eta^{AB}k_{kA}\big(a_i c_{jB} - a_j c_{iB}\big) + \frac{1}{2}A^{(1)}\eta^{AB}\big(k_{iA}c_{jB} - k_{jA}c_{iB}\big) \\ + \frac{1}{2}C^{(1)A}\big(c_{iB}D^B_{jA} - c_{jB}D^B_{iA}\big).\end{aligned} \tag{F.41}$$

Since neither $e^{-\eta}$ nor b_{ij} transform, τ does not transform either:

$$\delta\tau = 0, \tag{F.42}$$

and it has a trivial covariant derivative.

The scalars in the matrix M_{MN} transform as

$$\begin{aligned}
\delta M_{ij} =& -\Lambda\eta^{AB}\Big(k_{iA}M_{Bj} + k_{jA}M_{iB}\Big) + \lambda^A\left(D^B_{iA}M_{Bj} + D^B_{jA}M_{iB} - k_{iA}M_{6j} - k_{jA}M_{i6}\right), \\
\delta M_{\iota j} =& -\Lambda\eta^{AB}k_{jA}M_{\iota B} + \lambda^A\Big(D^B_{jA}M_{\iota B} - k_{jA}M_{\iota 6}\Big), \\
\delta M_{5j} =& \xi^k\eta^{AB}k_{kA}M_{Bj} - \Lambda\eta^{AB}k_{jA}M_{5B} + \lambda^A\Big(D^B_{jA}M_{5B} - \delta^{\iota k}k_{kA}M_{\iota j} - k_{jA}M_{56}\Big), \\
\delta M_{6j} =& -\Lambda\eta^{AB}k_{jA}M_{6B} + \lambda^A\Big(D^B_{jA}M_{6B} - k_{jA}M_{66}\Big), \\
\delta M_{Aj} =& \xi^k\Big(D^B_{kA}M_{Bj} - k_{kA}M_{6j}\Big) - \Lambda\Big(\eta^{BC}k_{jB}M_{AC} + \delta^{\iota k}k_{kA}M_{\iota j}\Big) \\
& + \lambda^C\Big(D^D_{jC}M_{AD} - k_{jC}M_{A6} - \delta^{\iota k}\eta_{AD}D^D_{kC}M_{\iota j}\Big), \\
\delta M_{\iota\iota'} =& 0, \\
\delta M_{5\iota} =& \xi^k\eta^{AB}k_{kA}M_{B\iota} - \lambda^A\delta^{\iota' k}k_{kA}M_{\iota'\iota}, \\
\delta M_{6\iota} =& 0, \\
\delta M_{A\iota} =& \xi^k\Big(D^B_{kA}M_{B\iota} - k_{kA}M_{6\iota}\Big) - \Lambda\delta^{\iota' k}k_{kA}M_{\iota'\iota} - \lambda^C\delta^{\iota' k}\eta_{AD}D^D_{kC}M_{\iota'\iota}, \\
\delta M_{55} =& \xi^k\eta^{AB}k_{kA}\Big(M_{B5} + M_{5B}\Big) - \lambda^A\delta^{\iota k}k_{kA}\Big(M_{\iota 5} + M_{5\iota}\Big), \\
\delta M_{56} =& \xi^k\eta^{AB}k_{kA}M_{B6} - \lambda^A\delta^{\iota k}k_{kA}M_{\iota 6}, \\
\delta M_{5A} =& \xi^k\Big(\eta^{BC}k_{kB}M_{CA} + D^B_{kA}M_{5B} - k_{kA}M_{56}\Big) - \Lambda\delta^{\iota i}k_{iA}M_{5\iota} \\
& - \lambda^B\delta^{\iota k}\Big(\eta_{AC}D^C_{kB}M_{5\iota} + k_{iB}M_{\iota A}\Big), \\
\delta M_{66} =& 0, \\
\delta M_{A6} =& \xi^k\Big(D^B_{kA}M_{B6} - k_{kA}M_{66}\Big) - \Lambda\delta^{\iota k}k_{kA}M_{\iota 6} - \lambda^C\delta^{\iota k}\eta_{AD}D^D_{kC}M_{\iota 6}, \\
\delta M_{AB} =& \xi^k\Big(D^C_{kA}M_{CB} + D^C_{kB}M_{AC} - k_{kA}M_{6B} - k_{kB}M_{6A}\Big) - \Lambda\delta^{\iota k}\Big(k_{kA}M_{\iota B} + k_{kB}M_{A\iota}\Big) \\
& - \lambda^C\delta^{\iota k}\Big(\eta_{AD}D^D_{kC}M_{\iota B} + \eta_{BD}D^D_{kC}M_{A\iota}\Big).
\end{aligned} \tag{F.43}$$

The covariant derivatives of the scalars in terms of the matrix elements are:

$$\begin{aligned}
\mathcal{D}M_{ij} =& dM_{ij} + A^{(1)}\eta^{AB}\Big(k_{iA}M_{Bj} + k_{jA}M_{iB}\Big) \\
&- C^{(1)A}\left(D^B_{iA}M_{Bj} + D^B_{jA}M_{iB} - k_{iA}M_{6j} - k_{jA}M_{i6}\right), \\
\mathcal{D}M_{\iota j} =& dM_{\iota j} + A^{(1)}\eta^{AB}k_{jA}M_{\iota B} - C^{(1)A}\Big(D^B_{jA}M_{\iota B} - k_{jA}M_{\iota 6}\Big), \\
\mathcal{D}M_{5j} =& dM_{5j} - G^{(1)k}\eta^{AB}k_{kA}M_{Bj} + A^{(1)}\eta^{AB}k_{jA}M_{5B} \\
&- C^{(1)A}\Big(D^B_{jA}M_{5B} - \delta^{\iota k}k_{kA}M_{\iota j} - k_{jA}M_{56}\Big), \\
\mathcal{D}M_{6j} =& dM_{6j} + A^{(1)}\eta^{AB}k_{jA}M_{6B} - C^{(1)A}\Big(D^B_{jA}M_{6B} - k_{jA}M_{66}\Big), \\
\mathcal{D}M_{Aj}+& dM_{Aj} - G^{(1)k}\Big(D^B_{kA}M_{Bj} - k_{kA}M_{6j}\Big) + A^{(1)}\Big(\eta^{BC}k_{jB}M_{AC} + \delta^{\iota k}k_{kA}M_{\iota j}\Big) \\
&- C^{(1)C}\Big(D^D_{jC}M_{AD} - k_{jC}M_{A6} - \delta^{\iota k}\eta_{AD}D^D_{kC}M_{\iota j}\Big), \\
\mathcal{D}M_{\iota\iota'} =& dM_{\iota\iota'}, \\
\mathcal{D}M_{5\iota} =& dM_{5\iota} - G^{(1)k}\eta^{AB}k_{kA}M_{B\iota} + C^{(1)A}\delta^{\iota' k}k_{kA}M_{\iota'\iota}, \\
\mathcal{D}M_{6\iota} =& dM_{6\iota}, \\
\mathcal{D}M_{A\iota} =& dM_{A\iota} - G^{(1)k}\Big(D^B_{kA}M_{B\iota} - k_{kA}M_{6\iota}\Big) - \Lambda\delta^{\iota' k}k_{kA}M_{\iota'\iota} + C^{(1)C}\delta^{\iota' k}\eta_{AD}D^D_{kC}M_{\iota'\iota}, \\
\mathcal{D}M_{55} =& dM_{55} - G^{(1)k}\eta^{AB}k_{kA}\Big(M_{B5} + M_{5B}\Big) + C^{(1)A}\delta^{\iota k}k_{kA}\Big(M_{\iota 5} + M_{5\iota}\Big), \\
\mathcal{D}M_{56} =& dM_{56} - G^{(1)k}\eta^{AB}k_{kA}M_{B6} + C^{(1)A}\delta^{\iota k}k_{kA}M_{\iota 6}, \\
\mathcal{D}M_{5A} =& dM_{5A} - G^{(1)k}\Big(\eta^{BC}k_{kB}M_{CA} + D^B_{kA}M_{5B} - k_{kA}M_{56}\Big) + A^{(1)}\delta^{\iota i}k_{iA}M_{5\iota} \\
&+ C^{(1)B}\delta^{\iota k}\Big(\eta_{AC}D^C_{kB}M_{5\iota} + k_{iB}M_{\iota A}\Big), \\
\mathcal{D}M_{66} =& dM_{66}, \\
\mathcal{D}M_{6A} =& dM_{6A} - G^{(1)k}\Big(D^B_{kA}M_{B6} - k_{kA}M_{66}\Big) + A^{(1)}\delta^{\iota k}k_{kA}M_{\iota 6} + C^{(1)C}\delta^{\iota k}\eta_{AD}D^D_{kC}M_{\iota 6}, \\
\mathcal{D}M_{AB} =& dM_{AB} - G^{(1)k}\Big(D^C_{kA}M_{CB} + D^C_{kB}M_{AC} - k_{kA}M_{6B} - k_{kB}M_{6A}\Big) \\
&+ A^{(1)}\delta^{\iota k}\Big(k_{kA}M_{\iota B} + k_{kB}M_{A\iota}\Big) + C^{(1)C}\delta^{\iota k}\Big(\eta_{AD}D^D_{kC}M_{\iota B} + \eta_{BD}D^D_{kC}M_{A\iota}\Big).
\end{aligned} \tag{F.44}$$

F.1.2 The Spectrum of IIA on Y_2

Here, we give the transformation rules, field strengths and covariant derivatives for the compactification of IIA supergravity on Y_2. We show only the fields in terms of which the supergravity is written, although we discuss the scalars both in- and outside the scalar matrix. Note that the transformations we show are valid on Y_2, the transformations on Y_1 are obtained by setting $D^k_{ij} = 0$ and those on $K3 \times T^2$ by setting $D^k_{ij} = D^B_{iA} = 0$.

The gauge bosons in the final electric frame transform as follows:

$$\begin{aligned}
\delta G^{(1)i} =& d\xi^i + \xi^k D^i_{kl} G^{(1)l} && \text{(F.45)}\\
\delta \tilde{B}^{(1)\iota} =& \xi^k D^l_{kl} \tilde{B}^{(1)\iota} - \lambda^A \delta^{i\iota} D^B_{iA} \eta_{BC} C^{(1)C} + \lambda \delta^{i\iota} D^l_{il} \tilde{C}^{(1)} && \text{(F.46)}\\
\delta A^{(1)} =& d\lambda && \text{(F.47)}\\
\delta \tilde{C}^{(1)} =& d\tilde{\lambda} + \xi^k D^l_{kl} \tilde{C}^{(1)} - \tilde{\lambda} D^l_{kl} G^{(1)k} && \text{(F.48)}\\
\delta C^{(1)A} =& d\lambda^A + \xi^k \Big(D^A_{kB} C^{(1)B} + D^l_{kl} C^{(1)A} \Big) && \text{(F.49)}\\
& - G^{(1)k} \Big(D^A_{kB} \lambda^B + D^l_{kl} \lambda^A \Big),
\end{aligned}$$

while the magnetic vector dual to $\tilde{B}^{(1)\iota}$ transforms as

$$\delta B^{(1)}_i = d\lambda_i - \xi^k D^l_{ik} B^{(1)}_l + \lambda_l D^l_{ik} G^{(1)k}. \tag{F.50}$$

The two-form field strengths for the final frame bosons are:

$$\begin{aligned}
\mathcal{F}^{(2)i+} :=& dG^{(1)i} - \frac{1}{2} G^{(1)k} \wedge G^{(1)l} D^i_{kl}, && \text{(F.51)}\\
\mathcal{F}^{(2)\iota+} :=& d\tilde{B}^{(1)\iota} + \frac{\delta^\iota_i \epsilon^{ij}}{2} D^k_{jk} B^{(2)++} && \text{(F.52)}\\
& + \frac{1}{2} \delta^{i\iota} \Big(D^k_{ik} G^{(1)m} \wedge \tilde{B}^{(1)\iota'} \delta_{m\iota'} + D^k_{ik} \tilde{C}^{(1)} \wedge A^{(1)} \\
& + \eta_{AC} D^C_{iB} C^{(1)A} \wedge C^{(1)B} \Big),\\
\mathcal{F}^{(2)5+} :=& dA^{(1)}, && \text{(F.53)}\\
\mathcal{F}^{(2)6+} :=& d\tilde{C}^{(1)} - G^{(1)k} \wedge D^l_{kl} \tilde{C}^{(1)}, && \text{(F.54)}\\
\mathcal{F}^{(2)A+} :=& dC^{(1)A} - G^{(1)k} \wedge \Big(D^A_{kB} C^{(1)B} + D^l_{kl} C^{(1)A} \Big), && \text{(F.55)}
\end{aligned}$$

while the one for the dual boson $B^{(1)}_i$ is

$$\mathcal{F}^{(2)-}_i := dB^{(1)}_i + G^{(1)k} \wedge D^l_{ki} B^{(1)}_l. \tag{F.56}$$

These fieldstrengths transform covariantly, meaning

$$\begin{aligned}
\delta \mathcal{F}^{(2)i+} =& \xi^k D^i_{kl} \mathcal{F}^{(2)l+},\\
\delta \mathcal{F}^{(2)\iota+} =& \xi^k \delta^{i\iota} \delta_{k\iota'} D^l_{il} \mathcal{F}^{(2)\iota'+} - \tilde{\lambda} \delta^{i\iota} D^l_{il} \mathcal{F}^{(2)5+} - \lambda_C \eta_{BC} \delta^{i\iota} D^B_{iA} \mathcal{F}^{(2)A+},\\
\delta \mathcal{F}^{(2)5+} =& 0, \qquad\qquad \text{(F.57)}\\
\delta \mathcal{F}^{(2)6+} =& \xi^k D^l_{kl} \mathcal{F}^{(2)6+} - \tilde{\lambda} D^l_{kl} \mathcal{F}^{(2)k+},\\
\delta \mathcal{F}^{(2)A+} =& \xi^k \Big(D^A_{kB} \mathcal{F}^{(2)B+} + D^l_{kl} \mathcal{F}^{(2)A+} \Big)\\
& - \mathcal{F}^{(2)k+} \Big(D^A_{kB} \lambda^B + D^l_{kl} \lambda^A \Big),
\end{aligned}$$

and

$$\delta\mathcal{F}_i^{(2)-} = -\xi^k D_{ik}^l \mathcal{F}_l^{(2)-} + \lambda_l D_{ik}^l \mathcal{F}^{(2)k+} \tag{F.58}$$

The two-form Lagrange-multiplier $B^{(2)++}$ transforms like

$$\begin{aligned}\delta_{\xi^i} B^{(2)++} =& \Big(d\xi^k - \xi^j D_{jl}^k G^{(1)l}\Big) \wedge \tilde{B}^{(1)k} & \text{(F.59)}\\ &+ \xi^k \left(A^{(1)} \wedge D_{kl}^l \tilde{C}^{(1)} + \eta_{AB} C^{(1)A} \wedge D_{kC}^B C^{(1)C} + 2\mathcal{F}^{(2)\iota+}\delta_{\iota k}\right),\\ \delta_{\tilde{\lambda}_i} B^{(2)++} =& \Big(d\tilde{\lambda}_k - \tilde{\lambda}_m D_{kn}^n G^{(1)m}\Big) \wedge G^{(1)k} - 2\tilde{\lambda}_i \mathcal{F}^{(2)i+}, & \text{(F.60)}\\ \delta_\Lambda B^{(2)++} =& -\Big(d\Lambda + \Lambda D_{kl}^l G^{(1)k}\Big) \wedge \tilde{C}^{(1)} - 2\Lambda\mathcal{F}^{(2)6+}, & \text{(F.61)}\\ \delta_{\tilde{\lambda}} B^{(2)++} =& -\Big(d\tilde{\lambda} - \tilde{\lambda} D_{kl}^l G^{(1)k}\Big) \wedge A^{(1)} - 2\tilde{\lambda}\mathcal{F}^{(2)5+}, & \text{(F.62)}\\ \delta_{\lambda_I} B^{(2)++} =& \Big(d\lambda^A + \lambda^C D_{kC}^A G^{(1)k}\Big) \wedge \eta_{AB} C^{(1)B} & \text{(F.63)}\\ &+ \lambda^A G^{(1)k} \wedge D_{kA}^C \eta_{CD} C^{(1)D} + 2\eta_{AB}\lambda^A \mathcal{F}^{(2)B+}.\end{aligned}$$

$B^{(2)++}$ also has its own gauge transformation. The only other field that transforms under this is $\tilde{B}^{(1)i}$:

$$\delta_\Xi B^{(2)++} = d\Xi^{++} - \frac{1}{2} G^{(1)k} \wedge D_{km}^m \Xi^{++}, \tag{F.64}$$

$$\delta_\Xi \tilde{B}^{(1)i} = -\frac{1}{2} D_{il}^l \Xi^{++}. \tag{F.65}$$

We now turn to the transformation of the scalars. The dilaton transforms like

$$\delta e^{-2\phi} = \xi^k D_{kl}^l e^{-2\phi}, \tag{F.66}$$

the scalars from the metric sector transform like

$$\delta e^{-\eta} = -\xi^k D_{kl}^l e^{-\eta} \tag{F.67}$$

$$\delta e^{-\rho} = \xi^k D_{kl}^l e^{-\rho} \tag{F.68}$$

$$\delta g_{ij} = -\xi^k \big(D_{ki}^l g_{lj} + D_{kj}^l g_{il}\big) \tag{F.69}$$

$$\delta H^A{}_B = \xi^k \big(D_{kB}^C H^A{}_C - D_{kC}^A H^C{}_B\big), \tag{F.70}$$

those coming from the form fields $A_{10}^{(1)}$, $B_{10}^{(2)}$ and $C_{10}^{(3)}$ transform like

$$\delta a_i = -\xi^k D_{ki}^l a_l \tag{F.71}$$

$$\delta b_{ij} = -\xi^k (D_{ki}^l b_{lj} + D_{kj}^k b_{il}) + \lambda_k D_{ij}^k \tag{F.72}$$

$$\delta b_A = -\xi^k D_{kA}^C b_C \tag{F.73}$$

$$\delta c_{iA} = -\xi^k (D_{ki}^l c_{kA} + D_{kA}^C c_{iC}) + \lambda^B D_{iA}^C \eta_{BC}, \tag{F.74}$$

whereas the fields that are the duals of two-form fields transform like

$$\delta\gamma_i = -\xi^k (D_{ki}^l \gamma_l - D_{kl}^l \gamma_i) - \tilde{\lambda} D_{ik}^k \tag{F.75}$$

$$\delta\beta_{ij} = -\xi^k (D_{ki}^l \beta_{lj} + D_{kj}^l \beta_{il} - D_{kl}^l \beta_{ij}) - \frac{1}{2}\lambda^A \left(c_{iB} D_{jA}^B - c_{jB} D_{iA}^B\right) \tag{F.76}$$

The covariant derivatives of the dilaton is:

$$\mathcal{D}e^{-2\phi} = de^{-2\phi} - G^{(1)k}D^l_{kl}e^{-2\phi}, \tag{F.77}$$

and those of the scalars from the metric sector is given by

$$\mathcal{D}e^{-\eta} = de^{-\eta} + G^{(1)k}D^l_{kl}e^{-\eta}, \tag{F.78}$$

$$\mathcal{D}e^{-\rho} = de^{-\rho} - G^{(1)k}D^l_{kl}e^{-\rho}, \tag{F.79}$$

$$\mathcal{D}g_{ij} := dg_{ij} + G^{(1)k}\left(D^l_{kj}g_{il} + D^l_{ki}g_{lj}\right), \tag{F.80}$$

$$\mathcal{D}H^A{}_B := dH^A{}_B + G^{(1)k}\left(D^C_{kB}H^A{}_C - D^A_{kC}H^C{}_B\right), \tag{F.81}$$

The scalars from the ten-dimensional form fields have the covariant derivatives

$$\mathcal{D}a_i := da_i + G^{(1)k}D^l_{ki}a_l, \tag{F.82}$$

$$\mathcal{D}b_{ij} := db_{ij} - B^{(1)}_k D^k_{ij} + G^{(1)k}\left(D^l_{kj}b_{il} + D^l_{ki}b_{lj}\right), \tag{F.83}$$

$$\mathcal{D}b_A := db_A + G^{(1)k}D^C_{kA}b_C, \tag{F.84}$$

$$\mathcal{D}c_{iA} := dc_{iA} - D^B_{iA}\eta_{BC}C^{(1)C} + G^{(1)k}\left(D^C_{kA}c_{iC} + D^l_{ki}c_{lA}\right), \tag{F.85}$$

and the scalars dual to the two-form fields have the covariant derivative

$$\mathcal{D}\gamma_i := d\gamma_i + D^k_{ik}\tilde{C}^{(1)} - G^{(1)l}D^k_{ik}\gamma_l, \tag{F.86}$$

$$\mathcal{D}\beta_{ij} := d\beta_{ij} - \frac{1}{2}\eta^{AB}\eta_{CD}C^{(1)D}(c_{iA}D^C_{jB} - c_{jA}D^C_{iB}). \tag{F.87}$$

For the transformation of τ, this means that that transforms like

$$\delta\tau = -\xi^k D^l_{kl}\tau - \frac{1}{4}\lambda_k\epsilon^{ij}D^k_{ij} \tag{F.88}$$

as follows from eqs. (F.27) and (F.32). Written in terms of $M_{\alpha\beta}$, this is

$$\delta M_{++} = \frac{1}{2}\lambda_k\epsilon^{ij}D^k_{ij}M_{+-} \tag{F.89}$$

$$\delta M_{+-} = -\frac{1}{4}\lambda_k\epsilon^{ij}D^k_{ij}M_{--} \tag{F.90}$$

$$\delta M_{--} = \xi^k D^l_{kl}M_{--} \tag{F.91}$$

The covariant derivative of the composite scalar τ is

$$\mathcal{D}\tau := d\tau + G^{(1)k}D^l_{kl}\tau + \frac{1}{4}B^{(1)}_k\epsilon^{ij}D^k_{ij}, \tag{F.92}$$

and again we can write this in terms of $M_{\alpha\beta}$ as well:

$$\mathcal{D}M_{--} := dM_{--} - G^{(1)k}D^l_{kl}M_{--}, \tag{F.93}$$

$$\mathcal{D}M_{+-} := dM_{-+} + B^{(1)}_k(\frac{1}{4}\epsilon^{ij}D^k_{ij})M_{--}, \tag{F.94}$$

$$\mathcal{D}M_{++} := dM_{++} + G^{(1)k}D^l_{kl}M_{++} + B^{(1)}_k(\frac{1}{2}\epsilon^{ij}D^k_{ij})M_{+-}. \tag{F.95}$$

The scalars in the matrix M_{MN} transform as

$$
\begin{aligned}
\delta M_{ij} =& -\xi^k\left(D^l_{ki}M_{lj}+D^l_{kj}M_{il}-D^l_{kl}M_{ij}\right)-\tilde{\lambda}\left(D^k_{ik}M_{j6}+D^k_{jk}M_{i6}\right)\\
&+\lambda^A\left(D^B_{iA}M_{Bj}+D^B_{jA}M_{iB}+D^k_{ik}M_{Aj}+D^k_{jk}M_{iA}\right),\\
\delta M_{\iota j} =& -\xi^k\left(D^l_{kj}M_{\iota l}-\delta_{m\iota}D^m_{kl}\delta^{l\iota'}M_{\iota' j}\right)-\tilde{\lambda}D^k_{jk}M_{6\iota}+\lambda^A\left(D^B_{jA}M_{\iota B}+D^k_{jk}M_{\iota A}\right),\\
\delta M_{5j} =& -\xi^k\left(D^l_{kj}M_{5l}-D^l_{kl}M_{5j}\right)-\tilde{\lambda}\left(D^k_{jk}M_{56}-\delta^{k\iota}D^l_{kl}M_{\iota j}\right)\\
&+\lambda^A\left(D^B_{jA}M_{5B}+D^k_{jk}M_{5A}\right),\\
\delta M_{6j} =& -\xi^k D^l_{kj}M_{6l}-\tilde{\lambda}D^k_{jk}M_{66}+\lambda^A\left(D^B_{jA}M_{6B}+D^k_{jk}M_{6A}\right),\\
\delta M_{Aj} =& -\xi^k\left(D^C_{kA}M_{Cj}+D^l_{kj}M_{Al}\right)-\tilde{\lambda}D^k_{jk}M_{A6}-\eta_{CD}\lambda^D\delta^{k\iota}D^C_{kA}M_{\iota j}\\
&+\lambda^B\left(D^C_{jB}M_{AC}+D^k_{jk}M_{AB}\right),\\
\delta M_{\iota\iota'} =&\xi^k\left(\delta_{m\iota}\delta^{l\iota''}D^m_{kl}M_{\iota'\iota''}+\delta_{m\iota'}\delta^{l\iota''}D^m_{kl}M_{\iota\iota''}-D^l_{kl}M_{\iota\iota'}\right),\\
\delta M_{5\iota} =&\xi^k\delta_{m\iota}\delta^{l\iota'}D^m_{kl}M_{5\iota'}+\tilde{\lambda}\delta^{k\iota'}D^l_{kl}M_{\iota\iota'},\\
\delta M_{6\iota} =&\xi^k\left(\delta_{m\iota}\delta^{l\iota'}D^m_{kl}M_{6\iota'}-D^l_{kl}M_{6\iota}\right),\\
\delta M_{A\iota} =& -\xi^k\left(D^C_{kA}M_{C\iota}-\delta_{m\iota}\delta^{l\iota'}D^m_{kl}M_{A\iota'}+D^l_{kl}M_{A\iota}\right)+\eta_{CD}\lambda^D\delta^{l\iota'}D^C_{lA}M_{\iota\iota'},\\
\delta M_{55} =&\xi^k D^l_{kl}M_{55}+2\tilde{\lambda}\delta^{k\iota'}D^l_{kl}M_{5\iota'},\\
\delta M_{56} =&\tilde{\lambda}\delta^{k\iota'}D^l_{kl}M_{6\iota'},\\
\delta M_{5A} =& -\xi^k D^C_{kA}M_{5C}+\tilde{\lambda}\delta^{k\iota'}D^l_{kl}M_{A\iota'}+\eta_{CD}\lambda^D\delta^{k\iota'}D^C_{kA}M_{5\iota'},\\
\delta M_{66} =& -\xi^k D^l_{kl}M_{66},\\
\delta M_{6A} =& -\xi^k\left(D^C_{kA}M_{6C}+D^l_{kl}M_{6A}\right)+\eta_{CD}\lambda^D\delta^{k\iota'}D^C_{kA}M_{6\iota'}\\
\delta M_{AB} =& -\xi^k\left(D^C_{kA}M_{CB}+D^C_{kB}M_{AC}+D^l_{kl}M_{AB}\right)\\
&+\eta_{CD}\lambda^D\delta^{k\iota'}\left(D^C_{kA}M_{B\iota'}+D^C_{kB}M_{A\iota'}\right).
\end{aligned}
\tag{F.96}
$$

The covariant derivatives of the scalars in terms of the matrix elements are:

$$\begin{aligned}
\mathcal{D}M_{ij} =& dM_{ij} + G^{(1)k}\left(D^l_{ki}M_{lj} + D^l_{kj}M_{il} - D^l_{kl}M_{ij}\right)\\
&+ \tilde{C}^{(1)}\left(D^k_{ik}M_{j6} + D^k_{jk}M_{i6}\right)\\
&+ C^{(1)A}\Big(D^B_{iA}M_{Bj} + D^B_{jA}M_{iB} + D^l_{il}M_{Aj} + D^l_{jl}M_{iB}\Big),\\
\mathcal{D}M_{\iota j} =& dM_{\iota j} + G^{(1)k}\left(D^l_{kj}M_{\iota l} - \delta_{m\iota}D^m_{kl}\delta^{l\iota'}M_{\iota' j}\right) + \tilde{C}^{(1)}D^k_{jk}M_{6\iota}\\
&+ C^{(1)A}\Big(D^B_{jA}M_{\iota B} + D^l_{jl}M_{\iota A}\Big),\\
\mathcal{D}M_{5j} =& dM_{5j} + G^{(1)k}\left(D^l_{kj}M_{5l} - D^l_{kl}M_{5j}\right) + \tilde{C}^{(1)}\left(D^k_{jk}M_{56} - \delta^{k\iota}D^l_{kl}M_{\iota j}\right)\\
&+ C^{(1)A}\Big(D^B_{jA}M_{5B} + D^l_{jl}M_{5A}\Big),\\
\mathcal{D}M_{6j} =& dM_{6j} + G^{(1)k}D^l_{kj}M_{6l} + \tilde{C}^{(1)}D^k_{jk}M_{66}\\
&- C^{(1)A}\Big(D^B_{jA}M_{6B} + D^l_{jl}M_{6A}\Big),\\
\mathcal{D}M_{Aj} +& dM_{Aj} + G^{(1)k}\left(D^C_{kA}M_{Cj} + D^l_{kj}M_{Al}\right) + \tilde{C}^{(1)}D^k_{jk}M_{A6}\\
&+ \eta_{CD}C^{(1)D}\delta^{k\iota}D^C_{kA}M_{\iota j} + C^{(1)B}\Big(D^C_{jB}M_{AC} + D^l_{jl}M_{AB}\Big),\\
\mathcal{D}M_{\iota\iota'} =& dM_{\iota\iota'} - G^{(1)k}\left(\delta_{m\iota}\delta^{l\iota''}D^m_{kl}M_{\iota'\iota''} + \delta_{m\iota'}\delta^{l\iota''}D^m_{kl}M_{\iota\iota''} - D^l_{kl}M_{\iota\iota'}\right),\\
\mathcal{D}M_{5\iota} =& dM_{5\iota} - G^{(1)k}\delta_{m\iota}\delta^{l\iota'}D^m_{kl}M_{5\iota'} - \tilde{C}^{(1)}\delta^{k\iota'}D^l_{kl}M_{\iota\iota'},\\
\mathcal{D}M_{6\iota} =& dM_{6\iota} - G^{(1)k}\left(\delta_{m\iota}\delta^{l\iota'}D^m_{kl}M_{6\iota'} - D^l_{kl}M_{6\iota}\right),\\
\mathcal{D}M_{A\iota} =& dM_{A\iota} + G^{(1)k}\left(D^C_{kA}M_{C\iota} - \delta_{m\iota}\delta^{l\iota'}D^m_{kl}M_{A\iota'} + D^l_{kl}M_{A\iota}\right)\\
&- \eta_{CD}C^{(1)D}\delta^{l\iota'}D^C_{lA}M_{\iota\iota'},\\
\mathcal{D}M_{55} =& dM_{55} - G^{(1)k}D^l_{kl}M_{55} - 2\tilde{C}^{(1)}\delta^{k\iota'}D^l_{kl}M_{5\iota'},\\
\mathcal{D}M_{56} =& dM_{56} - \tilde{C}^{(1)}\delta^{k\iota'}D^l_{kl}M_{6\iota'},\\
\mathcal{D}M_{5A} =& dM_{5A} + G^{(1)k}D^C_{kA}M_{5C} - \tilde{C}^{(1)}\delta^{k\iota'}D^l_{kl}M_{A\iota'} - \eta_{CD}C^{(1)D}\delta^{k\iota'}D^C_{kA}M_{5\iota'},\\
\mathcal{D}M_{66} =& dM_{66} + G^{(1)k}D^l_{kl}M_{66},\\
\mathcal{D}M_{6A} =& dM_{6A} + G^{(1)k}\left(D^C_{kA}M_{6C} + D^l_{kl}M_{6A}\right) - \eta_{CD}C^{(1)D}\delta^{k\iota'}D^C_{kA}M_{6\iota'}\\
\mathcal{D}M_{AB} =& dM_{AB} + G^{(1)k}\left(D^C_{kA}M_{CB} + D^C_{kB}M_{AC} + D^l_{kl}M_{AB}\right)\\
&- \eta_{CD}C^{(1)D}\delta^{k\iota'}\left(D^C_{kA}M_{B\iota'} + D^C_{kB}M_{A\iota'}\right).
\end{aligned} \tag{F.97}$$

F.2 The spectrum for the Scherk-Schwarz duality twist reduction

Here, we will present the spectrum of the Scherk-Schwarz duality twist reduction described in section 3.1. We will give the transformations and field strengths of the gauge fields, and the transformations and covariant derivatives of the scalars both as they are obtained from the reduction procedure and as they are in the matrix M_{MN}.

The gauge fields have the following transformations:

$$\delta G^{(1)i} = d\xi^i, \tag{F.98}$$

$$\delta \tilde{B}^{(1)\iota} = d\lambda^\iota + \lambda^K \delta^{i\iota} f_{iK}{}^I L_{IJ} A^J, \tag{F.99}$$

$$\delta A^{(1)I} = d\lambda^I - \xi^k f_{kK}{}^I A^K + \lambda^K f_{kK}{}^I G^{(1)k}. \tag{F.100}$$

Their field strengths, therefore, are

$$\mathcal{F}^{(2)i} = dG^{(1)i}, \tag{F.101}$$

$$\mathcal{F}^{(2)\iota} = d\tilde{B}^{(1)\iota} - \frac{1}{2} L_{JK} \delta^{i\iota} f_{iI}{}^K A^{(1)I} \wedge A^{(1)J}, \tag{F.102}$$

$$\mathcal{F}^{(2)I} = dA^{(1)I} + G^{(1)k} \wedge f_{kK}{}^I A^{(1)K}, \tag{F.103}$$

and these field strengths transform as

$$\delta \mathcal{F}^{(2)i} = 0, \tag{F.104}$$

$$\delta \mathcal{F}^{(2)\iota} = \lambda^K \delta^{i\iota} f_{iK}{}^I L_{IJ} \mathcal{F}^{(2)J}, \tag{F.105}$$

$$\delta \mathcal{F}^{(2)I} = -\xi^k f_{kK}{}^I \mathcal{F}^{(2)K} + \lambda^K f_{kK}{}^I \mathcal{F}^{(2)k}. \tag{F.106}$$

The scalars obtained from the reduction and the subsequent dualizations transform as

$$\delta g_{ij} = 0, \tag{F.107}$$

$$\delta e^{-2\phi} = 0, \tag{F.108}$$

$$\delta e^{-\eta} = 0, \tag{F.109}$$

$$\delta b_{ij} = 0, \tag{F.110}$$

$$\delta a_i^I = -\xi^k f_{kK}{}^I a_i^K + \lambda^K f_{iK}{}^I, \tag{F.111}$$

$$\delta \beta_{ij} = -\frac{1}{2} \lambda^K L_{IJ} \Big(f_{iK}{}^I a_j^J - f_{jK}{}^I a_i^J \Big), \tag{F.112}$$

$$\delta M_{IJ} = \xi^k \Big(f_{kI}^K M_{KJ} + f_{kJ}{}^L M_{IL} \Big). \tag{F.113}$$

We find that their covariant derivatives are given by

$$\mathcal{D} g_{ij} = dg_{ij}, \tag{F.114}$$

$$\mathcal{D} b_{ij} = db_{ij}, \tag{F.115}$$

$$\mathcal{D} M_{IJ} = dM_{IJ} - G^{(1)k} \Big(f_{kI}^K M_{KJ} + f_{kJ}{}^L M_{IL} \Big), \tag{F.116}$$

$$\mathcal{D} a_i^I = da_i^I - A^K f_{iK}{}^I + G^{(1)k} f_{kK}{}^I a_i^K, \tag{F.117}$$

$$\mathcal{D} \beta_{ij} = d\beta_{ij} + \frac{1}{2} A^K L_{IJ} \Big(f_{iK}{}^I a_j^J - f_{jK}{}^I a_i^J \Big), \tag{F.118}$$

$$\mathcal{D} e^{-2\phi} = de^{-2\phi}, \tag{F.119}$$

$$\mathcal{D} e^{-\eta} = de^{-\eta}. \tag{F.120}$$

The complex scalar

$$\tau := -\frac{1}{4}\epsilon^{ij}b_{ij} + \frac{i}{2}e^{-\eta} \tag{F.121}$$

does not not transform and therefore its covariant derivative is simply its regular derivative.

The scalar matrix M_{MN} that contains 132 of the 134 scalars is given by

$$\left(\begin{array}{c|c|c} e^{-2\phi}g_{ij} + \hat{M}_{IJ}a_i^I a_j^J + e^{2\phi}g^{kl}C_{ki}C_{lj} & -e^{2\phi}g^{kl}\delta_{k\iota}C_{lj} & -\hat{M}_{IK}a_j^K + e^{2\phi}g^{kl}L_{IK}a_k^K C_{lj} \\ \hline -e^{2\phi}g^{kl}\delta_{k\iota'}C_{li} & e^{2\phi}g^{kl}\delta_{k\iota}\delta_{l\iota'} & -e^{2\phi}g^{kl}\delta_{k\iota'}L_{IL}a_l^L \\ \hline -\hat{M}_{JK}a_i^K + e^{2\phi}g^{kl}L_{JK}a_k^K C_{li} & -e^{2\phi}g^{kl}\delta_{k\iota}L_{JL}a_l^L & \hat{M}_{IJ} + e^{2\phi}g^{kl}L_{IK}L_{JL}a_k^K a_l^L \end{array}\right) \tag{F.122}$$

Here, we have used the shorthand

$$C_{ij} := \beta_{ij} + \frac{1}{2}L_{IJ}a_i^I a_j^J. \tag{F.123}$$

Its inverse, $M^{MN} = L^{MO}M_{OP}L^{PN}$ is given by

$$\left(\begin{array}{c|c|c} e^{2\phi}g^{ij} & -e^{2\phi}g^{il}\delta^{k\iota'}C_{lk} & -e^{2\phi}g^{il}a_l^J \\ \hline -e^{2\phi}g^{jl}\delta^{k\iota}C_{lk} & \delta^{\iota k}\delta^{\iota' l}M_{kl} & -\hat{M}_{KL}a_l^L\delta^{l\iota}L^{JK} + e^{2\phi}g^{kl}a_k^J C_{lm}\delta^{m\iota} \\ \hline -e^{2\phi}g^{jl}a_l^I & -\hat{M}_{KL}a_l^L\delta^{l\iota'}L^{IK} + e^{2\phi}g^{kl}a_k^I C_{lm}\delta^{m\iota'} & L^{IK}L^{JL}\hat{M}_{KL} + e^{2\phi}g^{kl}a_k^K a_l^L \end{array}\right) \tag{F.124}$$

The transformations of the different components of M_{MN} are

$$\delta M_{ij} = \lambda^K\Big(f_{iK}{}^L M_{Lj} + f_{jK}{}^L M_{iL}\Big), \tag{F.125}$$

$$\delta M_{\iota j} = \lambda^K f_{jK}{}^L M_{\iota L}, \tag{F.126}$$

$$\delta M_{Ij} = \xi^k f_{kI}{}^K M_{Kj} + \lambda^K\Big(f_{iK}{}^L M_{IL} - f_{KI}{}^\iota M_{\iota j}\Big), \tag{F.127}$$

$$\delta M_{\iota\iota'} = 0, \tag{F.128}$$

$$\delta M_{I\iota'} = \xi^k f_{kI}{}^K M_{K\iota} - \lambda^K f_{KI}{}^\iota M_{\iota\iota'}, \tag{F.129}$$

$$\delta M_{IJ} = \xi^k\Big(f_{kI}{}^K M_{KJ} + f_{kJ}{}^K M_{IK}\Big) - \lambda^K\Big(f_{KI}{}^\iota M_{\iota J} + f_{KJ}{}^\iota M_{I\iota}\Big), \tag{F.130}$$

and their covariant derivatives are therefore given by

$$\mathcal{D}M_{ij} = dM_{ij} - A^K\Big(f_{iK}{}^L M_{Lj} + f_{jK}{}^L M_{iL}\Big), \tag{F.131}$$

$$\mathcal{D}M_{\iota j} = dM_{\iota j} - A^K f_{jK}{}^L M_{\iota L}, \tag{F.132}$$

$$\mathcal{D}M_{Ij} = dM_{Ij} - G^{(1)k} f_{kI}{}^K M_{Kj} - A^K\Big(f_{iK}{}^L M_{IL} - f_{KI}{}^\iota M_{\iota j}\Big), \tag{F.133}$$

$$\mathcal{D}M_{\iota\iota'} = dM_{\iota\iota'}, \tag{F.134}$$

$$\mathcal{D}M_{I\iota'} = dM_{I\iota'} - G^{(1)k} f_{kI}{}^K M_{K\iota} + A^K f_{KI}{}^\iota M_{\iota\iota'}, \tag{F.135}$$

$$\mathcal{D}M_{IJ} = dM_{IJ} - G^{(1)k}\Big(f_{kI}{}^K M_{KJ} + f_{kJ}{}^K M_{IK}\Big) + A^K\Big(f_{KI}{}^\iota M_{\iota J} + f_{KJ}{}^\iota M_{I\iota}\Big). \tag{F.136}$$

Appendix G

Spinor conventions for $SO(6)$

In this appendix, we summarize our Euclidean six-dimensional spinor conventions. We follow those laid out in [45] and [11].

The Clifford algebra is

$$\{\gamma_m \gamma_n\} = 2\delta_{mn}, \tag{G.1}$$

with the matrices γ_m hermitian. The antisymmetric products of the Clifford matrices γ_m, $\gamma_{m_1 \dots m_p}$ are defined as

$$\gamma_{m_1 \dots m_n} := \gamma_{[m_1} \gamma_{m_2} \dots \gamma_{m_p]} \tag{G.2}$$

for $p = 1, \dots 5$, with the sixfold antisymmetric matric γ_7 defined as

$$\gamma_7 := i\gamma_1 \dots \gamma_6, \tag{G.3}$$

such that

$$\gamma_7^2 = 1. \tag{G.4}$$

Furthermore, we choose the charge conjugation matrix C such that

$$C^T = C, \qquad \gamma_m^T = -C\gamma_m C^{-1}. \tag{G.5}$$

In six dimensions, the two Weyl representations η_- and η_+ of the Clifford algebra are complex conjugate to one another, meaning that

$$\eta_-^\dagger = \eta_+^T C, \tag{G.6}$$

or equivalently

$$\eta_+^\dagger = \eta_-^T C. \tag{G.7}$$

The spinors are normalized such that

$$\eta_+^\dagger \eta_+ = \eta_-^\dagger \eta_- = 1. \tag{G.8}$$

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