Kähler potentials of Type IIB orientifold projections

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Chapter 1

Introduction

The standard model of particle physics is a successful theory for the physics of the elementary particles. Nevertheless it has some severe drawbacks. A promising theory for eliminating these drawbacks is superstring theory. In the standard model, the elementary particles are described by point-like objects. In string theory, one chooses instead one dimensional objects - the strings. Consistency of the theory demands supersymmetry and more that 4 space-time dimensions. Actually, there are more than one string theory, namely the type I, the type IIA/B, the $E_8 \times E_8$ and the $SO(32)$ [4]. The several string theories differ, for example, by the number of supersymmetries. The type IIA/B theories have two generators of supersymmetry, whereas the remaining theories have one generator of supersymmetry.

Phenomenology is nowadays predominantly done with one supersymmetry, corresponding to $N = 1$, as it would be the simplest extension of the standard model of particle physics. When considering one of the type IIA/B theories, one has, in addition to perform the dimensional reduction, to get rid of the extra supersymmetry. The extra space dimensions are compactified on a Calabi-Yau manifold using the Kaluza-Klein mechanism. These manifolds are subject to two kinds of deformations. Deformations in the shape are described by the complex structure moduli, deformatins of the size by the Kähler moduli. These moduli have no affect on the topology of the Calabi-Yau manifold [4].

In order to reduce the supersymmetry, one uses the orientifold projection [1, 2, 3]. An orientifold projection is obtained by dividing a product of an orientation reversal and of a discrete symmetry group out of a Calabi-Yau manifold [2]. Actually, the type I theory is obtained from the type IIB theory by performing the orientifold projection [4]. Another evidence for the necessity of orientifolds stems from the fact that type II compactifications with Dp-branes, which are introduced to attain a supersymmetric and non-abelian gauge theory, are often inconsistent or at least unstable [5]. Both drawbacks, inconsistency and instability, can be eliminated with the help of orientifold projections.

The purpose of this work is to determine the Kähler potential of the hypermultiplets, appearing in type IIB orientifold projections. This Kähler potential describes the Kähler moduli of the Calabi-Yau manifold, which is used to compactify the extra space dimensions. In contrast to the earlier work performed on this topic [2], we use an alternative approach introduced in [6], and performed for the $\mathcal{O}3/\mathcal{O}7$ orientifold projection in [3]. We perform the calculations following [7, 8, 9, 3] by rederiving the general coupling function for the $N = 1$ tensor multiplets in type IIB string theory, using the contour integral approach of [6] involving projective tensor multiplets. Afterwards we perform the orientifold projection along the lines of [3] on the projective tensor multiplets, and rederive the Kähler potential for the $\mathcal{O}3/\mathcal{O}7$ orientifold projection. Then, we explicitly determine the Kähler potential for the $\mathcal{O}5/\mathcal{O}9$ orientifold projection, as this was not done in [3], and compare both Kähler potentials with those, derived in [2]. For this purpose, we also determine the redefinition equations for the involved hypermultiplet scalars. We find, after the redefining the hypermultiplet scalars, agreement with the Kähler potentials derived in [2] up to an overall factor of 2. The origin of this overall factor is presumably the different definitions of the Kähler metric used in the literature.

This work is organized as follows:

- In chapter two, we introduce the formalism of $N = 1$ supersymmetry and try to familiarize the reader with the basic topics, such as superfield formalism, the different multiplets and their actions.

- Chapter three is dedicated to the $N = 2$ formulation of supersymmetry. Again, we explain the $N = 2$ superfield formalism (though it is not used extensively), and present the multiplets with the corresponding actions. We also derive the constraints on the general coupling function F which appears in self-interacting tensor multiplet formulations. Afterwards, we determine the bosonic action of such a self-interacting model and perform the Legendre transformation on the scalar level.

- In Chapter four, we shortly review the type IIB theory and its action compactified on a Calabi-Yau threefold. We also present the emerging massless, bosonic spectrum.

- Finally, in chapter five, we determine the general coupling function F for type IIB tensor multiplets using the contour integral approach. Afterwards we derive the hyperkähler potential of the target space geometry by performing the Legendre transformation. Subsequently, we perform the orientifold projection using $\mathcal{O}3/\mathcal{O}7$ respectively $\mathcal{O}5/\mathcal{O}9$ planes and determine the corresponding Kähler potentials.

- In appendix A we present our notation and some identities, which are very useful when performing calculations with superfields.

- Appendix B is dedicated to the variation rules of superfields, which play a role when performing duality transformations.

- Furthermore, we present in appendix C some basic facts of nonlinear σ-models and Kähler geometry as well as an outline of hyperkähler geometry.

- And finally, in appendix D we introduce the notion of projective superfields, which we use in chapter 4 to perform the calculations.

Chapter 2

Rigid $N = 1$ supersymmetry

2.1 $N = 1$ supersymmetry algebra

This chapter is predominantly a summary of [10]. The notation and summation convention are the same as in [10] and some important facts are listed in appendix A.

The most simple supersymmetry algebra contains one set of fermionic generators of supersymmetry $Q_\alpha, \bar{Q}_{\dot{\alpha}}$. The algebra reads [10]:

$$\begin{aligned} &\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m, \\ &\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \\ &[P_m, Q_\alpha] = [P_m, \bar{Q}_{\dot{\alpha}}] = 0, \\ &[P_m, P_n] = 0, \end{aligned} \tag{2.1}$$

with $\alpha, \dot{\alpha} = 1, 2$ and $m = 0, \ldots, 3$. P_m denotes the energy-momentum four-vector. Fermionic objects carry spinor indices $\alpha, \dot{\alpha}$, bosonic objects either no index or Lorentz indices m.

2.2 $N = 1$ superfield formalism

After introducing anticommuting parameters $\xi^\alpha, \bar{\xi}_{\dot{\alpha}}$ which anticommute with all fermionic objects and commute with all bosonic ones, allocated

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, \xi_{\dot{\alpha}}\} = \{\xi^\alpha, Q_\beta\} = \{\xi^\alpha, \bar{Q}_{\dot{\alpha}}\} = [\xi^\alpha, P_m] = 0, \tag{2.2}$$

one can express the algebra (2.1) in terms of commutators:

$$\begin{aligned} &[\xi Q, \bar{\xi}\bar{Q}] = 2\xi\sigma^m\bar{\xi}P_m, \\ &[\xi Q, \xi Q] = [\bar{\xi}\bar{Q}, \bar{\xi}\bar{Q}] = 0, \\ &[P_m, \xi Q] = [P_m, \bar{\xi}\bar{Q}] = 0. \end{aligned} \tag{2.3}$$

This may be viewed as a Lie algebra with anticommuting parameters and one can define a group element, according to

$$G(x^m, \theta, \bar{\theta}) = e^{i[-x^m P_m + \theta Q + \bar{\theta}\bar{Q}]}. \tag{2.4}$$

The multiplication of two group elements induces a motion in the parameter space and one can express the $Q, \bar{Q}$ by differential operators. In the following we will use an equivalent set of differential operators, namely

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m, \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^m_{\alpha\dot{\alpha}}\partial_m. \end{aligned} \tag{2.5}$$

One can now introduce the superspace with the coordinates $(x, \theta, \bar{\theta})$ and define functions on this space. These functions are called *superfields* and depend on $(x, \theta, \bar{\theta})$. As the parameters $\theta, \bar{\theta}$ anticommute, it is obvious that powers of θ (or $\bar{\theta}$) higher than two vanish. Therefore it is convenient to expand superfields in power series in θ and $\bar{\theta}$:

$$\begin{aligned} f(x^m, \theta, \bar{\theta}) =& g(x) + \theta^\alpha \phi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(x) + \theta^\alpha\theta_\alpha m(x) + \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} n(x) + \\ & \theta^\alpha \sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} v_m(x) + \theta^\alpha\theta_\alpha\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta^\alpha \eta_\alpha(x) + \\ & \theta^\alpha\theta_\alpha\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} d(x). \end{aligned} \tag{2.6}$$

The coefficient functions depending on the space-time coordinates x are called *component fields*.

2.3 The constrained superfields

2.3.1 The chiral superfield

The chiral superfield is one of the irreducible representations of the supersymmetry algebra. Its defining relation is the action of the covariant derivative $\bar{D}_{\dot{\alpha}}$ introduced in (2.5) on a superfield Φ:

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{2.7}$$

Introducing the new variable

$$y^m = x^m + i\theta\sigma^m\bar{\theta}, \tag{2.8}$$

one observes that the action of $\bar{D}_{\dot{\alpha}}$ on y^m simply is[1]

[1]Differentiation with respect to anticommuting variables is introduced in appendix A.2.

$$\bar{D}_{\dot{\alpha}} y^m = 0. \tag{2.9}$$

In addition, the covariant derivation of θ vanishes, too:

$$\bar{D}_{\dot{\alpha}} \theta_\alpha = 0. \tag{2.10}$$

This means that any function, which depends only on y^m and θ, satisfies the constraint (2.7). The expansion of the chiral superfield in the coordinates y^m reads:

$$\Phi(y, \theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{2.11}$$

One can now express (2.11) in terms of x^m, θ and $\bar{\theta}$:

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) =& A(x) + i\theta\sigma^m\bar{\theta}\partial_m A(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A(x) \\ &+ \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_m\psi(x)\sigma^m\bar{\theta} + \theta\theta F(x)\end{aligned} \tag{2.12}$$

Analogously, we can define the antichiral superfield, using the constraint

$$D_\alpha \Phi^* = 0. \tag{2.13}$$

Its expansion in the variables $\theta, \bar{\theta}$ can be obtained by a complex conjugation of (2.11) respectively (2.12):

$$\begin{aligned}\Phi^*(x, \theta, \bar{\theta}) =& A^*(x) - i\theta\sigma^m\bar{\theta}\partial_m A^*(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A^*(x) \\ &+ \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^m\partial_m\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x)\end{aligned} \tag{2.14}$$

2.3.2 The vector superfield

The defining constraint of the vector superfield is

$$V = V^*. \tag{2.15}$$

The superfield expansion reads

$$
\begin{aligned}
V(x,\theta,\bar\theta) =& C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \frac{i}{2}\theta\theta\Big(M(x)+iN(x)\Big)+\\
&\frac{i}{2}\bar\theta\bar\theta\Big(M(x)-iN(x)\Big) - \theta\sigma^m\bar\theta v_m(x)+\\
&i\theta\theta\bar\theta\Big(\bar\lambda(x)+\frac{i}{2}\bar\sigma^m\partial_m\chi(x)\Big) - i\bar\theta\bar\theta\theta\Big(\lambda(x)+\frac{i}{2}\sigma^m\partial_m\bar\chi(x)\Big)+\\
&\frac{1}{2}\theta\theta\bar\theta\bar\theta\Big(D(x)+\frac{1}{2}\Box C(x)\Big).
\end{aligned}
\tag{2.16}
$$

Comparing the expressions (2.12) and (2.14) with (2.16) one realizes that the fields C, M, N and χ can be eliminated by choosing a special gauge:

$$
V \longrightarrow V + \Phi + \bar\Phi. \tag{2.17}
$$

This gauge is often called Wess-Zumino gauge[2] in the literature. The vector superfield then reduces to a very simple form

$$
V' = -\theta\sigma^m\bar\theta v_m(x) + i\theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x). \tag{2.18}
$$

One can also define chiral and antichiral field strengths which are invariant under (2.17):

$$
\begin{aligned}
W_\alpha =& -\frac{1}{4}\bar D\bar D D_\alpha V,\\
\bar W_{\dot\alpha} =& -\frac{1}{4}DD\bar D_{\dot\alpha} V.
\end{aligned}
\tag{2.19}
$$

The invariance of W_α or $\bar W_{\dot\alpha}$ under the gauge transformation (2.17) can be shown by using the the (anti)chirality constraints of Φ and $\bar\Phi$. The expansion in $\theta, \bar\theta$ is given by[3]

$$
W_\alpha = -i\lambda_\alpha + \Big[\delta_\alpha{}^\beta D - \frac{i}{2}(\sigma^m\sigma^n)_\alpha{}^\beta F_{mn}\Big]\theta_\beta + \theta\theta\sigma^m_{\alpha\dot\alpha}\partial_m\bar\lambda^{\dot\alpha}. \tag{2.20}
$$

Here, F_{mn} denotes the field strength

$$
F_{mn} = \partial_m v_n - \partial_n v_m. \tag{2.21}
$$

The antichiral field strength $\bar W_{\dot\alpha}$ can be obtained by complex conjugation.

[2] or WZ-gauge.
[3] when using the variable y^m.

2.3.3 The linear superfield

The linear superfield L [11, 12, 13, 14], also known as the real tensor multiplet or 2-form multiplet, is subject to the constraint

$$D^2L = \bar{D}^2L = 0. \tag{2.22}$$

Here, L is a real superfield. By virtue of this constraint, the linear superfield can be defined in another equivalent way using a chiral spinor multiplet[11, 13, 14]:

$$\bar{D}_{\dot{\alpha}}\Psi_\alpha = 0. \tag{2.23}$$

Using (A.25), one can show that the following definition of the linear superfield obeys the constraint (2.22):

$$L = \frac{1}{2}(D^\alpha\Psi_\alpha + \bar{D}_{\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}}). \tag{2.24}$$

The expansion in $\theta, \bar{\theta}$ reads

$$L = C+\theta\eta+\bar{\theta}\bar{\eta}+\theta\sigma^m\bar{\theta}v_m - \frac{i}{2}\theta\theta\bar{\theta}\bar{\sigma}^m\partial_m\eta - \frac{i}{2}\bar{\theta}\bar{\theta}\theta\sigma^m\partial_m\bar{\eta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box C. \tag{2.25}$$

As the vector v_m is subject to the constraint $\partial_m v^m = 0$, it can be written as

$$v^m = \frac{1}{2}\epsilon^{mnop}H_{nop} \tag{2.26}$$

with

$$H_{mno} = \partial_{[m}B_{no]} \tag{2.27}$$

Here, H^{mno} is the field strength of the antisymmetric tensor B_{mn}, which is natural a part of the chiral spinor superfield Ψ_α introduced in (2.23) and (2.24).

We now turn to the representation of the linear superfield L in terms of the chiral spinor multiplet Ψ_α. The expansion of Ψ_α in terms of $\theta, \bar{\theta}$ can be found analogous to the scalar chiral superfield Φ. Details are given in [13]. One finds[4]

$$\Psi_\alpha = \chi_\alpha - \theta_\beta\left(\frac{1}{2}\delta_\alpha{}^\beta(C+iE)+\frac{1}{4}(\sigma^m\bar{\sigma}^n)_\alpha{}^\beta B_{mn}\right)+\theta\theta(\eta_\alpha + i\sigma^m_{\alpha\dot{\alpha}}\partial_m\bar{\chi}^{\dot{\alpha}}) \tag{2.28}$$

The antisymmetric tensor B_{mn} is included directly as a component field in the $\theta, \bar{\theta}$ expansion, in contrast to the linear multiplet where the tensor appeared only through its field strength H^{mno}.

[4]The components are in terms of $y^m = x^m + i\theta\sigma^m\bar{\theta}$.

2.4 Supersymmetric Lagrangians

2.4.1 Lagrangian for chiral superfields

The most general and renormalizable Lagrangian for a theory with N_C chiral multiplets is given by [10]:

$$\mathcal{L} = \bar{\Phi}_i \Phi_i \Big|_{\theta\theta\bar{\theta}\bar{\theta}} + \left[\left(\frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k + \lambda_i \Phi_i \right) \Big|_{\theta\theta} + \text{h.c.} \right], \quad (2.29)$$

$i, j, k = 1, \ldots, N_C$. When introducing an arbitrary coupling function f, one is led to a nonlinear σ-model (cf. appendix C).

2.4.2 Lagrangians for the vector superfields

To motivate the construction of the Lagrangian (see [10]) for a free, massless vector multiplet, we recall that the field strengths $W_\alpha, \bar{W}_{\dot{\alpha}}$ of the vector multiplet contain only the gauge invariant components λ_α, D and F_{mn} of the vector multiplet V. They can therefore be used to construct a gauge invariant Lagrangian. Furthermore, the Lagrangian must be Lorentz invariant. Hence, we have to take a product of the form $W^\alpha W_\alpha$, in order to achieve a renormalizable Lagrangian. The supersymmetric, gauge invariant generalization of a Lagrangian for a free, massless vector field therefore is

$$\mathcal{L} = \frac{1}{4} \Big(W^\alpha W_\alpha \Big|_{\theta\theta} + \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \Big|_{\bar{\theta}\bar{\theta}} \Big). \quad (2.30)$$

The next step to more general Lagrangian is to allow N_V vector multiplets coupled by a symmetric coupling function f_{AB} with $A, B = 1, \ldots, N_V$. However, in the following we consider a more general scenario. We couple the N_V vector multiplets to N_C chiral multiplets using a chiral multiplet dependent coupling function $f_{AB}(\Phi_i)$, $i = 1, \ldots, N_C$ [13, 14]:

$$\mathcal{L} = \frac{1}{4} \int d^2\theta f_{AB}(\Phi) W^A W^B + \text{h.c.}. \quad (2.31)$$

The explicit form of the Lagrangian in components is determined in [14, 13]. The bosonic terms finally read:

$$\mathcal{L} = -\frac{1}{4} \mathrm{Re} f_{AB} F^A_{mn} F^{Bmn} + \frac{1}{8} \mathrm{Im} f_{AB} \epsilon^{mnop} F^A_{mn} F^B_{op}. \quad (2.32)$$

2.4.3 Lagrangians for the linear superfields

We restrict ourselves to the massless case [15, 16, 17, 12, 13, 14]. In a renormalizable scenario, the gauge invariant Lagrangian is given by the $\theta^2\bar{\theta}^2$ component of L^2, that is

$$\mathcal{L}_{kin} = -L^2\Big|_{\theta\theta\bar{\theta}\bar{\theta}}. \tag{2.33}$$

The tensor multiplet can also possess self-interactions, which we can include when writing the Lagrangian as an arbitrary, real function $F(L)$:

$$\mathcal{L}_{kin} = -F(L)\Big|_{\theta\theta\bar{\theta}\bar{\theta}}. \tag{2.34}$$

In components, this Lagrangian admits the following form (details are given in the appendix of [13]):

$$\begin{aligned}\mathcal{L}_{kin} =& -\frac{1}{4}\frac{\partial^2 F}{\partial C^2}\Big[\partial_m C\partial^m C + i(\eta\sigma^m\bar{\eta} + \bar{\eta}\bar{\sigma}^m\eta) + \frac{3}{2}H_{mno}H^{mno}\Big] \\ & - \frac{1}{8}\frac{\partial^3 F}{\partial C^3}\eta\sigma^m\bar{\eta}\epsilon_{mnop}H^{nop} - \frac{1}{48}\frac{\partial^4 F}{\partial C^4}\eta\eta\bar{\eta}\bar{\eta}.\end{aligned} \tag{2.35}$$

The linear multiplet is dual to the chiral multiplet. To show this, one introduces the first order action[5]

$$S = \int d^4x d^2\theta d^2\bar{\theta}\Big(-\frac{1}{2}V^2 + (\Phi + \bar{\Phi})V\Big) \tag{2.36}$$

where V is a general, real superfield. Using the rules for variation of superfields[6], we notice that variation of S with respect to V yields the following equation of motion

$$V = \Phi + \bar{\Phi}. \tag{2.37}$$

Inserting (2.37) back into (2.36) results in the action for a free chiral multiplet.

Now we vary with respect to the (anti-)chiral multiplets $\Phi, \bar{\Phi}$. As a consequence of the anticommuting character of the $D_\alpha, \bar{D}_{\dot{\alpha}}$, a chiral superfield can be written as

$$\Phi = \bar{D}^2\Lambda, \tag{2.38}$$

with Λ being a general superfield. The first order action reads, after integrating by parts,

$$S = \int d^4x d^2\theta d^2\bar{\theta}\Big(-\frac{1}{2}V^2 + (D^2V)\Lambda + (\bar{D}^2V)\bar{\Lambda}\Big). \tag{2.39}$$

The equations of motion for the general superfields $\Lambda, \bar{\Lambda}$ are

[5]Here we also introduced integration over anticommuting variables. An introduction to this topic is given in appendix A.

[6]The rules are given in the appendix B.

$$D^2V = \bar{D}^2V = 0, \tag{2.40}$$

which is exactly the constraint for a real, linear superfield. Putting (2.40) back into (2.36) reveals the action for a free linear multiplet.

Chapter 3

Rigid $N = 2$ supersymmetry

The most simple extension of the $N = 1$ supersymmetry is to introduce another generator of supersymmetry. We choose an algebra without any central charges. The algebra then is

$$\begin{aligned}
&\{Q^i_\alpha, \bar{Q}_{\dot{\alpha}j}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m \delta^i_j, \\
&\{Q^i_\alpha, Q^j_\beta\} = \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = 0, \\
&[P_m, Q^i_\alpha] = [P_m, \bar{Q}_{\dot{\alpha}i}] = 0, \\
&[P_m, P_n] = 0,
\end{aligned} \tag{3.1}$$

with $i, j = 1, 2$. All other indices are treated as in the case of $N = 1$, that is $\alpha, \dot{\alpha} = 1, 2$ and $m = 0, \ldots, 3$.

3.1 $N = 2$ superspace

In the $N = 1$ case, one can introduce the superspace with the coordinates $(x^m, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$ (see chapter 2.1). This formulation can be easily carried over to the $N = 2$ theory by introducing the coordinates

$$(x^m, \theta^\alpha_i, \bar{\theta}^i_{\dot{\alpha}}). \tag{3.2}$$

The spinor variables form a doublet under $SU(2)$ [18, 19], and as in the $N = 1$ case, they are totally anticommuting. One can also define a group element [17]:

$$G(x^m, \theta^\alpha_i, \bar{\theta}^i_{\dot{\alpha}}) = e^{i[-x^m P_m + \theta^\alpha_i Q^i_\alpha + \bar{\theta}^i_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}_i]}. \tag{3.3}$$

Under the action of a supersymmetry transformation $G(0, \xi^\alpha_i, \bar{\xi}^i_{\dot{\alpha}})$, a point in superspace $(x^m, \theta^\alpha_i, \bar{\theta}^i_{\dot{\alpha}})$ is transformed into

$$
\begin{aligned}
x^{m\prime} &= x^m + i\theta_i^\alpha \sigma^m_{\alpha\dot\alpha}\bar\xi^{\dot\alpha i} - i\xi_i^\alpha \sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i},\\
\theta_i^{\alpha\prime} &= \theta_i^\alpha + \xi_i^\alpha,\\
\bar\theta^{i\prime}_{\dot\alpha} &= \bar\theta^i_{\dot\alpha} + \bar\xi^i_{\dot\alpha}.
\end{aligned}
\tag{3.4}
$$

The generators of supersymmetry are in terms of the differential operators $Q^i_\alpha, \bar Q_{\dot\alpha i}$ given by [20]

$$
\begin{aligned}
Q^i_\alpha &= \frac{\partial}{\partial\theta_i^\alpha} - i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i}\partial_m,\\
\bar Q_{\dot\alpha i} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} + i\theta_i^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m.
\end{aligned}
\tag{3.5}
$$

A superfield F shall transform under a supersymmetry transformation as follows:

$$
\delta_\xi F = (\xi_i^\alpha Q^i_\alpha + \bar\xi^i_{\dot\alpha}\bar Q^{\dot\alpha}_i)F. \tag{3.6}
$$

The covariant derivatives $D^i_\alpha, \bar D_{\dot\alpha i}$ admit the following form [21, 20, 22]

$$
\begin{aligned}
D^i_\alpha &= \frac{\partial}{\partial\theta_i^\alpha} + i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i}\partial_m,\\
\bar D_{\dot\alpha i} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - i\theta_i^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m,
\end{aligned}
\tag{3.7}
$$

and the anticommutation relations for vanishing central charge are given by:

$$
\begin{gathered}
\{D^i_\alpha, \bar D_{\dot\alpha j}\} = -2i\delta^i_j\sigma^m_{\alpha\dot\alpha}\partial_m,\\
\{D^i_\alpha, D^j_\beta\} = 0,\\
\{\bar D_{\dot\alpha i}, \bar D_{\dot\beta j}\} = \{D^i_\alpha, Q^j_\beta\} = \{\bar D_{\dot\alpha i}, Q^j_\beta\} = \{D^i_\alpha, \bar Q_{\dot\beta j}\} = \{\bar D_{\dot\alpha i}, \bar Q_{\dot\beta j}\} = 0.
\end{gathered}
\tag{3.8}
$$

Superfields can be expanded in powers of the variables $\theta_i^\alpha, \bar\theta^i_{\dot\alpha}$. Examples for the superfield expansions are given in [21, 16]. However, this rather complicated description is not necessary. One can describe the $N = 2$ multiplets and interactions in terms of $N = 1$ superfields which describe their $N = 1$ submultiplets [16].

3.2 Scalar hypermultiplet

According to [23, 24, 16, 15], the hypermultiplet is described by a $SU(2)$ doublet of $N = 1$ chiral multiplets Φ^A, $A = 1, 2$. The off-shell field content

Field	Type	Degrees of freedom
A_A	complex scalars	4 bosonic
ψ_A	doublet of chiral fermions	8 fermionic
F_A	complex scalars	4 bosonic

Table 3.1: Component content of the $N = 2$ hypermultiplet

is shown in table 3.1. The off-shell multiplet content of the hypermultiplet is given in [25, 26].

One possibility for a free action is

$$S = \int d^4x d^4\theta \bar{\Phi}_A \Phi^A. \tag{3.9}$$

The chosen description admits an algebra which closes only on-shell, so that the supersymmetry transformations depend on the action. For the action above, the first supersymmetry is manifestly implemented by the integral over the anticommuting parameters, the second supersymmetry transformations are generated by [15], using the $N = 1$ covariant derivatives,

$$\begin{aligned} \delta\Phi_A &= \pm\bar{D}^2(\bar{\xi}\bar{\Phi}_A), \\ \delta\bar{\Phi}_A &= \pm D^2(\xi\Phi_A). \end{aligned} \tag{3.10}$$

The supersymmetry parameter ξ is space-time independent and chiral, i.e. $\bar{D}_{\dot{\alpha}}\xi = 0$ and has the following expansion:

$$\xi = z + \theta\zeta - \theta\theta q. \tag{3.11}$$

The parameter z generates the central charge transformations (which we set to zero), ζ generates the additional supersymmetry and q rotates the two supersymmetries into each other ($SU(2)$ symmetry of the two supersymmetries) [15, 16].

3.3 Vector multiplet

The component content of the $N = 2$ vector multiplet consists of $8 + 8$ off-shell degrees of freedom [26], namely a complex scalar, a doublet of chiral fermions, a vector gauge field and a triplet of real $SU(2)$ scalars. It is summarized in table 3.2. The real triplet of scalars satisfies $Y_{AB} = Y_{BA}$.

The $N = 2$ vector multiplet can be described by a $N = 1$ vector multiplet V and a $N = 1$ chiral multiplet Φ in the adjoint representation of the internal symmetry group [16, 27]. One supersymmetry is manifested in contrast to the other one, which has to be implemented explicitly in the transformation rules and mixes the superfields V and Φ:

Field	Type	Degrees of freedom
X	complex scalar	2 bosonic
ψ_A	doublet of chiral fermions	8 fermionic
v^m	gauge vector	3 bosonic
$Y_{(AB)}$	real $SU(2)$ triplet of scalars	3 bosonic

Table 3.2: Component content of the $N = 2$ vector multiplet

$$\begin{aligned} \delta\Phi &= -iW^\alpha D_\alpha \xi \\ e^{-V}\delta e^V &= \bar{\xi}\Phi + \xi\tilde{\Phi}. \end{aligned} \tag{3.12}$$

Here, $\tilde{\Phi} = e^{-V}\bar{\Phi}e^V$, $W^\alpha = i\bar{D}^2(e^{-V}D^\alpha e^V)$ and ξ is again a constant chiral superfield as defined in (3.11). The covariant derivative D_α is given by the covariant derivative of $N = 1$ case (2.5). This algebra closes without imposing any field equations, which means that the transformations are, in contrast to the hypermultiplet case, independent of the action. The action is given by [16]

$$S = \int d^4x d^4\theta \bar{\Phi}\Phi + \left\{ \frac{1}{4} \int d^4x d^2\theta W^\alpha W_\alpha + \text{h.c.} \right\}. \tag{3.13}$$

Of course, one can generalize this action to more than one vector multiplet. In that case, the $N = 2$ vector multiplets are described by n $N = 1$ vector multiplets V^i and n $N = 1$ chiral multiplets Φ^i in the adjoint representation of some group, with $i = 1 \ldots n$. The supersymmetry transformations are generalized in the following way [27]:

$$\begin{aligned} \delta\Phi &= -iW^\alpha D_\alpha \xi, \qquad & \Phi = \Phi^i T_i \\ e^{-V}\delta e^V &= \bar{\xi}\Phi + \xi\tilde{\Phi}, \qquad & V = V^i T_i, \end{aligned} \tag{3.14}$$

where $(T_i)^j{}_k = if_{ik}{}^j$,$\tilde{\Phi} = e^{-V}\bar{\Phi}e^V$, $W^{i\alpha} = i\bar{D}^2(e^{-V}D^\alpha e^V)^i$ and ξ is a constant chiral superfield (3.11). As before, this algebra closes without imposing any field equations. The action is given by [27]

$$S = \int d^4x d^4\theta K(\Phi, \tilde{\Phi}) + \left\{ \frac{1}{4} \int d^4x d^2\theta F_{ij} W^{i\alpha} W^j_\alpha + \text{h.c.} \right\}. \tag{3.15}$$

In order that (3.15) is invariant under $N = 2$ supersymmetry transformations, one has to require that the function K and the matrix F_{ij} can be expressed by one single holomorphic function $F(\Phi)$ [27]:

$$K = \frac{1}{2}(\bar{\Phi}^i F_i + \Phi^i \bar{F}_i),$$
$$F_i = \frac{\partial F}{\partial \Phi^i}, \tag{3.16}$$
$$F_{ij} = \frac{\partial^2 F}{\partial \Phi^i \partial \Phi^j}.$$

3.4 Tensor multiplet

We now discuss the $N = 2$ tensor multiplets and its actions [15, 16, 26]. A $N = 2$ tensor multiplet consists of a $SU(2)$ triplet of scalars, a doublet of spinors and a Lorentz vector, which is the subject to a constraint. The defining relation of the multiplet is, that the triplet of scalars transform under supersymmetry into the spinors. In table 3.3, the component content is shown.

Field	Type	Degrees of freedom
C_{AB}	real $SU(2)$ triplet of scalars	3 bosonic
ψ^A	doublet spinor	8 fermionic
v^m	Lorentz vector	3 bosonic
F	complex scalar	2 bosonic

Table 3.3: Component content of the $N = 2$ linear multiplet

The formulation of the $N = 2$ tensor multiplets in terms of $N = 1$ superfields is well known [15, 16]. They consist of a chiral superfield Φ and a spinor gauge field Ψ_α with the field strength $L = \frac{1}{2}(D^\alpha \Psi_\alpha + \bar{D}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}})$, satisfying the constraint of the $N = 1$ linear multiplet $D^2 L = \bar{D}^2 L = 0$ (2.25).

The $N = 2$ supersymmetry transformations read

$$\begin{aligned} \delta_\xi \Phi &= (\bar{D}_{\dot{\alpha}} L)(\bar{D}^{\dot{\alpha}} \bar{\xi}), \\ \delta_\xi \Psi_\alpha &= -\Phi D_\alpha \xi. \end{aligned} \tag{3.17}$$

As before, ξ is a constant and chiral superfield (3.11), describing the $N = 2$ supersymmetry transformations. But, in contrast to the transformations of the hypermultiplet, these transformations close off-shell. As a consequence, the sum of invariant actions is again invariant under the transformations given above.

The most simple, $N = 2$ invariant action is given by

$$S = \int d^4x d^4\theta \Big[-\frac{1}{2} L^2 + \bar{\Phi}\Phi \Big]. \tag{3.18}$$

The proof of invariance under (3.17) is straightforward. Thereby we assume $D^2\xi = 0$, which breaks the internal $SU(2)$ symmetry between the two supersymmetries [15, 16]:

$$\delta S = \int d^4x d^4\theta \Big[-L\delta L + \bar{\Phi}\delta\Phi + \Phi\delta\bar{\Phi} \Big].$$

The transformation for the linear multiplet L reads:

$$\delta L = -\frac{1}{2}\Big[(D^\alpha\Phi)(D_\alpha\xi) + (\bar{D}_{\dot{\alpha}}\bar{\Phi})(\bar{D}^{\dot{\alpha}}\bar{\xi}) \Big] \tag{3.19}$$

Converting the θ-integration into covariant derivatives using (A.22), one finds

$$\delta S = -\frac{1}{4}\int d^4x d^2\bar{\theta} D^2 \Big\{ \Big[L(D^\alpha\Phi) + \Phi(D^\alpha L) \Big] (D_\alpha\xi) \Big\}\Big|_{\theta=0} + \text{ h.c..}$$

This can be simplified using the product rule as follows

$$\delta S = -\frac{1}{4}\int d^4x d^2\bar{\theta} D^2 \Big\{ D^\alpha(L\Phi)(D_\alpha\xi) \Big\}\Big|_{\theta=0} + \text{ h.c.},$$

which is equal to zero, when using (A.24) and $D^2\xi = 0$.

A first step to generalize this simple action is to permit a general coupling function $F(L, \Phi, \bar{\Phi})$:

$$S = \int d^4x d^4\theta F(L, \Phi, \bar{\Phi}). \tag{3.20}$$

The proof of invariance under the transformations (3.17) can be performed in a similar way as before:

$$\delta S = \hat{\delta S} + \text{ h.c.},$$

with

$$\hat{\delta S} = \int d^4x d^4\theta \Big\{ \Big[-\frac{\partial F}{\partial L}(D^\alpha\Phi) + \frac{\partial F}{\partial\bar{\Phi}}(D^\alpha L) \Big] (D_\alpha\xi) \Big\}. \tag{3.21}$$

We now focus on the first part $\hat{\delta S}$ of δS, as the hermitian conjugated part can be treated analogously, and perform the θ-integration using the covariant derivatives:

$$\delta\hat{S} = -\frac{1}{4}\int d^4x d^2\bar{\theta} D^\beta \Big\{ \Big[-\frac{\partial^2 F}{\partial L \partial L}(D_\beta L)(D^\alpha \Phi) - \frac{\partial^2 F}{\partial L \partial \Phi}(D_\beta \Phi)(D^\alpha \Phi) - \frac{\partial F}{\partial L}(D_\beta D^\alpha \Phi) + \frac{\partial^2 F}{\partial \bar{\Phi} \partial L}(D_\beta L)(D^\alpha L) + \frac{\partial^2 F}{\partial \bar{\Phi} \partial \Phi}(D_\beta \Phi)(D^\alpha L) \Big] (D_\alpha \xi) \Big\} \Big|_{\theta=0}. \tag{3.22}$$

The second covariant derivative reveals[1]

$$\begin{aligned} \delta\hat{S} = -\frac{1}{4}\int d^4x d^2\bar{\theta} \Big\{ \Big[& - F_{LLL}(L^\beta L_\beta \Phi^\alpha) - F_{LL\Phi}(\Phi^\beta L_\beta \Phi^\alpha) \\ & + F_{LL}(L_\beta D^\beta \Phi^\alpha) - F_{L\Phi L}(L^\beta \Phi_\beta \Phi^\alpha) \\ & - F_{L\Phi\Phi}(\Phi^\beta \Phi_\beta \Phi^\alpha) - F_{L\Phi}(D^\beta \Phi_\beta)\Phi^\alpha \\ & + F_{L\Phi}(\Phi_\beta)(D^\beta \Phi^\alpha) \\ & - F_{LL}(L^\beta)(D_\beta \Phi^\alpha) - F_{L\Phi}(\Phi^\beta)(D_\beta \Phi^\alpha) \\ & + F_{\bar{\Phi}LL}(L^\beta L_\beta L^\alpha) + F_{\bar{\Phi}L\Phi}(\Phi^\beta L_\beta L^\alpha) \\ & + F_{\bar{\Phi}\Phi L}(L^\beta \Phi_\beta L^\alpha) + F_{\bar{\Phi}\Phi\Phi}(\Phi^\beta \Phi_\beta L^\alpha) \\ & + F_{\bar{\Phi}\Phi}(D^\beta \Phi_\beta)(L^\alpha) \Big] (D_\alpha \xi) \Big\} \Big|_{\theta=0}. \end{aligned} \tag{3.23}$$

Note, that the terms $\Phi^\beta \Phi_\beta \Phi^\alpha$ and $L^\beta L_\beta L^\alpha$ vanish due to the anticommutation relations spinor-indexed quantities (2.2).
The terms in (3.23) containing two linear multiplets and one chiral multiplet can be simplified as follows

$$\begin{aligned} & -F_{LLL}(L^\beta L_\beta \Phi^\alpha) + F_{\bar{\Phi}L\Phi}(\Phi^\beta L_\beta L^\alpha) + F_{\bar{\Phi}\Phi L}(L^\beta \Phi_\beta L^\alpha) = \\ & -F_{LLL}(L^\beta L_\beta \Phi^\alpha) + 2F_{\bar{\Phi}\Phi L}(\Phi^\beta L_\beta L^\alpha) = \\ & -F_{LLL}(L^\beta L_\beta \Phi^\alpha) - 2F_{\bar{\Phi}\Phi L}\frac{1}{2}\delta_\beta{}^\alpha(\Phi^\beta L^\gamma L_\gamma) = \\ & -\Big[F_{LLL} + F_{\bar{\Phi}\Phi L}\Big](L^\beta L_\beta \Phi^\alpha). \end{aligned} \tag{3.24}$$

Similarly, one can summarize the terms in (3.23) containing one linear multiplet and two chiral ones:

[1]We introduced the abbreviations $F_L = \frac{\partial F}{\partial L}$, $F_{L\Phi} = \frac{\partial^2 F}{\partial L \partial \Phi}$ etc. as well as $L^\alpha = D^\alpha L$ and $\Phi^\alpha = D^\alpha \Phi$.

$$
\begin{aligned}
&-F_{LL\Phi}(\Phi^\beta L_\beta \Phi^\alpha) - F_{L\Phi L}(L^\beta \Phi_\beta \Phi^\alpha) + F_{\bar\Phi\Phi\Phi}(\Phi^\beta \Phi_\beta L^\alpha) = \\
&\quad F_{\bar\Phi\Phi\Phi}(\Phi^\beta \Phi_\beta L^\alpha) - 2F_{LL\Phi}(L^\beta \Phi_\beta L^\alpha) = \\
&\quad F_{\bar\Phi\Phi\Phi}(\Phi^\beta \Phi_\beta L^\alpha) + 2F_{LL\Phi}\delta_\beta{}^\alpha \frac{1}{2}(L^\beta \Phi^\gamma \Phi_\gamma) = \\
&\quad \Big[F_{LL\Phi} + F_{\bar\Phi\Phi\Phi}\Big](\Phi^\beta \Phi_\beta L^\alpha).
\end{aligned}
\tag{3.25}
$$

The terms in (3.23) proportional to $F_{L\Phi}$ vanish

$$
\begin{aligned}
-F_{L\Phi}\Big[(D^\beta D_\beta \Phi)(D^\alpha \Phi) - (D_\beta \Phi)(D^\beta D^\alpha \Phi) + (D^\beta \Phi)(D_\beta D^\alpha \Phi)\Big] &= \\
-F_{L\Phi}\Big[(D^\beta D_\beta \Phi)(D^\alpha \Phi) + 2(D^\beta \Phi)(D_\beta D^\alpha \Phi)\Big] &= \\
-F_{L\Phi}\Big[(D^\beta D_\beta \Phi)(D^\alpha \Phi) - (D^\alpha \Phi)(D^\beta D_\beta \Phi)\Big] &= 0,
\end{aligned}
$$

and for the terms in (3.23) containing one chiral multiplet and one linear multiplet one finally finds

$$
\begin{aligned}
F_{LL}\Big[(D_\beta L)(D^\beta D^\alpha \Phi) - (D^\beta L)(D_\beta D^\alpha \Phi)\Big] + F_{\bar\Phi\Phi}(D^\beta D_\beta \Phi)(D^\alpha L) &= \\
-2F_{LL}(D^\beta L)(D_\beta D^\alpha \Phi) + F_{\bar\Phi\Phi}(D^\beta D_\beta \Phi)(D^\alpha L) &= \\
\Big[F_{LL} + F_{\bar\Phi\Phi}\Big](D^\beta D_\beta \Phi)(D^\alpha L).
\end{aligned}
\tag{3.26}
$$

One condition for invariance under the supersymmetry transformations therefore is

$$
F_{LL} + F_{\bar\Phi\Phi} = 0,
\tag{3.27}
$$

and in fact, this is the only condition as one can rewrite the remaining terms (3.24) and (3.25) in the following way:

$$
\begin{aligned}
-F_{LLL} - F_{\bar\Phi\Phi L} &= \frac{\partial}{\partial L}\Big[-F_{LL} - F_{\bar\Phi\Phi}\Big] = 0 \\
F_{LL\Phi} + F_{\bar\Phi\Phi\Phi} &= \frac{\partial}{\partial \Phi}\Big[F_{LL} + F_{\bar\Phi\Phi}\Big] = 0.
\end{aligned}
$$

The solutions for the differential equation (3.27) are known, as it is just the three-dimensional Laplace equation.

The final step of generalization is to allow N_T tensor multiplets. They are described by N_T chiral multiplets Φ^I and linear multiplets L^I, $I = 1, \ldots, N_T$. The supersymmetry transformations generalize as follows

$$
\begin{aligned}
\delta_\xi \Phi^I &= \bar{D}^2(\bar{\xi} L^I) \\
\delta_\xi \Phi^I_\alpha &= -\Phi^I D_\alpha \xi,
\end{aligned}
\tag{3.28}
$$

and the action simply reads

$$
S = \int d^4x d^4\theta F(L^I, \Phi^I, \bar{\Phi}^I). \tag{3.29}
$$

The constraint for the invariance of the action under the transformations (3.28) can be derived in a similar way as in the case for one tensor multiplet. The transformed action reads

$$
\delta S = \hat{\delta S} + \text{ h.c.},
$$

with

$$
\hat{\delta S} = \int d^4x d^4\theta \Big\{ \Big[-F_{L^I}(D^\alpha \Phi^I) + F_{\bar{\Phi}^I}(D^\alpha L^I) \Big] (D_\alpha \xi) \Big\}, \tag{3.30}
$$

and after rewriting the θ-integration in terms of covariant derivatives and performing the first covariant derivative, one obtains (up to the hermitean conjugated part)

$$
\begin{aligned}
\hat{\delta S} = -\frac{1}{4} \int d^4x d^2\bar{\theta} D^\beta \Big\{ \Big[&- F_{L^I L^J}(D_\beta L^J)(D^\alpha \Phi^I) \\
&- F_{L^I \Phi^J}(D_\beta \Phi^J)(D^\alpha \Phi^I) \\
&- F_{L^I}(D_\beta D^\alpha \Phi^I) + F_{\bar{\Phi}^I L^J}(D_\beta L^J)(D^\alpha L^I) \\
&+ F_{\bar{\Phi}^I \Phi^J}(D_\beta \Phi^J)(D^\alpha L^I) \Big] (D_\alpha \xi) \Big\} \Big|_{\theta=0}.
\end{aligned}
$$

After a circuitous calculation one arrives at

$$
\begin{aligned}
\delta\hat{S} = -\frac{1}{4}\int d^4x d^2\bar{\theta}\Big\{ \Big[& -F_{L^I L^J L^K}(D^\beta L^K)(D_\beta L^J)(D^\alpha \Phi^I) \\
& + 2F_{\bar{\Phi}^I \Phi^K L^J}(D^\beta \Phi^K)(D_\beta L^J)(D^\alpha L^I) \\
& + F_{\bar{\Phi}^I \Phi^J \Phi^K}(D^\beta \Phi^K)(D_\beta \Phi^J)(D^\alpha L^I) \\
& - 2F_{L^I L^K \Phi^K}(D^\beta L^K)(D_\beta \Phi^J)(D^\alpha \Phi^I) \\
& + F_{\bar{\Phi}^I L^J L^K}(D^\beta L^K)(D_\beta L^J)(D^\alpha L^I) \\
& - F_{L^I \Phi^J \Phi^K}(D^\beta \Phi^K)(D_\beta \Phi^J)(D^\alpha \Phi^I) \\
& + \Big(F_{L^I L^J} + F_{\bar{\Phi}^I \Phi^J}\Big)(D^\beta D_\beta \Phi^J)(D^\alpha L^I) \\
& + \Big(F_{L^J \Phi^I} - F_{L^I \Phi^J}\Big)(D^\beta D_\beta \Phi^J)(D^\alpha \Phi^I)\Big] \\
& \times (D_\alpha \xi)\Big\}\Big|_{\theta=0} .
\end{aligned}
$$

One condition for invariance therefore is

$$
F_{L^I L^J} + F_{\Phi^I \bar{\Phi}^J} = 0. \tag{3.31}
$$

By imposing the symmetry[2]

$$
F_{L^J \Phi^I} - F_{L^I \Phi^J} = 0, \tag{3.32}
$$

we show in the following, that the remaining terms vanish, too. Using the formula

$$
(\psi_1\psi_2)(\psi_3\psi_4) = -(\psi_1\psi_3)(\psi_2\psi_4) - (\psi_1\psi_4)(\psi_2\psi_3)
$$

for arbitrary spinors, we find

$$
\begin{aligned}
(D^\beta \Phi^K)(D_\beta \Phi^J)(D^\alpha \Phi^I)(D_\alpha \xi) = & - (D^\beta \Phi^K)(D_\beta \Phi^I)(D^\alpha \Phi^J)(D_\alpha \xi) \\
& - (D^\beta \Phi^K)(D_\beta \xi)(D^\alpha \Phi^J)(D_\alpha \Phi^I).
\end{aligned}
$$

With the help of the symmetry property (3.32), this relation reduces to

$$
(D^\beta \Phi^K)(D_\beta \Phi^J)(D^\alpha \Phi^I)(D_\alpha \xi) = -2(D^\beta \Phi^K)(D_\beta \Phi^I)(D^\alpha \Phi^J)(D_\alpha \xi)
$$

which is equal to zero.

In an analogous manner we can eliminate the term proportional to

[2]According to [16, 15], the term proportional to the symmetry constraint should vanish. However, one can choose $F_{L^I \Phi^J}$ to be symmetric in I, J as we will see in the following section. In this case the symmetry constraint is automatically fulfilled.

$$(D^\beta L^K)(D_\beta L^J)(D^\alpha L^I),$$

and the transformation of the action is reduced to

$$\begin{aligned}\delta\hat{S} = -\frac{1}{4}\int d^4x d^2\bar{\theta}\Big\{\Big[&- F_{L^I L^J L^K}(D^\beta L^K)(D_\beta L^J)(D^\alpha \Phi^I)\\
&+ 2F_{\bar{\Phi}^I \Phi^K L^J}(D^\beta \Phi^K)(D_\beta L^J)(D^\alpha L^I)\\
&+ F_{\bar{\Phi}^I \Phi^J \Phi^K}(D^\beta \Phi^K)(D_\beta \Phi^J)(D^\alpha L^I)\\
&- 2F_{L^I L^K \Phi^J}(D^\beta L^K)(D_\beta \Phi^J)(D^\alpha \Phi^I)\Big](D_\alpha \xi)\Big\}\Big|_{\theta=0}.\end{aligned} \tag{3.33}$$

For the term in (3.33) proportional to $F_{L^I L^J L^K}$, the following relation holds

$$\begin{aligned}&F_{L^I L^J L^K}(D^\beta L^K)(D_\beta L^J)(D^\alpha \Phi^I)(D_\alpha \xi) =\\
&\quad F_{L^I L^J L^K}\Big[- (D^\beta L^K)(D_\beta \Phi^I)(D^\alpha L^J)(D_\alpha \xi)\\
&\qquad\qquad - (D^\beta L^K)(D_\beta \xi)(D^\beta L^J)(D_\alpha \Phi^I)\Big] =\\
&\qquad\qquad - 2F_{L^I L^K L^J}(D^\beta L^J)(D_\beta \Phi^K)(D^\alpha L^I)(D_\alpha \xi).\end{aligned}$$

Analogously, one can rewrite the term in (3.33) proportional to $F_{\bar{\Phi}^I \Phi^J \Phi^K}$. The transformation of the action finally reads

$$\begin{aligned}\delta\hat{S} = -\frac{1}{4}\int d^4x d^2\bar{\theta}\Big\{\Big[&2\Big(F_{L^I L^J L^K} + F_{\bar{\Phi}^I \Phi^K L^J}\Big)(D^\beta \Phi^K)(D_\beta L^J)(D^\alpha L^I)\\
&- 2\Big(F_{\bar{\Phi}^I \Phi^J \Phi^K} + F_{L^I L^K \Phi^J}\Big)(D^\beta L^K)(D_\beta \Phi^J)(D^\alpha \Phi^I)\\
&+ \Big(F_{L^I L^J} + F_{\bar{\Phi}^I \Phi^J}\Big)(D^\beta D_\beta \Phi^J)(D^\alpha L^I)\\
&+ \Big(F_{L^J \Phi^I} - F_{L^I \Phi^J}\Big)(D^\beta D_\beta \Phi^J)(D^\alpha \Phi^I)\Big](D_\alpha \xi)\Big\}\Big|_{\theta=0},\end{aligned}$$

which is evidently equal to zero when imposing the constraints (3.31) and (3.32), because the remaining constraints are obviously fulfilled:

$$\begin{aligned}F_{L^I L^J L^K} + F_{\bar{\Phi}^I \Phi^K L^J} &= \frac{\partial}{\partial L^J}\Big[F_{L^I L^K} + F_{\bar{\Phi}^I \Phi^K}\Big] = 0\\
F_{\bar{\Phi}^I \Phi^J \Phi^K} + F_{L^I L^K \Phi^J} &= \frac{\partial}{\partial \Phi^J}\Big[F_{L^I L^K} + F_{\bar{\Phi}^I \Phi^K}\Big] = 0.\end{aligned}$$

To find the Lagrangian in components, one has to perform the full integration over θ and $\bar{\theta}$. Using (A.22), the action reads

$$S = \frac{1}{16}\int d^4x \Big\{ D^2 \bar{D}^2 F(L^I, \Phi^I, \bar{\Phi}^I) \Big\}\Big|_{\theta=\bar{\theta}=0},$$

and applying the covariant derivatives leads to

$$\begin{aligned} S = \frac{1}{16}\int d^4x \Big\{ D^2 \Big[& F_{L^I L^J} (\bar{D}_{\dot{\alpha}} L^J)(\bar{D}^{\dot{\alpha}} L^I) + F_{L^I \bar{\Phi}^J} (\bar{D}_{\dot{\alpha}} \bar{\Phi}^J)(\bar{D}^{\dot{\alpha}} L^I) \\ & + F_{\bar{\Phi}^I L^J} (\bar{D}_{\dot{\alpha}} L^J)(\bar{D}^{\dot{\alpha}} \bar{\Phi}^I) + F_{\bar{\Phi}^I \bar{\Phi}^J} (\bar{D}_{\dot{\alpha}} \bar{\Phi}^J)(\bar{D}^{\dot{\alpha}} \bar{\Phi}^I) \\ & + F_{\bar{\Phi}^I} (\bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{\Phi}^I) \Big] \Big\}\Big|_{\theta=\bar{\theta}=0}. \end{aligned}$$

In the following sections, we will restrict ourselves to the bosonic part of the $N = 2$ tensor multiplet Lagrangian. Higher order derivatives of the function F, i.e. terms containing for example $F_{L^I L^J L^K}$ or $F_{L^I L^J L^K L^M}$, will lead to terms containing fermions, which will be omitted in the following. The bosonic part of S is then given by

$$\begin{aligned} S = \frac{1}{16}\int d^4x \Big\{ & 2F_{L^I L^J} (D_\alpha \bar{D}_{\dot{\alpha}} L^J)(D^\alpha \bar{D}^{\dot{\alpha}} L^I) \\ & - 2F_{L^I \bar{\Phi}^J} (D^\alpha \bar{D}_{\dot{\alpha}} \bar{\Phi}^J)(D_\alpha \bar{D}^{\dot{\alpha}} L^I) \\ & - 2F_{\bar{\Phi}^I L^J} (D^\alpha \bar{D}_{\dot{\alpha}} L^J)(D_\alpha \bar{D}^{\dot{\alpha}} \bar{\Phi}^I) \\ & + 2F_{\bar{\Phi}^I \bar{\Phi}^J} (D_\alpha \bar{D}_{\dot{\alpha}} \bar{\Phi}^J)(D^\alpha \bar{D}^{\dot{\alpha}} \bar{\Phi}^I) \\ & + F_{\bar{\Phi}^I \Phi^J} (D^\alpha D_\alpha \Phi^J)(\bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{\Phi}^I) \\ & + F_{\bar{\Phi}^I} (D^\alpha D_\alpha \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{\Phi}^I) \Big\}\Big|_{\theta=\bar{\theta}=0}. \end{aligned} \tag{3.34}$$

Using the anticommutation relations of the D_α and $\bar{D}_{\dot{\alpha}}$ one determines

$$D_\alpha \bar{D}_{\dot{\alpha}} \bar{\Phi}^I \Big|_{\theta=\bar{\theta}=0} = \Big[-2i\sigma^m_{\alpha\dot{\alpha}} \partial_m \bar{\Phi}^I - \bar{D}_{\dot{\alpha}} \underbrace{D_\alpha \bar{\Phi}^I}_{=0} \Big]\Big|_{\theta=\bar{\theta}=0}, \tag{3.35}$$

that is in components, using (2.12),

$$D_\alpha \bar{D}_{\dot{\alpha}} \bar{\Phi}^I \Big|_{\theta=\bar{\theta}=0} = -2i\sigma^m_{\alpha\dot{\alpha}} \partial_m \bar{A}^I. \tag{3.36}$$

With (2.25) one can determine the similar expression for the linear multiplet

$$D_\alpha \bar{D}_{\dot{\alpha}} L^I \Big|_{\theta=\bar{\theta}=0} = \sigma^m_{\alpha\dot{\alpha}} \Big[v^I_m - i\partial_m C^I \Big]. \tag{3.37}$$

The remaining terms involving the chiral multiplets (2.12) give rise to

$$
\begin{aligned}
D^\alpha D_\alpha \Phi^I\Big|_{\theta=\bar\theta=0} &= 4F^I \\
\bar D_{\dot\alpha}\bar D^{\dot\alpha}\bar\Phi^I\Big|_{\theta=\bar\theta=0} &= 4\bar F^I \\
D^\alpha D_\alpha \bar D_{\dot\alpha}\bar D^{\dot\alpha}\bar\Phi^I\Big|_{\theta=\bar\theta=0} &= 16\Box\bar A^I .
\end{aligned}
\tag{3.38}
$$

Inserting the expressions (3.36)-(3.38) in (3.34), the action reads, after using the fact that $F_{L^I\bar\Phi^J}$ is symmetric in I, J

$$
\begin{aligned}
S = \int d^4x \Big\{ & F_{C^IC^J}\Big[-\frac{1}{4}v^{mI}v_m{}^J + \frac{i}{2}(\partial^m C^I)v_m{}^J + \frac{1}{4}(\partial^m C^I)(\partial_m C^J)\Big] + \\
& F_{C^I\bar A^J}\Big[i(\partial^m \bar A^J)v_m{}^I + (\partial^m \bar A^J)(\partial_m C^I)\Big] + \\
& F_{\bar A^I\bar A^J}(\partial^m \bar A^J)(\partial_m \bar A^I) + F_{\bar A^I A^J}F^J\bar F^I + F_{\bar A^I}\Box\bar A^I \Big\}.
\end{aligned}
\tag{3.39}
$$

Integrating the term containing $\Box\bar A^I$ by parts, one obtains up to boundary terms

$$
\begin{aligned}
-(\partial^m F_{\bar A^I})(\partial_m \bar A^I) = & -F_{\bar A^I C^J}(\partial^m C^J)(\partial_m \bar A^I) - \\
& F_{\bar A^I\bar A^J}(\partial^m \bar A^J)(\partial_m \bar A^I) - F_{\bar A^I A^J}(\partial^m A^J)(\partial_m \bar A^I).
\end{aligned}
$$

Keeping in mind that $\partial^m v_m = 0$, one finds

$$
\partial^m\Big[F_{C^J}v_m{}^J\Big] = F_{C^IC^J}(\partial^m C^I)v_m{}^J + F_{A^IC^J}(\partial^m A^I)v_m{}^J + F_{\bar A^IC^J}(\partial\bar A^I)v_m{}^J .
$$

Therefore, the final action reads

$$
\begin{aligned}
S = \int d^4x \Big\{ & F_{C^IC^J}\Big[(\partial^m A^J)(\partial_m \bar A^I) + \frac{1}{4}[(\partial^m C^I)(\partial_m C^J) - v^{mI}v_m{}^J]\Big] + \\
& \frac{i}{2}\Big[F_{C^I\bar A^J}(\partial^m \bar A^J) - F_{C^IA^J}(\partial^m A^J)\Big]v_m{}^I + F_{\bar A^IA^J}F^J\bar F^I \Big\}.
\end{aligned}
\tag{3.40}
$$

The last step is to eliminate the auxiliary fields F^I by using their equations of motion, which yield $F^I = 0$, and one rederives the action of [6].
When fixing the linear multiplets L^I by choosing constant chiral spinor multiplets Ψ^I_α, one finds a Lagrangian corresponding to a nonlinear σ-model (cf. (C.2)) with the Kähler potential $F(A^I, \bar A^I)$.

$$\mathcal{L} = -F_{\bar{A}^I A^J}\Big[(\partial^m A^J)(\partial_m \bar{A}^I) - F_{\bar{A}^I A^J} F^J \bar{F}^I\Big]. \tag{3.41}$$

For fixed A^I, one is led to the Lagrangian

$$\mathcal{L} = \frac{1}{4} F_{C^I C^J}\Big[(\partial^m C^I)(\partial_m C^J) - v^{mI} v_m{}^J\Big].$$

We can reexpress the vectors v^{mI} in terms of the field strength H^{nopI} with the help of

$$v_m{}^I = \frac{1}{2}\epsilon_{mnop} H^{nopI},$$

and the resulting Lagrangian reads due to the antisymmetry of H^{nop}

$$\mathcal{L} = \frac{1}{4} F_{C^I C^J}\Big[(\partial^m C^I)(\partial_m C^J) + \frac{3}{2} H^{nopI} H_{nop}{}^J\Big],$$

which is the generalization of the bosonic part of (2.35) to an arbitrary number N_T of linear multiplets.

We now study the duality properties of the $N = 2$ tensor multiplet. It turns out that the $N = 2$ tensor multiplet is dual to the hypermultiplet in the same way as the $N = 1$ linear multiplet is dual to the $N = 1$ chiral multiplet (see section 2.4.3).

We start from a first order action corresponding to (3.20) [15], with $\chi_I, \bar{\chi}_I$ being (anti)-chiral superfields:

$$S = \int d^4x d^4\theta \Big[F(V^I, \Phi^I, \bar{\Phi}^I) - V^I(\chi_I + \bar{\chi}_I)\Big]. \tag{3.42}$$

Due to the (anti)-chirality of $\chi_I, \bar{\chi}_I$, we can rewrite them in terms of a general superfield Σ_I:

$$\chi_I = \bar{D}^2 \Sigma_I \qquad \bar{\chi}_I = D^2 \bar{\Sigma}_I.$$

Performing the variation with respect to the fields $\Sigma_I, \bar{\Sigma}_I$ using the variation rule (B.2), one immediately finds that in order to get a stationary action, the following property holds for the real superfields V^I

$$D^2 V^I = 0.$$

An analogous property is found when varying with respect to the $\bar{\Sigma}_I$ and one realizes that the V^I are indeed linear multiplets. Reinserting in (3.42) latter yields to (3.29). The variation of (3.42) with respect to V^I reads, according to the variation rules of the appendix B,

$$\delta S = \int d^4x d^4\theta \Big\{ \Big[F_V^I - (\chi_I + \bar{\chi}_I)\Big]\delta V \Big\}.$$

Since the variation of the action should vanish for any variation of the superfield V, one is led to

$$(\chi_I + \bar{\chi}_I) = F^I_V. \tag{3.43}$$

Equation (3.43) defines the V^I in terms of $(\chi_I + \bar{\chi}_I)$ and $\Phi^I, \bar{\Phi}^I$. Thus, one finds the Legendre transformation of F with respect to the V^I.

3.5 Hyperkähler geometry and tensor multiplet Lagrangians

In the previous section we rederived the results of [15, 16, 6]. We determined the bosonic part of a general $N = 2$ supersymmetric Lagrangian for N_T tensor multiplets. We also showed, that in order to obtain an action that is invariant under the $N = 2$ supersymmetry transformations, the general coupling function $F(C^I, A^I, \bar{A}^I)$ is subject to the constraints (3.31) and (3.32). We then performed the duality transformation of the tensor multiplets to hypermultiplets and obtained the Legendre transformation of F with respect to L (3.43).

In this section we perform the duality transformation, following [6], in terms of the component fields in order to examine the geometry of the target space, spanned by the hypermultiplet scalars.

The bosonic part of a non-linear σ-model for N_T $N = 2$ tensor multiplets involves N_T real scalars C^I, N_T complex scalars A^I and $\bar{A}^I$ as well as the N_T field strengths ${v_m}^I$ of the tensor gauge fields ${B_{mn}}^I$:

$$\begin{aligned}\mathcal{L} =& F_{C^I C^J}\Big[(\partial^m A^J)(\partial_m \bar{A}^I) + \frac{1}{4}[(\partial^m C^I)(\partial_m C^J) - v^{mI}{v_m}^J]\Big] + \\ & \frac{i}{2}\Big[F_{C^I \bar{A}^J}(\partial^m \bar{A}^J) - F_{C^I A^J}(\partial^m A^J)\Big]{v_m}^I.\end{aligned} \tag{3.44}$$

We now accomplish the duality transformation on the level of the component fields along the same lines as in [6]. For this purpose, one introduces the term

$$-\frac{1}{2} Y_I \partial_m v^{mI} \tag{3.45}$$

in the Lagrangian. Here, the Y_I are real, Lagrangian multipliers. The Lagrangian then reads

$$\begin{aligned}\mathcal{L} =& F_{C^I C^J}\Big[(\partial^m A^J)(\partial_m \bar{A}^I) + \frac{1}{4}[(\partial^m C^I)(\partial_m C^J) - v^{mI}{v_m}^J]\Big] + \\ & \frac{i}{2}\Big[F_{C^I \bar{A}^J}(\partial^m \bar{A}^J) - F_{C^I A^J}(\partial^m A^J)\Big]{v_m}^I - \frac{1}{2} Y_I \partial_m v^{mI}.\end{aligned} \tag{3.46}$$

The equations of motion for the Y_I guarantee, that the additional constraint $\partial_m v^{mI} = 0$ is satisfied.

With the help of the field equations for the v^{mI}, one can express the vectors v^{mI} in terms of the $A^I, \bar{A}^I$ and Y_I:

$$v^{mI} = F^{C^I C^J}\Big\{ i\Big[F_{C^J \bar{A}^K}(\partial^m \bar{A}^K) - F_{C^J A^K}(\partial^m A^K)\Big] + \partial^m Y_K \Big\}. \tag{3.47}$$

Here, $F^{C^I C^J}$ denotes the inverse of $F_{C^I C^J}$. Inserting (3.47) in (3.46) yields to the Lagrangian

$$\begin{aligned}\mathcal{L} =& F_{C^I C^J}\Big[(\partial^m A^J)(\partial_m \bar{A}^I) + \frac{1}{4}(\partial^m C^I)(\partial_m C^J)\Big] + \\ & \frac{1}{4}F^{C^I C^J}\Big\{\partial^m Y_I + i\Big[F_{C^I \bar{A}^K}(\partial^m \bar{A}^K) - F_{C^I A^K}(\partial^m A^K)\Big]\Big\}\times \\ & \Big\{\partial_m Y_J + i\Big[F_{C^J \bar{A}^L}(\partial_m \bar{A}^L) - F_{C^J A^L}(\partial_m A^L)\Big]\Big\}\end{aligned} \tag{3.48}$$

when using the fact, that one can rewrite (3.45)

$$-\frac{1}{2}\partial_m\Big[Y_I v^{mI}\Big] = -\frac{1}{2}(\partial_m Y_I)v^{mI} - \frac{1}{2}Y_I \partial_m v^{mI}$$

and neglect the appearing total derivative in the Lagrangian (3.46) as it will not contribute to the action. Defining a new set of N_T complex fields B_I, one makes contact with the dualization to hypermultiplets performed in the previous section:

$$B_I = \frac{1}{2}\Big(iY_I + F_{C^I}\Big). \tag{3.49}$$

Thus, the C^I and Y_I are determined by the $A^I, \bar{A}^I, B_I$ and $\bar{B}_I$, and accordingly the sum $(B+\bar{B})_I$ is defined by

$$(B+\bar{B})_I = F_{C^I} \tag{3.50}$$

which yields the Legendre transformation of F with respect to the C^I. Varying the equation (3.49)

$$\delta B_I = \frac{1}{2}\Big(i\delta Y_I + F_{C^I C^J}\delta C^J + F_{C^I A^J}\delta A^J + F_{C^I \bar{A}^J}\delta \bar{A}^J\Big), \tag{3.51}$$

the variation of the real coordinates C^I, Y_I can be expressed in terms of the complex coordinates and one finds

$$
\begin{aligned}
\delta C^I =& F^{C^I C^J}\Big[(\delta B_J + \delta\bar{B}_J) - F_{C^J A^K}\delta A^K - F_{C^J \bar{A}^K}\delta\bar{A}^K\Big] \\
\delta Y_I =& i(\delta\bar{B}_I - \delta B_I).
\end{aligned}
\tag{3.52}
$$

With (3.52) one can now determine the metric in terms of the complex coordinates by varying (3.48):

$$
\begin{aligned}
g_{A^I\bar{A}^J} =& F_{C^I C^J} + F_{A^I C^K} F^{C^K C^L} F_{C^L \bar{A}^J}, \\
g_{A^I\bar{B}^J} =& - F_{A^I C^K} F^{C^K C^J}, \\
g_{B^I\bar{A}^J} =& - F^{C^I C^K} F_{C^K \bar{A}^J}, \\
g_{B^I\bar{B}^J} =& F^{C^I C^J}.
\end{aligned}
\tag{3.53}
$$

This is a kählerian[3] metric, because it can be derived from a Kähler potential which admits the following form:

$$
\chi(A^I, B^I, \bar{A}^I, \bar{B}^I) = -F(C^I, A^I, \bar{A}^I) + (B+\bar{B})_I C^I \tag{3.54}
$$

We illustrate this fact briefly by determining $g_{A^I\bar{A}^J}$:

$$
\frac{\partial\chi}{\partial\bar{A}^J} = -\frac{\partial F}{\partial\bar{A}^J} \underbrace{-\frac{\partial F}{\partial C^K}\frac{\partial C^K}{\partial\bar{A}^J} + (B+\bar{B})_K \frac{\partial C^K}{\partial\bar{A}^J}}_{=0}.
$$

Using the chain rule and the constraint (3.31) we obtain

$$
g_{A^I\bar{A}^J} = -F_{A^I\bar{A}^J} - F_{C^K\bar{A}^J}\frac{\partial C^K}{\partial A^I} = F_{C^I C^J} + F_{A^I C^K} F^{C^K C^L} F_{C^L\bar{A}^J}.
$$

The remaining components of the metric can be evaluated in an equivalent way. As the lagrangian (3.44), is $N=2$ supersymmetric, the target space described by (3.53) is hyperkählerian [6]. This is a non-trivial proof.

We now constrict the Lagrangian not only to be $N=2$ supersymmetric, but also invariant under $N=2$ superconformal transformations. In [6] was argued that, using scaling properties, the constraint (3.31) must be extended by

$$
\begin{aligned}
C^I F_{C^I} + A^I F_{A^I} + \bar{A}^I F_{\bar{A}^I} = F, \\
A^I F_{A^I} - \bar{A}^I F_{\bar{A}^I} = 0,
\end{aligned}
\tag{3.55}
$$

in order to guarantee a $N=2$ superconformal invariant action. In addition, the function $F_{C^I A^J}(C, A, \bar{A})$ can be chosen to be symmetric in the indices I and J. The constraints for $N=2$ superconformal invariance therefore are

[3] Some remarks concerning Kähler and Hyperkähler spaces are given in appendix C.

$$
\begin{aligned}
F_{C^I C^J} + F_{A^I \bar{A}^J} &= 0, \\
C^I F_{C^I} + A^I F_{A^I} + \bar{A}^I F_{\bar{A}^I} &= F, \\
A^I F_{A^I} - \bar{A}^I F_{\bar{A}^I} &= 0, \\
F_{C^I A^J} - F_{C^J A^I} &= 0.
\end{aligned}
\tag{3.56}
$$

We now turn to an equivalent description of the $N = 2$ tensor multiplets and its self-interacting Lagrangian by using the projective superfield formalism[4] presented in [28, 29, 30, 6].

A function F which satisfies the first constraint of (3.56) can be derived (cf. [29]) from a contour integral

$$
F = \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} H\Big(\eta^I(\zeta), \zeta\Big), \tag{3.57}
$$

where γ is an appropriate chosen, closed contour and $\eta^I(\zeta)$ is defined in the following way

$$
\eta^I(\zeta) = \frac{A^I}{\zeta} + C^I - \zeta \bar{A}^I. \tag{3.58}
$$

The η^I are the $N = 2$ tensor multiplets expressed in terms of the projective superfield formalism.

We now show, that a function determined by (3.57), indeed satisfies the constraints (3.31),(3.32):

$$
\begin{aligned}
F_{C^I C^J} =& \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial^2 H}{\partial\eta^K \partial\eta^L} \frac{\partial\eta^K}{\partial C^I} \frac{\partial\eta^L}{\partial C^J} = \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial^2 H}{\partial\eta^I \partial\eta^J}, \\
F_{A^I \bar{A}^J} =& \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial^2 H}{\partial\eta^K \partial\eta^L} \frac{\partial\eta^K}{\partial A^I} \frac{\partial\eta^L}{\partial \bar{A}^J} = -\mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial^2 H}{\partial\eta^I \partial\eta^J}.
\end{aligned}
$$

The sum of these terms vanish, so the constraint (3.31) is satisfied.

The symmetry of $F_{C^I A^J}$ in the indices I, J can be proven analogously

$$
F_{C^I A^J} = \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial^2 H}{\partial\eta^K \partial\eta^L} \frac{\partial\eta^K}{\partial C^I} \frac{\partial\eta^L}{\partial A^J} = \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta^2} \frac{\partial^2 H}{\partial\eta^I \partial\eta^J},
$$

which is obviously symmetric in I, J.

We now implement the remaining constraints for superconformal invariance in the function $H(\eta^I(\zeta), \zeta)$. The homogenity constraint of F is translated to

[4] A short introduction to the projective superfield formalism is given in appendix D.

$$C^I F_{C^I} + A^I F_{A^I} + \bar{A}^I F_{\bar{A}^I} = \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial H}{\partial \eta^J} \Big\{ \frac{\partial \eta^J}{\partial C^I} C^I + \frac{\partial \eta^J}{\partial A^I} A^I + \frac{\partial \eta^J}{\partial \bar{A}^I} \bar{A}^I \Big\},$$

which results in a homogenity property of H:

$$\eta^I \frac{\partial H}{\partial \eta^I} = H. \tag{3.59}$$

The last constraint of (3.56) ensures the $SO(2)$ invariance of F and requires H to have no explicit ζ-dependence:

$$\begin{aligned} 0 = A^I F_{A^I} - \bar{A}^I F_{\bar{A}^I} &= \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial H}{\partial \eta^I} \Big\{ \frac{A^I}{\zeta} + \zeta \bar{A}^I \Big\} \\ &= -\mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial H}{\partial \eta^I} \zeta \frac{\partial \eta^I}{\partial \zeta} \\ &= -\mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i} \Big\{ \frac{d}{d\zeta} H - \frac{\partial H}{\partial \zeta} \Big\}. \end{aligned}$$

The term $\frac{d}{d\zeta} H$ vanishes, as it is a total derivative and the contour is chosen to be closed, so the expression is zero if H is solely a function of $\eta^I(\zeta)$ and exhibits no explicit ζ-dependence.

We now turn to the determination of the hyperkähler potential (3.54) by performing a Legendre transformation with respect to the C^I. Using (3.54), (3.57) and the fact that the derivative $\frac{\partial}{\partial C^I}$ can be performed before the integration over the contour γ, one obtains the following expression:

$$\chi(A^I, \bar{A}^I, B_I + \bar{B}_I) = \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \Big\{ - H\Big(\eta^I(\zeta)\Big) + C^I \frac{\partial H}{\partial \eta^I} \Big\}. \tag{3.60}$$

The remaining step is to express the C^I in terms of the $A^I, \bar{A}^I$ and $(B+\bar{B})_I$ in virtue of

$$F_{C^I} = \mathrm{Im} \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial H}{\partial \eta^I} = (B + \bar{B})_I.$$

As a consequence of the homogenity property of F_{C^I}

$$F_{C^I}(\lambda C^I, \lambda A^I, \lambda \bar{A}^I) = F_{C^I}(C^I, A^I, \bar{A}^I),$$

the expression for the C^I is 'homogenious' in the sense of

$$C^I(\lambda A^I, \lambda \bar{A}^I, B_I + \bar{B}_I) = \lambda C^I(A^I, \bar{A}^I, B_I + \bar{B}_I), \tag{3.61}$$

when taking λ to be real. This homogenity property is carried over to the hyperkähler potential $\chi(A^I, \bar{A}^I, B_I + \bar{B}_I)$.

Chapter 4

Summary of type IIB supergravity compactified on Calabi-Yau threefolds

In this chapter, we briefly review the compactification of type IIB supergravity on a Calabi-Yau threefold, and present the massless, bosonic $D = 4$ spectrum, following [2, 4].

The massless spectrum of type IIB supergravity in $D = 10$ consists of the dilaton $\hat{\varphi}$, the metric $\hat{g}$ and a 2-form $\hat{B}_2$ in the NS-NS sector and the axion $\hat{l}$, a 2-form $\hat{C}_2$ and a 4-form $\hat{C}_4$ in the R-R sector[1]. The action decomposes into three parts

$$S_{IIB}^{(10)} = S_{NS} + S_R + S_{CS}. \tag{4.1}$$

In the $D = 10$ Einstein frame, these components read using the form notation

$$\begin{aligned} S_{NS} &= -\int \Big(\frac{1}{2}\hat{R} \star \mathbf{1} + \frac{1}{4} d\hat{\varphi} \wedge \star d\hat{\varphi} + \frac{1}{4} e^{-\hat{\varphi}} \hat{H}_3 \wedge \star \hat{H}_3\Big) \\ S_R &= -\frac{1}{4}\int \Big(e^{2\hat{\varphi}} d\hat{l} \wedge \star d\hat{l} + e^{\hat{\varphi}} \hat{F}_3 \wedge \star \hat{F}_3 + \frac{1}{2}\hat{F}_5 \wedge \star \hat{F}_5\Big) \\ S_{CS} &= -\frac{1}{4}\int \hat{C}_4 \wedge \hat{H}_3 \wedge \hat{F}_3, \end{aligned} \tag{4.2}$$

where $\star$ denotes the Hodge-$\star$ operator and the field strengths are defined as

[1] As in [2] the hats '$\hat{}$' denote the ten-dimensional fields. In addition, we stress, that the reader should not mix up the forms presented here with the component fields of the previous chapter.

$$
\begin{aligned}
\hat{H}_3 =& d\hat{B}_2 \\
\hat{F}_3 =& d\hat{C}_2 - \hat{l} d\hat{B}_2 \\
\hat{F}_5 =& d\hat{C}_4 - \frac{1}{2} d\hat{B}_2 \wedge \hat{C}_2 + \frac{1}{2}\hat{B}_2 \wedge d\hat{C}_2 .
\end{aligned}
\tag{4.3}
$$

A compactification of type IIB supergravity on a Calabi-Yau threefold M leads to a $D = 4$ theory with $N = 2$ supersymmetry. One takes the line-element to be of the following form

$$
ds^2 = g_{mn} dx^m dx^n + g_{i\bar{j}} dy^i d\bar{y}^{\bar{j}} . \tag{4.4}
$$

$g_{mn}, m, n = 0, \ldots, 3$ is a Minkowski metric and $g_{i\bar{j}}, i, \bar{j} = 0, \ldots, 3$ is the metric on the Calabi-Yau manifold M. The massless, bosonic spectrum is summarized in table 4.1.

gravity multiplet	1	(g_{mn}, V_m^0)
vector multiplets	$h^{(1,2)}$	(V_m^K, z^K)
hypermultiplets	$h^{(1,1)}$	(v^a, b^a, c^a, ρ_a)
double-tensor multiplets	1	(B_2, C_2, φ, l)

Table 4.1: Massless spectrum of type IIB superstring theory compactified on a Calabi-Yau threefold

Following [2], the action of type IIB compactified on a Calabi-Yau manifold reads:

$$
\begin{aligned}
S^{(4)}_{IIB} = \int & -\frac{1}{2}R\star\mathbf{1} + \frac{1}{4}\mathrm{Re}\mathcal{M}_{\hat{K}\hat{L}}F^{\hat{K}}\wedge F^{\hat{L}} + \frac{1}{4}\mathrm{Im}\mathcal{M}_{\hat{K}\hat{L}}F^{\hat{K}}\wedge\star F^{\hat{L}} \\
& - G_{KL}dz^K\wedge\star d\bar{z}^L - G_{ab}dv^a\wedge\star dv^b - \frac{1}{4}d\mathrm{ln}\kappa\wedge\star d\mathrm{ln}\kappa \\
& - \frac{1}{4}d\varphi\wedge\star d\varphi - \frac{1}{4}e^{2\varphi}dl\wedge\star dl - e^{-\varphi}G_{ab}db^a\wedge\star db^b \\
& - e^{\varphi}G_{ab}\Big(dc^a - ldb^a\Big)\wedge\star\Big(dc^b - ldb^b\Big) \\
& - \frac{9G^{ad}}{4\kappa^2}\Big(d\rho_a - \frac{1}{2}\kappa_{abc}(c^b db^c - b^b dc^c)\Big)\wedge \\
& \star\,\Big(d\rho_d - \frac{1}{2}\kappa_{def}(c^e db^f - b^e dc^f)\Big) \\
& - \frac{\kappa^2}{144}e^{-\varphi}dB_2\wedge\star dB_2 - \frac{\kappa^2}{144}e^{\varphi}\Big(dC_2 - ldB_2\Big)\wedge\star\Big(dC_2 - ldB_2\Big) \\
& + \frac{1}{2}\Big(db^a\wedge C_2 + c^a dB_2\Big)\wedge\Big(d\rho_a - \kappa_{abc}c^b db^c\Big) \\
& + \frac{1}{4}\kappa_{abc}c^a c^b dB_2\wedge db^c.
\end{aligned}
\tag{4.5}
$$

$F^{\hat{K}}$ is defined as $F^{\hat{K}} = dV^{\hat{K}}$. The gauge kinetic matrix $\mathcal{M}_{\hat{K}\hat{L}}$ is given in [2] and the metric of the complex structure deformations G_{KL} can be derived from the holomorphic prepotential $\mathcal{F}$ using

$$
G_{KL} = \frac{\partial}{\partial z^K}\frac{\partial}{\partial \bar{z}^L}K_{cs}, \tag{4.6}
$$

where K_{cs} is given by

$$
K_{cs} = -\mathrm{ln}\Big\{i\Big[\bar{X}^{\hat{K}}\mathcal{F}_{\hat{K}} - X^{\hat{K}}\bar{\mathcal{F}}_{\hat{K}}\Big]\Big\}. \tag{4.7}
$$

The special coordinates are defined as $X^{\hat{K}} = (1, z^K)$. The metric G_{KL} is a special Kähler metric that is entirely determined by the holomorphic prepotential $\mathcal{F}$, which is function homogenous of degree 2.

Turning to the complexified Kähler deformations, the corresponding metric of the space of Kähler deformations G_{ab} is given by [2]

$$
G_{ab} = -\frac{3}{2}\Big(\frac{\kappa_{ab}}{\kappa} - \frac{3}{2}\frac{\kappa_a\kappa_b}{\kappa^2}\Big) \tag{4.8}
$$

where κ_{abc} are intersection numbers of the Calabi-Yau manifold and

$$
\begin{aligned}
\kappa_{ab} =& \kappa_{abc}v^c, \\
\kappa_a =& \kappa_{abc}v^b v^c, \\
\kappa =& \kappa_{abc}v^a v^b v^c.
\end{aligned}
\tag{4.9}
$$

The v^a are the scalars of the hypermultiplets, mentioned in table 4.1 with $a = 2, \ldots, h^{(1,1)} + 1$. The volume of the Calabi-Yau manifold in this notation is given by $Vol(Y) = \frac{1}{6}\kappa$. Confining ourselves to the classical case without quantum corrections, the classical geometry is determined by [7, 8]

$$\mathcal{F} = \frac{1}{4!}\kappa_{abc}\frac{X^a X^b X^c}{X^1}. \tag{4.10}$$

Chapter 5

Orientifold projection

The aim of this chapter is, to determine the Kähler potentials of the orientifold compactifications of type IIB string theories derived in [2]. They use $\mathcal{O}3/\mathcal{O}7$ and $\mathcal{O}5/\mathcal{O}9$ orientifolds to break the $N = 2$ supersymmetry to $N = 1$. Using the approach introduced in chapter 3, we rederive the Kähler potential of the $\mathcal{O}3/\mathcal{O}7$ projection along the lines of [3] and rewrite the result in the variables of [2]. Afterwards we determine the Kähler potential of the $\mathcal{O}5/\mathcal{O}9$ orientifold in the variables of [2] since this was not performed explicitly in [3].

The question, which underlying function H leads to the correct tensor multiplet Lagrangians after performing the contour integral, was answered in [7]. It turns out, that using the classical prepotential $\mathcal{F}$ of the hypermultiplet geometry (4.10), evaluated as a function of the projective tensor multiplets η^I, one obtains a coupling function which leads to the correct supergravity Lagrangian for the bosonic part of tensor multiplet sector after introducing a compensator to establish the correct homogenity property of H [7].

5.1 The general coupling function of IIB tensor multiplets

To derive the superspace Lagrangian, one starts with the holomorphic prepotential $\mathcal{F}(X)$ (4.10) evaluated as a function of the projective tensor superfields cf. (3.58):

$$\eta^I(\zeta) = \frac{A^I}{\zeta} + C^I - \zeta \bar{A}^I.$$

The prepotential reads

$$\mathcal{F}(X^I) = \frac{1}{4!}\kappa_{abc}\frac{X^a X^b X^c}{X^1}, \tag{5.1}$$

with $X^I = \{X^1, X^a\}, a = 2, 3, \ldots, N_T$. In the later sections we will specialize $N_T = h^{(1,1)} + 1$. The κ_{abc} are the triple intersection numbers of the Calabi-Yau manifold. Recall that $H(\eta)$ must be homogenous of degree one in order to obtain a superconformal invariant action (cf. (3.59)). Therefore, we are forced to introduce a compensator η^0 to establish the correct homogenity property [7]:

$$H(\eta^\Lambda) = \frac{\mathcal{F}(\eta^I)}{\eta^0} \qquad \Lambda = 0, \ldots, N_T. \tag{5.2}$$

At this stage, we are able to determine the coupling function F in virtue of equation (3.57). Then, performing the Legendre transformation with respect to the real scalars C^I, we obtain the hyperkähler potential (3.54), which enables us to determine the metric of the target space.

We illustrate the described approach in the following along the lines of [7] by imposing a special gauge choice $A^0 = 0$ for the compensator η^0. With this choice the pole structure of the function H becomes very simple, and the determination of the general coupling function F is a straightforward calculation.

5.1.1 The general coupling function with the gauge choice $A^0 = 0$

Imposing the gauge choice $A^0 = 0$, the projective superfield for the compensator η^0 becomes

$$\eta^0(\zeta) = \eta^0 = C^0. \tag{5.3}$$

Inserting (5.3) in (5.2), the contour integral (3.57) reads

$$F(A^\Lambda, \bar{A}^\Lambda, C^\Lambda) = \frac{1}{C^0}\mathrm{Im}\oint_\gamma \frac{d\zeta}{2\pi i\zeta}\mathcal{F}(\eta^I) = \frac{1}{C^0}\mathrm{Im}\oint_\gamma \frac{d\zeta}{2\pi i}\frac{\mathcal{F}(\zeta\eta^I)}{\zeta^3}. \tag{5.4}$$

In the last step, the homogenity property of $\mathcal{F}$ is used. The contour γ is chosen around the origin $\zeta = 0$, and one finds that the product $\zeta\eta^I$ does not vanish for $\zeta = 0$:

$$\zeta\eta^I = A^I + \zeta C^I - \zeta^2\bar{A}^I \neq 0 \text{ for } A \neq 0. \tag{5.5}$$

This is an important fact, as one can now evaluate the contour integral using the residue at $\zeta = 0$, assuming $\mathcal{F}(\zeta\eta^I)$ has no poles in ζ inside the contour around the origin. For a function f with a singularity at z_0, the contour integral with a contour γ chosen to enclose z_0 reads

$$\oint_\gamma f(z)dz = 2\pi i Res f(z)|_{z=z_0}. \tag{5.6}$$

The singularity, that one faces here is of order three, so one can determine the residue using

$$Res\left[\frac{\mathcal{F}(\zeta\eta^I)}{\zeta^3}\right]_{\zeta=0} = \lim_{\zeta\to 0}\frac{1}{2!}\frac{d^2}{d\zeta^2}\left[\frac{\mathcal{F}(\zeta\eta^I)}{\zeta^3}\zeta^3\right] = \lim_{\zeta\to 0}\frac{1}{2!}\frac{d^2}{d\zeta^2}\Big[\mathcal{F}(\zeta\eta^I)\Big]. \quad (5.7)$$

Performing the differentiation, one obtains

$$\frac{1}{2}\lim_{\zeta\to 0}\Big[\mathcal{F}_{JK}(\zeta\eta^I)\big[C^J - 2\zeta\bar{A}^J\big]\big[C^K - 2\zeta\bar{A}^K\big] - \mathcal{F}_J(\zeta\eta^I)2\bar{A}^J\Big], \quad (5.8)$$

where $\mathcal{F}_I = \frac{\partial\mathcal{F}}{\partial X^J}$. Taking the limit $\zeta \to 0$, the residue reads

$$Res\left[\frac{\mathcal{F}(\zeta\eta^I)}{\zeta^3}\right]_{\zeta=0} = \frac{1}{2}\Big[\mathcal{F}_{JK}(A^I)\big[C^J C^K\big] - 2\mathcal{F}_J(A^I)\bar{A}^J\Big].$$

Therefore, the general coupling function F is just the imaginary part of the residue and reads

$$F(A^\Lambda, \bar{A}^\Lambda, C^\Lambda) = -\frac{1}{2C^0}\mathrm{Im}\Big\{\mathcal{F}_{JK}(A^I)C^J C^K - 2\mathcal{F}_J(A^I)\bar{A}^J\Big\}.$$

Using the abbreviations

$$\begin{aligned} N_{JK}(A^I, \bar{A}^I) =& 2\mathrm{Im}\Big[\mathcal{F}_{JK}(A^I)\Big] = i\Big(\mathcal{F}_{JK}(A^I) - \bar{\mathcal{F}}_{JK}(\bar{A}^I)\Big), \\ K(A^I, \bar{A}^I) =& 2\mathrm{Im}\Big[\bar{A}^J\mathcal{F}_J\Big] = i(\bar{A}^I\mathcal{F}_I - A^I\bar{\mathcal{F}}_I), \end{aligned} \quad (5.9)$$

the final expression for F is (cf. [7])

$$F(A^\Lambda, \bar{A}^\Lambda, C^\Lambda) = -\frac{1}{4C^0}\Big[N_{JK}(A^I, \bar{A}^I)C^J C^K - 2K(A^I, \bar{A}^I)\Big]. \quad (5.10)$$

5.1.2 The hyperkähler potential with gauge choice $A^0 = 0$

In the last subsection, we derived the general coupling function F of a self-interacting tensor multiplet lagrangian (3.44) in the special gauge for the compensator η^0, namely choosing $A^0 = 0$. Using this coupling function, one can perform the Legendre transformation with respect to the real scalars C^Λ in order to obtain the Kähler potential for the hyperkähler metric (3.54):

$$\chi(A^\Lambda, B_\Lambda, \bar{A}^\Lambda, \bar{B}_\Lambda) = -F(C^\Lambda, A^\Lambda, \bar{A}^\Lambda) + (B + \bar{B})_\Lambda C^\Lambda.$$

The general coupling function F with the gauge choice $A^0 = 0$ is given in (5.10). We remind the reader, that the new set of variables $(B+\bar{B})_\Lambda$ are obtained by taking the derivative of F with respect to C^Λ

$$(B+\bar{B})_\Lambda = \frac{\partial F}{\partial C^\Lambda} = F_{C^\Lambda}, \tag{5.11}$$

and solving this equation for C^Λ.

The appearing summands of the Legendre transformation can be determined using the symmetry of N_{JK} in its indices:

$$\begin{aligned} C^0 F_{C^0} &= -F \\ C^I F_{C^I} &= -\frac{2}{4C^0} N_{IJ} C^I C^J. \end{aligned} \tag{5.12}$$

When solving (5.11) for the scalars C^Λ, one obtains for the $(B+\bar{B})_I$ (cf. the second equation of (5.12)):

$$\frac{C^I}{C^0} = 2N^{IJ}(B+\bar{B})_J, \tag{5.13}$$

with $N^{IJ} = N^{-1}_{IJ}$. Inserting (5.13) in the expression for $(B+\bar{B})_0$

$$(B+\bar{B})_0 = \frac{1}{4(C^0)^2}\Big[N_{IJ}C^I C^J - 2K\Big] \tag{5.14}$$

leads to

$$(B+\bar{B})_0 = N_{IJ}N^{IK}N^{JL}(B+\bar{B})_K(B+\bar{B})_L - \frac{K(A^I,\bar{A}^I)}{2(C^0)^2}, \tag{5.15}$$

which can be used to express $(C^0)^2$ in terms of the $(B+\bar{B})_\Lambda$ and the $A^I, \bar{A}^I$:

$$(C^0)^2 = \frac{K(A^I,\bar{A}^I)}{2\Big[(B+\bar{B})_I N^{IJ}(B+\bar{B})_J - (B+\bar{B})_0\Big]}. \tag{5.16}$$

Finally, the hyperkähler potential obtained by the Legendre transformation reads:

$$\chi(A^I,\bar{A}^I,C^0) = -2F(A^\Lambda,\bar{A}^\Lambda,C^\Lambda) - \frac{2}{4C^0}N_{IJ}C^I C^J = -\frac{K(A^I,\bar{A}^I)}{C^0}. \tag{5.17}$$

When expressing C^0 with the help of (5.16) in terms of the $(B+\bar{B})_\Lambda$ and the $A^I, \bar{A}^I$, the hyperkähler potential becomes

$$\chi(A^I,\bar{A}^I,B_\Lambda,\bar{B}_\Lambda) \propto \sqrt{2}\sqrt{K(A^I,\bar{A}^I)}\sqrt{(B+\bar{B})_I N^{IJ}(B+\bar{B})_J-(B+\bar{B})_0}, \tag{5.18}$$

as mentioned in [7]. According to [7], the minus sign in (5.17) is irrelevant and can be neglected.

5.1.3 The general coupling function without gauge choice

We now relax the gauge condition $A^0 = 0$ and turn to the general case following [8, 9]. The first step is to evaluate the general coupling function F, performing the contour integral

$$F = \mathrm{Im}\oint_\gamma \frac{d\zeta}{2\pi i\zeta}\frac{\mathcal{F}(\eta^I(\zeta))}{\eta^0(\zeta)}, \qquad \Lambda = 0,\ldots,N_T.$$

Again, we use the homogenity property of $\mathcal{F}$ in order to reexpress the above relation in terms of $\zeta\eta^\Lambda(\zeta)$.

Without imposing any gauge choice, the pole structure in the complex plane becomes more complicated. In the case with the gauge choice, one has to evaluate the residue at the point $\zeta = 0$. Now, one has to find the zeros of the equation

$$\zeta\eta^0(\zeta) = A^0 + \zeta C^0 - \zeta^2\bar{A}^0 = 0$$

in order to perform the contour integral. The zeros are

$$\zeta_\pm = \frac{C^0 \mp \sqrt{(C^0)^2+4A^0\bar{A}^0}}{2\bar{A}^0}.$$

Evaluating the residue at the point $\zeta = \zeta_+$, one finds, using $\eta^I_+ = \eta^I(\zeta_+)$

$$\eta^I_\pm = C^I - \frac{C^0}{2}\Big(\frac{A^I}{A^0}+\frac{\bar{A}^I}{\bar{A}^0}\Big) \pm \frac{|\bar{r}^0|}{2}\Big(-\frac{A^I}{A^0}+\frac{\bar{A}^I}{\bar{A}^0}\Big), \tag{5.19}$$

for the residue

$$Res\Big[\frac{\mathcal{F}(\eta^I(\zeta))}{\zeta\eta^0(\zeta)}\Big]_{\zeta=\zeta_+} = \frac{\mathcal{F}(\eta^I_+)}{\sqrt{(C^0)^2+4A^0\overline{A}^0}}. \tag{5.20}$$

Here, the residue can be evaluated using

$$Res\Big[\frac{f(z)}{g(z)}\Big]\Big|_{z=z_0} = \frac{f(z_0)}{g'(z_0)},$$

where the $'$ denote differentiation with respect to the variable z. This rule can be used as the functions $\mathcal{F}(\eta^I(\zeta))$ and $\zeta\eta^0(\zeta)$ are analytic in $\zeta = \zeta_+$ and η_+ is a solution of the equation $\zeta\eta^0(\zeta) = 0$ whereas $(\zeta\eta^0(\zeta))'$ evaluated at

ζ_+ does not vanish.

The $N = 1$ scalars A^Λ, $\bar{A}^\Lambda$ and C^Λ, $\Lambda = 0, 1, 2, \ldots, N_T$ of the $N = 2$ theory can be grouped into vectors

$$\vec{r}^\Lambda = [-i(A^\Lambda - \bar{A}^\Lambda), A^\Lambda + \bar{A}^\Lambda, C^\Lambda], \tag{5.21}$$

which are invariant under the $SU(2)$ transformation of the second supersymmetry (cf. (3.11)), with the scalar product [8]

$$\vec{r}^\Lambda \cdot \vec{r}^\Sigma = 4A^{(\Lambda}\overline{A}^{\Sigma)} + C^\Lambda C^\Sigma. \tag{5.22}$$

The reformulated coupling function now reads (cf (5.20)):

$$F = \frac{1}{|\vec{r}^0|}\mathrm{Im}\Big\{\mathcal{F}(\eta_+^I)\Big\}. \tag{5.23}$$

5.1.4 The hyperkähler potential without gauge choice

We are now in the position to determine the hyperkähler potential for the hyperkähler metric by performing the Legendre transformation of (5.23) with respect to the real scalars C^Λ, following [8, 9]. One first determines the new variables $(B + \bar{B})_\Lambda$ in virtue of (5.11). Using (5.23) one determines analogously to the case with the gauge choice

$$F_{C^0} = -\frac{C^0}{|\vec{r}^0|^3}\mathrm{Im}\Big\{\mathcal{F}(\eta_+^I)\Big\} + \frac{1}{|\vec{r}^0|}\mathrm{Im}\Big\{\mathcal{F}_I(\eta_+^I)\frac{\partial \eta_+^I}{\partial C^0}\Big\} \tag{5.24}$$

$$F_{C^I} = \frac{1}{|\vec{r}^0|}\mathrm{Im}\Big\{\mathcal{F}_I(\eta_+^I)\Big\} \tag{5.25}$$

where $\mathcal{F}_I$ denotes $\frac{\partial \mathcal{F}}{\partial \eta^I}$. The appearing partial derivative $\frac{\partial \eta^I}{\partial C^0}$ reads

$$\frac{\partial \eta_+^I}{\partial C^0} = -\frac{1}{2}\Big[\frac{A^I}{A^0} + \frac{\bar{A}^I}{\bar{A}^0}\Big] + \frac{C^0}{2|\vec{r}^0|}\Big[-\frac{A^I}{A^0} + \frac{\bar{A}^I}{\bar{A}^0}\Big].$$

The expression for the Kähler potential then reads

$$\begin{aligned}\chi = &-\frac{(C^0)^2}{|\vec{r}^0|^3}\mathrm{Im}\Big[\mathcal{F}(\eta_+^\Lambda)\Big] + \frac{1}{|\vec{r}^0|}\mathrm{Im}\Big\{\mathcal{F}_1 q^1 + \mathcal{F}_a q^a\Big\} \\ &-\frac{1}{|\vec{r}^0|}\mathrm{Im}\Big\{\mathcal{F}(\eta_+^I)\Big\},\end{aligned}$$

with the abbreviation

$$q^I = \frac{C^0}{2}\Big[\frac{A^I}{A^0} + \frac{\bar{A}^I}{\bar{A}^0}\Big] + \frac{(C^0)^2}{2|\vec{r}^0|}\Big[-\frac{A^I}{A^0} + \frac{\bar{A}^I}{\bar{A}^0}\Big] + C^I. \tag{5.26}$$

Equation (5.26) can be simplified using η_+^I:

$$q^I = \eta_+^I - \frac{4A^0\bar{A}^0}{2|\vec{r}^0|}\Big[-\frac{A^I}{A^0}+\frac{\bar{A}^I}{\bar{A}^0}\Big].$$

This leads to the following expression for χ

$$\begin{aligned}\chi =& \Big[-\frac{(C^0)^2}{|\vec{r}^0|^3}-\frac{1}{|\vec{r}^0|}\Big]\mathrm{Im}\Big[\mathcal{F}(\eta_+^I)\Big]\\ &+\frac{1}{|\vec{r}^0|}\mathrm{Im}\Big[\underbrace{\mathcal{F}_1\eta_+^1+\mathcal{F}_a\eta_+^a}_{=2\mathcal{F}}\Big]\\ &-\frac{1}{|\vec{r}^0|}\mathrm{Im}\Big[\mathcal{F}_1\frac{4A^0\bar{A}^0}{2|\vec{r}^0|}\Big[-\frac{A^1}{A^0}+\frac{\bar{A}^1}{\bar{A}^0}\Big]+\mathcal{F}_a\frac{4A^0\bar{A}^0}{2|\vec{r}^0|}\Big[-\frac{A^a}{A^0}+\frac{\bar{A}^a}{\bar{A}^0}\Big]\Big].\end{aligned}$$

In the second line we made use of the homogenity properties of $\mathcal{F}$. A short calculation reveals

$$\begin{aligned}&\Big[-\frac{(C^0)^2}{|\vec{r}^0|^3}-\frac{1}{|\vec{r}^0|}\Big]\mathrm{Im}\Big[\mathcal{F}(\eta_+^I)\Big]+\frac{2}{|\vec{r}^0|}\mathrm{Im}\Big[\mathcal{F}(\eta_+^I)\Big]=\\ &\frac{4A^0\bar{A}^0}{|\vec{r}^0|^3}\mathrm{Im}\Big[\mathcal{F}(\eta_+^I)\Big]=\\ &\frac{4A^0\bar{A}^0}{2|\vec{r}^0|^3}\mathrm{Im}\Big[\eta^1\mathcal{F}_1+\eta^a\mathcal{F}_a\Big].\end{aligned}$$

The potential then reads

$$\chi = \frac{4A^0\bar{A}^0}{2|\vec{r}^0|^3}\mathrm{Im}\Big[\eta^1\mathcal{F}_1+\eta^a\mathcal{F}_a\Big]-\frac{4A^0\bar{A}^0}{2|\vec{r}^0|^2}\mathrm{Im}\Big\{\mathcal{F}_1\Big[\frac{\bar{A}^1}{\bar{A}^0}-\frac{A^1}{A^0}\Big]+\mathcal{F}_a\Big[\frac{\bar{A}^a}{\bar{A}^0}-\frac{A^a}{A^0}\Big]\Big\},$$

which can be further simplified using η_-^I

$$\chi = \frac{4A^0\bar{A}^0}{2|\vec{r}^0|^3}\mathrm{Im}\Big[\eta_-^1\mathcal{F}_1+\eta_-^a\mathcal{F}_a\Big].$$

Now, one introduces the new variable

$$z^a = \frac{\eta_+^a}{\eta_+^1} = b^a + iv^a, \qquad (5.27)$$

and uses the homogenity property of $\mathcal{F}_I$ to scale out η_+^1. After that operation, χ reads

$$\chi = \frac{4A^0\bar{A}^0}{2|\vec{r}^0|^3}\mathrm{Im}\Big\{\eta_-^1\eta_+^1\Big[\mathcal{F}_1(z)+\bar{z}^a\mathcal{F}_a(z)\Big]\Big\}.$$

	$\mathcal{O}3/\mathcal{O}7$		$\mathcal{O}5/\mathcal{O}9$	
gravity multiplet	1	g_{mn}	1	g_{mn}
vector multiplets	$h_+^{(1,2)}$	V^k	$h_-^{(1,2)}$	V^κ
chiral multiplets	$h_-^{(1,2)}$	X^κ	$h_+^{(1,2)}$	X^k
chiral multiplets	1 $h_-^{(1,1)}$	(φ, l) (b^α, c^α)	$-$ $h_+^{(1,1)}$	$-$ (v^μ, c^μ)
linear multiplets	$-$ $h_+^{(1,1)}$	$-$ (v^μ, ρ_μ)	1 $h_-^{(1,1)}$	(φ, C_2) (b^α, ρ_α)

Table 5.1: Spectrum of bosonic fields of Calabi-Yau orientifold compactifications with $\mathcal{O}3/\mathcal{O}7$ and $\mathcal{O}5/\mathcal{O}5$ planes

Using the identity (cf. [9])

$$\eta_+^I \eta_-^J + \eta_-^I \eta_+^J = \frac{1}{2v^0 \bar{v}^0} (\vec{r}^0 \times \vec{r}^I) \cdot (\vec{r}^0 \times \vec{r}^J), \tag{5.28}$$

the hyperkähler potential finally reads

$$\chi = \frac{|\vec{r}^0 \times \vec{r}^1|^2}{2|\vec{r}^0|^3} \mathrm{Im}\Big\{ F_1(z) - \bar{z}^a F_a(z) \Big\} = \frac{|\vec{r}^0 \times \vec{r}^1|^2}{2|\vec{r}^0|^3} V(t) \tag{5.29}$$

with $V(t) = \frac{1}{3!}\kappa_{abc} v^a v^b v^c$. This is the same result, as derived in [8].

The b^a and v^a introduced in (5.27) are in the following identified with the complexified Kähler moduli [8] (cf. table 4.1).

5.2 Superspace description of Calabi-Yau orientifolds of IIB superstrings

We no apply the result of [8], namely the Kähler potential of the hyperkähler metric (5.29), to the the orientifold projections following [3]. The number of tensor multiplets N_T is now specialized to $h^{(1,1)} + 1$. The orientifold projection of the compactified type IIB superstring theory is performed by the combined operation of an involution symmetry on the Calabi-Yau threefold with an orientation reversal on the worldsheet [3, 1]. The considered orientifold projections truncates the supersymmetry from $N = 2$ to $N = 1$.

The orientifold projections, which are performed in the following, lead to Calabi-Yau orientifolds with either $\mathcal{O}3/\mathcal{O}7$ or $\mathcal{O}5/\mathcal{O}9$ planes. In the first case, the involution operation transforms the holomorphic three-form as $\Omega \longrightarrow -\Omega$ and in the $\mathcal{O}5/\mathcal{O}9$ case it is transformed according to $\Omega \longrightarrow \Omega$ [1, 2]. The resulting spectrum of the two orientifolds is summarized in table 5.1.

The aim is to compare the resulting Kähler potentials with the one derived in [2]. For this purpose, one has to bring the effective action into the standard form. That is done by finding a complex structure for the Kähler space, spanned by the chiral fields. The holomorphic coordinates for the chiral fields are, according to [2, 3], given by

$$\tau \equiv l + ie^{-\varphi} \qquad \text{and} \qquad G^\alpha \equiv c^\alpha - \tau b^\alpha \tag{5.30}$$

for the case of a $\mathcal{O}3/\mathcal{O}7$ orientifold projection.

When considering the $\mathcal{O}5/\mathcal{O}9$ orientifold projection, the holomorphic coordinates are given by

$$\tau^\mu \equiv e^{-\varphi} v^\mu + ic^\mu. \tag{5.31}$$

Note the range of our indices. We defined, as indicated by table 5.1, $\alpha = 2, \ldots, h_-^{(1,1)} + 1$, $\mu = 2, \ldots, h_+^{(1,1)} + 1$, $k = 1, \ldots, h_+^{(1,2)}$ and $\kappa = 1, \ldots, h_-^{(1,2)}$. According to [3, 9, 8], the scalar fields of table 4.1 can be expressed by the scalar fields of the superconformal theory when taking the superconformal quotient. This is done in [9] and the result is

$$l + ie^{-\varphi} = \frac{1}{2\sqrt{2}|\vec{r}^0|^2}[\vec{r}^0 \cdot \vec{r}^1 + i|\vec{r}^0 \times \vec{r}^1|] \tag{5.32}$$

$$b^a + iv^a = \frac{\eta_+^a}{\eta_+^1} \tag{5.33}$$

$$lb^a - c^a = \frac{\vec{r}^0 \cdot \vec{r}^a}{2\sqrt{2}|\vec{r}^0|^2}, \tag{5.34}$$

with $a = 2, \ldots, h^{(1,1)} + 1$. The vectors $\vec{r}^\Lambda$ are the same vectors as defined in (5.21).

5.2.1 Truncation of the projective superfields

For applying the developed scheme, one has to find the right constraints for the projective superfields, in order to obtain the desired spectrum (cf. table 5.1). The considered orientifold projections can be easily performed in the projective superfield formalism by defining a parity operator Π on the complex coordinate ζ [3]. One then requires the projective superfields to be either even or odd under this parity operation:

$$\Pi\eta(\zeta) = \eta(-\zeta) = \eta(\zeta) \text{ parity-even} \tag{5.35}$$

$$\Pi\eta(\zeta) = \eta(-\zeta) = -\eta(\zeta) \text{ parity-odd.} \tag{5.36}$$

In the first case the $N = 1$ chiral multiplet is A is projected out while in the second case the $N = 1$ tensor multiplet is projected out:

$$\Pi\eta(\zeta) = \eta(-\zeta) = -\frac{A}{\zeta} + C + \zeta\bar{A} = \frac{A}{\zeta} + C - \zeta\bar{A} = \eta(\zeta) \Longleftrightarrow A = 0$$
$$\Pi\eta(\zeta) = \eta(-\zeta) = -\frac{A}{\zeta} + C + \zeta\bar{A} = -\frac{A}{\zeta} - C + \zeta\bar{A} = -\eta(\zeta) \Longleftrightarrow C = 0.$$

The projective superfields η^Λ may be subject to either one of these conditions, so that one ends up afterwards with a hyperkähler potential $\chi(v, \bar{v}, w + \bar{w})$ with an arbitrary number N_C of chiral multiplets and N_T tensor multiplets. The number N_C and N_T are chosen according to table 5.1.

5.2.2 Orientifolding to $\mathcal{O}3/\mathcal{O}7$

With the insight of the previous section, we examine now the truncation to $\mathcal{O}3/\mathcal{O}7$ [3]. Comparing the tables 4.1 and 5.1, one realizes that the fields $B_2, C_2, v^\alpha, \rho_\alpha, b^\mu$ and c^μ must be projected out. As far as concerning the double-tensor sector, this can be achieved by imposing η^0 and η^1 to be odd under the parity operator Π. The remaining $h_-^{(1,1)}$ η^α must also be odd, whereas the $h_+^{(1,1)}$ η^μ must be even under parity.

This leads to the following truncation of the $N = 1$ tensor multiplet scalars

$$C^0 = 0, \qquad C^1 = 0, \qquad C^\alpha = 0, \qquad A^\mu = 0. \tag{5.37}$$

The corresponding projective superfields read

$$\begin{aligned} \eta_+^1 &= \frac{|\bar{r}^0|}{2}\Big(- \frac{A^1}{A^0} + \frac{\bar{A}^1}{\bar{A}^0}\Big) \\ \eta^\alpha &= \frac{|\bar{r}^0|}{2}\Big(- \frac{A^\alpha}{A^0} + \frac{\bar{A}^\alpha}{\bar{A}^0}\Big) \\ \eta^\mu &= C^\mu. \end{aligned} \tag{5.38}$$

Inserting these truncated, projective superfields into the equations (5.32)-(5.34), one can determine the complex structures for the orientifolded theory.

- For the axion-dilaton system (5.32) one finds

$$\tau \equiv l + ie^{-\varphi} = \frac{1}{2\sqrt{2}}\frac{A^1}{A^0}. \tag{5.39}$$

- With the help of (5.33) and (5.38) one can now determine

$$b^\alpha = \frac{\bar{A}^\alpha A^0 - A^\alpha \bar{A}^0}{\bar{A}^1 A^0 - A^1 \bar{A}^0} \text{ and } v^\alpha = 0. \tag{5.40}$$

- Using (5.30), (5.34) and $v^\alpha = 0$, the expressions for the G^α read

$$G^\alpha = -ie^{-\varphi} - \frac{\vec{r}^0 \cdot \vec{r}^\alpha}{2\sqrt{2}|\vec{r}^0|^2} = -\frac{1}{2\sqrt{2}} \frac{A^\alpha}{A^0}. \tag{5.41}$$

- And finally, with the help of $A^\mu = 0$, one can determine

$$iv^\mu = \frac{\sqrt{A^0 \bar{A}^0} C^\mu}{\bar{A}^1 A^0 - A^1 \bar{A}^0} \text{ and } b^\mu = 0. \tag{5.42}$$

Since the scalar product $\vec{r}^0 \cdot \vec{r}^\mu = 0$, it follows from (5.34) that $c^\mu = 0$.

With these expressions one can determine the Kähler potential and compare it with the derived Kähler potentials of [2].

5.2.3 Orientifolding to $\mathcal{O}5/\mathcal{O}9$

Considering the orientifold projection using $\mathcal{O}5/\mathcal{O}9$ planes, one realizes after comparing the tables 4.1 and 5.1, that the fields $l, b^\mu, v^\alpha, c^\alpha$ and ρ^μ have to vanish. It is obvious, that in order to project out the axion l, the real part of (5.32) must vanish. That leads (in combination with the other constraints) to the following behaviour of the projective superfields under parity: η^0, η^μ must be odd under the parity operator Π, whereas η^1, η^α must be even.

$$C^0 = 0, \quad A^1 = 0, \quad A^\alpha = 0, \quad C^\mu = 0. \tag{5.43}$$

Now the projective superfields read

$$\begin{aligned} \eta_+^\mu &= \frac{|\vec{r}^0|}{2}\Big(-\frac{A^\mu}{A^0} + \frac{\bar{A}^\mu}{\bar{A}^0}\Big) \\ \eta^{1/\alpha} &= C^{1/\alpha}. \end{aligned} \tag{5.44}$$

After performing equivalent steps as before, one obtains the complex structures as follows:

- Calculating τ one finds that

$$\tau \equiv l + ie^{-\varphi} = \pm i \frac{1}{4\sqrt{2}} \frac{C^1}{\sqrt{A^0 \bar{A}^0}} \tag{5.45}$$

  and, indeed, l is projected out.

- With equation (5.33) one calculates

$$b^\alpha = \frac{C^\alpha}{C^1} \text{ and } v^\alpha = 0. \tag{5.46}$$

  In this case the v^α are projected out, as well as the c^α because the $c^\alpha \propto \vec{r}^0 \cdot \vec{r}^\alpha$ vanish.

- Turning to the μ-indexed quantities one first finds that

$$iv^\mu = \frac{\bar{A}^\mu A^0 - A^\mu \bar{A}^0}{C^1\sqrt{A^0\bar{A}^0}} \text{ and } b^\mu = 0. \tag{5.47}$$

- Now one is in the position to determine

$$c^\mu = -\frac{\vec{r}^0 \cdot \vec{r}^\mu}{2\sqrt{2}|\vec{r}^0|^2} = \frac{A^0\bar{A}^\mu + \bar{A}^0 A^\mu}{4\sqrt{2}A^0\bar{A}^0}. \tag{5.48}$$

- Using these expressions and (5.31) one finally finds

$$\tau^\mu = -\frac{i}{2\sqrt{2}}\frac{A^\mu}{A^0}. \tag{5.49}$$

5.3 Kähler potentials of the orientifolds

According to [3], the Kähler potential can be derived from the hyperkähler potential with the help of

$$\mathcal{K}(A^I, \bar{A}^I, B_I + \bar{B}_I) = -\log[\chi(A^I, \bar{A}^I, B_I + \bar{B}_I)]. \tag{5.50}$$

In our case, we derived in section 5.1.4 the hyperkähler potential for the tree-level. It is given by (compare (5.29))

$$\chi = 4|\vec{r}^0|e^{-2\varphi}V(v), \tag{5.51}$$

with $V(v) = \frac{1}{3!}\kappa_{abc}v^a v^b v^c$, where κ_{abc} are the triple intersection numbers of the Calabi-Yau manifold. With the help of (5.50) one can determine the Kähler potentials of the two orientifold projections, considered in the previous chapters.

We now remind the reader, that the dilaton is expressed in terms of the $A^\Lambda, \bar{A}^\Lambda, (B + \bar{B})_\Lambda$ as (5.32)

$$e^{-\varphi} = \frac{1}{2\sqrt{2}|\vec{r}^0|^2}|\vec{r}^0 \times \vec{r}^1|. \tag{5.52}$$

Furthermore, we have $|\vec{r}^0| = 2\sqrt{A^0\bar{A}^0}$. Inserting these expressions in the expression for the Kähler potential above, we find

$$\begin{aligned}\mathcal{K}(A^I, \bar{A}^I, B_I + \bar{B}_I) = \underbrace{-\log[8]}_{\text{irrelevant constant}} & \underbrace{-\frac{1}{2}\Big(\log A^0 + \log \bar{A}^0\Big)}_{\text{Kähler transformation}} - \\ & - 2\log[e^{-\varphi}] - \log V(v).\end{aligned}$$

Dropping all irrelevant terms, we can reduce the Kähler potential to

$$\mathcal{K}(A^\Lambda, \bar{A}^\Lambda, (B+\bar{B})_\Lambda) = -2\log[e^{-\varphi}] - \log V(v). \tag{5.53}$$

5.3.1 Kähler potential with $\mathcal{O}3/\mathcal{O}7$ orientifolds

We now determine the Kähler potential for the type IIB theory orientifolded theory using $\mathcal{O}3/\mathcal{O}7$ planes. Using the expression for τ (5.39), one can substitute the dilaton:

$$\mathcal{K} = -2\log\Big[-i(\tau - \overline{\tau})\Big] - \log V(v). \tag{5.54}$$

Now we must perform the Legendre transformation with respect to the v^μ in order to obtain a Kähler potential expressed solely in terms of chiral multiplet scalars. We introduce the new variables $(B+\bar{B})_\mu$ by differentiating the general coupling function (5.23) with respect to the v^μ. We obtain

$$(B+\bar{B})_\mu = F_{v^\mu} = -\frac{i}{2}\frac{1}{|\bar{r}^0|}\Big[\mathcal{F}_\mu - \bar{\mathcal{F}}_\mu\Big] \tag{5.55}$$

(Compare with (5.25)). Taking the derivative of (5.1) and evaluating the imaginary part, we find

$$(B+\bar{B})_\mu = \frac{3}{|\bar{r}^0|}\frac{\kappa_{\mu bc}}{4!}\frac{-\mathrm{Re}[\eta_+^b]\mathrm{Re}[\eta_+^c] + \mathrm{Im}[\eta_+^b]\mathrm{Im}[\eta_+^c]}{\mathrm{Im}[\eta_+^1]}. \tag{5.56}$$

The terms we dropped here are irrelevant due to vanishing of the real part of η_+^1. The appearing real and imaginary parts are

$$\begin{aligned}
\mathrm{Re}[\eta_+^\mu] &= C^\mu \\
\mathrm{Im}[\eta_+^\alpha] &= -i\frac{|\bar{r}^0|}{2}\Big[-\frac{A^\alpha}{A^0} + \frac{\bar{A}^\alpha}{\bar{A}^0}\Big].
\end{aligned}$$

Now, we express the C^μ, A^α and $\bar{A}^\alpha$ in terms of the variables (5.39) - (5.42) and we find

$$\begin{aligned}
C^\mu &= 2\sqrt{2}\sqrt{A^0\bar{A}^0}v^\mu\Big[\bar{\tau} - \tau\Big] \\
-i\frac{|\bar{r}^0|}{2}\Big[-\frac{A^\alpha}{A^0} + \frac{\bar{A}^\alpha}{\bar{A}^0}\Big] &= 2i\sqrt{2}\sqrt{A^0\bar{A}^0}\Big[\bar{G}^\alpha - G^\alpha\Big].
\end{aligned} \tag{5.57}$$

With a little algebra, we find using (5.57) in (5.56)

$$(B+\bar{B})_\mu = \frac{3\sqrt{2}}{4!}\Bigg\{\kappa_{\mu\alpha\beta}\frac{[G-\bar{G}]^\alpha[G-\bar{G}]^\beta}{-i(\tau-\bar{\tau})} + \kappa_{\mu\nu\rho}t^\nu t^\rho\Big[-i(\tau-\bar{\tau})\Big]\Bigg\}. \tag{5.58}$$

Comparing this with the expression for $T_\alpha + \overline{T}_\alpha$ using equation (3.48) and (3.49) of [2] with $\alpha = 1$ we can identify (up to a factor of $\frac{6}{\sqrt{2}}$) our the Kähler coordinates G^α with the G^a of [2]. The redefinition relation for the linear multiplets reads

$$v^\nu \equiv \sqrt{2}\frac{L^\alpha}{\sqrt{-i(\tau-\overline{\tau})}\sqrt{\mathcal{K}_{\alpha\beta\gamma}L^\alpha L^\beta L^\gamma}}.$$

Using this relation to express the Kähler potential (5.54) in terms of the linear multiplets L^α, the Kähler potential becomes

$$\hat{\mathcal{K}} = -\frac{3}{2}\log[2] - \frac{1}{2}\log\Big[-i(\tau-\bar{\tau})\Big] + \frac{1}{2}\log\Big[\frac{1}{3!}\kappa_{\alpha\beta\gamma}L^\alpha L^\beta L^\gamma\Big]. \tag{5.59}$$

Up to an additive constant, the result agrees with (3.52) of [2] when multiplied by a factor of 2. The appearing factor may be explained by the different definitions of the Kähler metric. In [31] for example, the metric is defined as

$$g_{i\bar{j}} = 2\partial_i\partial_{\bar{j}}K(z,\bar{z}) \tag{5.60}$$

5.3.2 Kähler potential with $\mathcal{O}5/\mathcal{O}9$ orientifolds

After the orientifold projection, the hyperkähler potential reads

$$\chi = \frac{(C^1)^2}{|A^0|}V(v). \tag{5.61}$$

This result is obtained by inserting (5.45) into the relation for χ (5.51).

We now have to dualize the tensor superfield C^1 to hypermultiplets, namely $\frac{\partial F}{\partial C^1} = (B+\bar{B})_1$ and we find using (5.25) and the symmetry properties of κ_{abc} after taking the imaginary part

$$C^1 = \frac{2|A^0|(B+\bar{B})_1}{\frac{1}{4!}\kappa_{abc}(v^a v^b v^c - 3b^a b^b v^c)}. \tag{5.62}$$

Now we are able to dualize remaining tensor superfields C^α using again (5.25). The imaginary part of $\mathcal{F}_{C^\alpha}$ reads

$$\mathrm{Im}\{\mathcal{F}_{C^\alpha}\} = \frac{6}{4!}\kappa_{\alpha\beta\mu}\frac{C^\beta\frac{|\bar{r}^0|}{2}(-\frac{A^\mu}{A^0}+\frac{\bar{A}^\mu}{\bar{A}^0})}{C^1} \tag{5.63}$$

Using the expressions for the b^α and v^μ, namely (5.46) and (5.47), we find:

$$F_{C^\alpha} = \frac{i}{\sqrt{2}}e^{-\varphi}\kappa_{\alpha\beta\mu}b^\beta v^\mu \equiv (B+\bar{B})_\alpha. \tag{5.64}$$

As we want dualize the linear multiplet, we introduce the matrix

$$\Theta_{\alpha\beta} = \frac{i}{\sqrt{2}} e^{-\varphi} \kappa_{\alpha\beta\mu} v^{\mu}, \tag{5.65}$$

and we can solve (5.64) for the b^{α} when introducing the inverse matrix $\Theta^{\alpha\beta}$ of $\Theta_{\alpha\beta}$

$$b^{\alpha} = \Theta^{\alpha\beta}(B + \bar{B})_{\beta}. \tag{5.66}$$

Having done this step, we are now enabled to express $(B + \bar{B})_1$ completely in terms of chiral multiplets and the dilaton by replacing the appearing b^{α} in (5.62) using (5.66) as well as (5.45), and solving (5.62) for $(B + \bar{B})_1$. We find

$$(B + \bar{B})_1 = \frac{2\sqrt{2}}{4!} e^{-\varphi} \kappa_{\mu\nu\rho} v^{\mu} v^{\nu} v^{\rho} - \frac{1}{2i}(B + \bar{B})_{\alpha} \Theta^{\alpha\beta} (B + \bar{B})_{\beta}. \tag{5.67}$$

Taking the logarithm of (5.61), and substituting C^1 using the variables (5.62), (5.66) and (5.67), the Kähler potential reads (up to constants and a Kähler transformation)

$$\mathcal{K} = -2\ln\left[(B+\bar{B})_1 + \frac{1}{2i}(B+\bar{B})_{\alpha}\Theta^{\alpha\beta}(B+\bar{B})_{\beta}\right] + \ln\left[\frac{1}{3!}\kappa_{\mu\nu\rho}v^{\mu}v^{\nu}v^{\rho}\right]. \tag{5.68}$$

After redefining the v^{μ} in terms of new variables $\tilde{\tau}^{\mu}$

$$v^{\mu} = \frac{1}{2} e^{\varphi} [\tilde{\tau} + \bar{\tilde{\tau}}]^{\mu} \tag{5.69}$$

we find a Kähler potential of the following form

$$\begin{aligned} \mathcal{K} = &-\frac{1}{2}\ln\left[(B + \bar{B})_1 + \frac{1}{2i}(B + \bar{B})_{\alpha}\Theta^{\alpha\beta}(B + \bar{B})_{\beta}\right] \\ &-\frac{1}{2}\ln\left[\frac{1}{3!}\kappa_{\mu\nu\rho}(\tilde{\tau} + \bar{\tilde{\tau}})^{\mu}(\tilde{\tau} + \bar{\tilde{\tau}})^{\nu}(\tilde{\tau} + \bar{\tilde{\tau}})^{\rho}\right] \end{aligned} \tag{5.70}$$

which differs by an overall factor of 2 from equation (4.17) of [2]. Again, we encounter here an overall factor of 2. As in the previous chapter, the appearing factor may be explained by the different definition of the Kähler metrics used in [2].

Chapter 6

Conclusion

In this work we rederived the constraints for a general coupling function F stated in [29], which appears in self-interactions of $N = 2$ supersymmetric tensor multiplet models. After the dualization of the tensor multiplets to hypermultiplets, we showed, that the space spanned by the scalars is at least a Kähler space. We also showed, following [31, 6] that the coupling function F can be derived by a complex contour integral of a more general function. This method was used in the following to derive the Kähler potentials of IIB supersymmetric string theory compactified on a Calabi-Yau threefold with N_T $N = 2$ tensor multiplets.

With the help of the coupling function F, we determined the hyperkähler potential for the type IIB theory by performing a Legendre transformation along the lines of [8, 9]. Afterwards, we determined the Kähler potential for two different orientifold projections involving $\mathcal{O}3/\mathcal{O}7$ as in [3] and determined the explicit form of the Kähler potential involving $\mathcal{O}5/\mathcal{O}9$ planes, since this was not done in [3]. We then compared the derived Kähler potentials with those given [2] and found agreement after redefining our scalars up to an overall factor of 2. This factor may be explained by the different conventions used for the Kähler metric in the literature, which differ exactly by an overall factor 2.

An interesting future step would be to apply this method to orientifolds of type IIA supersymmetric string theories.

Appendix A

Facts in supersymmetry

A.1 Conventions and spinor algebra

This work heavily uses the notation of [10].

Spinors are two-component Weyl spinors which can be composed into one Dirac spinor

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix}. \tag{A.1}$$

Spinors have dotted and undotted greek indices from the beginning of the alphabet.

- Minkowski metric
$$\eta_{mn} = \mathrm{diag}(-1, 1, 1, 1) \tag{A.2}$$

- ϵ-symbol
$$\epsilon^{12} = \epsilon_{21} = 1, \qquad \epsilon^{\dot 1 \dot 2} = \epsilon_{\dot 2 \dot 1} = 1 \tag{A.3}$$
$$\epsilon^{\alpha\beta}\epsilon_{\gamma\alpha} = \delta^\beta_\gamma, \qquad \epsilon^{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\alpha} = \delta^{\dot\beta}_{\dot\gamma} \tag{A.4}$$
$$\epsilon_{0123} = -1 \tag{A.5}$$

- σ-matrices
$$\sigma^m = (-\mathbf{1}_2, \sigma^i), \qquad \bar\sigma^m = (-\mathbf{1}_2, -\sigma^i) \tag{A.6}$$

- With the help of the ϵ-symbol the spinor indices can be pulled up and down
$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \tag{A.7}$$
$$\bar\sigma^{m\alpha\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^m_{\beta\dot\beta} \tag{A.8}$$

  Equivalent formulae hold for the dotted quantities.

- Products of spinors

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \tag{A.9}$$

$$(\chi\psi)^\dagger = (\chi^\alpha\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\psi\bar\chi = \bar\chi\bar\psi \tag{A.10}$$

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta, \qquad \theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \tag{A.11}$$

$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta, \qquad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta \tag{A.12}$$

$$\psi_\alpha\chi_\beta = \psi_\beta\chi_\alpha + \epsilon_{\alpha\beta}\psi\chi \tag{A.13}$$

$$\bar\psi_{\dot\alpha}\bar\chi_{\dot\beta} = \bar\psi_{\dot\beta}\bar\chi_{\dot\alpha} - \epsilon_{\dot\alpha\dot\beta}\bar\psi\bar\chi \tag{A.14}$$

$$(\psi_1\psi_2)(\psi_3\psi_4) = -(\psi_1\psi_3)(\psi_2\psi_4) - (\psi_1\psi_4)(\psi_2\psi_3) \tag{A.15}$$

- Rules for the σ matrices

$$\sigma^m_{\alpha\dot\alpha}\bar\sigma^{n\dot\alpha\beta} = -\eta^{mn}\delta^\beta_\alpha + 2(\sigma^{mn})_\alpha{}^\beta \tag{A.16}$$

$$(\sigma^{mn})_\alpha{}^\beta = \frac{1}{4}(\sigma^m_{\alpha\dot\alpha}\bar\sigma^{n\dot\alpha\dot\beta} - \sigma^n_{\alpha\dot\alpha}\bar\sigma^{m\dot\alpha\dot\beta}) \tag{A.17}$$

$$tr(\sigma^m\bar\sigma^n) = -2\eta^{mn} \tag{A.18}$$

A.2 Integration and differentiation with respect to anticommuting variables

Differentiation with respect to an anticommuting variable is defined in the natural way

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta^\beta_\alpha, \tag{A.19}$$

and similar for the dotted indices.

The integration is defined with the help of

$$\begin{aligned} \int d\theta_\alpha\theta^\beta &= \delta_\alpha{}^\beta \\ \int d\bar\theta^{\dot\alpha}\bar\theta_{\dot\beta} &= \delta^{\dot\alpha}{}_{\dot\beta}. \end{aligned} \tag{A.20}$$

The volume elements in superspace are defined as follows

$$\begin{aligned} d^2\theta &= -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta} \\ d^2\bar\theta &= -\frac{1}{4}d\bar\theta_{\dot\alpha}d\bar\theta_{\dot\beta}\epsilon^{\dot\alpha\dot\beta} \\ d^4\theta &= d^2\theta d^2\bar\theta. \end{aligned} \tag{A.21}$$

Due to the anticommuting properties of the variables, and hence of the derivatives and integrals, we can express the integration over $\theta, \bar{\theta}$ in terms of the covariant derivatives

$$
\begin{aligned}
\int d^2\theta d^2\bar{\theta} v(x,\theta,\bar{\theta}) =& \Big(-\frac{1}{4} \int d^2\theta \bar{D}^2 v(x,\theta,\bar{\theta}) \Big)\Big|_{\bar{\theta}=0} = \\
& \Big(\frac{1}{16} D^2 \bar{D}^2 v(x,\theta,\bar{\theta}) \Big)\Big|_{\theta=\bar{\theta}=0} \\
\int d^2\theta d^2\bar{\theta} v(x,\theta,\bar{\theta}) =& \Big(-\frac{1}{4} \int d^2\bar{\theta} D^2 v(x,\theta,\bar{\theta}) \Big)\Big|_{\theta=0} = \\
& \Big(\frac{1}{16} \bar{D}^2 D^2 v(x,\theta,\bar{\theta}) \Big)\Big|_{\theta=\bar{\theta}=0}
\end{aligned} \tag{A.22}
$$

when leaving out total derivative-like terms.

A.3 Covariant derivative rules

We present now some identities, which often occur when performing calculations with superfields

$$D_\alpha D_\beta = \frac{1}{2}\epsilon_{\alpha\beta} D^2, \qquad \bar{D}_{\dot{\alpha}} \bar{D}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}} \bar{D}^2 \tag{A.23}$$

$$D_\alpha D_\beta D_\gamma = 0, \qquad \bar{D}_{\dot{\alpha}} \bar{D}_{\dot{\beta}} \bar{D}_{\dot{\gamma}} = 0 \tag{A.24}$$

$$[D^2, \bar{D}_{\dot{\alpha}}] = -4i\sigma^m_{\alpha\dot{\alpha}} \partial_m D^\alpha, \qquad [\bar{D}^2, D_\alpha] = 4i\sigma^m_{\alpha\dot{\alpha}} \partial_m D^{\dot{\alpha}} \tag{A.25}$$

Appendix B

Variation rules for superfields

A well written introduction to variation rules of superfields can be found in [12]. We only present the most important results.

We start with a functional $S[V]$ of a superfield. The principle of the extremal action demands, that for any variation δV

$$\delta S[V] = S[V+\delta V] - S[V] = \int d^4x d^4\theta \frac{\delta S[V]}{\delta V(x,\theta,\bar{\theta})} = 0 \tag{B.1}$$

is fulfilled. This constraint yields the equation of motion. The occurring derivatives depend on the involved superfield. The following relations hold:

- general superfields

$$\frac{\delta\Sigma(x',\theta',\bar{\theta}')}{\delta\Sigma(x,\theta,\bar{\theta})} = -\frac{1}{4}\delta^4(x-x')\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}') \tag{B.2}$$

- chiral superfields

$$\begin{aligned} \frac{\delta\Phi(x',\theta',\bar{\theta}')}{\delta\Phi(x,\theta,\bar{\theta})} &= -\frac{1}{4}\bar{D}^2\delta^4(x-x')\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}') \\ \frac{\delta\bar{\Phi}(x',\theta',\bar{\theta}')}{\delta\bar{\Phi}(x,\theta,\bar{\theta})} &= -\frac{1}{4}D^2\delta^4(x-x')\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}') \end{aligned} \tag{B.3}$$

- vector superfields

$$\frac{\delta V(x',\theta',\bar{\theta}')}{\delta V(x,\theta,\bar{\theta})} = \delta^4(x-x')\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}') \tag{B.4}$$

- chiral spinor superfields

$$\frac{\delta\Phi^\beta(x',\theta',\bar{\theta}')}{\delta\Phi^\alpha(x,\theta,\bar{\theta})} = -\frac{1}{4}\delta_\alpha{}^\beta\bar{D}^2\delta^4(x-x')\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}') \tag{B.5}$$

Appendix C

Kähler and Hyperkähler geometry

In this appendix we provide an introduction (based on [32, 12, 33]) to complex, hermitian manifolds, and show how Kähler manifolds arise in an $N = 1$ supersymmetric context. Afterwards, we give a short overview over Hyperkähler manifolds, as these arise in the context of $N = 2$ supersymmetric Lagrangians of hypermultiplets.

C.1 Kähler geometry and $N = 1$ supersymmetry

A $2n$ dimensional manifold $\mathcal{M}$ with a complex structure

$$J : \mathcal{TM} \to \mathcal{TM}, \qquad J^2 = -\mathbb{1},$$

where $\mathcal{TM}$ denotes the tangent space of $\mathcal{M}$, is called a hermitian manifold, if the metric g on $\mathcal{M}$ is hermitian with respect to J, that is

$$g(Jx, Jy) = g(x, y).$$

Defining a tensor field Ω with the action

$$\Omega(x, y) = g(Jx, y),$$

one immediately notices that Ω is antisymmetric:

$$\Omega(x, y) = g(Jx, y) = g(J^2 x, Jy) = -g(Jy, x) = -\Omega(y, x).$$

This form is called the Kähler form of the hermitian metric g. The components of Ω are, due to the hermiticity of g, given by

$$\Omega_{i\bar{j}} = g_{i\bar{j}} J_{\bar{k}}{}^{\bar{j}}.$$

A Kähler manifold is a complex, hermitian manifold $\mathcal{M}$ with a closed Kähler form:

$$d\Omega = 0.$$

This can be interpreted as a differential equation for $g_{i\bar{j}}$ with the general solution:

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(z, \bar{z}).$$

Here, K is a real function $K = \bar{K}$, and is called the Kähler potential of g. It is defined up to Kähler transformations

$$K'(z, \bar{z}) = K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}).$$

It can be shown [12, 16, 10], that a $N = 1$ supersymmetric σ-model

$$S[\Phi, \bar{\Phi}] = \int d^4x d^4\theta K(\Phi, \bar{\Phi}) \tag{C.1}$$

is described by a Kähler space with the Kähler potential $K(\Phi, \bar{\Phi})$. Expressing the action in terms of component fields, one finds for the Lagrangian $\mathcal{L}$ after performing the integration over superspace (we confine ourselves to the bosonic part of $\mathcal{L}$)

$$\mathcal{L} = -K_{i\bar{j}} \Big(\partial_m \bar{A}^{\bar{j}} \partial^m A^i - \bar{F}^{\bar{j}} F^i \Big). \tag{C.2}$$

C.2 Hyperkähler geometry

Let $\mathcal{HM}$ be a real manifold of dimension $4m$ with a metric g

$$ds^2 = g_{uv}(q) dq^u dq^v$$

with $u, v = 1, \ldots, 4m$ and *three* complex structures

$$J^i : \mathcal{T}(\mathcal{HM}) \to \mathcal{T}(\mathcal{HM}), \qquad i = 1, 2, 3$$

with respect to which the metric is hermitian

$$g(J^i x, J^i y) = g(x, y), \qquad i = 1, 2, 3.$$

The complex structures are subject to the quaternionic algebra

$$J^i J^j = -\delta^{ij} \mathbb{1} + \epsilon^{ijk} J^k.$$

One can introduce the three 2-forms

$$\Omega^i = \Omega^i_{uv} dq^u \wedge dq^v$$
$$\Omega^i_{uv} = g_{uw} (J^i)^w_v.$$

This triplet of 2-forms is $SU(2)$ Lie-algebra valued and named *Hyperkähler* form.

In the complex case, the Kähler form must be closed for $\mathcal{M}$ being not just a hermitian manifold but a Kähler manifold. Therefore, one expects that a similar condition arises in the context of Hyperkähler manifolds. It can be shown that Kähler manifolds belong to the rigid case of $N = 1$ supersymmetry and, in a similar way, the Hyperkähler manifolds correspond to rigid $N = 2$ supersymmetry [32]. In the local $N = 1$ description, one encounters Hodge-Kähler manifolds and the Kähler 2-form can be identified with the curvature of a line bundle, which vanishes in the rigid case [32]. Analogous steps can be performed in the $N = 2$ case [32].

Let $\mathcal{SU}$ be a principal $SU(2)$-bundle and Γ^i a connection on such a bundle. One has to demand, that the Hyperkähler 2-form is covariantly closed with respect to the connection Γ^i

$$\nabla \Omega^i \equiv d\Omega^i + \epsilon^{ijk} \Gamma^j \wedge \Omega^k = 0.$$

One now defines a Hyperkähler manifold as a $4m$-dimensional manifold with the above structure, such that the $\mathcal{SU}$-curvature vanishes [32]:

$$C^i \equiv d\Gamma^i + \frac{1}{2} \epsilon^{ijk} \Gamma^j \Gamma^k = 0.$$

Appendix D

Projective superfield formalism

This appendix is basically an overview of the material presented in the appendix of [6]. The $N = 2$ algebra is

$$\begin{aligned}
&\{Q^i_\alpha, \bar{Q}_{\dot{\alpha}j}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m \delta^i_j \\
&\{Q^i_\alpha, Q^j_\beta\} = \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}^j_{\dot{\beta}}\} = 0, \\
&i, j = 1, 2.
\end{aligned} \tag{D.1}$$

We can define an abelian subspace of the $N = 2$ superspace, which is parameterized by a complex coordinate ζ and spanned by the covariant derivatives

$$\begin{aligned}
\mathcal{D}_\alpha(\zeta) &= D_{1\alpha} + \zeta D_{2\alpha} \\
\bar{\mathcal{D}}_{\dot{\alpha}}(\zeta) &= \bar{D}^2_{\dot{\alpha}} - \zeta \bar{D}^1_{\dot{\alpha}}.
\end{aligned} \tag{D.2}$$

In order to simplify the notation we write in this chapter $D_{1\alpha} = D_\alpha, D_{2\alpha} = Q_\alpha$.

We construct the conjugate of any object of this subspace by the composite of the antipodal map on the Riemann sphere with hermitean conjugation

$$\zeta^* \longrightarrow -\frac{1}{\zeta} \tag{D.3}$$

and multiplying with an appropriate factor.

Projective superfields in this space are subject to the constraint

$$\mathcal{D}_\alpha \Upsilon = 0 = \bar{\mathcal{D}}_{\dot{\alpha}} \Upsilon, \tag{D.4}$$

and we can construct a restricted measure in order to integrate Lagrangians over this subspace from any differential operator which is linearly independent of $\mathcal{D}$ and $\bar{\mathcal{D}}$. A generic choice is the usual $N = 1$ measure

$$S = \oint_\gamma \frac{d\zeta}{2\pi i\zeta} d^4x D^2 \bar{D}^2 G(\Upsilon, \bar{\Upsilon}, \zeta), \tag{D.5}$$

where the integration contour γ generally depends on G. The above constraints (D.4) ensure that the action is $N = 2$ supersymmetric.

Projective superfields can be classified as [6]:

- $O(k)$ multiplets
- rational multiplets
- analytical multiplets

We concentrate on the $O(k)$ multiplets which are polynomials in ζ. The minimal power of ζ is 0, the maximal power is k. For even $k = 2p$ one can impose a reality condition with respect of the above introduced conjugation mapping (D.3). With $\eta^{(2p)}$ we denote a real finite order $O(k)$ multiplet. The reality condition yields

$$\begin{aligned} \eta^{(2p)}(\zeta) &= \frac{1}{\zeta^p} \sum_{n=1}^{2p} \eta_n^{(2p)} \zeta^n \\ \eta^{(2p)} &= \bar{\eta}^{(2p)}. \end{aligned} \tag{D.6}$$

Obviously, the reality constraint relates different coefficients of the ζ expansion of η

$$\eta_{2p-n} = (-)^{p-n} \bar{\eta}_n. \tag{D.7}$$

We now examine the constraints (D.4). They relate different ζ-coefficient superfields

$$\begin{aligned} D_\alpha \Upsilon_{n+1} &= -Q_\alpha \Upsilon_n \\ \bar{D}_{\dot\alpha} \Upsilon_n &= \bar{Q}_{\dot\alpha} \Upsilon_{n+1}. \end{aligned} \tag{D.8}$$

As the important example for this work we present the $O(2)$ multiplet. The expansion reads

$$\eta^{(2)} = \frac{\bar{A}}{\zeta} + C - \zeta A. \tag{D.9}$$

The field A obeys $\bar{D}_{\dot\alpha} = Q_\alpha = 0$ while C is real and obeys $\bar{D}^2 C = Q^2 C = 0$. Hence, A is projected to a chiral superfield and C to a linear one. That means, that the projective superfield $\eta^{(2)}$ describes an $N = 2$ tensor multiplet.

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Erklärung gemäß Diplomprüfungsordnung

Hiermit erkläre ich, diese Arbeit selbstständig angefertigt und nur die aufgelisteten Quellen und Hilfsmittel verwendet zu haben. Ich gestatte die Veröffentlichung dieser Arbeit.

Hamburg, den 22.11.2007

Zusammenfassung

Betrachtet man Typ II Stringtheorien, so muss man, zusätzlich zur dimensionalen Reduktion durch die Kaluza-Klein-Kompaktifizierung mit Hilfe von Calabi-Yau Mannigfaltigkeiten, sogenannte Orientifold Projektionen durchführen. Diese Orientifold Projektionen ermöglichen einerseits konsistente beziehungsweise stabile Typ II Theorien mit Dp-Branes, andererseits reduzieren sie die Anzahl der Superladungen von 32 auf 16. Dies entspricht dem Übergang von einer $N = 2$ Theorie zu einer $N = 1$ Theorie. Der Grund warum man Theorien mit Dp-Branes betrachtet ist, dass Strings, die auf den Dp-Branen enden, eine Yang-Mills Quantenfeldtheorie, wie das Standardmodell eine ist, hervorrufen.

Calabi-Yau Mannigfaltigkeiten unterliegen speziellen Deformationen, sogenannten Moduli, welche den Calabi-Yau-Bedingungen genügen müssen. Diese Deformationen bezüglich der Form und der Größe der Mannigfaltigkeit heißen *komplexe Struktur Moduli* beziehungsweise *Kähler Moduli*.

In dieser Arbeit bestimmen wir die Kähler Potentiale der Hypermultiplets von Typ IIB Orientifold Projektionen. Im Gegensatz zu früheren Arbeiten zu diesem Thema nutzen wir einen Formalismus aus der Literatur, welcher auf sogenannten *projektiven Superfeldern* basiert. Dazu bestimmen wir zuallererst, wie in der Literatur bereits geschehen, die allgemeine Kopplungsfunktion der $N = 2$ Tensor Multiplets mit Hilfe eines komplexen Kurvenintegrals über das klassische Präpotential der Hypermultiplets. Im Anschluss können wir die Bedingungen der zwei betrachteten Orientifold Projektionen an die Skalare der Tensor Multiplets stellen. Daraufhin führen wir die Dualisierung der Tensor Multiplets zu Hypermultiplets durch, und erhalten dann die Kähler Potentiale der beiden Projektionen. Dabei ist hervorzuheben, dass das Kähler Potential, das man im Falle von $\mathcal{O}5/\mathcal{O}9$ Orientifolds mit diesem Formalismus erhält, nicht in der Literatur angegeben war. Abschließend vergleichen wir die betrachteten Kähler Potentiale mit denen, die man mit dem konventionellen Formalismus erhält und führen in diesem Zusammenhang eine Umdefinierung unserer Variablen durch. Wir erhalten Kähler Potentiale, welche sich um einen Faktor 2 von den bereits bestimmten unterscheiden. Die Ursache dieses Faktors ist wahrscheinlich in den verschiedenen Definitionen für das Kähler Potential zu suchen, da diese sich ebenfalls um einen Faktor 2 unterscheiden.