3-Form-Multiplets in Four Dimensional $\mathcal{N} = 1$ Supersymmetric Theories

Master Thesis

by Jason Sommerfeld

II. Institut für Theoretische Physik der Universität Hamburg, Luruper Chaussee 149,
22761 Hamburg, Germany

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Summary

In this master thesis $\mathcal{N} = 1$ supersymmetric actions involving the massless and massive 3-form gauge field in four spacetime dimensions are constructed and analyzed in a systematic way. The issue of boundary terms and boundary conditions for the massless 3-form is discussed, including a supersymmetric formulation. Special emphasis is put on non-renormalizable sigma model actions and the Poincaré dualization of these actions. The resulting scalar target space geometries are Kähler where the respective field variables and superspace couplings are related by a Legendre transformation. The transition between the on-shell action and dual action is analyzed on the component level and shown to constitute a local field redefinition in the massless case.

Zusammenfassung

Diese Masterarbeit behandelt die Konstruktion von $\mathcal{N} = 1$ supersymmetrischen Wirkungen des masselosen und massiven 3-Form-Eichfelds in vier Raumzeit-Dimensionen auf systematische Weise. Die Bedeutung von Randtermen und Randbedingungen für die masselose 3-Form wird diskutiert und in die supersymmetrische Formulierung übertragen. Insbesondere werden nicht-renormierbare sigma-Modelle und deren Poincaré-Dualisierung untersucht. Die dabei gefundenen Skalarfeld-Geometrien in Wirkung und dualer Wirkung sind Kähler, wobei die auftretenden Feldvariablen und Superraum-Kopplungen durch eine Legendre-Transformation zusammenhängen. Der Übergang zwischen on-shell-Wirkung und dualer Wirkung wird auf der Ebene der Komponentenfelder analysiert. Im masselosen Fall stellt dieser eine lokale Feldredefinition dar.

Contents

1 Introduction

Supersymmetry is one of the most attractive concepts of theoretical physics for several reasons. First of all, it is the only possible nontrivial extension of the Poincaré symmetry group for the S matrix under reasonable physical assumptions [1, 2, 3]. As spacetime possesses all the Poincaré symmetries it seems to be most natural that this additional symmetry is realized in nature as well. Furthermore, supersymmetry could resolve most of the problems of the standard model, e.g. the hierarchy problem and the missing candidate for a dark matter particle [4, 5]. In its local form as supergravity, supersymmetry could even provide the framework for a consistent quantum gravity [6].

Supersymmetry is also an essential feature of string theory, which is today the most promising candidate for a unified fundamental theory of particle physics. In the low energy limit of string theory (which is supergravity), the vibrational modes of the string appear as familiar particles such as gauge bosons, matter fermions and scalars but also as less familiar particles such as 2-, 3- and 4-form fields [7, 8]. The latter live in a 10 dimensional spacetime as this is the number of dimensions of consistent superstring theories with geometrical interpretation. To find the particles that can be observed in our four-dimensional world, one has to compactify the extra dimensions e.g. by the Kaluza-Klein technique [9]. In this process there arise also p-forms that live in four dimensions [7, 10]. Therefore it is important to study the properties of these fields in supersymmetric theories.

In this work we investigate supersymmetric actions containing the 3-form gauge field. This field appears in the Kaluza-Klein compactification of type IIA supergravity [10, 11]. Apart from that, it can be used to describe Chern-Simons terms in super Yang-Mills theories [12]. The 3-form has the interesting property of not possessing any on-shell degrees of freedom in the massless case. This can be seen either from its equation of motion, which makes its field strength a constant, or by finding the Poincaré dual action which explicitly contains a constant field. These constants that appear in the (on-shell) action and dual action respectively can be considered as a natural origin of a cosmological constant which is needed to explain the observed acceleration in the expansion of the universe [13, 14]. This is another reason for the increased interest in theories including 3-forms over the last decades. In the massive case however, the 3-form acquires one on-shell degree of freedom by 'eating' a 2-form field. Then it is dual to a massive scalar.

In this thesis we do not restrict to renormalizable actions but rather concentrate on the general case of a sigma model action with 3-form supermultiplets. We will compute the component form of this action which has the structure of a Kähler geometry and eliminate the auxiliary fields to find the effective on-shell action. Special emphasis will be put on the dualization of this action and the relation between the on-shell component action and dual action. We will find that the scalar fields of the dual action come with the same Kähler geometry but expressed in terms of the Legendre transform of the original

Kähler potential.

This work is structured as follows. Section 2 recalls some aspects of supersymmetry that are needed throughout this work. In Section 3 the supermultiplet containing the 3-form and the associated field strength multiplet is introduced. We discuss gauge and supersymmetry transformations of this multiplet. In an intermezzo we study the actions of the massless and massive 3-form and their dualization, including the discussion of boundary terms and boundary conditions for the massless 3-form. Then we proceed to the supersymmetric case and give the renormalizable actions for the 3-form multiplet (massless and massive). When we dualize these actions, we will find a new supermultiplet which in the massless case is closely related to the complex linear one [15, 16]. In Section 4 we generalize the analysis to non-renormalizable actions. Here we confine our attention to the bosonic part of the action and focus on the appearing scalar geometries. In Section 5 we couple massive 3-form multiplets to chiral ones first in renormalizable, then in generic theories. We conclude in Section 6. Some additional material is given in four appendices. Appendix A summarizes the conventions of this work, together with some useful relations. In Appendix B some generic formulas for eliminating auxiliary fields are derived. Appendix C gives a brief introduction to the Legendre transformation, while in Appendix D we analyze the transition of the massive sigma model action to the dual action on the component level.

This master thesis is based upon the diploma thesis of K. Groh [17]. There has been a discrepancy between his result for the dual sigma model action of the 3-form multiplet and a result stated in [18] for the sigma model action with complex linear multiplets which is a special case of the former (see Sec. 4). The aim of this master thesis is to clarify this discrepancy and to answer the question whether the dual scalar geometry is Kähler again. Beyond this, the work of K. Groh has been completely revised and extended in many ways.

The results found by K. Groh for the renormalizable actions of the massless and massive 3-form multiplet are stated in Section 3. Here the new contribution of this work is, besides the supersymmetry variation of the 3-form multiplet, the discussion of boundary terms and boundary conditions for the 3-forms first on the component level, then in the supersymmetric generalization in terms of superfields. The main new developments of this work are presented in Sections 4 and 5: For the sigma model action of the massless 3-form multiplet we will again derive the supersymmetric boundary terms that have to be added to the action in order to eliminate the massless 3-forms in a consistent way. In the dualization of this action with the technique proposed in [17] elimination of the auxiliary fields is performed in a systematic and straightforward way. The correct on-shell dual component action is found and shown to be equal to the original on-shell action by use of the duality relations between the physical fields appearing these actions, thereby providing a consistency check for the correctness of the result. Dualization of the massive sigma model action is demonstrated and in App. D the on-shell component action is translated back into the original action using the duality relations between the component fields together with their equations of motion. The same check account is done for the special case of Kähler potentials with a shift symmetry that is treated in Section 4.4. For the coupling of 3-form multiplets to chiral ones, we will depart from a more general ansatz than ref. [17] both for the renormalizable as well as for the generic case. Here complete new contributions are given by the analysis of the conditions for spontaneous supersymmetry breaking and the mass spectrum in the case of vanishing superpotentials. In the generic case we will eliminate the auxiliary fields from the action and determine the scalar potential and the metric for the 3-form field strengths.

The results of [17] and this work will also be presented in a paper [19] to be published soon.

2 Some aspects of supersymmetry

This work does not give an introduction to supersymmetry; for this we refer to the literature (see e.g. [20, 21, 22, 23]). However, let us briefly recall some ideas of supersymmetry that are needed for the understanding of the following sections.

2.1 Supersymmetry algebra, superfields, covariant derivatives, supersymmetric actions

The generators of simple ($\mathcal{N}=1$) supersymmetry Q_α satisfy the anticommutation relations

$$\begin{aligned} &\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^m_{\alpha\dot\beta}P_m = -2i\sigma^m_{\alpha\dot\beta}\partial_m, \qquad \alpha, \dot\beta = 1,2, \quad m = 0,\ldots,3, \\ &\{Q_\alpha, Q_\beta\} = 0, \qquad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0, \end{aligned} \tag{2.1}$$

where $\sigma^0 = -\mathbf{1}$ and σ^i with $i = 1,2,3$ are the Pauli matrices. On superspace (the space which is parameterized by the four spacetime variables x^m plus two complex Grassmann valued coordinates θ^α), they can be represented by[1]

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - \sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m, \qquad \bar{Q}_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m. \tag{2.2}$$

The concept of superspace makes it very easy to find supermultiplets, i.e. representations of the supersymmetry algebra (2.1). The operators (2.2) already define a (highly reducible) representation on the space of superfields (functions that are defined on super-

[1] Both Q_α and $\bar{Q}_{\dot\alpha}$ as defined here differ by a factor of $-i$ from those defined in [20]. With these conventions there is no sign discrepancy between (2.1) and (2.2). However, the supersymmetry variation defined in (2.6) (which is the only place where Q_α and $\bar{Q}_{\dot\alpha}$ enter in this work) is in agreement with the conventions of [20] where it is defined as $\delta_\xi F = (\xi Q + \bar\xi\bar{Q})F$.

space). To find subrepresentations one defines the covariant superspace derivatives[2]

$$D_\alpha := \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m, \qquad \bar D_{\dot\alpha} := \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m. \tag{2.3}$$

They satisfy

$$\begin{aligned} \{D_\alpha, Q_\beta\} &= 0 = \{D_\alpha, \bar Q_{\dot\beta}\}, \qquad \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 = \{\bar D_{\dot\alpha}, Q_\beta\}, \\ \{D_\alpha, D_\beta\} &= 0 = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\}, \qquad \{D_\alpha, \bar D_{\dot\beta}\} = 2i\sigma^m_{\alpha\dot\beta}\partial_m. \end{aligned} \tag{2.4}$$

Since D_α and $\bar D_{\dot\alpha}$ anticommute with Q_α and $\bar Q_{\dot\alpha}$, they can be used to construct supersymmetry invariant conditions on superfields (e.g. $\bar D_{\dot\alpha}F = 0$) and thus subrepresentations of the supersymmetry algebra on the space of superfields.

A superfield F can always be expanded in terms of component functions as

$$\begin{aligned} F(x,\theta,\bar\theta) \; &= f(x) + \theta\eta(x) + \bar\theta\bar\chi(x) + \theta^2 h(x) + \bar\theta^2 n(x) \\ &\quad + \theta\sigma^m\bar\theta v_m(x) + \theta^2\bar\theta\bar\lambda(x) + \bar\theta^2\theta\psi(x) + \theta^2\bar\theta^2 d(x). \end{aligned} \tag{2.5}$$

(In the following the arguments of the superfields and component fields will be suppressed.) It follows from (2.1) that the Q_α have mass dimension $\frac{1}{2}$, and consequently θ^α must have mass dimension $-\frac{1}{2}$. Therefore the components of F have different mass dimensions depending on the mass dimension of F itself.

From the supersymmetry variation of F

$$\delta_\xi F = i(\xi Q + \bar\xi\bar Q)F, \tag{2.6}$$

where ξ is a constant 2-component spinor, the variations of the component fields can be derived. Since these have to be linear in the component fields (the Q_αs are linear operators) and d is the component of F with the highest mass dimension, it is clear that the supersymmetry variation of d has to be a total divergence. Indeed, it is given by

$$\delta_\xi d = \tfrac{i}{2}\partial_m\big(\psi\sigma^m\bar\xi + \xi\sigma^m\bar\lambda\big). \tag{2.7}$$

As total divergences typically vanish under the spacetime integral, supersymmetric actions can be constructed simply by taking the $\theta^2\bar\theta^2$-component of any (real) combination of superfields and integrating over the Minkowski space. To write these actions in an elegant way, one defines integration over the Grassmann valued coordinates as

$$\int d^2\theta \; := \tfrac{1}{4}\int \varepsilon^{\alpha\beta}d\theta_\alpha d\theta_\beta, \qquad \int d^2\bar\theta \; := \tfrac{1}{4}\int \varepsilon^{\dot\beta\dot\alpha}d\bar\theta_{\dot\alpha}d\bar\theta_{\dot\beta}. \tag{2.8}$$

[2]Here again we choose slightly different conventions than [20]; namely $\bar D_{\dot\alpha}$ differs by a factor of -1. This does not affect any of the results stated in this work.

Here the Berezin integral for Grassmann numbers is used, which is defined by

$$\int d\eta\,\eta \;:=\; 1, \qquad \int d\eta\,1 \;:=\; 0 \tag{2.9}$$

(where η is a single Grassmann variable) together with the requirement of linearity. The prefactors in (2.8) where chosen such that

$$\int d^2\theta\,\theta^2 = 1 = \int d^2\bar\theta\,\bar\theta^2, \qquad \Rightarrow \int d^2\theta\,d^2\bar\theta\,F = d, \tag{2.10}$$

i.e., to integrate over θ and $\bar\theta$ means to extract the $\theta^2\bar\theta^2$-component of the integrand. For convenience let us also define

$$\int d^8z \;:=\; \int d^4x \int d^2\theta\,d^2\bar\theta\,. \tag{2.11}$$

Then a supersymmetric action for N complex superfields F^i, $i=1,\ldots,N$, may be constructed as

$$S = \int d^8z\,K(F,\bar F), \tag{2.12}$$

where K is a real function. Depending on possible constraints on the F^i, the equations of motion can be given in terms of the superfields by varying the action with respect to the F^i:

$$\delta S = \int d^8z \left(\frac{\partial K}{\partial F^i}\delta F^i + \frac{\partial K}{\partial \bar F^{\bar\jmath}}\delta \bar F^{\bar\jmath}\right). \tag{2.13}$$

This has to vanish for every allowed variation δF. If the F^i are unconstrained superfields, also the variation δF^i is unconstrained (except for boundary conditions) and then every component of $\partial K/\partial F^i$ is multiplied by a component of δF^i to contribute to the $\theta^2\bar\theta^2$-component of the integrand, so that one finds the superfield equations of motion

$$\frac{\partial K}{\partial F^i} = 0, \qquad \frac{\partial K}{\partial \bar F^{\bar\jmath}} = 0. \tag{2.14}$$

When the F^i are real superfields, then $\partial K/\partial F^i$ is also real and by the same argument one has

$$\frac{\partial K}{\partial F^i} = 0. \tag{2.15}$$

When the F^i are constrained superfields, e.g. by the condition $\bar D_{\dot\alpha}F^i = 0$, one has to find the most general solution to the constraints in terms of generic superfields U^i, e.g. $F^i = \bar D^2 U^i$, substitute this into the action and then vary with respect to U^i. Then integration by parts with the superspace derivatives has to be applied in order to derive the superfield equations of motion. This is possible because for D_α and $\bar D_{\dot\alpha}$ the Leibniz rule holds,

$$D_\alpha(FG) = (D_\alpha F)G + F D_\alpha G, \quad \bar D_{\dot\alpha}(FG) = (\bar D_{\dot\alpha}F)G + F\bar D_{\dot\alpha}G, \tag{2.16}$$

and terms of the form

$$\begin{aligned}\int d^8z\, D_\alpha(FG) &= \int d^4x \int d^2\theta\, d^2\bar{\theta}\, i\sigma^m_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_m(FG),\\ \int d^8z\, \bar{D}_{\dot{\alpha}}(FG) &= \int d^4x \int d^2\theta\, d^2\bar{\theta}\, i\theta^\beta\sigma^m_{\beta\dot{\alpha}}\partial_m(FG)\end{aligned} \tag{2.17}$$

are indeed boundary terms in the usual sense, which can be dropped in most cases (but not in all as we will see below).

2.2 Supersymmetry multiplets

Let us now recall some of the $\mathcal{N}=1$ supermultiplets which will be referred to in the following sections.

2.2.1 Chiral multiplet

The supermultiplet which is used most frequently is the chiral multiplet. The corresponding superfield Φ is defined by

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{2.18}$$

It can always be expressed in terms of an unconstrained superfield F as $\Phi = \bar{D}^2F$ (since $\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}} = 0$ a superfield defined in this way is chiral) and has the generic form

$$\begin{aligned}\Phi =&\ A + \sqrt{2}\theta\psi + i\theta\sigma^m\bar{\theta}\partial_m A + \theta^2 F + \tfrac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^m\partial_m\psi + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box A,\\ \bar{\Phi} =&\ A^* + \sqrt{2}\bar{\theta}\bar{\psi} - i\theta\sigma^m\bar{\theta}\partial_m A^* + \bar{\theta}^2F^* + \tfrac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\psi} + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box A^*,\end{aligned} \tag{2.19}$$

containing a complex scalar A, a Weyl fermion ψ and an auxiliary field F. Its renormalizable kinetic action is given by

$$S = \int d^8z\, \Phi\bar{\Phi} = \int d^4x \Big(- \partial_m A\, \partial^m A^* - i\psi\sigma^m\partial_m\bar{\psi} + FF^*\Big). \tag{2.20}$$

Therefore Φ must have mass dimension 1 so that A and ψ have mass dimension 1 and 1/2 respectively. The components of the chiral multiplet transform under supersymmetry as follows

$$\delta_\xi A = \sqrt{2}\xi\psi, \quad \delta_\xi\psi = i\sqrt{2}\sigma^m\bar{\xi}\,\partial_m A + \sqrt{2}\xi F, \quad \delta_\xi F = i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m\psi. \tag{2.21}$$

By inserting $\Phi = \bar{D}^2F$ into the action (2.20) one can vary with respect to F to find the superfield equation of motion

$$\bar{D}^2\bar{\Phi} = 0. \tag{2.22}$$

2.2.2 Vector multiplet

The vector multiplet is represented by a real superfield $V = \bar{V}$. Its θ-expansion can be written as

$$\begin{aligned} V = {} & B + i\theta\chi - i\bar{\theta}\bar{\chi} + \theta^2 M^* + \bar{\theta}^2 M + 2\theta\sigma^m\bar{\theta} v_m \\ & + \theta^2\bar{\theta}\big(\sqrt{2}\bar{\lambda} + \tfrac{1}{2}\bar{\sigma}^m\partial_m\chi\big) + \bar{\theta}^2\theta\big(\sqrt{2}\lambda - \tfrac{1}{2}\sigma^m\partial_m\bar{\chi}\big) + \theta^2\bar{\theta}^2\big(D - \tfrac{1}{4}\Box B\big), \end{aligned} \tag{2.23}$$

with real scalars B and D, a complex scalar M, a real vector v_m and Weyl spinors $\chi,\ \lambda$. The vector multiplet is used for the description of supersymmetric gauge theories with v_m being the gauge boson. A gauge transformation is implemented as

$$V \to\ V + \Phi + \bar{\Phi}, \quad v_m \to\ v_m + \tfrac{i}{2}\partial_m(A - A^*), \tag{2.24}$$

with a chiral superfield Φ. The field strength multiplet of V is defined by

$$W_\alpha = -\tfrac{1}{4}\bar{D}^2 D_\alpha V, \tag{2.25}$$

and contains the 2-form field strength $v_{mn} = \partial_m v_n - \partial_n v_m$. W_α is invariant under (2.24). By choosing a special gauge, called Wess-Zumino gauge, it is possible to set the components B, χ and M in (2.23) to zero.

2.2.3 Linear multiplet

A real multiplet L that satisfies the additional constraint

$$D^2 L = 0 \tag{2.26}$$

is called linear multiplet [21]. It is of the form [24, 25]

$$L = E + i\theta\eta - i\bar{\theta}\bar{\eta} + \tfrac{1}{2}\theta\sigma^m\bar{\theta}\varepsilon_{mnpq}\partial^{[n}B^{pq]} + \tfrac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^m\partial_m\eta - \tfrac{1}{2}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\eta} - \tfrac{1}{4}\theta^2\bar{\theta}^2\Box E, \tag{2.27}$$

containing a real scalar E, a 2-form B_{pq} and a Weyl spinor η. Its action is given by

$$S = -\int d^8z\, L^2 = \int d^4x \Big(-\tfrac{1}{2}\partial_m E\partial^m E - i\eta\sigma^m\partial_m\bar{\eta} - \tfrac{3}{4}\partial_{[n}B_{pq]}\partial^n B^{pq}\Big). \tag{2.28}$$

The 3-form field strength $F_{npq} := 3\partial_{[n}B_{pq]} = \partial_n B_{pq} + \partial_p B_{qn} + \partial_q B_{np}$ is invariant under gauge transformations

$$B_{pq}\ \to\ B_{pq} + \partial_{[p}a_{q]}. \tag{2.29}$$

Here the parameter a_q is itself a gauge field and has three gauge invariant degrees of freedom, so of the six independent components of B_{pq} three are gauge invariant. Thus the linear multiplet carries four bosonic and four fermionic off-shell degrees of freedom. There is a duality between the chiral and linear multiplet that corresponds to the physical equivalence of a scalar field and a 2-form [26].

2.2.4 Complex linear multiplet

The complex linear multiplet Σ is defined by a similar condition as the linear multiplet but with no reality condition [18, 27]:

$$D^2\bar{\Sigma} = 0, \qquad \bar{D}^2\Sigma = 0. \tag{2.30}$$

This is solved by the superfield with the component expansion

$$\begin{aligned}\Sigma &= f + \theta\psi + \sqrt{2}\bar{\theta}\bar{\varphi} + \theta^2 h + \theta\sigma^m\bar{\theta}w_m + \theta^2\bar{\theta}\bar{\vartheta} - \tfrac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\varphi} \\ &\quad + \theta^2\bar{\theta}^2\big(- \tfrac{i}{2}\partial_m w^m - \tfrac{1}{4}\Box f\big), \\ \bar{\Sigma} &= f^* + \bar{\theta}\bar{\psi} + \sqrt{2}\theta\varphi + \bar{\theta}^2 h^* + \theta\sigma^m\bar{\theta}w^*_m + \bar{\theta}^2\theta\vartheta - \tfrac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^m\partial_m\varphi \\ &\quad + \theta^2\bar{\theta}^2\big(\tfrac{i}{2}\partial_m w^{m*} - \tfrac{1}{4}\Box f^*\big),\end{aligned} \tag{2.31}$$

where f and h are complex scalars, w_m is a complex vector and ψ, φ, ϑ are Weyl spinors. These are 12 bosonic and 12 fermionic off-shell degrees of freedom. Note that a chiral multiplet also satisfies (2.30), so that the complex linear multiplet carries a reducible representation of the supersymmetry algebra (2.1). The action for the complex linear multiplet reads

$$\begin{aligned}S &= -\int d^8z\, \Sigma\bar{\Sigma} \\ &= \int d^4x \Big(\tfrac{i}{2}f^*\partial_m w^m - \tfrac{i}{2}f\partial_m w^{m*} + \tfrac{1}{2}f\Box f^* + \tfrac{1}{2}\psi\vartheta + \tfrac{1}{2}\bar{\psi}\bar{\vartheta} \\ &\qquad\qquad - i\varphi\sigma^m\partial_m\bar{\varphi} - hh^* + \tfrac{1}{2}w^*_m w^m\Big).\end{aligned} \tag{2.32}$$

The on-shell action after elimination of the auxiliary fields $w_m,\ h,\ \vartheta$ and ψ is given by

$$S = \int d^4x \Big(- \partial_m f\partial^m f^* - i\varphi\sigma^m\partial_m\bar{\varphi}\Big). \tag{2.33}$$

Like the action of the chiral multiplet, it describes a complex scalar and a Weyl spinor. Therefore the chiral multiplet can alternatively be dualized to a complex linear multiplet [27].

3 The 3-form multiplet

3.1 Components, field strength, gauge and supersymmetry transformations

To find a supermultiplet containing the 3-form C_{npq} one makes use of the fact that its Hodge dual is a vector

$$v_m = \tfrac{1}{6}\,\varepsilon_{mnpq}C^{npq}, \tag{3.1}$$

which can reside in a vector multiplet. The latter is represented by a real superfield[3] [12, 29]

$$\begin{aligned} U = {} & B + i\theta\chi - i\bar\theta\bar\chi + \theta^2 M^* + \bar\theta^2 M + \tfrac{1}{3}\theta\sigma^m\bar\theta\varepsilon_{mnpq}C^{npq} \\ & + \theta^2\bar\theta\big(\sqrt{2}\bar\lambda + \tfrac{1}{2}\bar\sigma^m\partial_m\chi\big) + \bar\theta^2\theta\big(\sqrt{2}\lambda - \tfrac{1}{2}\sigma^m\partial_m\bar\chi\big) + \theta^2\bar\theta^2\big(D - \tfrac{1}{4}\Box B\big). \end{aligned} \tag{3.2}$$

Like the vector multiplet given in (2.23), U carries 8 fermionic (χ, λ) and 8 bosonic (B, D, M and C_{npq}) off-shell degrees of freedom. The difference between the 3-form multiplet and the vector multiplet is only visible in the definitions of their associated field strength multiplets. A vector field as a 1-form has a 2-form field strength, with another 2-form as its dual. The field strength of the 3-form on the other hand is a 4-form

$$H_{mnpq} = 4\partial_{[m}C_{npq]} = \partial_m C_{npq} - \partial_n C_{pqm} + \partial_p C_{qmn} - \partial_q C_{mnp}, \tag{3.3}$$

with a 0-form (i.e., a scalar) H as its dual,

$$H = \tfrac{1}{4!}\varepsilon^{mnpq}H_{mnpq} = \tfrac{1}{6}\varepsilon^{mnpq}\partial_m C_{npq}, \qquad H_{mnpq} = -\varepsilon_{mnpq}H. \tag{3.4}$$

For this reason a field strength for the 3-form multiplet cannot be constructed like the vector multiplet's defined in (2.25). Instead the field strength multiplet for U is defined by [30, 31]

$$S = -\tfrac{1}{4}\bar D^2 U, \qquad \bar S = -\tfrac{1}{4}D^2\bar U. \tag{3.5}$$

Since $\bar D_{\dot\alpha}\bar D_{\dot\beta}\bar D_{\dot\gamma} = 0$, this definition implies that S is a chiral superfield,

$$\bar D_{\dot\alpha}S = 0. \tag{3.6}$$

Its expansion in component fields reads

$$\begin{aligned} S = {} & M + \sqrt{2}\theta\lambda + \theta^2(D + iH) + i\theta\sigma^m\bar\theta\partial_m M + \tfrac{i}{\sqrt{2}}\theta^2\bar\theta\bar\sigma^m\partial_m\lambda + \tfrac{1}{4}\theta^2\bar\theta^2\Box M, \\ \bar S = {} & M^* + \sqrt{2}\bar\theta\bar\lambda + \bar\theta^2(D - iH) - i\theta\sigma^m\bar\theta\partial_m M^* + \tfrac{i}{\sqrt{2}}\bar\theta^2\theta\sigma^m\partial_m\bar\lambda + \tfrac{1}{4}\theta^2\bar\theta^2\Box M^*. \end{aligned} \tag{3.7}$$

[3] Compared to the usual normalization we have rescaled U in (3.2) and (3.5) by a factor of 16 to avoid some clumsy factors in the massive action where the superfield Lagrangian depends directly on U.

S contains only 4 bosonic and fermionic degrees of freedom off-shell, because the fields B and χ do not contribute and the field strength H as a scalar field carries only one degree of freedom. Since S is built from a real superfield, it is not a general chiral field. Indeed, the imaginary part of the θ^2-component of S is the dual field strength H, which, being a total divergence, is not an unconstrained field.

Since S is a chiral superfield, we assign to it mass dimension 1. Consequently U must have mass dimension 0. From this it follows that the fields B and χ have both one mass dimension less than a normal scalar and Weyl fermion, namely 0 and 1/2 respectively. Therefore, when they enter the massive 3-form action as described in Sec. 3.4, they will have to be rescaled by parameters of mass dimension 1 in order to get kinetic terms of the standard form.

It follows directly form the definition (3.5) that S is invariant under the gauge transformation

$$U \to U - L, \tag{3.8}$$

where L is a linear multiplet, i.e., a real superfield that satisfies $D^2L = 0 = \bar{D}^2L$. According to (2.27), L contains the field strength of a 2-form B_{pq} in its vector component, that is needed to describe the gauge transformation of the 3-form. At the level of component fields, (3.8) reads

$$\begin{aligned} B \to B - E, \quad \chi \to \chi - \eta, \quad M \to M, \\ C_{npq} \to C_{npq} - \partial_{[n}B_{pq]}, \quad \lambda \to \lambda, \quad D \to D. \end{aligned} \tag{3.9}$$

From (3.3) we see that the field strength H_{mnpq} remains invariant. As shown in the previous section, the gauge parameter B_{pq} carries three gauge invariant degrees of freedom. Thus the 3-form, having four independent components, carries one gauge invariant degree of freedom which is represented by its dual field strength H. In analogy to the the Wess-Zumino gauge of the vector multiplet, the components B and χ can be set to zero by use of the gauge freedom (3.9). However, unlike the θ^2-component of the vector multiplet, the scalar M is gauge invariant and will turn out to be a physical field.

The supersymmetry transformation of the 3-form multiplet is given by [30, 31]

$$\begin{aligned} \delta_\xi M &= \sqrt{2}\xi\lambda, \\ \delta_\xi \lambda &= \sqrt{2}i\sigma^m\bar{\xi}\partial_m M + \sqrt{2}\xi(D + iH), \\ \delta_\xi C_{npq} &= \varepsilon_{mnpq}\xi\left(\tfrac{1}{\sqrt{2}}\sigma^m\bar{\lambda} + \sigma^{ml}\partial_l\chi\right) + \text{h.c.}, \\ \Rightarrow \delta_\xi H &= \mathrm{Im}\big(i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m\lambda\big), \\ \delta_\xi B &= i\xi\chi - i\bar{\xi}\bar{\chi}, \\ \delta_\xi \chi &= -2i\xi M^* + \sigma^m\bar{\xi}\left(-\tfrac{i}{3}\varepsilon_{mnpq}C^{npq} + \partial_m B\right). \end{aligned} \tag{3.10}$$

Note that the gauge invariant components of S transform among themselves as in an ordinary chiral multiplet, see (2.21). The second term in the supersymmetry variations of the 3-form drops out of $\delta_\xi H$ so it has to be pure gauge (this term does not appear in the references [30, 31]). Using

$$\sigma^{ml} = -\tfrac{1}{4}\varepsilon^{mlrs}\varepsilon_{rsto}\sigma^{to}, \tag{3.11}$$

one can show directly that it constitutes a gauge transformation by writing it as

$$\varepsilon_{mnpq}\xi\sigma^{ml}\partial_l\chi = \tfrac{3}{2}\partial_{[n}\varepsilon_{pq]ml}\xi\sigma^{ml}\chi. \tag{3.12}$$

Curiously, the supersymmetry variations of χ involves the field M with no derivatives, so that supersymmetry can be unbroken only if M has a vanishing vev. We will meet this issue again in Section 5.

3.2 The massless and massive 3-form action

Before we turn to the supersymmetric case let us get familiar with the renormalizable actions and equations of motion of the massless and massive 3-form itself. In the massless case the 3-form does not carry any on-shell degrees of freedom, so that it can be eliminated from the action, thereby taking the form of a constant potential (i.e., an effective cosmological constant). However, there is an issue with boundary terms and boundary conditions that shall be discussed in the following. After finding the correct action for the massless 3-form, the Poincaré dual action will be derived. In the massive case we will analyze the equations of motion to determine the number of on-shell degrees of freedom.

The canonical renormalizable action of the massless 3-form is[4]

$$S_3 = -\tfrac{1}{24}\int d^4x\, H^{mnpq}H_{mnpq} = \int d^4x\, H^2. \tag{3.13}$$

Using $H = \frac{1}{6}\varepsilon^{mnpq}\partial_m C_{npq}$, the equations of motion for the 3-form are found to be

$$-\tfrac{1}{3}\varepsilon^{mnpq}\partial_m H = 0. \tag{3.14}$$

They force the field strength to be a constant, $H = c$ with $c \in \mathbb{R}$, or

$$H^{mnpq} = -c\,\varepsilon^{mnpq}. \tag{3.15}$$

For this reason the massless 3-form has been studied in the context of the cosmological constant problem [32, 33, 34, 35, 36]. However, the action (3.13) is not the full story. To see the problem, one has to focus on the boundary term in the variation

$$\delta S_3 = \tfrac{1}{3}\int d^4x\, \partial_m\big(H\varepsilon^{mnpq}\delta C_{npq}\big) - \tfrac{1}{3}\int d^4x\, (\partial_m H)\varepsilon^{mnpq}\delta C_{npq}. \tag{3.16}$$

[4]The usual normalization includes another factor of 1/2 which we omit for convenience.

In order to make this boundary term vanish, one has to impose the condition

$$\delta C_{npq}\Big|_{\partial\mathcal{M}} = 0, \tag{3.17}$$

where $\partial\mathcal{M}$ denotes the boundary of the integration volume $\mathcal{M}$.[5] One might already doubt whether this is a good boundary condition as it is not gauge invariant. Moreover, it has been pointed out by Duff [33] that substituting the solution (3.15) back into (3.13) yields a wrong sign in the correction of the bare cosmological constant Λ_0. The correct value of the effective cosmological constant can be found by coupling the 3-form to gravity via the action

$$\begin{aligned} S_{3,\mathrm{EH}} &= \tfrac{1}{16\pi G}\int d^4x\,\sqrt{-g}(R-2\Lambda_0) - \tfrac{1}{24}\int d^4x\,\sqrt{-g}H^{mnpq}H_{mnpq} \\ &= \tfrac{1}{16\pi G}\int d^4x\,\sqrt{-g}(R-2\Lambda_0) + \int d^4x\,\sqrt{-g}H^2, \end{aligned} \tag{3.18}$$

where in the second step we made use of the generalized Hodge duality relations

$$H = \tfrac{1}{4!\sqrt{-g}}\varepsilon^{mnpq}H_{mnpq}, \quad H_{mnpq} = -\tfrac{1}{\sqrt{-g}}\varepsilon_{mnpq}H. \tag{3.19}$$

(For this, notice our conventions (A.1) which imply e.g. that $\varepsilon^{mnpq}\varepsilon_{mnpq} = 24g$. The definition of H was chosen such that it is a Lorentz scalar.) Then one solves the equations of motion of the 3-form,

$$\varepsilon^{mnpq}\partial_m H = 0 \;\Rightarrow\; H = c, \tag{3.20}$$

and inserts the solution into the stress energy tensor

$$T^{mn} = -g^{mn}H^2 = -c^2 g^{mn}, \tag{3.21}$$

which appears in the Einstein equations

$$R^{mn} - \tfrac{1}{2}g^{mn}R = -\Lambda_0 g^{mn} + 8\pi G T^{mn} \equiv -\Lambda g^{mn}. \tag{3.22}$$

The effective cosmological constant is then $\Lambda = \Lambda_0 + 8\pi G c^2$. On the other hand, substituting (3.20) into the action (3.18) yields $\Lambda = \Lambda_0 - 8\pi G c^2$. This discrepancy is clearly a result of the incompatibility of the variational constraint (3.17) with the solution (3.20). Namely, (3.20) is equivalent to

$$\sqrt{-g} = \tfrac{1}{24c}\varepsilon^{mnpq}\partial_m C_{npq}. \tag{3.23}$$

Thus, in order to implement the constraint (3.17) in the on-shell action, one could allow

[5] Alternatively one could demand $\delta C_{npq}(x) \to 0$ for $x \to \infty$ sufficiently fast when one integrates over the whole Minkowski space.

only for variations of the metric for which

$$\int_{\mathcal{M}} d^4x\, \delta\sqrt{-g} = \tfrac{1}{24c}\int_{\mathcal{M}} d^4x\, \partial_m(\varepsilon^{mnpq}\delta C_{npq}) = 0, \tag{3.24}$$

which is surely not a reasonable constraint. In fact, there is no way at all to implement the constraint $\delta C_{npq}|_{\partial\mathcal{M}} = 0$ in the on-shell action since for given δg_{mn} the on-shell relation (3.23) fixes δC_{npq} only up to a gauge transformation. Thus it is not possible to derive a consistent on-shell action from the action (3.18). The way to cure this disease is to impose a different variational constraint on the 3-form,

$$\delta H\Big|_{\partial\mathcal{M}} = 0. \tag{3.25}$$

This condition is automatically fulfilled by (3.20), so there is no issue of implementing it in the on-shell action. In order to apply the new boundary condition, one adds a boundary term to the action (3.13) [34, 35],

$$S_3' = \int d^4x\, H^2 - \tfrac{1}{3}\int d^4x\, \partial_m\big(H\varepsilon^{mnpq}C_{npq}\big), \tag{3.26}$$

which does not alter the equations of motion. Indeed, the variation of this action is given by

$$\delta S_3' = -\tfrac{1}{3}\int d^4x\, (\partial_m H)\varepsilon^{mnpq}\delta C_{npq} - \tfrac{1}{3}\int d^4x\, \partial_m\big(\delta H\varepsilon^{mnpq}C_{npq}\big). \tag{3.27}$$

Substituting the solution $H = c$ into the action S_3', we find that the boundary term gives negative twice the value of the kinetic term,

$$S'_{3,\text{ on-shell}} = \int d^4x\, c^2 - 2\int d^4x\, c^2 = \int d^4x\, (-c^2), \tag{3.28}$$

leading to the correct positive sign in the contribution to the cosmological constant.

Poincaré duality denotes a relation between two actions S and S_{dual} with the same number of on-shell degrees of freedom but different field content. This relation is established via a so-called first order action S_{first} (i.e., an action which is first order in 'velocities'), which couples the fields of the action S to those of the dual action S_{dual} [37]. When the fields of S_{dual} are eliminated from S_{first} by their equations of motion one recovers the original action S. On the other hand the fields of S can be eliminated from S_{first} to obtain the dual action S_{dual}. The equations of motion of action and dual action are equivalent by the duality relations that are contained in the Euler-Lagrange equations of the first order action. Thus action and dual action are physically equivalent on the classical level.

The action (3.26) including the boundary term can be dualized via the first order action

$$
\begin{aligned}
S_{\text{first}} &= \int d^4x \left(-\phi^2 + 2\phi H \right) - \tfrac{1}{3} \int d^4x\, \partial_m \left(\phi\, \varepsilon^{mnpq} C_{npq} \right) \\
&= \int d^4x \left(-\phi^2 - \tfrac{1}{3} (\partial_m \phi) \varepsilon^{mnpq} C_{npq} \right),
\end{aligned}
\tag{3.29}
$$

where the scalar field strength H was coupled to a real scalar ϕ. Inserting the equation of motion for ϕ (we impose the boundary condition $\delta\phi|_{\partial\mathcal{M}} = 0$)

$$
\phi = H \tag{3.30}
$$

into this first order action one finds that it reproduces (3.26) correctly, including the boundary term. On the other hand, the equations of motion for the C_{npq}

$$
\varepsilon^{mnpq} \partial_m \phi = 0 \tag{3.31}
$$

constrain ϕ to be a constant, $\phi = \hat{c}$ with $\hat{c} \in \mathbb{R}$. Like in the original action (3.26), the boundary term makes it possible to obtain these equations without imposing that δC_{npq} should vanish on the boundary. Here in the first order action the necessity of adding a boundary term becomes even more obvious, since only this boundary term makes it possible to eliminate the 3-form from the action, leading to the dual action[6]

$$
S_{\text{dual}} = \int d^4x \, \left(-\hat{c}^2 \right). \tag{3.32}
$$

The action of the massive 3-form is given by

$$
S_3 = \int d^4x \left(- \tfrac{1}{24} H_{mnpq} H^{mnpq} - \tfrac{1}{6} m^2 C_{npq} C^{npq} \right). \tag{3.33}
$$

Note that the mass term breaks the gauge invariance of the action. (The gauge invariance can be preserved by the Stückelberg mechanism which we will use for the supersymmetric case in Section 3.4.) The equation of motion that follows from (3.33) is

$$
4 \partial^m \partial_{[m} C_{npq]} - m^2 C_{npq} = 0. \tag{3.34}
$$

Thus the massive 3-form is dynamic and there is no need to add boundary terms to the action (3.33) because the 3-form will not be eliminated from it. In order to determine the number of on-shell degrees of freedom, that is, the number of linearly independent polarizations, we make the ansatz

$$
C_{npq}(x) = \epsilon_{npq} e^{ipx}, \tag{3.35}
$$

[6] Variations of the dual action with respect to the constant field $\hat{c}$ are not allowed since we impose the constraint $\delta\hat{c}|_{\partial\mathcal{M}} = 0$, i.e., $\delta\hat{c} = 0$.

where ϵ_{npq} is a constant antisymmetric polarization tensor. Inserting (3.35) into (3.34) yields

$$-(p^2+m^2)\epsilon_{npq}+3p^m p_{[n}\epsilon_{pq]m}=0. \tag{3.36}$$

Since the indices n,p,q in (3.36) are antisymmetrized, this equation constitutes four independent conditions. One of them is the mass-shell condition $p^2=-m^2$, so there are three independent conditions on the four independent elements of the polarization tensor. Thus the massive 3-form has one physical polarization, i.e., one on-shell degree of freedom.[7] This can also be seen by dualizing the action (3.33) via the first-order action

$$S_{\text{first}}=\int d^4x\left(-\phi^2-\tfrac{1}{3}(\partial_m\phi)\varepsilon^{mnpq}C_{npq}-\tfrac{1}{6}m^2C_{npq}C^{npq}\right). \tag{3.37}$$

Here we simply added the mass term to the first-order action (3.29), so the equation of motion for ϕ is not altered and (3.33) is correctly reproduced. Eliminating the 3-form from (3.37) by its equation of motion

$$m^2C^{npq}=-\varepsilon^{mnpq}\partial_m\phi \tag{3.38}$$

one finds the dual action

$$S_{\text{dual}}=\int d^4x\left(-\phi^2+\frac{1}{6m^2}\varepsilon^{mnpq}\varepsilon_{lnpq}\partial_m\phi\partial^l\phi\right)=\int d^4x\left(-\frac{1}{m^2}\partial_m\phi\partial^m\phi-\phi^2\right). \tag{3.39}$$

After rescaling $\phi\to\frac{1}{\sqrt{2}}m\phi$ one obtains the canonical action for a real scalar of mass m which, like the massive 3-form it is dual to, carries one physical degree of freedom.

3.3 Renormalizable action of the massless 3-form multiplet

The gauge invariant action for N massless 3-form multiplets U^a ($a=1,\dots,N$) with field strengths S^a is given by[8] [16]

$$S_3=\int d^8z\,\delta_{a\bar b}S^a\bar S^{\bar b}=\int d^4x\,\delta_{a\bar b}\Big[-\partial_mM^a\,\partial^mM^{*\bar b}-i\lambda^a\sigma^m\partial_m\bar\lambda^{\bar b}+D^aD^{\bar b}+H^aH^{\bar b}\Big], \tag{3.40}$$

where $D^{\bar b}=D^b$ and $H^{\bar b}=H^b$ since both are real. The auxiliary fields D^a vanish by their equations of motion $D^a=0$. The action (3.40) contains the correct kinetic term for the 3-form,

$$H^aH^a=-\tfrac{1}{24}H^a_{mnpq}H^{amnpq}. \tag{3.41}$$

[7] This can be shown with more rigor by going to the rest frame where $(p^m)=(\sqrt{-p^2},0,0,0)$ (after proving that there is no solution with $p^2=0$) and then inserting different combinations for the indices n,p,q.

[8] To obtain the component action integration by parts was applied for the fields M^a and λ^a which is not completely correct since we cannot assume that boundary terms involving the 3-form field strengths H^a vanish and the latter transform into these fields under supersymmetry. However, as boundary terms do not affect the equations of motion, it is legitimate to drop them here.

According to the discussion in the previous section, one has to add a boundary term to the action (3.40) in order to impose the gauge invariant boundary condition $\delta H^a|_{\partial\mathcal{M}} = 0$ instead of $\delta C^a_{npq}|_{\partial\mathcal{M}} = 0$. The supersymmetric generalization of this idea can be found by considering the variation of the action (3.40) with respect to U^a,

$$\begin{aligned}\delta S_3 &= \int d^8z \left(\bar{S}^a \delta S^a + S^a \delta \bar{S}^a\right) \\ &= -\tfrac{1}{4}\int d^8z\, \bar{S}^a \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}\delta U^a + \text{h.c.} \\ &= -\tfrac{1}{4}\int d^8z \left(\bar{D}_{\dot\alpha}\big(\bar{S}^a\bar{D}^{\dot\alpha}\delta U^a\big) - \bar{D}_{\dot\alpha}S^a\bar{D}^{\dot\alpha}\delta U^a\right) + \text{h.c.} \\ &= -\tfrac{1}{4}\int d^8z \left[\bar{D}_{\dot\alpha}\big(\bar{S}^a\bar{D}^{\dot\alpha}\delta U^a - (\bar{D}^{\dot\alpha}\bar{S}^a)\delta U^a\big) + (\bar{D}^2\bar{S}^a)\delta U^a\right] + \text{h.c.}\end{aligned} \tag{3.42}$$

(Here the Leibniz rule (2.16) was applied. In the last step, note the two minus signs from moving $D_{\dot\alpha}$ past $D^{\dot\alpha}\bar{S}^a$ and switching the index positions that cancel each other.) Thus we see that the action (3.40) would require the supersymmetric boundary conditions $\delta U^a|_{\partial\mathcal{M}} = 0$ and $D_\alpha(\delta U^a)|_{\partial\mathcal{M}} = 0$. As we want to impose gauge invariant (and supersymmetric) boundary conditions we naturally choose

$$\delta S^a\Big|_{\partial\mathcal{M}} = 0, \qquad D_\alpha(\delta S^a)\Big|_{\partial\mathcal{M}} = 0. \tag{3.43}$$

In order to apply these, we add to the action the boundary terms

$$\mathcal{B} = \tfrac{1}{4}\int d^8z\, \bar{D}_{\dot\alpha}\big(\bar{S}^a\bar{D}^{\dot\alpha}U^a - (\bar{D}^{\dot\alpha}\bar{S}^a)U^a\big) + \text{h.c.} \tag{3.44}$$

These were chosen such that the terms that arise when the δ-operator acts on one of the the U^a in (3.44) cancel the boundary terms in (3.42). In exchange one gets boundary terms proportional to δS^a and $D_\alpha(\delta S^a)$ (and their complex conjugates); namely the variation of the new action

$$S_3' = S_3 + \mathcal{B} \tag{3.45}$$

with respect to U is given by

$$\delta S_3' = \tfrac{1}{4}\int d^8z \Big[-(\bar{D}^2\bar{S}^a)\delta U^a + \bar{D}_{\dot\alpha}\big((\delta\bar{S}^a)\bar{D}^{\dot\alpha}U^a - \bar{D}^{\dot\alpha}(\delta\bar{S}^a)U^a\big) + \text{h.c.}\Big]. \tag{3.46}$$

Thus for S_3' we can apply the variational constraints (3.43). As U^a is real but otherwise unconstrained, one finds the superfield equations of motion

$$D^2S^a + \bar{D}^2\bar{S}^a = 0. \tag{3.47}$$

The boundary terms for the 3-forms that are contained in (3.44) can be found by writing it as

$$\begin{aligned}\mathcal{B} &= \mathrm{Re}\int d^8z\, \bar{D}_{\dot\alpha}\left(\bar{S}^a\bar{D}^{\dot\alpha}U^a\right) - \tfrac{1}{2}\mathrm{Re}\int d^8z\, \bar{D}^2\left(\bar{S}^aU^a\right)\\ &= -\mathrm{Im}\int d^8z\, \partial_m(\theta\sigma^m)_{\dot\alpha}\left(\bar{S}^a\bar{D}^{\dot\alpha}U^a\right) + \tfrac{1}{2}\mathrm{Re}\int d^8z\, \theta^2\Box\left(\bar{S}^aU^a\right),\end{aligned} \tag{3.48}$$

where all θ-derivatives under the full superspace integral have been dropped. Here we are interested only in boundary terms involving the 3-forms C^a_{npq} without derivatives. The only such term originates from multiplying

$$\tfrac{1}{3}(\bar{\sigma}^m\theta)^{\dot\alpha}\varepsilon_{mnpq}C^{anpq}, \tag{3.49}$$

which is contained in $\bar{D}^{\dot\alpha}U^a$, with the $\bar{\theta}^2$-component of $\bar{S}^a$

$$\bar{\theta}^2(D^a - iH^a) = -i\bar{\theta}^2H^a, \tag{3.50}$$

where we used the trivial equation of motion for D^a. Thus we see that the first term in the second line of (3.48) contains the correct boundary term for the 3-forms (cf. (3.26))

$$-\tfrac{1}{3}\int d^4x\, \partial_m\left(H^a\varepsilon^{mnpq}C^a_{npq}\right). \tag{3.51}$$

The equations of motion for the C^a_{npq}

$$-\tfrac{1}{3}\varepsilon^{mnpq}\partial_mH^a = 0 \tag{3.52}$$

imply that the 3-form field strengths become constants, $H^a = c^a$ with $c^a \in \mathbb{R}$. Thus the action (3.40) describes $2N$ bosonic and $2N$ fermionic degrees of freedom on-shell. By virtue of the boundary term (3.51), it is seen that the 3-forms create a constant *positive* potential

$$\mathcal{V} = c^ac^a, \tag{3.53}$$

which corresponds to a positive correction of the bare cosmological constant. Since the ground state has a non-vanishing energy expectation value, supersymmetry must be spontaneously broken. Indeed, the supersymmetry variation of λ given in (3.10) shows that it transforms inhomogeneously when H becomes a non-vanishing constant, so that it can be identified as the Goldstone fermion.

3.4 Renormalizable action of the massive 3-form multiplet

One can add a gauge invariant mass term to the action (3.40) with the help of the Stückelberg mechanism [38]. To this end one introduces N additional linear multiplets L'^a with the transformation law

$$L'^a \to L'^a - L^a, \tag{3.54}$$

where the transformation parameters L^a are also linear superfields. Then $U^a - L'^a$ is gauge invariant and one can add to the action the terms [29]

$$S_{\text{mass}} = \int d^8z \, \left(-\tfrac{1}{2}m^2_{ab}(U^a - L'^a)(U^b - L'^b) + \xi_a(U^a - L'^a)\right), \tag{3.55}$$

where it is understood that $m^2_{ab} = m_{ac}m_{cb}$ with a symmetric mass matrix $m_{ab} = m_{ba}$ and the ξ_a parameterize possible Fayet-Iliopoulos terms.[9] The additional degrees of freedom that where introduced in the form of the L'^a can be absorbed into the U^a by fixing the gauge to $L'^a = 0$ (which is obtained by choosing $L^a = L'^a$) so that the L'^a drop out of the action. In the following we will always work in this gauge. Furthermore, we take out the massless modes (which can be treated as described in section 3.3) so that we can assume without loss of generality that m_{ab} is invertible. We then find the action[10]

$$\begin{aligned}
S_3 &= \int d^8z \left(\delta_{a\bar{b}} S^a \bar{S}^{\bar{b}} - \tfrac{1}{2}m^2_{ab}U^aU^b + \xi_a U^a\right) \\
&= \int d^4x \Big[-\partial_m M^a \, \partial^m M^{a*} - i\lambda^a\sigma^m\partial_m\bar{\lambda}^a + D^aD^a + H^aH^a \\
&\qquad - \tfrac{1}{2}m^2_{ab}\Big(i\chi^a\sigma^m\partial_m\bar{\chi}^b - \sqrt{2}i\chi^a\lambda^b + \sqrt{2}i\bar{\chi}^a\bar{\lambda}^b + 2M^aM^{b*} \\
&\qquad + 2B^aD^b - \tfrac{1}{2}B^a\Box B^b + \tfrac{1}{3}C^a_{npq}C^{bnpq}\Big) + \xi_a\big(D^a - \tfrac{1}{4}\Box B^a\big)\Big].
\end{aligned} \tag{3.56}$$

The auxiliary fields D^a can be eliminated by their equations of motion

$$2\delta_{ab}D^b - m^2_{ab}B^b + \xi_a = 0. \tag{3.57}$$

This is done most conveniently by "completing the square" as described in Appendix B, leading to the on-shell action

$$\begin{aligned}
S_3 = \int d^4x \Big[&-\partial_m M^a \, \partial^m M^{a*} - i\lambda^a\sigma^m\partial_m\bar{\lambda}^a + H^aH^a \\
&- m^2_{ab}\Big(\tfrac{i}{2}\chi^a\sigma^m\partial_m\bar{\chi}^b - \tfrac{i}{\sqrt{2}}\chi^a\lambda^b + \tfrac{i}{\sqrt{2}}\bar{\chi}^a\bar{\lambda}^b + M^aM^{b*} \\
&+ \tfrac{1}{4}\partial^m B^a\partial_m B^b + \tfrac{1}{6}C^a_{npq}C^{bnpq}\Big) - \tfrac{1}{4}\big(m^2_{ab}B^b - \xi_a\big)\big(m^2_{ac}B^c - \xi_a\big)\Big].
\end{aligned} \tag{3.58}$$

As already mentioned in Sec. 3.1, the fields B^a and χ^a have non-canonical mass dimension and therefore have to be rescaled in order to obtain kinetic terms of the canonical form. This is done by the field redefinitions

$$B'^a := \tfrac{1}{2}\big(\delta^{ab}m_{bc}B^c - m^{-1ab}\xi_b\big), \qquad \chi'^a := -\tfrac{i}{\sqrt{2}}\delta^{ab}m_{bc}\chi^c. \tag{3.59}$$

[9] Terms cubic in U would render the action non-renormalizable.

[10] To be on the safe side, on could add the boundary terms (3.44) also to the massive action. However, these are not really needed since we are not going to eliminate the massive 3-forms from the action.

Here the fields B^a where also shifted to absorb their vacuum expectation values $\langle B^a\rangle = m^{-2ab}\xi_b$. Note that the Fayet-Iliopoulos term in (3.55) has not broken supersymmetry due to the mass term also present in (3.55). Then the on-shell action becomes

$$\begin{aligned} S_3 = \int d^4x \Big[&- \partial^m M^a\, \partial_m M^{a*} - m^2_{ab} M^a M^{b*} - \partial^m B'^a \partial_m B'^a - m^2_{ab} B'^a B'^b \\ &- i\lambda^a \sigma^m \partial_m \bar\lambda^a - i\chi'^a \sigma^m \partial_m \bar\chi'^a - m_{ab}\chi'^a \lambda^b - m_{ab} \bar\chi'^a \bar\lambda^b \\ &+ H^a H^a - \tfrac{1}{6} m^2_{ab} C^a_{npq} C^{bnpq} \Big]. \end{aligned} \tag{3.60}$$

We see that the fermions λ^a and χ'^a form N massive Dirac spinors corresponding to $4N$ fermionic degrees of freedom. As shown in Section 3.2, the massive 3-forms contribute one on-shell degree of freedom each. Together with the N complex scalars M^a and the N real scalars B'^a we thus also have $4N$ bosonic on-shell degrees of freedom.

3.5 Dualization of the massless action

The massless action (3.45) can be reproduced from the first order action [16]

$$S_{\text{first}} = \int d^8z\, (-\delta^{a\bar b} F_a \bar F_{\bar b} + F_a S^a + \bar F_{\bar a} \bar S^{\bar a}) + \mathcal{B}_{\text{first}}, \tag{3.61}$$

with the boundary terms

$$\mathcal{B}_{\text{first}} = \tfrac{1}{4} \int d^8z \left[\bar D_{\dot\alpha} \left(F_a \bar D^{\dot\alpha} U^a - \bar D^{\dot\alpha} F_a U^a \right) + \text{h.c.} \right]. \tag{3.62}$$

Here the F_a are unconstrained, i.e., generic superfields. For later convenience we write their component field expansion as

$$\begin{aligned} F_a = {} & f_a + \theta\psi_a + \sqrt{2}\bar\theta\bar\varphi_a + \theta^2 h_a + \bar\theta^2 n_a + \theta\sigma^m\bar\theta w_{am} \\ & + \theta^2\bar\theta\bar\vartheta_a + \bar\theta^2\theta(\zeta_a - \tfrac{i}{\sqrt{2}}\sigma^m\partial_m\bar\varphi_a) + \theta^2\bar\theta^2(d_a - \tfrac14 \Box f_a - \tfrac{i}{2}\partial_m w_a^m), \\ \bar F_{\bar b} = {} & f^*_{\bar b} + \bar\theta\bar\psi_{\bar b} + \sqrt{2}\theta\varphi_{\bar b} + \bar\theta^2 h^*_{\bar b} + \theta^2 n^*_{\bar b} + \theta\sigma^m\bar\theta w^*_{\bar b m} \\ & + \bar\theta^2\theta\vartheta_{\bar b} + \theta^2\bar\theta(\bar\zeta_{\bar b} - \tfrac{i}{\sqrt{2}}\bar\sigma^m\partial_m\varphi_{\bar b}) + \theta^2\bar\theta^2(d^*_{\bar b} - \tfrac14\Box f^*_{\bar b} + \tfrac{i}{2}\partial_m w^{m*}_{\bar b}), \end{aligned} \tag{3.63}$$

where $f_a,\ h_a,\ n_a$ and d_a are complex scalars, w_{am} is a complex vector and $\psi_a,\ \varphi_a, \vartheta_a$ and ζ_a are Weyl spinors. Eliminating the F_a from (3.61) by inserting their equations of motion

$$\delta^{a\bar b}\bar F_{\bar b} = S^a, \tag{3.64}$$

one recovers the kinetic action of the massless 3-form multiplet (3.40) as well as the correct boundary terms (3.44).

In order to eliminate the fields U^a from (3.61) one inserts the definition (3.5) for their field strengths S^a to rewrite the action as

$$\begin{aligned} S_{\text{first}} &= \int d^8z \Big(-\tfrac{1}{2}\delta^{a\bar{b}} F_a \bar{F}_{\bar{b}} - \tfrac{1}{4} F_a \bar{D}^2 U^a + \tfrac{1}{4}\bar{D}_{\dot{\alpha}} \left(F_a \bar{D}^{\dot{\alpha}} U^a - \bar{D}^{\dot{\alpha}} F_a U^a \right) + \text{h.c.} \Big) \\ &= \int d^8z \Big(-\delta^{a\bar{b}} F_a \bar{F}_{\bar{b}} - \tfrac{1}{4} \left(\bar{D}^2 F_a + D^2 \bar{F}_a \right) U^a \Big), \end{aligned} \tag{3.65}$$

where again the Leibniz rule (2.16) was applied. As shown on the component level for the 3-form in Sec. 3.2, the first order action takes a simple form by virtue of the boundary terms (3.62). In this form, varying with respect to U^a immediately and without dropping any boundary term yields the constraint for F_a

$$\begin{aligned} 0 \;&=\; -\tfrac{1}{4}\left(\bar{D}^2 F_a + D^2 \bar{F}_a \right) \\ &= n_a + n_a^* + \theta\zeta_a + \bar{\theta}\bar{\zeta}_a + \theta^2 d_a + \bar{\theta}^2 d_a^* + i\theta\sigma^m\bar{\theta}\partial_m(n_a - n_a^*) \\ &\quad + \tfrac{i}{2}\theta^2\bar{\theta}\bar{\sigma}^m\partial_m\zeta_a + \tfrac{i}{2}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\zeta}_a + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box(n_a + n_a^*) \ . \end{aligned} \tag{3.66}$$

As usual, Poincaré duality has exchanged the equation of motion with the constraint with respect to the duality relation (3.64) (cf. (3.47)). We will see below that the condition (3.66) is special for the massless case in that it reduces the number of degrees of freedom in F_a while in the massive case the F_a remain unconstrained superfields. (3.66) implies that ζ_a and d_a vanish whereas n_a becomes a purely imaginary constant,

$$\zeta_a = 0, \qquad d_a = 0, \qquad n_a = i\hat{c}_a \qquad \text{with } \hat{c}_a \in \mathbb{R}, \tag{3.67}$$

so that F_a takes the form

$$\begin{aligned} F_a = &\ f_a + \theta\psi_a + \sqrt{2}\bar{\theta}\bar{\varphi}_a + \theta^2 h_a + i\bar{\theta}^2\hat{c}_a + \theta\sigma^m\bar{\theta}w_{am} + \theta^2\bar{\theta}\bar{\vartheta}_a - \tfrac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\varphi}_a \\ &+ \theta^2\bar{\theta}^2(-\tfrac{1}{4}\Box f_a - \tfrac{i}{2}\partial_m w_a^m), \\ \bar{F}_{\bar{b}} = &\ f_{\bar{b}}^* + \bar{\theta}\bar{\psi}_{\bar{b}} + \sqrt{2}\theta\varphi_{\bar{b}} + \bar{\theta}^2 h_{\bar{b}}^* - i\theta^2\hat{c}_{\bar{b}} + \theta\sigma^m\bar{\theta}w_{\bar{b}m}^* + \bar{\theta}^2\theta\vartheta_{\bar{b}} - \tfrac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^m\partial_m\varphi_{\bar{b}} \\ &+ \theta^2\bar{\theta}^2(-\tfrac{1}{4}\Box f_{\bar{b}}^* + \tfrac{i}{2}\partial_m w_{\bar{b}}^{m*}). \end{aligned} \tag{3.68}$$

It contains 12 bosonic and 12 fermionic off-shell degrees of freedom. Using (3.66) and (3.68) we obtain as the dual component action

$$\begin{aligned} S_{\text{dual}} &= \int d^8z \,(-\delta^{a\bar{b}} F_a \bar{F}_{\bar{b}}) \\ &= \int d^4x \Big[f_a^*(\tfrac{i}{2}\partial_m w_a^m + \tfrac{1}{4}\Box f_a) + \tfrac{1}{2}\psi_a\vartheta_a - \tfrac{i}{2}\varphi_a\sigma^m\partial_m\bar{\varphi}_a \\ &\qquad - \tfrac{1}{2}h_a h_a^* - \tfrac{1}{2}\hat{c}_a\hat{c}_a + \tfrac{1}{4}w_{am}^* w_a^m + \text{h.c.} \Big]. \end{aligned} \tag{3.69}$$

After eliminating the auxiliary fields ψ_a, ϑ_a, h_a and w_{am} this becomes

$$S_{\text{dual}} = \int d^4x \Big(-\partial_m f_a \partial^m f_a^* - i\varphi_a\sigma^m\partial_m\bar{\varphi}_a - \hat{c}_a\hat{c}_a \Big). \tag{3.70}$$

Just like the original action (3.40), the dual action describes N complex scalars and N Weyl spinors. The field strengths of the 3-forms are represented by the constants $\hat{c}_a$, that also create a constant positive potential. In fact, the superfield equation of motion (3.64) includes the duality relation

$$H^a = \delta^{a\bar{b}} \mathrm{Im} n^*_{\bar{b}} = -\delta^{ab} \hat{c}_b, \tag{3.71}$$

so that the cosmological constants of action and dual action coincide.

Before we proceed let us note that in the dualization of the massless action a new multiplet F appeared. It differs from the complex linear multiplet, whose component expansion is given in (2.31), only by the free constant $\hat{c}$ (for $\hat{c} = 0$ they coincide). This difference arises from the fact that S is not a general chiral superfield but constructed from a real superfield U via (3.5). If U was complex then $\bar{D}^2 F_a$ and $D^2 \bar{F}_a$ had to vanish separately in (3.66) as in the duality between the chiral and the complex linear multiplet [27].

3.6 Dualization of the massive action

In the massive case the first order action is given by

$$S_{\text{first}} = \int d^8z \left(-\delta^{a\bar{b}} F_a \bar{F}_{\bar{b}} + F_a S^a + \bar{F}_{\bar{a}} \bar{S}^{\bar{a}} - \tfrac{1}{2} m^2_{ab} U^a U^b + \xi_a U^a \right), \tag{3.72}$$

where the mass and Fayet-Iliopoulos terms were simply added to (3.61) and the boundary term was dropped. Since the equations of motion for the F_a are the same as in the massless case, the massive action (3.56) is correctly reproduced when the F_a are eliminated from (3.72).

In order to find the dual action one has to rewrite (3.72) as in (3.65) (now dropping all boundary terms)

$$S_{\text{first}} = \int d^8z \left(-\delta^{a\bar{b}} F_a \bar{F}_{\bar{b}} - \tfrac{1}{4} U^a \left(\bar{D}^2 F_a + D^2 \bar{F}_a \right) - \tfrac{1}{2} m^2_{ab} U^a U^b + \xi_a U^a \right), \tag{3.73}$$

and then again eliminate the 3-form multiplets U^a by their equations of motion

$$-\tfrac{1}{4}(\bar{D}^2 F_a + D^2 \bar{F}_a) - m^2_{ab} U^b + \xi_a = 0. \tag{3.74}$$

In contrast to the massless case (3.66) the superfields F_a now remain unconstrained. Therefore the complex scalars d_a, n_a and the Weyl spinors ζ_a no longer drop out of the dual action. Substituting (3.74) into (3.73) and using the abbreviation

$$\Omega_a := -\tfrac{1}{4} \left(\bar{D}^2 F_a + D^2 \bar{F}_a \right), \tag{3.75}$$

one obtains the dual action

$$S_{\mathrm{dual}} = \int d^8z \Big(- \delta^{a\bar{b}} F_a \bar{F}_{\bar{b}} + \tfrac{1}{2} m^{-2ab} (\Omega_a + \xi_a)(\Omega_b + \xi_b) \Big). \tag{3.76}$$

With the θ-expansions of F_a and Ω_a as given in (3.63) and (3.66) respectively, and after applying integration by parts, one finds the component form

$$\begin{aligned} S_{\mathrm{dual}} = \int d^4x \Big(& - f_a d_a^* - f_a^* d_a - \tfrac{1}{2}\partial_m f_a \partial^m f_a^* + \tfrac{i}{2}\partial_m f_a w_a^{m*} - \tfrac{i}{2} w_a^m \partial_m f_a^* + \tfrac{1}{2} w_a^m w_{am}^* \\ & + \tfrac{1}{2}\psi_a \vartheta_a + \tfrac{1}{2}\bar{\psi}_a \bar{\vartheta}_a + \tfrac{1}{\sqrt{2}} \varphi_a \zeta_a + \tfrac{1}{\sqrt{2}} \bar{\varphi}_a \bar{\zeta}_a - i\varphi_a \sigma^m \partial_m \bar{\varphi}_a - h_a h_a^* - n_a n_a^* \\ & + m^{-2ab} \left(-\partial_m n_a \partial^m n_b^* - \tfrac{i}{2} \zeta_a \sigma^m \partial_m \bar{\zeta}_b + d_a d_b^* \right) \Big). \end{aligned} \tag{3.77}$$

Note that the ξ_a have dropped out of the action due to the fact that the highest component of Ω_a is a total spacetime divergence (remember that they also dropped out of the original action (3.56) by a field redefinition). The action (3.77) still contains the auxiliary fields h_a, ψ_a, ϑ_a, d_a and w_a^m. Eliminating them by their equations of motion yields the on-shell action

$$\begin{aligned} S_{\mathrm{dual}} = \int d^4x \Big(& - \partial_m f_a \, \partial^m f_a^* - m_{ab}^2 f_a f_b^* - m^{-2ab} \partial_m n_a \partial^m n_b^* - n_a n_a^* \\ & - i\varphi_a \sigma^m \partial_m \bar{\varphi}_a - \tfrac{i}{2} m^{-2ab} \zeta_a \sigma^m \partial_m \bar{\zeta}_b + \tfrac{1}{\sqrt{2}} \varphi_a \zeta_a + \tfrac{1}{\sqrt{2}} \bar{\varphi}_a \bar{\zeta}_a \Big). \end{aligned} \tag{3.78}$$

By the field redefinitions

$$n'_a := \delta_{ab} \, m^{-1bc} n_c \ , \qquad \zeta'_a := -\tfrac{1}{\sqrt{2}} \delta_{ab} \, m^{-1bc} \zeta_c, \tag{3.79}$$

the kinetic terms for the n_a and ζ_a take the standard form,

$$\begin{aligned} S_{\mathrm{dual}} = \int d^4x \Big(& - \partial_m f_a \, \partial^m f_a^* - m_{ab}^2 f_a f_b^* - \partial_m n'_a \partial^m n'^*_a - m_{ab}^2 n'_a n'^*_b \\ & - i\varphi_a \sigma^m \partial_m \bar{\varphi}_a - i\zeta'_a \sigma^m \partial_m \bar{\zeta}'_a - m_{ab} (\varphi_a \zeta'_b + \bar{\varphi}_a \bar{\zeta}'_b) \Big). \end{aligned} \tag{3.80}$$

The action (3.80) is dual to the renormalizable massive action of the 3-form multiplet given in (3.60) and describes the dynamics of $2N$ massive complex scalars f_a, n'_a and N massive Dirac spinors formed by φ_a, ζ'_a. The massive 3-forms C^a_{npq} and real scalars B^a that appear in (3.60) are represented in the dual action by the complex scalars n'_a, so that action and dual action again contain an equal number of on-shell degrees of freedom.

4 Non-renormalizable action

4.1 From the superfield Lagrangian to the on-shell action

In this section we drop the requirement of renormalizability and consider an action with arbitrary real functions G, K of the 3-forms U^a and their field strengths S^a respectively,[11]

$$S_3 = \int d^8z \Big(K(S, \bar{S}) - G(U - L') \Big). \tag{4.1}$$

Such non-renormalizable actions are called non-linear sigma models. They can arise as low energy limits of string theories or higher dimensional supergravities.

The action S_3 is invariant under gauge transformations (3.8), (3.54) and as before we choose the gauge $L' = 0$. For simplicity we restrict our analysis to the bosonic part of the action by setting all fermionic components to zero. After defining the usual abbreviations for the derivatives

$$\begin{aligned} K_{a_1 \dots a_n \bar{b}_1 \dots \bar{b}_m}(M, M^*) &:= \left. \frac{\partial K}{\partial S^{a_1} \dots \partial S^{a_n} \partial \bar{S}^{\bar{b}_1} \dots \partial \bar{S}^{\bar{b}_m}} \right|_{\theta = \bar{\theta} = 0}, \\ G_{a_1 \dots a_n}(B) &:= \left. \frac{\partial G}{\partial U^{a_1} \dots \partial U^{a_n}} \right|_{\theta = \bar{\theta} = 0}, \end{aligned} \tag{4.2}$$

we Taylor expand $K(S, \bar{S})$ around (M, M^*),

$$K(S, \bar{S}) = K(M, M^*) + K_a \Delta_S^a + K_{\bar{b}} \bar{\Delta}_S^{\bar{b}} + \tfrac{1}{2} K_{ab} \Delta_S^a \Delta_S^b + \tfrac{1}{2} K_{\bar{a}\bar{b}} \bar{\Delta}_S^{\bar{a}} \bar{\Delta}_S^{\bar{b}} + K_{a\bar{b}} \Delta_S^a \bar{\Delta}_S^{\bar{b}}, \tag{4.3}$$

where

$$\begin{aligned} \Delta_S^a &:= S^a - M^a = i\theta\sigma^m\bar{\theta}\partial_m M^a + \theta^2(D^a + iH^a) + \tfrac{1}{4}\theta^2\bar{\theta}^2 \Box M^a, \\ \bar{\Delta}_S^{\bar{a}} &:= \bar{S}^{\bar{a}} - M^{*\bar{a}} = -i\theta\sigma^m\bar{\theta}\partial_m M^{*\bar{a}} + \bar{\theta}^2(D^{\bar{a}} - iH^{\bar{a}}) + \tfrac{1}{4}\theta^2\bar{\theta}^2 \Box M^{*\bar{a}}. \end{aligned} \tag{4.4}$$

Note that (4.3) is exact since products of three Δ's vanish. Using (4.3) one can easily perform the θ-integration of $K(S, \bar{S})$ to find

$$\begin{aligned} \int d^8z K = \int d^4x \Big[& \tfrac{1}{4} K_a \Box M^a + \tfrac{1}{4} K_{\bar{b}} \Box M^{*\bar{b}} + \tfrac{1}{4} K_{ab} \partial_m M^a \partial^m M^b + \tfrac{1}{4} K_{\bar{a}\bar{b}} \partial_m M^{*\bar{a}} \partial^m M^{*\bar{b}} \\ & + K_{a\bar{b}} \Big(-\tfrac{1}{2} \partial_m M^a \partial^m M^{*\bar{b}} + (D^a + iH^a)(D^{\bar{b}} - iH^{\bar{b}}) \Big) \Big] \\ = \int d^4x \Big[& K_{a\bar{b}} \big(-\partial_m M^a \partial^m M^{*\bar{b}} + D^a D^b + H^a H^b \big) - i(K_{a\bar{b}} - K_{b\bar{a}}) D^a H^b \Big], \end{aligned} \tag{4.5}$$

where in the second step integration by parts was applied and the chain rule

$$\partial_m K_a = K_{ab} \partial_m M^b + K_{a\bar{b}} \partial_m M^{*\bar{b}} \tag{4.6}$$

[11] We choose a minus sign for the G-term in order to have $G_{ab}|_{B=0} = m_{ab}^2$, cf. (3.56), so that a positive definite G_{ab} corresponds to a ghost free theory.

(and similar for $\partial_m K_{\bar{b}}$) was used. The complex scalar fields M^a can be viewed as coordinates of a Kähler manifold with the metric $K_{a\bar{b}}$ derived from the Kähler potential K [20, 39].

Using

$$U^a - B^a = \theta^2 M^{a*} + \bar{\theta}^2 M^a + \tfrac{1}{3}\theta\sigma^m\bar{\theta}\varepsilon_{mnpq}C^{anpq} + \theta^2\bar{\theta}^2\big(D^a - \tfrac{1}{4}\Box B^a\big), \tag{4.7}$$

one can perform the θ-integration of $G(U)$ with the same technique,

$$-\int d^8z\, G(U) = -\int d^4x \left[G_a\left(D^a - \tfrac{1}{4}\Box B^a\right) + \tfrac{1}{2}G_{ab}\left(2M^aM^{b*} + \tfrac{1}{3}C^{anpq}C^b_{npq}\right)\right]. \tag{4.8}$$

Because K is real and partial derivatives commute, the Kähler metric $K_{a\bar{b}}$ is hermitian:

$$(K_{b\bar{a}})^* = K_{\bar{b}a} = K_{a\bar{b}}. \tag{4.9}$$

Since $K_{a\bar{b}}$ is the coefficient matrix of the kinetic terms for the fields M^a (and C^a_{npq}), we demand here that it is also positive definite to exclude unphysical ghost fields with kinetic terms of the wrong sign. Then it follows from

$$0 \;<\; K_{a\bar{b}}\, x^a x^b \;=\; \tfrac{1}{2}(K_{a\bar{b}} + K_{b\bar{a}})\, x^a x^b \quad \text{for all } x \in \mathbb{R}^N\backslash\{0\} \tag{4.10}$$

that also the symmetric, i.e., real part of the Kähler metric is positive definite and in particular invertible. Thus the equations of motion for the auxiliary fields D^a

$$2(\mathrm{Re}K_{a\bar{b}})D^b + 2(\mathrm{Im}K_{a\bar{b}})H^b - G_a = 0 \tag{4.11}$$

have the solution

$$D^a = \tfrac{1}{2}(\mathrm{Re}K)^{-1ab}\left(G_b - 2(\mathrm{Im}K)_{bc}H^c\right), \tag{4.12}$$

where $(\mathrm{Im}K)_{bc} = -(\mathrm{Im}K)_{cb}$ denotes the imaginary part of the Kähler metric while $(\mathrm{Re}K)^{-1ab}$ denotes the inverse of the real part of the Kähler metric. Thus the D^a can be eliminated from the action by "completing the square" as in (B.4) (with $M_{ab} = K_{a\bar{b}}$ and $J_a = 2(\mathrm{Im}K)_{ab}H^b - G_a$) to obtain the on-shell action

$$\begin{aligned} S_3 = \int d^4x \Big[&K_{a\bar{b}}\Big(-\partial^m M^a\partial_m M^{*\bar{b}} + H^aH^{\bar{b}}\Big) - G_{ab}\Big(M^aM^{b*} + \tfrac{1}{6}C^{anpq}C^b_{npq}\Big) \\ &+ \tfrac{1}{4}G_a\Box B^a - \tfrac{1}{4}\Big(G_a + 2H^c(\mathrm{Im}K)_{ca}\Big)(\mathrm{Re}K)^{-1ab}\Big(G_b - 2(\mathrm{Im}K)_{bd}H^d\Big)\Big] \\ = \int d^4x \Big[&-K_{a\bar{b}}\,\partial^m M^a\partial_m M^{*\bar{b}} - G_{ab}\Big(\tfrac{1}{4}\partial^m B^a\partial_m B^b + M^aM^{b*} + \tfrac{1}{6}C^{anpq}C^b_{npq}\Big) \\ &+ g_{ab}H^aH^b + G_a(\mathrm{Re}K)^{-1ab}(\mathrm{Im}K)_{bc}H^c - \tfrac{1}{4}G_a(\mathrm{Re}K)^{-1ab}G_b\Big], \end{aligned} \tag{4.13}$$

where in the second step integration by parts was used and a real metric

$$g_{ab} := (\mathrm{Re}K)_{ab} + (\mathrm{Im}K)_{ac}(\mathrm{Re}K)^{-1cd}(\mathrm{Im}K)_{db} \tag{4.14}$$

for the 3-form scalar field strengths H^a was defined. Using the symbol K for the Kähler metric rather than the Kähler potential, g can be written in matrix notation as

$$\begin{aligned} g &= \mathrm{Re}K - \tfrac{1}{2}(K-K^*)(K+K^*)^{-1}(K-K^*) \\ &= \mathrm{Re}K - \tfrac{1}{2}\big((K-K^*) - 2(K-K^*)(K+K^*)^{-1}K^*\big) \\ &= \mathrm{Re}K - \tfrac{1}{2}\big(K - K^* + 2K^* - 4K(K+K^*)^{-1}K^*\big) \\ &= K(\mathrm{Re}K)^{-1}K^*. \end{aligned} \tag{4.15}$$

The last expression for g shows explicitly that it is positive definite and that its inverse is given by

$$g^{-1ab} = \big[K^{*-1}(\mathrm{Re}K)K^{-1}\big]^{ab} = \mathrm{Re}(K^{-1a\bar{b}}). \tag{4.16}$$

The scalar potential of the action (4.13) is

$$\mathcal{V} = G_{ab}M^aM^{b*} + \tfrac{1}{4}G_a(\mathrm{Re}K)^{-1ab}G_b. \tag{4.17}$$

Depending on the choice of the functions K and G it can lead to non-vanishing vacuum expectation values of the fields B^a and M^a like in the renormalizable case (3.58).

As $H^a = \frac{1}{6}\varepsilon^{mnpq}\partial_m C^a_{npq}$, the equations of motion for the 3-forms that follow from (4.13) are

$$-\tfrac{1}{3}G_{ab}C^{bnpq} = \tfrac{1}{3}\varepsilon^{mnpq}\partial_m\big(g_{ab}H^b - \tfrac{1}{2}(\mathrm{Im}K)_{ab}(\mathrm{Re}K)^{-1bc}G_c\big). \tag{4.18}$$

In the massless case $G = 0$ they force $g_{ab}H^b$ to be constant,

$$g_{ab}H^b = c_a \text{ with } c_a \in \mathbb{R}. \tag{4.19}$$

Then it is important to add appropriate boundary terms to the action (4.1) as one wants to eliminate the 3-forms by (4.19). These terms should cancel all boundary terms in the variation δS_3 containing δU^a's in favor of boundary terms containing δS^a's, which can be assumed to vanish. Since

$$\begin{aligned} \delta S_3 &= \int d^8z\,\Big(K_a(S,\bar{S})\delta S^a + K_{\bar{b}}(S,\bar{S})\delta\bar{S}^{\bar{b}}\Big) \\ &= -\tfrac{1}{4}\int d^8z\,\Big(\bar{D}_{\dot\alpha}\left(K_a(S,\bar{S})\bar{D}^{\dot\alpha}\delta U^a\right) - \bar{D}_{\dot\alpha}K_a(S,\bar{S})\bar{D}^{\dot\alpha}\delta U^a + \text{h.c.}\Big) \\ &= -\tfrac{1}{4}\int d^8z\,\Big(\bar{D}_{\dot\alpha}\left(K_a\bar{D}^{\dot\alpha}\delta U^a - (\bar{D}^{\dot\alpha}K_a)\delta U^a\right) + (\bar{D}^2K_a)\delta U^a + \text{h.c.}\Big), \end{aligned} \tag{4.20}$$

we add to the action (4.1) the boundary terms

$$\mathcal{B} = \tfrac{1}{4}\int d^8z\, \bar{D}_{\dot{\alpha}}\Big(K_a(S,\bar{S})\bar{D}^{\dot{\alpha}}U^a - (\bar{D}^{\dot{\alpha}}K_a(S,\bar{S}))U^a\Big) + \text{h.c.}\ . \tag{4.21}$$

Then the variation

$$\delta(S_3+\mathcal{B}) = \tfrac{1}{4}\int d^8z\,\Big(\bar{D}_{\dot{\alpha}}\left(\delta K_a\bar{D}^{\dot{\alpha}}U^a - (\bar{D}^{\dot{\alpha}}\delta K_a)U^a\right) - (\bar{D}^2K_a)\delta U^a + \text{h.c.}\Big) \tag{4.22}$$

contains only boundary terms proportional to δS^a or $\bar{D}^{\dot{\alpha}}\delta\bar{S}^{\bar{a}}$ (and their complex conjugates) that can be dropped. As a byproduct of this calculation we have found the massless superfield equations of motion

$$\bar{D}^2K_a(S,\bar{S}) + D^2K_{\bar{a}}(S,\bar{S}) \;=\; 0. \tag{4.23}$$

To extract the relevant 3-form boundary terms from $\mathcal{B}$ we rewrite it as (cf. (3.48)),

$$\begin{aligned}\mathcal{B} &= \text{Re}\int d^8z\,\bar{D}_{\dot{\alpha}}\left(K_a(S,\bar{S})\bar{D}^{\dot{\alpha}}U^a\right) - \tfrac{1}{2}\text{Re}\int d^8z\,\bar{D}^2\left(K_a(S,\bar{S})U^a\right)\\ &= -\text{Im}\int d^8z\,\partial_m(\theta\sigma^m)_{\dot{\alpha}}\left(K_a(S,\bar{S})\bar{D}^{\dot{\alpha}}U^a\right) + \tfrac{1}{2}\text{Re}\int d^8z\,\theta^2\Box\left(K_a(S,\bar{S})U^a\right).\end{aligned} \tag{4.24}$$

To further evaluate these expressions we use the θ-expansions of U^a, $\bar{D}^{\dot{\alpha}}U^a$ and $K_a(S,\bar{S})$,

$$\begin{aligned}U^a &= B^a + \theta^2M^{a*} + \bar{\theta}^2M^a - \tfrac{1}{3}\bar{\theta}\bar{\sigma}^m\theta\varepsilon_{mnpq}C^{anpq} + \theta^2\bar{\theta}^2\big(D^a - \tfrac{1}{4}\Box B^a\big),\\ \bar{D}^{\dot{\alpha}}U^a &= -2\bar{\theta}^{\dot{\alpha}}M^a + \tfrac{1}{3}(\bar{\sigma}^m\theta)^{\dot{\alpha}}\varepsilon_{mnpq}C^{anpq} - i(\bar{\sigma}^m\theta)^{\dot{\alpha}}\partial_m(B^a+\bar{\theta}^2M^a) + \theta^2(\dots),\\ K_a(S,\bar{S}) &= K_a(M,M^*) + i\theta\sigma^m\bar{\theta}\big(K_{ab}\partial_mM^b - K_{a\bar{b}}\partial_mM^{*\bar{b}}\big) + \bar{\theta}^2K_{a\bar{b}}(D^{\bar{b}} - iH^{\bar{b}}) + \theta^2(\dots).\end{aligned} \tag{4.25}$$

The terms in $\bar{D}^{\dot{\alpha}}U^a$ and $K_a(S,\bar{S})$ that contain at least two θ's have not been written out as they do not contribute in (4.24). Here we are interested only in boundary terms involving the 3-forms C^a_{npq} without derivatives, which are all contained in the first term in the second line of (4.24) (and originate from the product of the second term in $\bar{D}^{\dot{\alpha}}U^a$ with the $\bar{\theta}^2$-component of $K_a(S,\bar{S})$). They read

$$\mathcal{B}_3 = -\tfrac{1}{3}\int d^4x\,\partial_m\Big(\left((\text{Re}K)_{ab}H^b - (\text{Im}K)_{ab}D^b\right)\varepsilon^{mnpq}C^a_{npq}\Big). \tag{4.26}$$

Inserting the solution for D^b (4.12) with $G=0$ into this expression it becomes

$$\begin{aligned}\mathcal{B}_3 &= -\tfrac{1}{3}\int d^4x\,\partial_m\Big(\left((\text{Re}K)_{ab} + (\text{Im}K)_{ac}(\text{Re}K)^{-1cd}(\text{Im}K)_{db}\right)H^b\varepsilon^{mnpq}C^a_{npq}\Big)\\ &= -\tfrac{1}{3}\int d^4x\,\partial_m\Big(g_{ab}H^a\varepsilon^{mnpq}C^b_{npq}\Big).\end{aligned} \tag{4.27}$$

Now we are ready to eliminate the 3-forms from the massless sigma model action. When the 3-forms are on-shell, i.e., when they satisfy (4.19), the boundary term (4.27) becomes negative twice the kinetic term contained in (4.13)

$$g_{ab}H^aH^b = g^{-1ab}c_ac_b = \mathrm{Re}\big(K^{-1a\bar{b}}\big)c_ac_b, \tag{4.28}$$

so that the massless sigma model action becomes

$$S_3 = \int\! d^4x \,\Big(- K_{a\bar{b}}\, \partial^m M^a \partial_m M^{*\bar{b}} - \mathrm{Re}\big(K^{-1\bar{a}b}\big)c_ac_b\Big). \tag{4.29}$$

The second term in (4.29) has the form of a (positive) potential for the fields M^a which can contain a positive contribution to the cosmological constant. Furthermore, the M^a can acquire masses by spontaneous supersymmetry breaking. As we will see below, the same phenomenon occurs in the massless dual action.

4.2 Dual action in the massless case

We now want to find a dual action for (4.1) in the massless case where $G = 0$. For the first order action we make the ansatz

$$S_{\text{first}} = \int\! d^8z \,\Big(- \hat{K}(F,\bar{F}) + F_aS^a + \bar{F}_{\bar{a}}\bar{S}^{\bar{a}}\Big), \tag{4.30}$$

where $\hat{K}$ is real. The equations of motion for the F_a then read

$$\frac{\partial \hat{K}}{\partial F_a} = S^a, \qquad \frac{\partial \hat{K}}{\partial \bar{F}_{\bar{a}}} = \bar{S}^{\bar{a}}. \tag{4.31}$$

In order to reproduce (4.1) (with $G = 0$), $\hat{K}$ has to fulfill the equation

$$K\big(\frac{\partial \hat{K}}{\partial F}, \frac{\partial \hat{K}}{\partial \bar{F}}\big) = F_a\frac{\partial \hat{K}}{\partial F_a} + \bar{F}_{\bar{a}}\frac{\partial \hat{K}}{\partial \bar{F}_{\bar{a}}} - \hat{K}(F,\bar{F}), \tag{4.32}$$

i.e., K has to be the Legendre transform of $\hat{K}$ (and vice versa, because the Legendre transformation is its own inverse, see App. C). Thus (4.31) is equivalent to

$$F_a = \frac{\partial K}{\partial S^a} \equiv K_a(S,\bar{S}). \tag{4.33}$$

In order to eliminate the U^a from the first order action (4.30) without dropping any boundary term, we again have to add the boundary terms (3.62). Note that these exactly reproduce the terms given in (4.21) for $F_a = K_a(S,\bar{S})$. Just like in (3.65), we can then write the action in the form

$$S_{\text{first}} + \mathcal{B}_{\text{first}} = \int\! d^8z \,\Big(-\hat{K}(F,\bar{F}) - \tfrac{1}{4}U^a(\bar{D}^2F_a + D^2\bar{F}_a)\Big). \tag{4.34}$$

Variation with respect to the U^a yields here the same condition on the F_a as in the renormalizable case,

$$\bar{D}^2 F_a + D^2 \bar{F}_a \;=\; 0, \tag{4.35}$$

and thus F_a again takes the form (3.68). With the fermionic components set to zero, we have

$$\begin{aligned}\Delta^F_a &:= F_a - f_a = \;\theta^2 h_a + i\bar\theta^2 \hat c_a + \theta\sigma^m\bar\theta w_{am} + \theta^2\bar\theta^2\left(-\tfrac{1}{4}\Box f_a - \tfrac{i}{2}\partial_m w_a^m\right),\\ \bar\Delta^F_{\bar b} &:= \bar F_{\bar b} - f^*_{\bar b} = \bar\theta^2 h^*_{\bar b} - i\theta^2 \hat c_{\bar b} + \theta\sigma^m\bar\theta w^*_{\bar b m} + \theta^2\bar\theta^2\left(-\tfrac{1}{4}\Box f^*_{\bar b} + \tfrac{i}{2}\partial_m w^{m*}_{\bar b}\right).\end{aligned} \tag{4.36}$$

To compute the component form of the dual action we extract the $\theta^2\bar\theta^2$-component of $\hat K(F,\bar F)$ by Taylor expanding around (f, f^*),

$$\hat K(F,\bar F) = \;\hat K(f,f^*) + \hat K^a\Delta^F_a + \hat K^{\bar a}\bar\Delta^F_{\bar a} + \tfrac{1}{2}\hat K^{ab}\Delta^F_a\Delta^F_b + \tfrac{1}{2}\hat K^{\bar a\bar b}\bar\Delta^F_{\bar a}\bar\Delta^F_{\bar b} + \hat K^{a\bar b}\Delta^F_a\bar\Delta^F_{\bar b}. \tag{4.37}$$

Using $\hat K^{ab} = \hat K^{ba}$ and $\hat K^{\bar a\bar b} = \hat K^{\bar b\bar a}$ one easily finds

$$\begin{aligned} S_{\text{dual}} &= -\int d^8z\, \hat K(F,\bar F)\\ &= -\int d^4x\, \Big[\hat K^a\Big(-\tfrac{1}{4}\Box f_a - \tfrac{i}{2}\partial_m w^m_a\Big) + \hat K^{\bar a}\Big(-\tfrac{1}{4}\Box f^*_{\bar a} + \tfrac{i}{2}\partial_m w^{m*}_{\bar a}\Big)\\ &\qquad + \hat K^{ab}\Big(-\tfrac{1}{4}w_{am}w^m_b + i\hat c_a h_b\Big) + \hat K^{\bar a\bar b}\Big(-\tfrac{1}{4}w^*_{\bar a m}w^{m*}_{\bar b} - i\hat c_{\bar a}h^*_{\bar b}\Big)\\ &\qquad + \hat K^{a\bar b}\Big(-\tfrac{1}{2}w_{am}w^{m*}_{\bar b} + h_a h^*_{\bar b} + \hat c_a\hat c_{\bar b}\Big)\Big].\end{aligned} \tag{4.38}$$

The fields h_a and w_{am} have purely algebraic equations of motion and can thus be eliminated. This is most conveniently done for the h_a by completing the square as in (B.7) (with $J^a = -i\hat K^{ab}\hat c_b$) resulting in[12]

$$\begin{aligned} S_{\text{dual}} = \int d^4x\, \Big[&\tfrac{1}{4}\hat K^{ab}w^m_a w_{bm} + \tfrac{1}{4}\hat K^{\bar a\bar b}w^{m*}_{\bar a}w^*_{\bar b m} + \tfrac{1}{2}\hat K^{a\bar b}w^m_a w^*_{\bar b m}\\ &- \partial_m\hat K^a\left(\tfrac{1}{4}\partial^m f_a + \tfrac{i}{2}w^m_a\right) - \partial_m\hat K^{\bar a}\left(\tfrac{1}{4}\partial^m f^*_{\bar a} - \tfrac{i}{2}w^{m*}_{\bar a}\right)\\ &+ \left(\hat K^{ac}\hat K^{-1}_{c\bar d}\hat K^{\bar d\bar b} - \hat K^{a\bar b}\right)\hat c_a\hat c_{\bar b}\Big],\end{aligned} \tag{4.39}$$

where integration by parts was applied for the terms in the first line of (4.38). The w^m_a appear in the action (4.39) in the most general quadratic form for complex auxiliary fields given in (B.8). Following the prescription given in Appendix B, we can complete

[12] As K is the Legendre transform of $\hat K$, it is implicit in formula (C.11) — with K and $\hat K$ exchanged — that $\hat K^{a\bar b}$ is invertible.

the square with respect to the w_a^m by writing the action as

$$\begin{aligned} S_{\text{dual}} = \int\! d^4x \Bigg[& \tfrac{1}{4}\big((w_a^m + u_a^m)\ (w_{\bar a}^{m*} + u_{\bar a}^{m*})\big) \begin{pmatrix} \hat K^{ab} & \hat K^{a\bar b} \\ \hat K^{\bar a b} & \hat K^{\bar a\bar b} \end{pmatrix} \begin{pmatrix} w_{bm} + u_{bm} \\ w_{\bar b m}^* + u_{\bar b m}^* \end{pmatrix} \\ & - \tfrac{1}{4}\big(u_a^m\ u_{\bar a}^{m*}\big) \begin{pmatrix} \hat K^{ab} & \hat K^{a\bar b} \\ \hat K^{\bar a b} & \hat K^{\bar a\bar b} \end{pmatrix} \begin{pmatrix} u_{bm} \\ u_{\bar b m}^* \end{pmatrix} \\ & - \tfrac{1}{4}\partial^m f_a \partial_m \hat K^a - \tfrac{1}{4}\partial^m f_{\bar a}^* \partial_m \hat K^{\bar a} - \big(\hat K^{a\bar b} - \hat K^{ac}\hat K^{-1}_{c\bar d}\hat K^{\bar d\bar b}\big)\hat c_a \hat c_{\bar b} \Bigg], \end{aligned} \tag{4.40}$$

where the u_a^m have to solve the equations

$$\begin{pmatrix} \hat K^{ab} & \hat K^{a\bar b} \\ \hat K^{\bar a b} & \hat K^{\bar a\bar b} \end{pmatrix} \begin{pmatrix} u_{bm} \\ u_{\bar b m}^* \end{pmatrix} = -i \begin{pmatrix} \partial_m \hat K^a \\ -\partial_m \hat K^{\bar a} \end{pmatrix}. \tag{4.41}$$

Here the Hesse matrix of the Legendre transformed Kähler potential $\hat K(f, f^*)$ has appeared,

$$\text{Hess}\,\hat K \ = \begin{pmatrix} \hat K^{ab} & \hat K^{a\bar b} \\ \hat K^{\bar a b} & \hat K^{\bar a\bar b} \end{pmatrix}. \tag{4.42}$$

As derived in Appendix C, it is the inverse of the Hesse matrix of $K(M, M^*)$. However, let us ignore this fact for the moment and only notice that $\text{Hess}\,\hat K$ is invertible with its inverse given by (cf. (B.13))

$$(\text{Hess}\,\hat K)^{-1} = \begin{pmatrix} C & D \\ D^* & C^* \end{pmatrix} \qquad \text{where} \qquad \begin{aligned} D_{a\bar b} &= \Big(\hat K^{\bar b a} - \hat K^{\bar b\bar c}\hat K^{-1}_{\bar c d}\hat K^{da}\Big)^{-1}, \\ C_{ab} &= -\hat K^{-1}_{a\bar c}\hat K^{\bar c\bar d} D^*_{\bar d b}. \end{aligned} \tag{4.43}$$

The equations of motion for the w_{am} and $w_{\bar a m}^*$ imply that the term in the first line of (4.40) (the "square") vanishes and we write the term in the second line as

$$-\tfrac{1}{4}\big(u_a^m\ u_{\bar a}^{m*}\big) \begin{pmatrix} \hat K^{ab} & \hat K^{a\bar b} \\ \hat K^{\bar a b} & \hat K^{\bar a\bar b} \end{pmatrix} \begin{pmatrix} u_{bm} \\ u_{\bar b m}^* \end{pmatrix} = \tfrac{1}{4}\big(\partial^m \hat K^a\ \ -\partial^m \hat K^{\bar a}\big) \begin{pmatrix} C_{ab} & D_{a\bar b} \\ D^*_{\bar a b} & C^*_{\bar a\bar b} \end{pmatrix} \begin{pmatrix} \partial_m \hat K^b \\ -\partial_m \hat K^{\bar b} \end{pmatrix}. \tag{4.44}$$

Thus we obtain the on-shell action (note that D is hermitian, $D^*_{\bar a b} = D_{b\bar a}$)

$$\begin{aligned} S_{\text{dual}} = \int\! d^4x \Bigg[& -\tfrac{1}{2}\partial^m \hat K^a D_{a\bar b}\partial_m \hat K^{\bar b} + \tfrac{1}{4}\partial^m \hat K^a C_{ab}\partial_m \hat K^b + \tfrac{1}{4}\partial^m \hat K^{\bar a} C^*_{\bar a\bar b}\partial_m \hat K^{\bar b} \\ & -\tfrac{1}{4}\partial^m f_a \partial_m \hat K^a - \tfrac{1}{4}\partial^m f^*_{\bar a}\partial_m \hat K^{\bar a} - \text{Re}\left(D^{-1\bar b a}\right)\hat c_a \hat c_b \Bigg]. \end{aligned} \tag{4.45}$$

To simplify this expression, we use the hermicity of $D_{a\bar b}$ and $\hat K^{-1}_{a\bar b}$ to write the matrix C that is defined in (4.43) as

$$C_{ab} = -D_{b\bar c}\hat K^{\bar c\bar d}\hat K^{-1}_{\bar d a}. \tag{4.46}$$

Then we substitute this expression to rewrite the terms in (4.45) that depend on C,

$$\begin{aligned}
\partial^m \hat{K}^a C_{ab} \partial_m \hat{K}^b &= -\partial_m \hat{K}^b D_{b\bar{c}} \hat{K}^{\bar{c}\bar{d}} \hat{K}^{-1}_{\bar{d}a} \left(\hat{K}^{ae}\, \partial^m f_e + \hat{K}^{a\bar{e}}\, \partial^m f^*_{\bar{e}} \right) \\
&= -\partial_m \hat{K}^b D_{b\bar{c}} \Big((\hat{K}^{\bar{c}e} - D^{-1\bar{c}e}) \partial^m f_e + \hat{K}^{\bar{c}\bar{e}} \partial^m f^*_{\bar{e}} \Big) \\
&= -\partial_m \hat{K}^a D_{a\bar{b}}\, \partial^m \hat{K}^{\bar{b}} + \partial_m \hat{K}^a \partial^m f_a,
\end{aligned} \tag{4.47}$$

where in the second step we made use of the expression for $D_{a\bar{b}}$ given in (4.43). By (4.47) and its complex conjugate the dual action (4.45) can be written as[13]

$$S_{\text{dual}} = \int d^4x \left(-D_{a\bar{b}}\, \partial^m \hat{K}^a \partial_m \hat{K}^{\bar{b}} - \operatorname{Re}\left(D^{-1\bar{a}b} \right) \hat{c}_a \hat{c}_b \right). \tag{4.48}$$

This is the moment to remember that $(\operatorname{Hess} \hat{K})^{-1} = \operatorname{Hess} K$, and in particular

$$D_{a\bar{b}}(f, f^*) = K_{a\bar{b}}(M, M^*), \tag{4.49}$$

where $K_{a\bar{b}}$ has to be evaluated at

$$M^a = \hat{K}^a(f, f^*), \qquad M^{*\bar{a}} = \hat{K}^{\bar{a}}(f, f^*). \tag{4.50}$$

Equation (4.50) is just the lowest component of the Legendre relation (4.31), that appeared as the equation of motion for F_a in the first order action (4.30). Thus we see that the kinetic term of the dual on-shell action (4.48) is, upon using the Legendre relation (4.50), equal to that of the original action (4.5). In particular the "new" metric

$$D_{a\bar{b}} = \left(\hat{K}^{\bar{b}a} - \hat{K}^{\bar{b}\bar{c}} \hat{K}^{-1}_{\bar{c}d} \hat{K}^{da} \right)^{-1} \tag{4.51}$$

appearing in the dual action is again Kähler with respect to the variables $\hat{K}^a$ and $\hat{K}^{\bar{b}}$.

The constants $\hat{c}_a$ that are dual to the 3-forms appear in (4.48) with the matrix $\operatorname{Re}\left(D^{-1\bar{a}b}\right)$, which is a function of the f_a. Thus this term, as the second term in (4.29) which it corresponds to, has the form of a potential that can lead to an effective cosmological constant or to mass terms for the f_a, even though no mass term was introduced a priori. In the following we will show that also these potential terms are equal with respect to the duality relation of the constants c_a and $\hat{c}_a$ which is contained in (4.31). To find this relation, we have to determine the θ^2 and $\bar{\theta}^2$-components of (4.31) with constrained F_a (i.e., $n_a = i\hat{c}_a$). While the θ-expansion of S^a can be read off from (4.4), that of $\hat{K}^a(F, \bar{F})$ can be found by the usual Taylor expansion technique,

$$\frac{\partial \hat{K}}{\partial F_a} = \hat{K}^a + \hat{K}^{ab} \Delta^F_b + \hat{K}^{a\bar{b}} \bar{\Delta}^F_{\bar{b}} + \tfrac{1}{2} \hat{K}^{abc} \Delta^F_b \Delta^F_c + \tfrac{1}{2} \hat{K}^{a\bar{b}\bar{c}} \bar{\Delta}^F_{\bar{b}} \bar{\Delta}^F_{\bar{c}} + \hat{K}^{ab\bar{c}} \Delta^F_b \bar{\Delta}^F_{\bar{c}}, \tag{4.52}$$

[13] Here an error in the corresponding result of ref. [17] has been corrected. For the case of the complex linear multiplet, i.e. for $\hat{c}_a = 0$, ref. [18] also gives a different result due to an error in that work.

and $\Delta_a^F = F_a - f_a$ is given in (4.36). Thus the θ^2 and $\bar\theta^2$-components of (4.31) with constrained F_a read

$$\begin{aligned} \hat K^{ab} h_b - i\hat K^{a\bar b}\hat c_{\bar b} &= D^a + iH^a, \\ \hat K^{a\bar b} h^*_{\bar b} + i\hat K^{ab}\hat c_b &= 0. \end{aligned} \tag{4.53}$$

The second equation in (4.53) is just the equation of motion for the auxiliary field h_a which we already used to compute the on-shell action (4.48). Inserting the solution for h_b,

$$h_b = i\hat K^{-1}_{b\bar c}\hat K^{\bar c\bar d}\hat c_{\bar d}, \tag{4.54}$$

into the first equation in (4.53), one finds the on-shell duality relation

$$H^a = \mathrm{Re}\big(\hat K^{ab}\hat K^{-1}_{b\bar c}\hat K^{\bar c\bar d} - \hat K^{a\bar d}\big)\hat c_d = -\mathrm{Re}\big(D^{-1a\bar d}\big)\hat c_d. \tag{4.55}$$

From the last equation one can derive the relation between the constants c_a and $\hat c_a$ appearing in the on-shell action and dual action respectively:

$$c_a = g_{ab}H^b = -\hat c_a, \tag{4.56}$$

where (4.16) and (4.49) were used. Now we see that the sigma model action of the massless 3-form multiplet (4.29), is indeed equal to its dual action (4.48) by use of the two duality relations (4.50) and (4.56).

Should one really be surprised about the equality of action and dual action with respect to the duality relations? Or is it rather something one could have expected from the beginning? To answer this question, let us briefly recall what we have done in this section: We began by writing down a first order action S_{first} from which the original action S_3 could be reproduced by using the Euler-Lagrange equations for the new superfields F_a given in (4.31). In mathematical language, this simply reads

$$\frac{\partial\hat K}{\partial F_a} = S^a \quad\Rightarrow\quad S_{\text{first}} = S_3. \tag{4.57}$$

Then we eliminated the 3-form superfields U^a from S_{first} by their Euler-Lagrange equations (4.35) to obtain the dual action S_{dual}, that is

$$\bar D^2 F_a + D^2\bar F_a = 0 \quad\Rightarrow\quad S_{\text{first}} = S_{\text{dual}}. \tag{4.58}$$

Now elementary logic tells us that

$$\left(\frac{\partial\hat K}{\partial F_a} = S^a \quad\wedge\quad \bar D^2 F_a + D^2\bar F_a = 0\right) \quad\Rightarrow\quad S_3 = S_{\text{dual}}. \tag{4.59}$$

In other words, action and dual action are equal when both the Euler-Lagrange equations for the F_a as well as those of the U^a are used. However, the situation is not that simple because these two superfield equations contain all the equations of motion of the first order action, in particular also those of the physical fields M^a and f_a appearing in the on-shell action and dual action. Therefore one can not expect that these actions can be translated into each other only by use of the duality relations (4.50) and (4.56). It has to be considered as coincidence that this is nevertheless possible for the massless case. It certainly has to do with the fact that the number of off-shell degrees of freedom of the action (4.29) and dual action (4.48) coincide so that the duality relations (4.50) and (4.56) constitute only a field redefinition. Using this field redefinition to re-express the dual action in terms of the fields M^a and constants c_a one will find an action whose equations of motion are equivalent to those of the dual action by (4.50). The massless 3-form action (4.29) has the same property, therefore it is not a big surprise that they coincide. We will see below that in the massive case there is no field redefinition that relates the dual action to the 3-form action and one has to make use of the equations of motion of the 3-forms (which are then dynamical fields that can not be eliminated from the action) to translate them into each other.

4.3 Dual action in the massive case

Let us now turn to the massive case where the potential $G(U)$ is non-trivial. We simply add this term to the first order action (4.30),

$$S_{\text{first}} = \int d^8z \Big(- \hat{K}(F, \bar{F}) + F_a S^a + \bar{F}_{\bar{b}} \bar{S}^{\bar{b}} - G(U) \Big), \tag{4.60}$$

where $\hat{K}$ is again the Legendre transform of K. Since the Euler-Lagrange equations (4.31) do not change, the original action (4.1) is reproduced correctly. We then rewrite the action as[14]

$$S_{\text{first}} = \int d^8z \Big(- \hat{K}(F, \bar{F}) + \Omega_a U^a - G(U) \Big), \tag{4.61}$$

where Ω_a was defined in (3.75), and determine the Euler-Lagrange equation for U^a to be

$$\frac{\partial G}{\partial U^a} \;=\; \Omega_a. \tag{4.62}$$

To eliminate the U^a from the action we have to assume that there is a Legendre transform $\hat{G}$ of G, i.e., a function which satisfies

$$\hat{G}(\frac{\partial G}{\partial U}) \;=\; U^a \frac{\partial G}{\partial U^a} - G(U). \tag{4.63}$$

When this is the case, we find as a dual action

$$S_{\text{dual}} = \int d^8z \Big(-\hat{K}(F, \bar{F}) + \hat{G}(\Omega) \Big). \tag{4.64}$$

[14]Again, in the massive case it is legitimate to drop boundary terms.

The superfields F_a remain unconstrained as in the massive renormalizable case so that their bosonic part is given by

$$\begin{aligned} F_a &= f_a + \theta^2 h_a + \bar\theta^2 n_a + \theta\sigma^m\bar\theta w_{am} + \theta^2\bar\theta^2(d_a - \tfrac{1}{4}\Box f_a - \tfrac{i}{2}\partial_m w_a^m), \\ \bar F_{\bar a} &= f_{\bar a}^* + \theta^2 n_{\bar a}^* + \bar\theta^2 h_{\bar a}^* + \theta\sigma^m\bar\theta w_{\bar a m}^* + \theta^2\bar\theta^2(d_{\bar a}^* - \tfrac{1}{4}\Box f_{\bar a}^* + \tfrac{i}{2}\partial_m w_{\bar a}^{m*}). \end{aligned} \tag{4.65}$$

For the K-part of the action we now obtain

$$\begin{aligned} S_{\mathrm{K}} &= -\int d^8z\, \hat K(F,\bar F) \\ &= -\int d^4x \Big[\hat K^a \left(d_a - \tfrac{1}{4}\Box f_a - \tfrac{i}{2}\partial_m w_a^m\right) + \hat K^{ab}\left(-\tfrac{1}{4} w_a^m w_{bm} + h_a n_b\right) + \text{h.c.} \\ &\qquad + \hat K^{a\bar b}\left(-\tfrac{1}{2} w_a^m w_{\bar b m}^* + h_a h_{\bar b}^* + n_a n_{\bar b}^*\right)\Big]. \end{aligned} \tag{4.66}$$

Note that compared to (4.38), the complex scalars d_a and n_a also appear in the massive dual action since the F_a are unconstrained. For the G-term we use the θ-expansion of Ω_a as given in (3.66) but with fermionic components set to zero,

$$\Omega_a = 2\mathrm{Re}(n_a) + \theta^2 d_a + \bar\theta^2 d_a^* - 2\theta\sigma^m\bar\theta\partial_m \mathrm{Im}(n_a) + \tfrac{1}{2}\Box\mathrm{Re}(n_a). \tag{4.67}$$

Now perform the θ-integration of $\hat G(\Omega)$,

$$\begin{aligned} S_{\mathrm{G}} &= \int d^8z\, \hat G(\Omega) \\ &= \int d^4x \left[\tfrac{1}{2}\hat G^a \Box\mathrm{Re}(n_a) + \tfrac{1}{2}\hat G^{ab}\big(d_a d_b^* + d_a^* d_b - 2\partial_m \mathrm{Im}(n_a)\partial^m \mathrm{Im}(n_b)\big)\right] \\ &= \int d^4x\, \hat G^{ab}\big(d_a d_b^* - \partial_m \mathrm{Re}(n_a)\partial^m\mathrm{Re}(n_b) - \partial_m\mathrm{Im}(n_a)\partial^m\mathrm{Im}(n_b)\big) \\ &= \int d^4x\, \hat G^{ab}\big(d_a d_b^* - \partial_m n_a \partial^m n_b^*\big), \end{aligned} \tag{4.68}$$

where the derivatives $\hat G^a$ and $\hat G^{ab}$ are defined in the usual way and in the third step we made use of

$$\partial_m \hat G^a = 2\hat G^{ab}\partial_m \mathrm{Re}(n_b). \tag{4.69}$$

After assembling the whole action $S_{\mathrm{dual}} = S_{\mathrm{K}} + S_{\mathrm{G}}$, the part of the Lagrangian containing the auxiliary fields d_a and h_a is

$$\mathcal{L}_{\mathrm{d,h}} = \hat G^{ab} d_a d_b^* - \hat K^a d_a - \hat K^{\bar a} d_{\bar a}^* - \hat K^{a\bar b} h_a h_{\bar b}^* - \hat K^{ab} n_a h_b - \hat K^{\bar a\bar b} n_{\bar a} h_{\bar b}^*. \tag{4.70}$$

As derived in Appendix B, by the equations of motion for d_a and h_a (4.70) becomes

$$-\hat K^{\bar b}\hat G^{-1}_{ba}\hat K^a + n_a \hat K^{ac}\hat K^{-1}_{c\bar d}\hat K^{\bar d\bar b} n_{\bar b}^*, \tag{4.71}$$

where $\hat{G}^{-1}_{ab} = G_{ab}|_{B^a=\hat{G}^a}$ since G_{ab} is the Hesse matrix of $G(B)$ and $\hat{G}^{ab}$ is that of $\hat{G}(2\mathrm{Re}(n))$. Thus we obtain the intermediate result

$$\begin{aligned} S_{\text{dual}} = \int d^4x \Big[& \tfrac{1}{4}\hat{K}^{ab} w_a^m w_{bm} + \tfrac{1}{4}\hat{K}^{\bar{a}\bar{b}} w_{\bar{a}}^{m*} w_{\bar{b}m}^* + \tfrac{1}{2}\hat{K}^{a\bar{b}} w_a^m w_{\bar{b}m}^* \\ & - \partial_m \hat{K}^a \left(\tfrac{1}{4}\partial^m f_a + \tfrac{i}{2} w_a^m\right) - \partial_m \hat{K}^{\bar{a}} \left(\tfrac{1}{4}\partial^m f_{\bar{a}}^* - \tfrac{i}{2} w_{\bar{a}}^{m*}\right) \\ & - D^{-1a\bar{b}} n_a n_{\bar{b}}^* - \hat{K}^{\bar{b}} \hat{G}^{-1}_{ba} \hat{K}^a - \hat{G}^{ab} \partial^m n_a \partial_m n_b^* \Big] , \end{aligned} \tag{4.72}$$

where $D^{-1a\bar{b}} = \hat{K}^{a\bar{b}} - \hat{K}^{ac}\hat{K}^{-1}_{c\bar{d}}\hat{K}^{\bar{d}b}$. Note that the action (4.72) differs from the massless action (4.39) only by the three terms in the last line of (4.72). Therefore the auxiliary fields w_a^m can be eliminated using the same steps as in the massless case and resulting on-shell actions will differ by the same terms, namely

$$S_{\text{dual}} = \int d^4x \Big(- D_{a\bar{b}} \partial^m \hat{K}^a \partial_m \hat{K}^{\bar{b}} - \hat{G}^{ab} \partial^m n_a \partial_m n_b^* - D^{-1a\bar{b}} n_a n_{\bar{b}}^* - \hat{K}^a \hat{G}^{-1}_{ab} \hat{K}^{\bar{b}} \Big). \tag{4.73}$$

Just as for the renormalizable action discussed in section 3.6, the massive 3-forms are no longer dual to constants but replaced, together with the real scalars B_a, by the complex scalars n_a. The kinetic term for these scalars also takes the form of a Kähler geometry with the Legendre transform of G as Kähler potential. The last two terms in (4.73) can give rise to masses for the scalars f_a and n_a. In Appendix D it is shown that the massive dual on-shell action (4.73) is equal to the 3-form action (4.13) upon using the duality relations that are contained in (4.31) and (4.62) and the equations of motion of the 3-forms.

4.4 Dualization in the case of Kähler potentials with a shift symmetry

Not every sigma model action with 3-form multiplets can be dualized in the way described in sections 4.2 and 4.3. Let us consider the specific class of Kähler potentials that have a shift symmetry $S^a \to S^a + iR^a$, where R^a is a real superfield. Such Kähler potentials only depend on the real parts of the superfields S^a, i.e.

$$K(S, \bar{S}) = K(S + \bar{S}). \tag{4.74}$$

Then one has

$$K_{a\bar{b}} = K_{ab} = K_{\bar{a}b} \;\Rightarrow\; K_{a\bar{b}} \in \mathbb{R}. \tag{4.75}$$

Thus the massive component action (4.13) becomes

$$S_3 = \int d^4x \Big[- K_{ab}\,\partial^m M^a \partial_m M^{b*} - G_{ab}\Big(\tfrac{1}{4}\partial^m B^a \partial_m B^b + M^a M^{b*} + \tfrac{1}{6}C^{anpq}C^b_{npq}\Big) + K_{ab}H^a H^b - \tfrac{1}{4}G_a K^{-1ab} G_b \Big], \tag{4.76}$$

and the massless action (4.29) is

$$\begin{aligned} S_3 &= \int d^4x \Big(- K_{ab}\,\partial^m M^a \partial_m M^{b*} - K^{-1ab} c_a c_b \Big) \\ &= \int d^4x \Big(- K_{ab}\,\big(\partial^m(\mathrm{Re}M^a)\partial_m(\mathrm{Re}M^b) + \partial^m(\mathrm{Im}M^a)\partial_m(\mathrm{Im}M^b)\big) - K^{-1ab}c_a c_b \Big). \end{aligned} \tag{4.77}$$

Note that in the massless as well as in the massive case the fields $\mathrm{Im}M^a$ have no self-interaction terms, in particular they are massless.

The actions (4.76) and (4.77) can not be dualized as described in the previous sections because the arguments of the Legendre transform $\hat{K}$ of K

$$F_a = \frac{\partial K}{\partial S^a} = \frac{\partial K}{\partial \bar{S}^{\bar{a}}} = \bar{F}_{\bar{a}} \tag{4.78}$$

have to be real.[15] Therefore in the massless case a first order action is given by [17]

$$S_{\text{first}} = \int d^8z \Big(- \hat{K}(F) + F_a(S^a + \bar{S}^a) \Big) + \mathcal{B}_{\text{first}}, \tag{4.79}$$

with boundary terms as before (cf. (3.62))

$$\mathcal{B}_{\text{first}} = \tfrac{1}{4}\int d^8z \Big[\bar{D}_{\dot{\alpha}}\left(F_a \bar{D}^{\dot{\alpha}} U^a - (\bar{D}^{\dot{\alpha}} F_a) U^a \right) + \text{h.c.} \Big]. \tag{4.80}$$

and $\hat{K}$ is the Legendre transform of the function K that takes only one real argument. Since the F_a are real, their component expansion can be written as

$$F_a = f_a + \theta^2 n_a + \bar{\theta}^2 n_a^* + \theta\sigma^m\bar{\theta} w_{am} + \theta^2\bar{\theta}^2\big(d_a - \tfrac{1}{4}\Box f_a\big), \tag{4.81}$$

where f_a, d_a and w_{am} are real. Varying the action (4.79) with respect to F^a yields

$$S^a + \bar{S}^a \;=\; \frac{\partial \hat{K}}{\partial F_a}. \tag{4.82}$$

This is the Legendre relation that leads back to the original action

$$S_3 = \int d^8z\, K(S + \bar{S}). \tag{4.83}$$

[15] To be more precise, there is no Legendre transform of K in the sense of (4.32) because the Hesse matrix of K with S^a and $\bar{S}^{\bar{a}}$ considered as independent variables is not invertible.

Substituting $S^a + \bar{S}^a = -\frac{1}{4}(D^2 + \bar{D}^2)U^a$ in (4.79) the Euler-Lagrange equation for U^a is found to be

$$\begin{aligned} 0 &= -\tfrac{1}{4}(D^2 + \bar{D}^2)F_a \\ &= (n_a + n_a^*) + \theta^2 \left(d_a + \tfrac{i}{2}\partial_m w_a^m\right) + \bar{\theta}^2 \left(d_a - \tfrac{i}{2}\partial_m w_a^m\right) \\ &\quad - i\theta\sigma^m\bar{\theta}\partial_m \left(n_a - n_a^*\right) + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box(n_a + n_a^*). \end{aligned} \tag{4.84}$$

This equation, which is used to eliminate the 3-form multiplets from the action and find a dual action, imposes the constraints

$$d_a = 0, \quad \partial_m w_a^m = 0, \quad n_a = i\hat{c}_a, \; \hat{c}_a \in \mathbb{R} \tag{4.85}$$

on the components of F_a. The second condition is solved by $w_{am} = \varepsilon_{mnpq}\partial^n B_a^{pq}$ with a 2-form B^{pq}, so that F_a takes the form

$$F_a = f_a + i\theta^2\hat{c}_a - i\bar{\theta}^2\hat{c}_a + \theta\sigma^m\bar{\theta}\varepsilon_{mnpq}\partial^n B_a^{pq} - \tfrac{1}{4}\theta^2\bar{\theta}^2\Box f_a. \tag{4.86}$$

Therefore we find as a dual action

$$\begin{aligned} S_{\text{dual}} &= -\int d^8z\, \hat{K}(F) \\ &= -\int d^4x \Big(-\tfrac{1}{4}\hat{K}^a\Box f_a + \tfrac{1}{2}\hat{K}^{ab}\left(2\hat{c}_a\hat{c}_b - \tfrac{1}{2}\varepsilon_{mnpq}\partial^n B_a^{pq}\varepsilon^{mlrs}\partial_l B_{brs}\right)\Big) \\ &= \int d^4x\, \hat{K}^{ab}\Big(-\tfrac{1}{4}\partial^m f_a\partial_m f_b - \tfrac{3}{2}\partial^{[n}B_a^{pq]}\partial_n B_{bpq} - \hat{c}_a\hat{c}_b\Big). \end{aligned} \tag{4.87}$$

The $2N$ bosonic on-shell degrees of freedom contained in the M^a are distributed in the dual action among the real scalars f_a and the 2-forms B_a^{pq}, which carry one degree of freedom each. As shown in Appendix C, equation (4.82) is equivalent to

$$F_a = \frac{\partial K}{\partial(S^a + \bar{S}^a)}. \tag{4.88}$$

This equation contains the duality relations between the components of the massless 3-form multiplet U and 2-form multiplet F. Using

$$S^a + \bar{S}^a = 2\text{Re}M^a - 2\theta\sigma^m\bar{\theta}\partial_m(\text{Im}M^a) + \theta^2(D^a + iH^a) + \bar{\theta}^2(D^a - iH^a) + \tfrac{1}{2}\theta^2\bar{\theta}^2\Box(\text{Re}M^a), \tag{4.89}$$

one finds from the lowest component of (4.88)

$$f_a = K_a(2\text{Re}M) \;\Rightarrow\; \partial_m f_a = 2K_{ab}\partial_m(\text{Re}M^b), \tag{4.90}$$

while the $\theta\sigma^m\bar{\theta}$-component reads

$$\varepsilon_{mnpq}\partial^n B_a^{pq} = -2K_{ab}\partial_m(\text{Im}M^b) \;\Rightarrow\; \partial^{[n}B_a^{pq]} = \tfrac{1}{3}\varepsilon^{mnpq}K_{ab}\partial_m(\text{Im}M^b), \tag{4.91}$$

and finally from the θ^2-component

$$i\hat{c}_a = K_{ab}(D^b + iH^b) \;\Rightarrow\; \hat{c}_a = K_{ab}H^b = g_{ab}H^b = c_a. \tag{4.92}$$

(The last equation also contains the equation of motion of D^a, $K_{ab}D^b = 0$.) Let us now check whether the dual component action (4.87) can be translated into the massless 3-form action (4.77) by these duality relations. First of all (4.90) implies $\hat{K}^{ab} = K^{-1ab}$. Thus we see that the last term in (4.87) coincides with the potential term in (4.77). With the second equation in (4.90) and (4.91) we can also translate the kinetic terms to obtain

$$S_{\text{dual}} \cong \int d^4x \Big(- K_{ab} \left(\partial^m(\text{Re}M^a)\partial_m(\text{Re}M^b) - \partial^m(\text{Im}M^a)\partial_m(\text{Im}M^b)\right) - K^{-1ab}c_a c_b \Big). \tag{4.93}$$

This equals (4.77) up to the sign of the kinetic term for $\text{Im}M^a$. To understand where this difference arises from, let us consider the relations (4.90), (4.91) and (4.92) as field redefinitions we choose without knowing anything about Poincaré duality. When we express the action (4.87) in terms of the new fields $\text{Re}M^a$ and $\text{Im}M^a$ we have to keep in mind that $\text{Im}M^a$ is not a free field, because according to (4.91) it satisfies

$$\partial_m\big(K_{ab}\partial^m(\text{Im}M^b)\big) = 0. \tag{4.94}$$

This is just the equation of motion for $\text{Im}M^a$ that follows from (4.77).[16] Therefore one should not be surprised about the fact that (4.93) does not coincide with (4.77). The $\theta\sigma^m\bar{\theta}$-component of (4.88) relates $\text{Im}M^a$ to the constrained fields w_a^m, and the constraint corresponds to the equation of motion for $\text{Im}M^a$ which follows from the 3-form action with *unconstrained* $\text{Im}M^a$. Of course the expression (4.93) equals (4.77) upon using (4.94) because by this equation the kinetic term for $\text{Im}M^a$ simply vanishes. But action and dual action cannot be translated into each other by a field redefinition as in the complex massless case. This is already clear from the fact that they do not contain an equal number of off-shell degrees of freedom as the imaginary parts of the M^a are dual to the 2-forms B_a^{pq} that possess three gauge invariant off-shell degrees of freedom each.

As an example consider the Kähler potential

$$K(S,\bar{S}) \;=\; -\log(S+\bar{S}). \tag{4.95}$$

The massless action including the boundary term for the 3-form is then given by

$$S = \int d^4x \, \frac{1}{(2\text{Re}M)^2}\Big[- \partial_m M \partial^m M^* + H^2 \Big] - \tfrac{1}{3}\int d^4x \, \partial_m \Big[\frac{1}{(2\text{Re}M)^2} H \varepsilon^{mnpq} C_{npq} \Big]. \tag{4.96}$$

[16] Interestingly, the equation of motion for B_a^{pq}, $0 = \partial_n\big(\hat{K}^{ab}\partial^{[n}B_b^{pq]}\big) \sim \varepsilon^{mnpq}\partial_n\partial_m(\text{Im}M^a)$, is also automatically fulfilled by the duality relations (4.91) and (4.90).

To find $\hat{K}(F)$, use the Legendre relation $F = \partial K/\partial S = -(S+\bar{S})^{-1}$ and

$$\hat{K}(F) \;=\; -K(S,\bar{S}) + F(S+\bar{S}) \;=\; \log(-F) - 1. \tag{4.97}$$

Thus the dual action is given by

$$\begin{aligned} S_{\rm dual} &= -\int d^8z\, \hat{K}(F) \\ &= \int d^4x\, \frac{1}{f^2}\Big(-\tfrac{1}{4}\partial^m f \partial_m f - \tfrac{3}{2}\partial^{[n}B^{pq]}\partial_n B_{pq} - \hat{c}^2\Big). \end{aligned} \tag{4.98}$$

The duality relations read here

$$f = -\frac{1}{2{\rm Re}M}, \quad \partial^{[n}B^{pq]} = \tfrac{1}{12}({\rm Re}M)^{-2}\varepsilon^{mnpq}\partial_m({\rm Im}M), \quad \hat{c} = \frac{H}{(2{\rm Re}M)^2}. \tag{4.99}$$

5 Coupling to chiral fields

5.1 Renormalizable coupling

In this section we want to study the coupling of N 3-form multiplets U^a to N_c chiral fields Φ^i. We start with the massive action (3.56) and add interaction terms and kinetic terms for the chiral superfields. The action is then of the form[17]

$$\begin{aligned} S = \int d^4x \Big[\int d^2\theta\, d^2\bar{\theta}\, \Big(S^a\bar{S}^a + \Phi^i\bar{\Phi}^i - \tfrac{1}{2}m^2_{ab}U^aU^b + \xi_a U^a\Big) \\ + \int d^2\theta\, W(S,\Phi) + \int d^2\bar{\theta}\, W^*(\bar{S},\bar{\Phi})\Big]. \end{aligned} \tag{5.1}$$

Generally the function W can contain both interaction terms that couple the superfields S^a to the Φ^i, as well as potential terms which depend only on the S^a or Φ^i respectively:

$$W(S,\Phi) = W^{\rm int.}(S,\Phi) + W^S(S) + W^\Phi(\Phi)\ . \tag{5.2}$$

In order for the action to be renormalizable W can be at most cubic in the superfields. To determine the component form of (5.1) we only have to add to (3.56) the $\theta^2\bar{\theta}^2$-component of $\Phi^i\bar{\Phi}^i$ as given in (2.20) and the θ^2-component of $W(S,\Phi)$ which can be found with the usual Taylor expansion technique. With the shorthands for the derivatives of W

$$W_a(M,A) := \frac{\partial W}{\partial S^a}\Big|_{\theta=\bar{\theta}=0}, \quad W_i(M,A) := \frac{\partial W}{\partial \Phi^i}\Big|_{\theta=\bar{\theta}=0} \quad \text{etc.} \tag{5.3}$$

[17] Note that this is a gauge fixed action, with the gauge specified in section 3.4. Furthermore, for renormalizable theories m^2_{ab} has to be constant and we do not consider the possibility of a ΦU coupling.

this yields

$$
\begin{aligned}
S = \int d^4x \Big[& - \partial^m M^a \, \partial_m M^{a*} - i\lambda^a \sigma^m \partial_m \bar{\lambda}^a - \partial_m A^i \, \partial^m A^{i*} - i\psi^i \sigma^m \partial_m \bar{\psi}^i + F^i F^{i*} \\
& + D^a D^a + H^a H^a - m^2_{ab} \Big(\tfrac{i}{2} \chi^a \sigma^m \partial_m \bar{\chi}^b - \tfrac{i}{2} \left(\chi^a \lambda^b - \bar{\chi}^a \bar{\lambda}^b \right) + M^a M^{b*} \\
& \qquad + B^a D^b - \tfrac{1}{4} B^a \Box B^b + \tfrac{1}{6} C^a_{npq} C^{bnpq} \Big) + \xi_a \left(D^a - \tfrac{1}{4} \Box B^a \right) \\
& + \Big(W_a \left(D^a + iH^a \right) + W_i F^i - \tfrac{1}{2} W_{ij} \psi^i \psi^j - \tfrac{1}{2} W_{ab} \lambda^a \lambda^b - W_{ai} \lambda^a \psi^i + \text{h.c.} \Big) \Big] .
\end{aligned}
\tag{5.4}
$$

By completing the square for the auxiliary fields D^a and F^i they can be easily eliminated from the action (5.1) to get

$$
\begin{aligned}
S = \int d^4x \Big[& - \partial^m M^a \, \partial_m M^{a*} - i\lambda^a \sigma^m \partial_m \bar{\lambda}^a - \partial_m A^i \, \partial^m A^{i*} - i\psi^i \sigma^m \partial_m \bar{\psi}^i \\
& - m^2_{ab} \Big(\tfrac{i}{2} \chi^a \sigma^m \partial_m \bar{\chi}^b - \tfrac{i}{\sqrt{2}} \left(\chi^a \lambda^b - \bar{\chi}^a \bar{\lambda}^b \right) + M^a M^{b*} + \tfrac{1}{6} C^a_{npq} C^{bnpq} \Big) \\
& - \tfrac{1}{4} m^2_{ab} \partial^m B^a \, \partial_m B^b + H^a H^a - \tfrac{1}{4} \Big(m^2_{ab} B^b - \xi_a - 2\text{Re}(W_a) \Big)^2 - W_i W_i^* \\
& + \Big(iW_a H^a - \tfrac{1}{2} W_{ij} \psi^i \psi^j - \tfrac{1}{2} W_{ab} \lambda^a \lambda^b - W_{ai} \lambda^a \psi^i + \text{h.c.} \Big) \Big] .
\end{aligned}
\tag{5.5}
$$

Here again the fields B and χ should be redefined as in (3.59) to give their kinetic terms the standard form and to absorb the vacuum expectation value of B. Then one obtains the action

$$
\begin{aligned}
S = \int d^4x \Big[& - \partial^m M^a \, \partial_m M^{a*} - i\lambda^a \sigma^m \partial_m \bar{\lambda}^a - i\chi'^a \sigma^m \partial_m \bar{\chi}'^a - \partial_m A^i \, \partial^m A^{i*} - i\psi^i \sigma^m \partial_m \bar{\psi}^i \\
& - \Big(m_{ab} \chi'^a \lambda^b + W_{ai} \lambda^a \psi^i + \tfrac{1}{2} W_{ab} \lambda^a \lambda^b + \tfrac{1}{2} W_{ij} \psi^i \psi^j + \text{h.c.} \Big) \\
& - \partial^m B'^a \partial_m B'^a + H^a H^a - \tfrac{1}{6} m^2_{ab} C^a_{npq} C^{bnpq} - 2\text{Im}(W_a) H^a - \mathcal{V} \ \Big]
\end{aligned}
\tag{5.6}
$$

with a scalar potential

$$
\mathcal{V} \ = \ m^2_{ab} M^a M^{b*} + W_i W_i^* + \left(m_{ab} B'^b - \text{Re}W_a \right)^2 \ \geq \ 0. \tag{5.7}
$$

The form of this potential implies that supersymmetry is unbroken if and only if there is a field configuration for which the equations

$$
m_{ab} B'^b - \text{Re}W_a = 0, \qquad W_i = 0, \qquad M^a = 0 \tag{5.8}
$$

are fulfilled, because then and only then is $\langle \mathcal{V} \rangle = 0$. (This can also be seen directly from the supersymmetry variations of the chiral multiplet (2.21) and the 3-form multiplet (3.10) using the equations of motion for F^i and D^a.) Since m_{ab} is invertible, the first equation in (5.8) has always a solution that fixes only the B^a. For renormalizable interactions the function $W^{\text{int.}}$ in (5.2) is of the form

$$W^{\text{int.}}(S,\Phi) = \mu_{a,i} S^a \Phi^i + \rho_{a,ij} S^a \Phi^i \Phi^j + \gamma_{ab,i} S^a S^b \Phi^i, \tag{5.9}$$

where $\rho_{a,ij} = \rho_{a,ji}$ and $\gamma_{ab,i} = \gamma_{ba,i}$, so that the second and third equation in (5.8) can be summarized to

$$0 = W_i|_{M=0} = \left(W_i^{int.}(M,A) + W_i^{\Phi}(A)\right)|_{M=0} = W_i^{\Phi}(A). \tag{5.10}$$

From this it is seen that in the class of models considered here supersymmetry is broken for exactly the same superpotentials W^{Φ} as in the well known chiral theories, the O'Raifeartaigh models, because in those theories the scalar potential is simply given by $\mathcal{V} = W_i W_i^*$ when $W(\Phi)$ is the superpotential [41]. It is particularly interesting that the superpotential W^S for the 3-form field strength multiplets cannot break supersymmetry due to the presence of the non-singular mass matrix m_{ab}, which also prevented the Fayet-Iliopoulos term from breaking supersymmetry.

Let us now analyze the mass spectrum for the case of vanishing superpotentials $W^S(S) = 0 = W^{\Phi}(\Phi)$. By (5.9) one then has

$$\begin{aligned} W_i &= \mu_{a,i} M^a + 2\rho_{a,ij} M^a A^j + \gamma_{ab,i} M^a M^b, \\ W_a &= \mu_{a,i} A^i + \rho_{a,ij} A^i A^j + 2\gamma_{ab,i} M^b A^i. \end{aligned} \tag{5.11}$$

Thus the potential (5.7) vanishes (i.e., is minimized) for $M^a = B'^a = A^i = 0$ so that supersymmetry is unbroken. Since all fields have vanishing vacuum expectation values, contributions to the mass matrices only come from terms that are quadratic in the fields. For the scalars M^a these can be found in

$$m_{ab}^2 M^a M^{b*} + W_i W_i^* = \left(m_{ab}^2 + \mu_{a,i}^* \mu_{b,i}\right) M^{a*} M^b + \ \ldots, \tag{5.12}$$

where the dots denote terms that are at least cubic. Thus the M^a mix among themselves to form mass eigenstates and their mass matrix is given by

$$\hat{m}_{ab}^2 = m_{ab}^2 + \mu_{a,i}^* \mu_{b,i}. \tag{5.13}$$

Next collect the mass terms for the scalars A^i and B^a (from now on we drop the prime on B'^a and χ'^a) which are entirely contained in the term

$$\left(m_{ab} B^b - \text{Re}(W_a)\right)^2 \ = \ \left(m_{ab} B^b - \tfrac{1}{2}\mu_{a,i} A^i - \tfrac{1}{2}\mu_{a,i}^* A^{i*}\right)^2 + \ \ldots. \tag{5.14}$$

To analyze the corresponding mass matrix it is useful to define

$$(\tilde{A}^\alpha) := \left(B^a \ A^i \ A^{*\bar{j}}\right), \quad (\mu_{a,\alpha}) := (m_{ab}, \ -\tfrac{1}{2}\mu_{a,i}, \ -\tfrac{1}{2}\mu_{a,\bar{j}}^*), \quad \alpha = 1, \ldots, N + 2N_c, \tag{5.15}$$

so that the mass terms in (5.14) can be written as

$$\left| \sum_{\alpha=1}^{N+2N_c} \mu_{a,\alpha} \tilde{A}^\alpha \right|^2 \;=\; \mu^*_{a,\alpha}\mu_{a,\beta}\tilde{A}^{\alpha *}\tilde{A}^\beta. \tag{5.16}$$

Thus we have determined the quadratic mass matrix $m^{2B,A}_{\alpha\beta} = \mu^*_{a,\alpha}\mu_{a,\beta}$ for the fields $\tilde{A}^\alpha$. The special structure of this matrix, namely that it is the product of an $N \times (N+2N_c)$-matrix $(\mu_{a,\alpha})$ with its hermitian conjugate, allows for two significant statements about its eigenvalues. First, there are (at least) $2N_c$ massless states, which correspond to the $2N_c$ linearly independent vectors $\vec{v}^r \in \mathbb{C}^{N+2N_c}$, $r = 1, \dots, 2N_c$ that satisfy the N linear equations

$$\mu_{a,\alpha} v^r_\alpha = 0, \qquad a = 1, \dots, N. \tag{5.17}$$

Second, the remaining N eigenvalues of $m^{2B,A}$ coincide with the eigenvalues q_a of the hermitian matrix

$$Q_{ab} \;:=\; \mu^*_{a,\beta}\mu_{b,\beta} \;=\; m^2_{ab} + \tfrac{1}{2}\mathrm{Re}\left(\mu_{a,i}\mu^*_{b,i}\right). \tag{5.18}$$

To see this, denote by U_{ab} the elements of the unitary matrix that diagonalizes Q,

$$U_{ac}Q_{cd}U^*_{bd} \;=\; q_a\delta_{ab}, \qquad U^*_{ca}U_{cb} = \delta_{ab}. \tag{5.19}$$

Then the $N + 2N_c$ component vector

$$\vec{u}_a = \left(U_{ab}\mu^*_{b,\alpha}\right)_{\alpha=1,\dots,N+2N_c}, \tag{5.20}$$

is eigenvector of $m^{2B,A}$ with eigenvalue q_a:

$$m^{2B,A}_{\alpha\beta}(\vec{u}_a)_\beta = \mu^*_{c,\alpha}\mu_{c,\beta}U_{ab}\mu^*_{b,\beta} = \mu^*_{c,\alpha}U_{ab}Q_{bc} = (\vec{u}_d)_\alpha U^*_{dc}U_{ab}Q_{bc} = q_a(\vec{u}_a)_\alpha. \tag{5.21}$$

As an example consider the simplest case $N = N_c = 1$ where the mass matrix $m^{2B,A}$ reads

$$m^{2B,A} = (\mu^*_\alpha\mu_\beta) = \begin{pmatrix} m \\ -\frac{1}{2}\mu^* \\ -\frac{1}{2}\mu \end{pmatrix} \begin{pmatrix} m & -\frac{1}{2}\mu & -\frac{1}{2}\mu^* \end{pmatrix}. \tag{5.22}$$

The two massless states correspond to the two linearly independent solutions $\vec{v}_1, \vec{v}_2$ of

$$0 = \mu_\beta v_{r\beta} = m v_{r1} - \tfrac{1}{2}\mu v_{r2} - \tfrac{1}{2}\mu^* v_{r3}, \quad r = 1, 2. \tag{5.23}$$

These should be chosen to be orthonormal, for example

$$\vec{v}_1 := \frac{-i}{\sqrt{2}|\mu|}\begin{pmatrix}0\\ \mu^*\\ -\mu\end{pmatrix}, \qquad \vec{v}_2 := \frac{1}{|\mu|\sqrt{\mu\mu^*+2m^2}}\begin{pmatrix}\mu\mu^*\\ m\mu^*\\ m\mu\end{pmatrix}, \tag{5.24}$$

where $\vec{v}_2$ was found by taking the cross product of $\vec{\mu} = (\mu_\alpha)$ with $\vec{v}_1^*$. The massless fields are then given by

$$A_1 := \frac{i}{\sqrt{2}|\mu|}\Big(\mu A - \mu^* A^*\Big), \qquad A_2 := \frac{1}{|\mu|\sqrt{\mu\mu^*+2m^2}}\Big(\mu\mu^* B + m\mu A + m\mu^* A^*\Big). \tag{5.25}$$

Note that these are two real fields, so that each carries only one degree of freedom. The third eigenvector of $m^2_{\alpha\beta}$ is (μ^*_α) with eigenvalue

$$\tilde{m}^2 = \mu_\beta \mu^*_\beta = m^2 + \tfrac{1}{2}\mu\mu^*. \tag{5.26}$$

The field that has this mass is given by

$$A_3 := \frac{1}{\sqrt{m^2 + \frac{1}{2}\mu\mu^*}}\Big(mB - \tfrac{1}{2}\mu A - \tfrac{1}{2}\mu^* A^*\Big). \tag{5.27}$$

The action (5.6) contains only one mass term for the 3-forms C^a_{npq} and their mass matrix is simply m^2_{ab}. Thus we have (in principal) determined the $4N + 2N_c$ mass eigenvalues of all the bosonic fields of the theory.

One might also be interested in the fermion masses, as they should be in some way related to the boson masses by supersymmetry. The mass m of a fermion Ψ can be defined from its second order equation of motion,

$$\Box\Psi \;=\; m^2\Psi + \;\ldots, \tag{5.28}$$

where the dots denote terms that are at least quadratic in the fields. Let us consider first the case of one chiral and one 3-form multiplet and then generalize to arbitrary N and N_c. The part of the Lagrangian that defines the fermion masses (i.e., that is purely fermionic and quadratic in fields) is then given by

$$\mathcal{L}^{\text{ferm.}}_{\text{mass}} = -i\lambda\sigma^m\partial_m\bar{\lambda} - i\chi\sigma^m\partial_m\bar{\chi} - i\psi\sigma^m\partial_m\bar{\psi} - m(\chi\lambda + \bar{\chi}\bar{\lambda}) - \mu\lambda\psi - \mu^*\bar{\lambda}\bar{\psi}. \tag{5.29}$$

Thus the equations of motion of the fermions λ^a, χ^a and ψ^i can be written in matrix notation as

$$-i\sigma^m\partial_m\begin{pmatrix}\bar{\lambda}\\ \bar{\chi}\\ \bar{\psi}\end{pmatrix} = \begin{pmatrix}0 & m & \mu\\ m & 0 & 0\\ \mu & 0 & 0\end{pmatrix}\begin{pmatrix}\lambda\\ \chi\\ \psi\end{pmatrix} + \;\ldots\;. \tag{5.30}$$

With the last equation and its complex conjugate the action of the d'Alambert operator on the fermionic fields is found to be

$$\begin{aligned}\Box\begin{pmatrix}\lambda\\ \chi\\ \psi\end{pmatrix} &= -\sigma^m\bar{\sigma}^n\partial_m\partial_n\begin{pmatrix}\lambda\\ \chi\\ \psi\end{pmatrix} = \begin{pmatrix}0 & m & \mu^*\\ m & 0 & 0\\ \mu^* & 0 & 0\end{pmatrix}\begin{pmatrix}0 & m & \mu\\ m & 0 & 0\\ \mu & 0 & 0\end{pmatrix}\begin{pmatrix}\lambda\\ \chi\\ \psi\end{pmatrix} + \ldots\\ &= \begin{pmatrix}m^2+\mu^*\mu & 0 & 0\\ 0 & m^2 & m\mu\\ 0 & \mu^* m & \mu^*\mu\end{pmatrix}\begin{pmatrix}\lambda\\ \chi\\ \psi\end{pmatrix} + \ldots .\end{aligned} \tag{5.31}$$

With the last expression we have found the quadratic mass matrix for λ, χ and ψ. The spinor λ is a mass eigenstate with mass $\hat{m}^2 = m^2 + \mu\mu^*$. This coincides with the mass of the field M which is the superpartner of λ. The quadratic mass matrix for χ and ψ can be written as

$$m^{2\chi,\psi} = \begin{pmatrix}m^2 & m\mu\\ \mu^* m & \mu^*\mu\end{pmatrix} = \begin{pmatrix}m\\ \mu^*\end{pmatrix}(m\ \mu). \tag{5.32}$$

It has a similar structure as $m^{2B,A}$ in the bosonic sector, cf. (5.22). But note that $m^{2\chi,\psi}$ is a mass matrix for two complex Weyl spinors, i.e. four physical states while $m^{2B,A}$ describes the masses of three real scalars, i.e. three physical states and the fourth bosonic state is that of the 3-form C_{npq}. The fermion mass matrix $m^{2\chi,\psi}$ has two eigenvectors

$$\vec{w}_1 = \frac{1}{\sqrt{m^2+\mu\mu^*}}\begin{pmatrix}\mu\\ -m\end{pmatrix}, \qquad \vec{w}_2 = \frac{1}{\sqrt{m^2+\mu\mu^*}}\begin{pmatrix}m\\ \mu^*\end{pmatrix}. \tag{5.33}$$

with eigenvalues 0 and $\hat{m}^2 = m^2 + \mu\mu^*$ respectively. They correspond to the linear combinations

$$\eta_1 = \frac{1}{\hat{m}}(\mu^*\chi - m\psi), \qquad \eta_2 = \frac{1}{\hat{m}}(m\chi + \mu\psi). \tag{5.34}$$

The results for the $1+1$ dimensional case can be summarized as follows:

boson	mass^2	fermion	mass^2
M	$\hat{m}^2 = m^2 + \mu\mu^*$	λ	$\hat{m}^2$
A_1, A_2	0	η_1	0
A_3	$\tilde{m}^2 = m^2 + \frac{1}{2}\mu\mu^*$	η_2	$\hat{m}^2$
C_{npq}	m^2		

One might naively expect that the bosonic and fermionic mass spectra coincide in a theory with unbroken supersymmetry. The argument would be as follows: For any field ϕ of the theory with mass m the equation of motion

$$0 = (-\Box + m^2)\phi + \ \ldots, \tag{5.35}$$

where the dots again denote terms at least quadratic in the fields, must be supersymmetry invariant. So if ψ is the superpartner of ϕ, $\delta_\xi \phi \sim \xi\psi$, the supersymmetry variation of (5.35) shows that ψ also must have mass m because the supersymmetry variation of the higher order terms is again of higher order if no field transforms inhomogeneously (i.e., supersymmetry is unbroken). So why do the fermionic and bosonic masses differ in the theory at hand? The answer is that the dots in (5.35) could also include a term linear in fields but with derivatives. Namely, the action (5.6) contains a term which is quadratic in fields but cannot be treated as a mass term:

$$-2(\mathrm{Im}W_a)H^a \;=\; i(\mu_{a,i}A^i - \mu^*_{a,i}A^{i*})H^a + \;\dots\; . \tag{5.36}$$

This term appears in the equations of motion for A and A^*:

$$\begin{aligned} \Box A &= -\mu^*(mB - \tfrac{1}{2}\mu A - \tfrac{1}{2}\mu^* A^*) + i\mu^* H + \;\dots, \\ \Box A^* &= -\mu(mB - \tfrac{1}{2}\mu A - \tfrac{1}{2}\mu^* A^*) - i\mu H + \;\dots\; . \end{aligned} \tag{5.37}$$

Now take for example the massless field A_1 whose supersymmetry variation is, according to (5.25) and with $\delta_\xi A = \sqrt{2}\xi\psi$ given by

$$\delta_\xi A_1 = -\frac{2}{|\mu|}\mathrm{Im}\big(\mu\xi\psi\big), \tag{5.38}$$

although ψ is not even a mass eigenstate. But from (5.37) one finds

$$\Box A_1 = -\sqrt{2}|\mu|H + \;\dots\; . \tag{5.39}$$

This explains why A_1 does not have a massless superpartner: The supersymmetry variation of the left hand side of (5.39) is according to (5.38) and (5.31)

$$\Box(\delta_\xi A_1) = -\frac{2}{|\mu|}\mathrm{Im}\big(\mu\xi\Box\psi\big) = -2|\mu|\mathrm{Im}\big(\xi(m\chi + \mu\psi)\big), \tag{5.40}$$

while acting with δ_ξ on the right hand side of (5.39) yields

$$-\sqrt{2}|\mu|\delta_\xi H = -2|\mu|\mathrm{Im}\big(-i\xi\sigma^m\partial_m\bar{\lambda}\big) = -2|\mu|\mathrm{Im}\big(\xi(m\chi + \mu\psi)\big), \tag{5.41}$$

where $\delta_\xi H$ is given in (3.10) and in the last step the equation of motion for λ as given in (5.30) was used. Thus we have checked that the equation of motion (5.39) is indeed supersymmetry invariant (to first order in fields, but of course it is invariant to all orders) when all fields are on-shell.

It is not difficult to generalize this analysis to the case of N_c chiral and N 3-form multiplets. The second order equation of motion for the spinors is then given by

$$\Box \begin{pmatrix} \lambda^a \\ \chi^a \\ \psi^i \end{pmatrix} = \begin{pmatrix} m^2_{ab} + \mu^*_{a,i}\mu_{b,i} & 0 & 0 \\ 0 & m^2_{ab} & m_{ac}\mu_{c,j} \\ 0 & \mu^*_{c,i}m_{cb} & \mu^*_{c,i}\mu_{c,j} \end{pmatrix} \begin{pmatrix} \lambda^b \\ \chi^b \\ \psi^j \end{pmatrix} + \ldots . \tag{5.42}$$

Thus the spinors λ^a mix among themselves to form mass eigenstates and their mass matrix coincides with that of the M^a. This is what we expected as these fields are superpartners, being contained in the same chiral supermultiplet S^a. Again, the quadratic mass matrix for the χ^a and ψ^i can be analyzed in exactly the same way as $m^2_{\alpha\beta}$ in the bosonic sector, since it is also of the form

$$m^{2\chi,\psi}_{\varrho\sigma} = \hat{\mu}^*_{a,\varrho}\hat{\mu}_{a,\sigma} \quad \text{where} \quad (\hat{\mu}_{a,\varrho}) := (m_{ab},\ \mu_{a,i}), \quad \rho, \sigma = 1, \ldots, N + N_c. \tag{5.43}$$

From this it is clear that there are N_c zero eigenvalues (which correspond to $2N_c$ massless states) and the remaining N eigenvalues coincide with those of the matrix

$$\hat{Q}_{ab} := \hat{\mu}^*_{a,\varrho}\hat{\mu}_{b,\varrho} \ = \ m^2_{ab} + \mu^*_{a,i}\mu_{b,i}. \tag{5.44}$$

(Interestingly, this is identical with the mass matrix $\hat{m}^2_{ab}$ of the M^a and λ^a.) Notice the slight difference to (5.18) and remember that $\hat{Q}$ gives the mass spectrum for $2N$ fermionic states while Q defines the masses of only N bosonic states. From the supersymmetry transformation of the 3-form multiplet (3.10) one sees that the spinor χ has as superpartners B as well as C_{npq}. Therefore supersymmetry relates the mass eigenstates formed by the χ^a and ψ^i to those formed by the B^a, A^i and C^a_{npq} in a rather complex way. As we have seen for the $1+1$ dimensional case, one cannot even say that the $2N_c$ massless bosonic and fermionic states are superpartners. This is due to the presence of the term linear in H^a in the action (5.6) which is also responsible for the difference between the bosonic and fermionic mass spectrum.

By a superpotential W^Φ, the fields A^i can get non-vanishing vevs, and in the case of spontaneously broken supersymmetry the same applies to the M^a. In general, the mass spectrum of the theory gets much more complicated when superpotentials with terms linear in S or Φ are included in the action, since then W_a or W_i respectively contain constants that give rise to additional mass terms (including terms that mix the M^a with the A^i) and even more mass terms arise when the scalars acquire non-vanishing vevs.

5.2 Non-renormalizable coupling

In the non-renormalizable case we allow for arbitrary couplings $G(U, \Phi, \bar{\Phi})$ between the 3-form and chiral superfields, as well as non-renormalizable kinetic terms that mix the fields of the chiral with those of the 3-form multiplets. Thus we start with the expression

$$S = \int d^4x \left[\int d^2\theta \, d^2\bar{\theta} \left(K(S, \bar{S}, \Phi, \bar{\Phi}) - G(U, \Phi, \bar{\Phi}) \right) + \int d^2\theta \, W(S, \Phi) + \int d^2\bar{\theta} \, W^*(\bar{S}, \bar{\Phi}) \right]. \tag{5.45}$$

We again consider only the bosonic part of the action. Since both the S^a and the Φ^i are chiral superfields the θ-integration of $K(S, \bar{S}, \Phi, \bar{\Phi})$ can be performed in the same manner as that of $K(S, \bar{S})$ in (4.5), leading to

$$\begin{aligned}\int d^8z \, K = \int d^4x \Big[& K_{a\bar{b}} \Big(- \partial_m M_a \, \partial^m M^{*\bar{b}} + D^a D^{\bar{b}} + H^a H^{\bar{b}} \Big) + 2(\mathrm{Im} K_{a\bar{b}}) H^a D^b \\ & + K_{i\bar{j}} \Big(- \partial_m A^i \partial^m A^{*\bar{j}} + F^i F^{*\bar{j}} \Big) + K_{\bar{a}i} \Big(- \partial_m M^{*\bar{a}} \partial^m A^i + (D^{\bar{a}} - iH^{\bar{a}}) F^i \Big) \\ & + K_{a\bar{j}} \Big(- \partial_m M^a \partial^m A^{*\bar{j}} + (D^a + iH^a) F^{*\bar{j}} \Big) \Big].\end{aligned} \tag{5.46}$$

Using the component expansions (4.7) and (2.19) (with fermionic components set to zero) one finds for the G-part of the action

$$\begin{aligned}& \int d^4x \Big[- G_a \Big(D^a - \tfrac{1}{4} \Box B^a \Big) - \tfrac{1}{4} G_i \Box A^i - \tfrac{1}{4} G_{\bar{j}} \Box A^{*\bar{j}} - \tfrac{1}{2} G_{ab} \Big(2 M^a M^{b*} + \tfrac{1}{3} C^{anpq} C^b_{\ npq} \Big) \\ & \qquad - \tfrac{1}{4} G_{ij} \partial_m A^i \partial^m A^j - \tfrac{1}{4} G_{\bar{i}\bar{j}} \partial_m A^{*\bar{i}} \partial^m A^{*\bar{j}} - G_{i\bar{j}} \Big(F^i F^{*\bar{j}} - \tfrac{1}{2} \partial_m A^i \, \partial^m A^{*\bar{j}} \Big) \\ & \qquad - G_{ai} \Big(M^a F^i - \tfrac{1}{6} i \varepsilon^{mnpq} C^a_{\ npq} \partial_m A^i \Big) - G_{a\bar{j}} \Big(M^a F^{*\bar{j}} - \tfrac{1}{6} i \varepsilon^{mnpq} C^a_{\ npq} \partial_m A^{*\bar{j}} \Big) \Big] \\ = & \int d^4x \Big[- G_{i\bar{j}} \Big(- \partial_m A^i \, \partial^m A^{*\bar{j}} + F^i F^{*\bar{j}} \Big) - G_{ab} \Big(\tfrac{1}{4} \partial_m B^a \partial^m B^b + M^a M^{b*} + \tfrac{1}{6} C^a_{\ npq} C^{bnpq} \Big) \\ & \qquad - G_a D^a - G_{ai} M^a F^i - G_{a\bar{j}} M^{a*} F^{*\bar{j}} + \tfrac{1}{6} i \Big(G_{ai} \partial^m A^i - G_{a\bar{j}} \partial^m A^{*\bar{j}} \Big) \varepsilon_{mnpq} C^{anpq} \Big],\end{aligned} \tag{5.47}$$

where the terms with d'Alambert operators were rewritten using integration by parts and the chain rule

$$\partial_m G_a = G_{ab} \partial_m B^b + G_{ai} \partial_m A^i + G_{a\bar{j}} \partial_m A^{*\bar{j}} \tag{5.48}$$

(and similar for $\partial_m G_i$). Note that the 'mixed kinetic' terms proportional to $G_{ai} \partial_m A^i \partial^m B^a$ (and complex conjugate), that arise with this step, cancel due to the different signs of the $\Box$-terms in U and Φ. Let us take a moment to argue why such a term cannot appear in a supersymmetric action. To this end, consider the simplest case $N = N_c = 1$ and the terms

$$\mu \, \partial_m A \, \partial^m B + \mu^* \, \partial_m A^* \partial^m B, \tag{5.49}$$

where μ is a constant of mass dimension 1 which for simplicity we take to be real. The supersymmetry variation of (5.49) is, according to (3.10) and (2.21), given by

$$\delta_\xi \big(\mu \, \partial_m (A + A^*) \partial^m B \big) = \sqrt{2} \mu \, \partial_m (\xi \psi + \bar{\xi} \bar{\psi}) \partial^m B + i \mu \, \partial_m (A + A^*) \partial^m (\xi \chi - \bar{\xi} \bar{\chi}). \tag{5.50}$$

The only term whose supersymmetry variation could cancel this is a mixed kinetic term for the superpartners ψ and χ of A and B,

$$\sqrt{2}\mu\,\psi\sigma^m\partial_m\bar{\chi} \;+\; \text{h.c.}\;. \tag{5.51}$$

For simplicity, suppose that χ and ψ transform as

$$\delta_\xi\chi = \sigma^m\bar{\xi}\,\partial_m B, \quad \delta_\xi\psi = \sqrt{2}i\sigma^m\bar{\xi}\,\partial_m A, \tag{5.52}$$

neglecting the other terms given in (3.10) and (2.21) (this is legitimate as one wants to collect all terms whose supersymmetry variation could cancel (5.50)). Then the supersymmetry variation of (5.51) would be

$$\begin{aligned}\sqrt{2}\mu\,\delta_\xi\big(\psi\sigma^m\partial_m\bar{\chi}\big) + \text{h.c.} &= -2i\mu\,(\partial_n A)\bar{\xi}\bar{\sigma}^n\sigma^m\partial_m\bar{\chi} - \sqrt{2}\mu\,\psi\sigma^m\bar{\sigma}^n\xi(\partial_m\partial_n B) + \text{h.c.}\\ &= 2i\mu\,(\partial_n A)\bar{\xi}\partial^n\bar{\chi} - \sqrt{2}\mu\,\partial_m(\xi\psi)\partial^m B + \text{h.c.} + \partial_m(\dots),\end{aligned} \tag{5.53}$$

where in the second step we made use of $\bar{\sigma}^{(m}\sigma^{n)} = -\eta^{mn}$ and $\partial_m(\dots)$ denotes a total divergence as we applied integration by parts twice on the first term and once on the second term. Thus the sum of (5.50) and (5.53) is, modulo a total divergence, given by

$$i\mu\partial_m(A - A^*)\partial^m(\xi\chi + \bar{\xi}\bar{\chi}). \tag{5.54}$$

Roughly speaking, the supersymmetry variation of (5.49) cannot be canceled since there is no spinor whose supersymmetry variation would be proportional to $A+A^*$. The argument presented here, though far from being rigorous, is supported by the fact that when the fermionic components of the supermultiplets are included, the ψ, χ mixed kinetic terms $\sim G_{a\bar{j}}\chi^a\sigma^m\partial_m\bar{\psi}^{\bar{j}}$ also cancel, as one can easily check. Therefore it is appropriate to say that these interactions are forbidden by supersymmetry.

Assembling the different parts of the action (5.45) (the last two terms where already computed in Sec. 5.1) we obtain the component form

$$\begin{aligned}S = \int\! d^4x\,\Big[&K_{a\bar{b}}\Big(-\partial_m M_a\,\partial^m M^{*\bar{b}} + D^a D^{\bar{b}} + H^a H^{\bar{b}}\Big) + 2\mathrm{Im}(K_{a\bar{b}})H^a D^b\\ &+ \Big(K_{\bar{a}i}\Big(-\partial_m M^{*\bar{a}}\partial^m A^i + (D^{\bar{a}} - iH^{\bar{a}})F^i\Big) + \text{h.c.}\Big) + P_{i\bar{j}}\Big(-\partial_m A^i\partial^m A^{*\bar{j}} + F^i F^{*\bar{j}}\Big)\\ &- G_{ab}\Big(\tfrac{1}{4}\partial_m B^a\partial^m B^b + M^a M^{b*} + \tfrac{1}{6}C^a_{npq}C^{bnpq}\Big) - G_a D^a\\ &- G_{ai}M^a F^i - G_{a\bar{j}}M^{a*}F^{*\bar{j}} + \tfrac{i}{6}\Big(G_{ai}\partial_m A^i - G_{a\bar{j}}\partial_m A^{*\bar{j}}\Big)\varepsilon^{mnpq}C^a_{npq}\\ &+ W_i F^i + W^*_{\bar{i}}F^{*\bar{i}} + W_a\big(D^a + iH^a\big) + W^*_{\bar{a}}\big(D^{\bar{a}} - iH^{\bar{a}}\big)\Big],\end{aligned} \tag{5.55}$$

were we defined $P_{i\bar{j}} := K_{i\bar{j}} - G_{i\bar{j}}$. The action still contains the auxiliary fields F^i and D^a. To eliminate them, we again follow the prescription given in Appendix B and first consider only the part of the Lagrangian containing the F^i. With the definitions

$$Z_i := -iK_{i\bar{a}}H^{\bar{a}} - G_{ai}M^a + W_i, \qquad J_i := D^{\bar{a}}K_{\bar{a}i} + Z_i, \tag{5.56}$$

this is[18]

$$\begin{aligned}\mathcal{L}_F &= F^i P_{i\bar{j}} F^{*\bar{j}} + J_i F^i + J^*_{\bar{j}} F^{*\bar{j}} \\ &= \big(F^i + J^*_{\bar{k}} P^{-1\bar{k}i}\big) P_{i\bar{j}} \big(F^{*\bar{j}} + P^{-1\bar{j}k} J_k\big) - J^*_{\bar{j}} P^{-1\bar{j}i} J_i.\end{aligned} \tag{5.57}$$

By completing the square for the F^i in this way, it can be immediately seen how the action changes by elimination of the F^i: only the second term in the second line of (5.57) survives. Then the part of the Lagrangian containing the fields D^a becomes (note that J_i also depends on the D^a)

$$\begin{aligned}\mathcal{L}_D &= D^a(K_{a\bar{b}} - K_{a\bar{j}}P^{-1\bar{j}i}K_{i\bar{b}})D^{\bar{b}} + Q_a D^a \\ &= (D^a + \tfrac{1}{2}Q_c R^{-1ca})R_{ab}(D^b + \tfrac{1}{2}R^{-1bd}Q_d) - \tfrac{1}{4}Q_a R^{-1ab} Q_b\end{aligned} \tag{5.58}$$

where

$$\begin{aligned}Q_a &:= -2\mathrm{Im}(K_{a\bar{b}})H^b - G_a + 2\mathrm{Re}\big(W_a - K_{a\bar{j}}P^{-1\bar{j}i}Z_i\big), \\ R_{ab} &:= \mathrm{Re}\big(K_{a\bar{b}} - K_{a\bar{j}}P^{-1\bar{j}i}K_{i\bar{b}}\big).\end{aligned} \tag{5.59}$$

Now the D^a can also be eliminated from the action: again the first term in the second line of (5.58) vanishes and one finds that all the terms in the Lagrangian containing the auxiliary fields F^i and D^a are replaced by

$$-Z^*_{\bar{j}} P^{-1\bar{j}i} Z_i - \tfrac{1}{4} Q_a R^{-1ab} Q_b. \tag{5.60}$$

This expression contains potential terms for the scalars M^a, B^a and A^i as well as terms involving the field strengths H^a. To separate them, let us define

$$\tilde{Z}_i := -G_{ai}M^a + W_i, \qquad \tilde{Q}_a := -G_a + 2\mathrm{Re}\big(W_a - K_{a\bar{j}}P^{-1\bar{j}i}\tilde{Z}_i\big). \tag{5.61}$$

Then (5.60) can be written as

$$-\big(iH^cK_{c\bar{j}} + \tilde{Z}^*_{\bar{j}}\big)P^{-1\bar{j}i}\big(-iK_{i\bar{c}}H^{\bar{c}} + \tilde{Z}_i\big) - \tfrac{1}{4}\big(2H^c\mathrm{Im}(K_{\bar{c}a}) + \tilde{Q}_a\big)R^{-1ab}\big(-2\mathrm{Im}(K_{b\bar{c}})H^c + \tilde{Q}_b\big) \tag{5.62}$$

[18]In a ghost free theory $P_{i\bar{j}}$ is invertible, as it is the coefficient matrix of the kinetic term for the scalars A^i.

and the on-shell action becomes

$$
\begin{aligned}
S = \int d^4x \Bigg[& - K_{a\bar{b}} \partial_m M^a \partial^m M^{*\bar{b}} - P_{i\bar{j}}\, \partial_m A^i \partial^m A^{*\bar{j}} - K_{a\bar{i}} \partial_m M^a \partial^m A^{*\bar{i}} - K_{\bar{a}i} \partial_m M^{*\bar{a}} \partial^m A^i \\
& - G_{ab} \Big(\tfrac{1}{4} \partial_m B^a \partial^m B^b + \tfrac{1}{6} C^a_{npq} C^{bnpq} \Big) + \tfrac{i}{6} \Big(G_{ai} \partial_m A^i - G_{a\bar{j}} \partial_m A^{*\bar{j}} \Big) \varepsilon^{mnpq} C^a_{npq} \\
& + \hat{g}_{ab} H^a H^b + \mathrm{Im} \left(\tilde{Q}_a R^{-1ab} K_{b\bar{c}} - 2 \tilde{Z}^*_{\bar{j}} P^{-1\bar{j}i} K_{i\bar{c}} - 2 W_c \right) H^c - \mathcal{V} \Bigg],
\end{aligned}
\tag{5.63}
$$

where the 3-form field strengths come with the metric

$$
\hat{g}_{ab} = \mathrm{Re}\big(K_{a\bar{b}} - K_{a\bar{j}} P^{-1\bar{j}i} K_{i\bar{b}} \big) + \mathrm{Im}(K_{a\bar{c}}) R^{-1cd} \mathrm{Im}(K_{d\bar{b}}) \tag{5.64}
$$

and the scalar potential is given by

$$
\mathcal{V} = G_{ab} M^a M^{b*} + \tilde{Z}^*_{\bar{j}} P^{-1\bar{j}i} \tilde{Z}_i + \tfrac{1}{4} \tilde{Q}_a R^{-1ab} \tilde{Q}_b. \tag{5.65}
$$

6 Conclusion

We have seen in this work how to construct $\mathcal{N} = 1$ supersymmetric, gauge invariant actions of the 3-form multiplet in four dimensions, and how to dualize these actions. First we briefly discussed the renormalizable case, and then proceeded to study generic sigma model actions and their appropriate dualization. We introduced supersymmetric boundary terms in order to eliminate the massless 3-forms from the action and find a consistent on-shell action. First we dualized the massless action with the help of a Legendre transformation of the Kähler potential as proposed in [17]. Elimination of auxiliary fields has been demonstrated for the dual action and the scalar geometry of the resulting on-shell action has been shown to be identical with that of the original action. Even more, the dualization of the massless sigma model action has been recognized as a simple field redefinition given by the duality relations between the physical fields of action and dual action. In the massive case with a generic potential for the 3-form superfields, the dual action has been constructed by a Legendre transformation of this potential. Thereby arises an additional scalar field, whose kinetic term also has the structure of a Kähler geometry. We also discussed the special case of Kähler potentials with a shift symmetry including a simple example.

Finally we coupled the massive 3-form multiplets to chiral ones, first in a renormalizable, then in a generic theory. In the renormalizable case we discussed the condition for spontaneous supersymmetry breaking and thereby showed that it is identical with that of a simple O'Raifeartaigh model. Furthermore we analyzed the mass spectrum for the case of vanishing superpotentials and showed that for each chiral multiplet there are two bosonic and two fermionic massless states. Here we saw that the bosonic and fermionic mass spectra, even though supersymmetry is unbroken, do not coincide due to a term in the action that is linear in the 3-form scalar field strengths. In the case of non-renormalizable couplings we briefly argued why supersymmetry forbids a particular interaction between a real and a complex scalar. Lastly we showed how to eliminate the auxiliary fields from the action to obtain the on-shell action and derive the scalar potential and the metric for the 3-form field strengths.

Future projects that follow this work may include the coupling of the 3-form multiplet to gravity, the 3-form in extended supersymmetry, as well as applications of the results to string theory.

Appendix

A Conventions and useful relations

The Minkowski metric is taken to be $\eta = \mathrm{diag}(-1,1,1,1)$ and we fix the totally antisymmetric tensor ε^{mnpq} by

$$\varepsilon^{0123} = 1, \quad \varepsilon_{0123} = \det g = -1 \text{ for } g = \eta. \tag{A.1}$$

Spinor indices are raised and lowered with the antisymmetric tensor $\varepsilon_{\alpha\beta}$ as follows

$$\begin{gathered} \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \qquad \psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \\ \varepsilon^{12} = \varepsilon_{21} = 1, \quad \varepsilon^{21} = \varepsilon_{12} = -1, \quad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma. \end{gathered} \tag{A.2}$$

Two spinors can form Lorentz invariant products by contraction of their indices:

$$\psi\chi := \psi^\alpha\chi_\alpha = \chi\psi, \qquad \bar\psi\bar\chi := \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\chi\bar\psi, \qquad \theta^2 := \theta\theta, \qquad \bar\theta^2 := \bar\theta\bar\theta \tag{A.3}$$

The Pauli matrices σ^m and $\bar\sigma^m$ are defined by

$$\sigma^m_{\alpha\dot\alpha} := (-\mathbf{1}, \sigma^1, \sigma^2, \sigma^3)_{\alpha\dot\alpha}, \quad \bar\sigma^{m\dot\alpha\alpha} := \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^m_{\beta\dot\beta}. \tag{A.4}$$

They satisfy

$$\begin{aligned} \bar\sigma^{(m}\sigma^{n)} &= -\eta^{mn} &&\Rightarrow\ \bar\sigma^m\sigma^n\partial_m\partial_n = -\Box, \\ \varepsilon^{\alpha\beta}\left(\sigma^m\bar\chi\right)_\beta &= -(\bar\chi\bar\sigma^m)^\alpha &&\Rightarrow\ \psi\sigma^m\bar\chi = -\bar\chi\bar\sigma^m\psi. \end{aligned} \tag{A.5}$$

When computing the component form of a supersymmetric action, one often has to rewrite various combinations of θs using the following relations

$$\begin{aligned} (\theta\psi)(\theta\chi) &= -\tfrac{1}{2}\theta^2(\psi\chi), \qquad &(\bar\theta\bar\psi)(\bar\theta\bar\chi) &= -\tfrac{1}{2}\bar\theta^2(\bar\psi\bar\chi), \\ \theta\sigma^m\bar\theta\cdot\theta\sigma^n\bar\theta &= -\tfrac{1}{2}\theta^2\bar\theta^2\eta^{mn}, \qquad &\theta\sigma^m\bar\sigma^n\theta &= -\theta^2\eta^{mn}. \end{aligned} \tag{A.6}$$

When working with θ-derivatives, one should keep in mind that

$$\begin{aligned} &\frac{\partial}{\partial\theta^\alpha} = -\varepsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta}, \qquad &&\frac{\partial}{\partial\bar\theta^{\dot\alpha}} = -\varepsilon_{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta_{\dot\beta}}, \qquad &&\frac{\partial}{\partial\theta^\alpha}\theta^2 = 2\theta_\alpha, \qquad &&\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta^2 = 2\bar\theta^{\dot\alpha}, \\ &\frac{\partial^2}{\partial\theta^2} := \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha}, \qquad &&\frac{\partial^2}{\partial\bar\theta^2} := \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}, \qquad &&\frac{\partial^2}{\partial\theta^2}\theta^2 = 4, \qquad &&\frac{\partial^2}{\partial\bar\theta^2}\bar\theta^2 = 4. \end{aligned} \tag{A.7}$$

The generators of supersymmetry were chosen to be represented by[19]

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m, \qquad \bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m, \tag{A.8}$$

while the covariant superspace derivatives were defined as

$$D_\alpha := \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m, \qquad \bar D_{\dot\alpha} := \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m. \tag{A.9}$$

They satisfy

$$\begin{aligned}
\{Q_\alpha, \bar Q_{\dot\beta}\} &= -2i\sigma^m_{\alpha\dot\beta}\partial_m, \qquad && \{D_\alpha, \bar D_{\dot\beta}\} = 2i\sigma^m_{\alpha\dot\beta}\partial_m,\\
\{D_\alpha, D_\beta\} &= 0 = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\}, && \Rightarrow\ D_\alpha D_\beta D_\gamma = 0 = \bar D_{\dot\alpha}\bar D_{\dot\beta}\bar D_{\dot\gamma},\\
D^2 = -\frac{\partial^2}{\partial\theta^2} - 2i\frac{\partial}{\partial\theta_\alpha}(\sigma^m\bar\theta)_\alpha\partial_m - \bar\theta^2\Box, & && \bar D^2 = -\frac{\partial^2}{\partial\bar\theta^2} - 2i(\theta\sigma^m)_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\partial_m - \theta^2\Box.
\end{aligned} \tag{A.10}$$

B Elimination of auxiliary fields

Most of the known supermultiplets contain besides the physical fields also auxiliary fields, i.e. fields that do not carry any on-shell degrees of freedom. These can be eliminated from the action by using their purely algebraic equations of motion. The equations of motion of the on-shell action are of course equivalent to those of the original action. Elimination of N auxiliary fields can become computationally involved in the case of complex fields. However, when they occur in a quadratic form, which is often the case, a generalization of the well known technique of "completing the square" simplifies this task a lot. Since we use this technique frequently throughout this work, we want to describe the general procedure here.

Suppose we have N real auxiliary fields D^a that occur in the Lagrangian as

$$\mathcal{L} = D^a M_{ab} D^b + J_a D^a + C, \tag{B.1}$$

where M_{ab}, J_a and C are arbitrary functions of all other fields contained in the action. In order for the Lagrangian to be real, J_a and C have to be real and M has to be a hermitian matrix, even though only its symmetric, i.e. real part contributes in (B.1), which we take to be invertible here.[20] To eliminate the D^a, we could simply insert their equations of motion

$$2(\mathrm{Re}M_{ab})D^b + J_a = 0 \qquad \Rightarrow \qquad D^b = -\tfrac{1}{2}(\mathrm{Re}M)^{-1ba}J_a \tag{B.2}$$

[19] Our conventions for the supersymmetry generators and covariant superspace derivatives differ slightly from those of [20]. For a discussion see the footnotes in Sec. 2.1.

[20] If $\mathrm{Re}M$ was not invertible, each of its zero eigenvalues would account for a constraint on the fields that couple to the D^a, $v^b J^b = 0$, where v is a corresponding zero eigenvector.

into the Lagrangian (B.1). However the same can be achieved in a more elegant way by first shifting the D^a,

$$\tilde{D}^a := D^a + \tfrac{1}{2}(\mathrm{Re}M)^{-1ac} J_c, \tag{B.3}$$

and then rewriting the Lagrangian as

$$\mathcal{L} = \tilde{D}^a(\mathrm{Re}M_{ab})\tilde{D}^b - \tfrac{1}{4} J_a (\mathrm{Re}M)^{-1ab} J_b + C. \tag{B.4}$$

Now we immediately see that the first term in (B.4) (the "square") vanishes by the equations of motion for the D^a (or $\tilde{D}^a$ respectively) and we can easily read off the final Lagrangian.

Now suppose that there are complex auxiliary fields F^a that occur in the Lagrangian as

$$\mathcal{L} = F^a K_{a\bar{b}} F^{*\bar{b}} + J_a F^a + J^*_{\bar{b}} F^{*\bar{b}} + C, \tag{B.5}$$

where K is an invertible hermitian matrix, J^a is a complex and C a real function of the other fields. Here the square is completed by shifting

$$\tilde{F}^a := F^a + J^*_{\bar{c}} K^{-1\bar{c}a}, \tag{B.6}$$

so that the Lagrangian becomes

$$\mathcal{L} = \tilde{F}^a K_{a\bar{b}} \tilde{F}^{*\bar{b}} - J^*_{\bar{a}} K^{-1\bar{a}b} J_b + C. \tag{B.7}$$

Again the "square" vanishes by the equations of motion for the F^a and $F^{*\bar{b}}$.

Note that (B.5) does not give the most general form of quadratic terms for N complex auxiliary fields; one could also have terms proportional to FF and F^*F^* multiplied by a symmetric matrix M and its complex conjugate,

$$\mathcal{L} = F^a M_{ab} F^b + F^{*\bar{a}} M^*_{\bar{a}\bar{b}} F^{*\bar{b}} + 2F^a K_{a\bar{b}} F^{*\bar{b}} + J_a F^a + J^*_{\bar{b}} F^{*\bar{b}} + C. \tag{B.8}$$

Now the task of completing the square is more complicated than for (B.5). However, the quadratic structure of the Lagrangian (B.8) becomes clearer when writing it in the form

$$\mathcal{L} = \begin{pmatrix} F^T & F^\dagger \end{pmatrix} \begin{pmatrix} M & K \\ K^* & M^* \end{pmatrix} \begin{pmatrix} F \\ F^* \end{pmatrix} + \begin{pmatrix} F^T & F^\dagger \end{pmatrix} \begin{pmatrix} J \\ J^* \end{pmatrix} + C. \tag{B.9}$$

Then, by making the ansatz

$$\mathcal{L} = \begin{pmatrix} (F+T)^T & (F+T)^\dagger \end{pmatrix} \begin{pmatrix} M & K \\ K^* & M^* \end{pmatrix} \begin{pmatrix} F+T \\ F^*+T^* \end{pmatrix} - \begin{pmatrix} T^T & T^\dagger \end{pmatrix} \begin{pmatrix} M & K \\ K^* & M^* \end{pmatrix} \begin{pmatrix} T \\ T^* \end{pmatrix} + C, \tag{B.10}$$

one finds that the T^a have to satisfy (note that the block matrix in (B.10) is symmetric)

$$2\begin{pmatrix} M & K \\ K^* & M^* \end{pmatrix}\begin{pmatrix} T \\ T^* \end{pmatrix} = \begin{pmatrix} J \\ J^* \end{pmatrix}. \tag{B.11}$$

Provided that K and the matrix

$$H := \begin{pmatrix} M & K \\ K^* & M^* \end{pmatrix} \tag{B.12}$$

are invertible, the inverse is of the form

$$H^{-1} = \begin{pmatrix} N & G \\ G^* & N^* \end{pmatrix}, \quad \text{where} \quad \begin{array}{l} G = \big(K^* - M^* K^{-1} M\big)^{-1}, \\ N = -\big(K^{-1} M G\big)^*. \end{array} \tag{B.13}$$

Then the on-shell Lagrangian becomes (note that G is hermitian)

$$\begin{aligned} \mathcal{L}_{\text{on-shell}} &= -\tfrac{1}{4}\big(J^T \;\, J^\dagger\big) H^{-1} \begin{pmatrix} J \\ J^* \end{pmatrix} + C \\ &= -\tfrac{1}{4}\big(J_a N^{ab} J_b + J^*_{\bar a} N^{*\bar a \bar b} J^*_{\bar b}\big) - \tfrac{1}{2} J_a G^{a\bar b} J^*_{\bar b} + C. \end{aligned} \tag{B.14}$$

C Legendre transformation

The Legendre transform of a function $K:\ \mathbb{R}^n \to \mathbb{R}$ is defined by [40]

$$\hat K(p) = \max_{x\in\mathbb{R}^n}\Big(p_i x^i - K(x)\Big). \tag{C.1}$$

The existence of the maximum for all p is equivalent to the invertibility of the relation

$$p_i = \frac{\partial K}{\partial x^i}(x) \tag{C.2}$$

to give a function $x(p)$. The latter is unambiguous as long as K is convex (or concave). Otherwise (C.2) does not necessarily describe a maximum (or minimum). Given a function $x(p)$ that satisfies (C.2), the Legendre transform can also be written as

$$\hat K(p) = p_i x^i(p) - K(x(p)). \tag{C.3}$$

Thus its derivative is given by

$$\frac{\partial \hat K}{\partial p_i}(p) \;=\; p_j \frac{\partial x^j}{\partial p_i} + x^i - \frac{\partial K}{\partial x^j}(x(p))\frac{\partial x^j}{\partial p_i} \;=\; x^i(p). \tag{C.4}$$

If we denote the variable of the double Legendre transform $\hat{\hat{K}}$ as $\tilde{x}$, the function $p(\tilde{x})$ is defined by

$$\frac{\partial \hat{K}}{\partial p_i}(p(\tilde{x})) = \tilde{x}^i. \tag{C.5}$$

Thus equation (C.4) shows that $x(p(\tilde{x})) = \tilde{x}$ (i.e. $\tilde{x}$ is really the original variable x) which implies that the Legendre transformation is its own inverse:

$$\begin{aligned} \hat{\hat{K}}(\tilde{x}) &= \tilde{x}^i p_i(\tilde{x}) - \hat{K}(p(\tilde{x})) \\ &= \tilde{x}^i p_i(\tilde{x}) - \Big(x^i(p(\tilde{x})) p_i(\tilde{x}) - K(x(p(\tilde{x}))) \Big) \ = \ K(\tilde{x}). \end{aligned} \tag{C.6}$$

From the relations (C.2) and (C.4) it follows that the second derivatives obey

$$\frac{\partial^2 K}{\partial x^i \partial x^j} = \frac{\partial p_i}{\partial x^j}, \qquad \frac{\partial^2 \hat{K}}{\partial p_j \partial p_k} = \frac{\partial x^j}{\partial p_k}, \tag{C.7}$$

which implies

$$\frac{\partial^2 K}{\partial x^i \partial x^j} \frac{\partial^2 \hat{K}}{\partial p_j \partial p_k} = \delta_i^k, \qquad \text{or} \qquad \mathrm{Hess}\, K = \big(\mathrm{Hess}\, \hat{K} \big)^{-1}. \tag{C.8}$$

Here the derivatives of $\hat{K}$ have to be evaluated at $p(x) = \partial K / \partial x$ when those of K are evaluated at x.

In the case of a Kähler potential $K(z, \bar{z})$ with Kähler metric

$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j} \tag{C.9}$$

one has

$$\mathrm{Hess}\, K = \begin{pmatrix} K_{ij} & K_{i\bar{j}} \\ K_{\bar{i}j} & K_{\bar{i}\bar{j}} \end{pmatrix}, \qquad \mathrm{Hess}\, \hat{K} = \begin{pmatrix} \hat{K}^{ij} & \hat{K}^{i\bar{j}} \\ \hat{K}^{\bar{i}j} & \hat{K}^{\bar{i}\bar{j}} \end{pmatrix}. \tag{C.10}$$

The inverse of a block matrix of the form of Hess $\hat{K}$ is given in equation (B.13). Thus we obtain the formula

$$K_{i\bar{j}} = \Big(\hat{K}^{\bar{j}i} - \hat{K}^{\bar{j}\bar{k}} \hat{K}^{-1}_{\bar{k}l} \hat{K}^{li} \Big)^{-1}. \tag{C.11}$$

D Translation of the massive dual sigma model action on the component level

In this appendix we translate the massive dual on-shell action (4.73) back into the 3-form action (4.13) by use of the relations between the physical fields f_a, n_a and M^a, B^a, C^a_{npq} that are contained in (4.31) and (4.62). As indicated at the end of section 4.2, we will also have to make use of the equations of motion for the 3-forms (4.18) to achieve this.

As the relation between the f_a and M^a is the same as in the massless case,

$$\hat{K}^a(f,f^*) = M^a, \qquad \hat{K}^{\bar{a}}(f,f^*) = M^{*\bar{a}}, \tag{D.1}$$

the kinetic terms for the f_a (the first term in (4.73)) and M^a (the first term in (4.13)) are again seen to be equal by these relations. For the other terms in (4.73) we have to express real and imaginary part of n_a in terms of the fields B^a and C^a_{npq} (or H^a resp.). For the real part of n_a we use the lowest component of (4.62),

$$2\mathrm{Re}(n_a) = G_a(B). \tag{D.2}$$

This is the Legendre relation between the coordinates of $G(B)$ and $\hat{G}(2\mathrm{Re}n)$ and thus implies $\hat{G}^{-1}_{ab} = G_{ab}$. Thus we can translate the last term in (4.73),

$$-\hat{K}^a \hat{G}^{-1}_{ab} \hat{K}^{\bar{b}} = -G_{ab} M^a M^{b*}, \tag{D.3}$$

and the term on the right hand side is indeed present in (4.13). For the imaginary part of n_a we can use again the θ^2 and $\bar{\theta}^2$-components of (4.31) like in (4.53) but now with unconstrained F_a (whose bosonic part is given in (4.65)):

$$\begin{aligned} \hat{K}^{ab} h_b + \hat{K}^{a\bar{b}} n^*_{\bar{b}} &= D^a + iH^a, \\ \hat{K}^{a\bar{b}} h^*_{\bar{b}} + \hat{K}^{ab} n_b &= 0. \end{aligned} \tag{D.4}$$

Inserting $h^*_{\bar{b}}$ from the second equation into the complex conjugate of the first one we find

$$D^{\bar{a}} - iH^{\bar{a}} = \big(\hat{K}^{\bar{a}b} - \hat{K}^{\bar{a}\bar{c}} \hat{K}^{-1}_{\bar{c}d} \hat{K}^{db}\big) n_b = D^{-1\bar{a}b} n_b \tag{D.5}$$

Using $D_{a\bar{b}} = K_{a\bar{b}}$ and taking the imaginary part of this equation we can solve for $\mathrm{Im}(n_a)$,

$$\mathrm{Im}(n_a) = -(\mathrm{Re}(K^{-1}))^{-1}_{ab} \big(\mathrm{Im}(K^{-1\bar{b}c})\mathrm{Re}(n_c) + H^b\big) = -g_{ab}\big(\tfrac{1}{2}\mathrm{Im}(K^{-1\bar{b}c}) G_c + H^b\big), \tag{D.6}$$

where in the second step we used (4.16) and (D.2). With these expressions for the real and imaginary part of n_a we readily translate the third term in (4.73),

$$\begin{aligned} -D^{-1\bar{b}a} n_a n^*_{\bar{b}} &= -\mathrm{Re}(D^{-1\bar{b}a})\big(\mathrm{Re}n_a \mathrm{Re}n_b + \mathrm{Im}n_a \mathrm{Im}n_b\big) + 2\mathrm{Im}(D^{-1\bar{b}a})\mathrm{Im}n_a \mathrm{Re}n_b \\ &= -\tfrac{1}{4} G_b \mathrm{Re}(K^{-1\bar{b}a}) G_a - \big(H^c - \tfrac{1}{2} G_d \mathrm{Im}(K^{-1\bar{d}c})\big) g_{cb} \big(\tfrac{1}{2}\mathrm{Im}(K^{-1\bar{b}a}) G_a + H^b\big) \\ &\quad - G_b \mathrm{Im}(K^{-1\bar{b}a}) g_{ac} \big(\tfrac{1}{2}\mathrm{Im}(K^{-1\bar{c}d}) G_d + H^c\big) \\ &= -\tfrac{1}{4} G_b \big(\mathrm{Re}(K^{-1\bar{b}a}) + \mathrm{Im}(K^{-1\bar{b}c}) g_{cd} \mathrm{Im}(K^{-1\bar{d}a})\big) G_a - g_{ab} H^a H^b, \end{aligned} \tag{D.7}$$

where in the second step we used again (4.16). Now a quick auxiliary calculation is required where one uses four different forms the metric g can take,

$$g = K(\mathrm{Re}K)^{-1}K^* = 2K - K(\mathrm{Re}K)^{-1}K = K^*(\mathrm{Re}K)^{-1}K = 2K^* - K^*(\mathrm{Re}K)^{-1}K^*, \quad \text{(D.8)}$$

to rewrite

$$\begin{aligned} \mathrm{Im}(K^{-1})\, g\, \mathrm{Im}(K^{-1}) &= -\tfrac{1}{4}(K^{-1} - K^{*-1})\, g\, (K^{-1} - K^{*-1}) \\ &= \tfrac{1}{4}(K^{-1}gK^{*-1} - K^{-1}gK^{-1} + K^{*-1}gK^{-1} - K^{*-1}gK^{*-1}) \\ &= (\mathrm{Re}K)^{-1} - \mathrm{Re}(K^{-1}). \end{aligned} \quad \text{(D.9)}$$

Thus the term (D.7) becomes

$$-D^{-1\bar{b}a} n_a n^*_{\bar{b}} = -\tfrac{1}{4} G_b (\mathrm{Re}K)^{-1ba} G_a - g_{ab} H^a H^b. \quad \text{(D.10)}$$

These terms are also present in (4.13), but the second term apparently has the wrong sign. But before we interpret the result let us proceed and translate also the second term in (4.73). Here it is obvious that we need another expression for $\mathrm{Im} n_a$ than (D.6) to get terms that look similar to those in (4.13). This can be found in the $\theta\sigma^m\bar{\theta}$-component of (4.62), which reads

$$-2\partial_m \mathrm{Im}(n_a) = \tfrac{1}{3} G_{ab} \varepsilon_{mnpq} C^{bnpq}. \quad \text{(D.11)}$$

One can easily check that this equation combined with (D.6) yields the equation of motions of the 3-forms that we already stated in (4.18). (Therefore we are already implicitly using the equations of motion of the 3-forms when using both equation (D.6) and (D.11).) With the help of (D.11) and (D.2) which implies

$$\partial_m \mathrm{Re}(n_a) = \tfrac{1}{2} G_{ab} \partial_m B^b \quad \text{(D.12)}$$

one finds

$$-\hat{G}^{ab} \partial^m n_a \partial_m n^*_b = -G_{ab}\big(\tfrac{1}{4}\partial_m B^a \partial^m B^b - \tfrac{1}{6} C^a_{\ npq} C^{bnpq}\big). \quad \text{(D.13)}$$

Putting everything together we have translated the massive dual action (4.73) as

$$\begin{aligned} \int d^4x \Big[&- K_{a\bar{b}}\, \partial^m M^a \partial_m M^{*\bar{b}} - G_{ab}\Big(\tfrac{1}{4}\partial^m B^a \partial_m B^b + M^a M^{b*} - \tfrac{1}{6} C^{anpq} C^b_{\ npq}\Big) \\ &- g_{ab} H^a H^b - \tfrac{1}{4} G_a (\mathrm{Re}K)^{-1ab} G_b \Big]. \end{aligned} \quad \text{(D.14)}$$

This is quite similar to the 3-form action (4.13), but not exactly the same: the terms $\sim C^2$ and H^2 have the wrong sign and the term linear in H^a is missing. However, the expression (D.14) is not an action for the 3-forms since in its construction the equations of motion of the 3-forms have been used. Nevertheless it may serve as a check account for the correctness of (4.13) and (4.73) to use these again to bring (4.13) to the form (D.14). It is then most elegant to include in (4.13) the boundary terms for the 3-forms which we ignored in the massive case. Before elimination of the D^a they are given by (4.26). After inserting the solution (4.12) for D^a and with the abbreviation

$$N_a := g_{ab}H^b - \tfrac{1}{2}(\mathrm{Im}K)_{ab}(\mathrm{Re}K)^{-1bc}G_c \tag{D.15}$$

the boundary terms become

$$\mathcal{B}_3 = -\tfrac{1}{3}\int d^4x\,\varepsilon^{mnpq}\partial_m\big(N_a \cdot C^a_{npq}\big) = -\tfrac{1}{3}\int d^4x\,\Big(\varepsilon^{mnpq}\partial_m N_a \cdot C^a_{npq} + 6N_aH^a\Big). \tag{D.16}$$

Now the equations of motion for the 3-forms (4.18) can be written as

$$\varepsilon^{mnpq}\partial_m N_a = -G_{ab}C^{bnpq}. \tag{D.17}$$

Inserting them into (D.16) one finds

$$\mathcal{B}_3 = \int d^4x\,\Big(\tfrac{1}{3}G_{ab}C^a_{npq}C^{bnpq} - 2g_{ab}H^aH^b - G_c(\mathrm{Re}K)^{-1cb}(\mathrm{Im}K)_{ba}H^a\Big). \tag{D.18}$$

When these terms are added to (4.13), one gets exactly the expression (D.14)! Thus we have translated the massive sigma model action and dual action into each other on the component level. However we had to use the equations of motion of the 3-forms for this, which means that the transition from action to dual action can not be performed by a field redefinition as in the massless case. This is already clear from the fact that action and dual action do not contain an equal number of off-shell degrees of freedom as the massive 3-form possesses four off-shell degrees of freedom while the scalars n_a (which are dual to C^a_{npq} and B^a) contain only two off-shell degrees of freedom each. Nevertheless the equations of motion of action and dual action are of course equivalent with respect to the duality relations stated here.

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Eidesstattliche Erklärung

Hiermit bestätige ich, dass die vorliegende Arbeit von mir selbstständig verfasst wurde und ich keine anderen als die angegebenen Hilfsmittel — insbesondere keine im Quellenverzeichnis nicht benannten Internetquellen — benutzt habe und die Arbeit von mir vorher nicht in einem anderen Prüfungsverfahren eingereicht wurde. Die eingereichte Fassung entspricht der auf dem elektronischen Speichermedium.

Ich bin damit einverstanden, dass die Masterarbeit veröffentlicht wird.

Hamburg, ______________ ____________________________

Jason Sommerfeld