Lecture notes

Introduction to Supersymmetry and Supergravity

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ABSTRACT

These lectures give a basic introduction to supersymmetry and supergravity.

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Contents

1 The Supersymmetry Algebra

Supersymmetry is an extension of the Poincare algegebra which relates states or fields of different spin. By now it has ample applications in particle physics, quantum field theories, string theory, mathematics, stastical mechanics, solid state physics and many more.[1]

1.1 Review of Poincare Algebra

Let $x^\mu, \mu = 0, \dots, 3$, be the coordinates of Minkowski space $M_{1,3}$ with metric

$$(\eta_{\mu\nu}) = \text{diag}(-1, 1, 1, 1) \ . \tag{1.1}$$

Lorentz transformations are rotations in $M_{1,3}$ and thus correspond to the group $O(1, 3)$

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu_\nu x^\nu \ . \tag{1.2}$$

$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ is invariant for

$$\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma} \ , \qquad \text{or in matrix form} \qquad \Lambda^T \eta \Lambda = \eta \ . \tag{1.3}$$

This generalizes the familiar orthogonal transformation $O^T O = \mathbf{1}$ of $O(4)$.

Λ depends on $4 \cdot 4 - (4 \cdot 4)_s = 16 - 10 = 6$ parameters. $\Lambda_R := \Lambda^i_j, i, j = 1, 2, 3$ satisfies $\Lambda_R^T \Lambda_R = \mathbf{1}$ corresponding to the $O(3)$ subgroup of three-dimensional space rotations. Λ_R depends on 3 rotation angles. $\Lambda_B := \Lambda^0_j$ corresponds to Lorentz boosts depending on 3 boost velocities.

One expands Λ infinitesimally near the identity as

$$\Lambda = \mathbf{1} - \tfrac{i}{2} \omega_{[\mu\nu]} L^{[\mu\nu]} + \dots \ , \tag{1.4}$$

where $\omega_{[\mu\nu]}$ are the 6 parameters of the transformation. The $L^{[\mu\nu]}$ are the generators of the Lie algebra $SO(1, 3)$ which satisfy

$$[L^{\mu\nu}, L^{\rho\sigma}] = -i\big(\eta^{\nu\rho} L^{\mu\sigma} - \eta^{\mu\rho} L^{\nu\sigma} - \eta^{\nu\sigma} L^{\mu\rho} + \eta^{\mu\sigma} L^{\nu\rho}\big) \ . \tag{1.5}$$

The Poincare group includes in addition the (constant) translations

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu_\nu x^\nu + a^\mu \ , \tag{1.6}$$

generated by the momentum operator $P_\mu = -i\partial_\mu$. The algebra of the Lorentz generators (1.5) is augmented by

$$[P_\mu, P_\nu] = 0 \ , \qquad [P_\mu, L_{\nu\rho}] = i(\eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu) \ . \tag{1.7}$$

The Poincare group has two Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ where $W_\mu = \epsilon_{\mu\nu\rho\sigma} L^{\nu\rho} P^\sigma$ is the Pauli-Lubanski vector. Both commute with $P_\mu, L_{\mu\nu}$. Thus the representations can be characterized by the eigenvalues of P^2 and W^2, i.e., the mass m and the spin s.

[1] Textbooks of supersymmetry and supergravity include [1–5]. For review lectures see, for example, [7–9] .

1.2 Representations of the Lorentz Group

First of all the Lorentz group has (n,m) tensor representations with tensor which transform according to

$$T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m} \to T'^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m} = \Lambda^{\mu_1}_{\rho_1}\cdots\Lambda^{\mu_n}_{\rho_n}\, T^{\rho_1\cdots\rho_n}_{\sigma_1\cdots\sigma_m}\, \Lambda^{\sigma_1}_{\nu_1}\cdots\Lambda^{\sigma_m}_{\nu_m}\ . \tag{1.8}$$

In addition all $SO(n,m)$ groups also have spinor representations.[2] They are constructed from Dirac matrices γ^μ satisfying the Clifford/Dirac algebra[3]

$$\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu}\ . \tag{1.9}$$

From the γ^μ one constructs the operators

$$S^{\mu\nu} := -\tfrac{i}{4}\,[\gamma^\mu,\gamma^\nu]\ , \tag{1.10}$$

which satisfy (1.5) and thus are generators of (the spinor representations of) $SO(1,3)$.

The γ matrices are unique (up to equivalence transformations) and a convenient choice in the following is the chiral representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\ , \qquad \text{where} \qquad \sigma^\mu = (-\mathbf{1},\sigma^i)\ ,\quad \bar\sigma^\mu = (-\mathbf{1},-\sigma^i)\ . \tag{1.11}$$

Here σ^i are the Pauli matrices which satisfy $\sigma^i\sigma^j = \delta^{ij}\mathbf{1} + i\epsilon^{ijk}\sigma^k$. Inserted into (1.10) one finds

$$S^{\mu\nu} = i\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix}\ , \quad \text{where} \quad \sigma^{\mu\nu} = \tfrac14(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\ , \quad \bar\sigma^{\mu\nu} = \tfrac14(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)\ . \tag{1.12}$$

For the boosts and rotations one has explicitly

$$S^{0i} = \tfrac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}\ , \qquad S^{ij} = \tfrac12\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}\ . \tag{1.13}$$

Since they are block-diagonal the smallest spinor representation is the two-dimensional Weyl spinor. In the Van der Waerden notation one decomposes a four-component Dirac spinor Ψ_D as

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix}\ , \qquad \alpha,\dot\alpha = 1,2\ , \tag{1.14}$$

where χ_α and $\bar\psi^{\dot\alpha}$ are two independent two-component complex Weyl spinors. The dotted and undotted spinors transform differently under the Lorentz group. Concretely one has

$$\begin{aligned} \delta\chi_\alpha &= \tfrac12\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha^\beta\chi_\beta = \tfrac12(\omega_{0i}\sigma^i + i\omega_{ij}\epsilon^{ijk}\sigma^k)\chi\ , \\ \delta\bar\psi^{\dot\alpha} &= \tfrac12\omega_{\mu\nu}(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\dot\beta}\bar\psi^{\dot\beta} = \tfrac12(-\omega_{0i}\sigma^i + i\omega_{ij}\epsilon^{ijk}\sigma^k)\bar\psi\ , \end{aligned} \tag{1.15}$$

where we used (1.12) and (1.13). These transformation laws are often referred to as $(\frac12,0)$ and $(0,\frac12)$ representations respectively. Note that the two spinors transforms identically under the rotation subgroup while they transform with opposite sign under the boosts.

[2] They are two-valued in $SO(n,m)$ but single valued in the double cover denoted by $Spin(n,m)$.
[3] Here we use the somewhat unconventional convention of [5].

The spinor indices are raised and lowered using the Lorentz-invariant ϵ-tensor

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \ , \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \ , \tag{1.16}$$

where

$$\epsilon_{21} = -\epsilon_{12} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha^\beta \ .$$

For dotted indices the analogous equations hold. One can check that σ^μ carries the indices $\sigma^\mu_{\alpha\dot\alpha}$ and $\bar\sigma^{\mu\alpha\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}$. Complex conjugation interchanges the two representations, i.e., $(\chi_\alpha)^* = \bar\chi_{\dot\alpha}$.

1.3 Supersymmetry Algebra

The supersymmetry algebra is an extension of the Poincare algebra. One augments the Poincare algebra by a fermionic generator Q_α which transforms as a Weyl spinor of the Lorentz group. Haag, Lopuszanski and Sohnius showed that the following algebra is the only extension compatibly with the requirements of a QFT [5, 6]

$$\begin{aligned}
&\{Q_\alpha, \bar Q_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}\, P_\mu \, , && \{Q_\alpha, Q_\beta\} = 0 = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} \, , \\
&[\bar Q_{\dot\alpha}, P_\mu] = 0 = [Q_\alpha, P_\mu] \, , && \\
&[Q_\alpha, L^{\mu\nu}] = \tfrac{1}{2}\,(\sigma^{\mu\nu})_\alpha^\beta Q_\beta \, , && [\bar Q_{\dot\alpha}, L^{\mu\nu}] = \tfrac{1}{2}\,(\bar\sigma^{\mu\nu})_{\dot\alpha}^{\dot\beta} \bar Q_{\dot\beta} \, .
\end{aligned} \tag{1.17}$$

The only further generalization which we will discuss later on is the possibility of having N supersymmetric generators $Q^I_\alpha, I = 1, \ldots, N$ – a situation which is referred to as N-extended supersymmetry.

Eq. (1.17) implies

$$[P^2, Q_\alpha] = 0 \ , \qquad [W^2, Q_\alpha] \neq 0 \ , \tag{1.18}$$

and thus the representations of (1.17) are labelled by the mass m but not by the spin s. From (1.17) one can further show that for any finite-dimensional representations the number of bosonic states n_B and fermionic states n_F coincides and one has

$$\mathrm{Tr}\left((-)^{N_F}\right) = n_B - n_F = 0 \ . \tag{1.19}$$

Here the fermion number operator $(-)^{N_F}$ is defined by

$$(-)^{N_F}|B\rangle = |B\rangle \ , \qquad (-)^{N_F}|F\rangle = -|F\rangle \ , \tag{1.20}$$

where $|B\rangle$ ($|F\rangle$) denotes any bosonic (fermionic) state. Due to (1.20) and the fermionic nature of Q_α one has $(-)^{N_F} Q_\alpha = -Q_\alpha (-)^{N_F}$.

The cyclicity of the trace then implies

$$\mathrm{Tr}\left((-)^{N_F}\{Q_\alpha, \bar Q_{\dot\alpha}\}\right) = \mathrm{Tr}\left(-Q_\alpha(-)^{N_F}\bar Q_{\dot\alpha} + Q_\alpha(-)^{N_F}\bar Q_{\dot\alpha}\right) = 0 \ . \tag{1.21}$$

Inserting (1.17) yields

$$\mathrm{Tr}\left((-)^{N_F} 2\sigma^\mu_{\alpha\dot\beta}\, P_\mu\right) = 2\sigma^\mu_{\alpha\dot\beta}\, P_\mu \mathrm{Tr}\left((-)^{N_F}\right) = 0 \ , \tag{1.22}$$

where in the first step the trace was evaluated for fixed P_μ. This proves (1.19).

As for the Poincare group, the representations (supermultiplets) of the algebra (1.17) are distinct for different values of the Casimir operator P^2.

2 Representations of the supersymmetry algebra and the Chiral Multiplet

In this lecture we discuss the representations of the supersymmetry algebra (1.17).

2.1 Massive representations

For massive representations ($P^2 = -m^2$, $m > 0$) one goes to the rest frame $P_\mu = (-m, 0, 0, 0)$ such that the superalgebra (1.17) becomes

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta} \ , \qquad \{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} \ . \tag{2.1}$$

Then one defines the operators

$$a_\alpha := \frac{1}{\sqrt{2m}}\, Q_\alpha \ , \qquad (a_\alpha)^\dagger := \frac{1}{\sqrt{2m}}\, \bar{Q}_{\dot\alpha} \tag{2.2}$$

such that (2.1) becomes

$$\{a_\alpha, (a_\beta)^\dagger\} = \delta_{\alpha\dot\beta} \qquad \{a_\alpha, a_\beta\} = 0 = \{a^\dagger_{\dot\alpha}, a^\dagger_{\dot\beta}\} \, . \tag{2.3}$$

This is the algebra of two fermionic harmonic oscillators and thus its representations can be constructed as in quantum mechanics. One defines a “ground state” (Clifford vacuum) $|0\rangle$ by the condition

$$a_\alpha|0\rangle = 0 \ , \tag{2.4}$$

and constructs the multiplet by acting with $a^\dagger_\alpha$

$$|0\rangle \ , \quad (a_\alpha)^\dagger|0\rangle \ , \quad (a_1)^\dagger(a_2)^\dagger|0\rangle \ . \tag{2.5}$$

By acting with the spin operator L^2 one determines that the first and the last state have spin $s = 0$ while the two other states have $s = 1/2$. We therefore have $n_B = n_F = 2$ and this representation is called the chiral multiplet.

Other multiplets can be constructed in a similar way if one also assigns spin to the Clifford vacuum. In this case one finds the multiplet

$$|s\rangle \ , \quad (a_\alpha)^\dagger|s\rangle \ , \quad (a_1)^\dagger(a_2)^\dagger|s\rangle \ , \tag{2.6}$$

corresponding to the spins $(s, s \pm \frac{1}{2}, s)$ and the multiplicities $2s+1, 2(s \pm \frac{1}{2})+1, 2s+1$. Thus altogether one has $n_B = n_F = 4s + 2$. The different multiplets are summarized in Table 2.1.

Since P^2 commutes with Q it also is a Casimir operator of the supersymmetry algebra. Therefore all members of a supermultiplet have the same mass and in particular bosonic states are mass degenerate with fermionic states

$$m_{\mathrm{B}} = m_{\mathrm{F}} \quad \forall \text{ states} \ . \tag{2.7}$$

Hence, supersymmetry has to be broken, if realized in nature.

Spin	$\lvert 0\rangle$	$\lvert \frac{1}{2}\rangle$	$\lvert 1\rangle$	$\lvert \frac{3}{2}\rangle$
0	2	1		
$\frac{1}{2}$	1	2	1	
1		1	2	1
$\frac{3}{2}$			1	2
2				1
$n_B = n_F$	2	4	6	8
	chiral multiplet	vector multiplet	spin $\frac{3}{2}$ multiplet	spin 2 multiplet

Table 2.1: Massive $N = 1$ multiplets.

2.2 Massless representations

For massless representations one goes again to a light-like frame, $P_\mu = (-E, 0, 0, E)$. Inserted into (1.17) one obtains

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2E(-\sigma^0 + \sigma^1)_{\alpha\dot\beta} = 2E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \qquad \{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} . \tag{2.8}$$

We see that the algebra is trivial for Q_2. Inserting

$$a := \frac{1}{\sqrt{2E}}\, Q_1 , \qquad a^\dagger := \frac{1}{\sqrt{2E}}\, \bar{Q}_1 , \tag{2.9}$$

into (11.11) one arrives at

$$\{a, a^\dagger\} = 1 , \qquad \{a, a\} = 0 = \{a^\dagger, a^\dagger\} , \tag{2.10}$$

which is the algebra of a single fermionic oscillator. In the massless case the representations are labeled by the helicity λ and a multiplet has only the two states

$$\lvert\lambda\rangle , \qquad a^\dagger\lvert\lambda\rangle , \tag{2.11}$$

corresponding to the helicities $\lambda, \lambda + \frac{1}{2}$. However, due to the CPT theorem of quantum field theories a massless particle with helicity corresponds to two states with helicities $\pm\lambda$. Therefore in quantum field theoretic applications one has to double the multiplets (2.11) appropriately. The relevant massless multiplets are summarized in Table 2.2.

2.3 The chiral multiplet in QFTs

The chiral multiplet has in the massive and massless case two states with spin/helicity zero and two states with spin/helicity 1/2. In a QFT this can be realized as a complex scalar $\phi(x)$ and a Weyl fermion $\chi_\alpha(x)$. However, with χ being complex it has initially (off-shell) four degrees of freedom (d.o.f.) and only after using the equation of motion (the Weyl equation) it carries two d.o.f. on-shell.

The next step is to find the supersymmetry transformation of the chiral multiplet. To this end we define

$$\delta_\xi := \xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha} \bar{Q}^{\dot\alpha} , \tag{2.12}$$

λ	$\|0\rangle$ $\|-\frac{1}{2}\rangle$	$\|\frac{1}{2}\rangle$ $\|-1\rangle$	$\|1\rangle$ $\|-\frac{3}{2}\rangle$	$\|\frac{3}{2}\rangle$ $\|-2\rangle$
-2				1
$-\frac{3}{2}$			1	1
-1		1	1	
$-\frac{1}{2}$	1	1		
0	1 1			
$+\frac{1}{2}$	1	1		
$+1$		1	1	
$+\frac{3}{2}$			1	1
$+2$				1
$n_B = n_F$	2	2	2	2
	chiral multiplet	vector multiplet	gravitino multiplet	graviton multiplet

Table 2.2: The massless multiplets for $N = 1$.

where the parameters of the transformation ξ_α are constant, complex anti-commuting Grassmann parameters obeying

$$\xi_\alpha \xi_\beta = -\xi_\beta \xi_\alpha \,. \tag{2.13}$$

The supersymmetry algebra (1.17) implies

$$[\delta_\eta, \delta_\xi] = -2i(\eta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\eta})\partial_\mu \,. \tag{2.14}$$

One demands that (2.14) holds on all fields of a supermultiplet. For the chiral multiplet this is satisfied for

$$\delta_\xi \phi = \sqrt{2}\xi^\alpha \chi_\alpha \ , \qquad \delta_\xi \chi_\alpha \ = i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_\mu\phi \ , \tag{2.15}$$

if the equation of motion $\bar{\sigma}^\mu\partial_\mu\chi = 0$ holds.

This set of transformation can be promoted to an off-shell realization by introducing an auxiliary complex scalar field $F(x)$ and the transformations

$$\begin{aligned} \delta_\xi \phi &= \sqrt{2}\xi\chi \ , \\ \delta_\xi \chi &= \sqrt{2}\xi F + i\sqrt{2}\sigma^\mu\bar{\xi}\partial_\mu\phi \ , \\ \delta_\xi F &= i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\chi \ , \end{aligned} \tag{2.16}$$

which satisfy (2.14) without using any equation of motion. Note that $F = 0$ demands $\bar{\sigma}^\mu\partial_\mu\chi = 0$ and the transformation reduce to the previous case. Thus the off-shell chiral multiplet reads

$$\big(\phi(x), \chi_\alpha(x), F(x)\,\big) \,, \tag{2.17}$$

and has $n_B = n_F = 4$.

The supersymmetric Lagrangian for the kinetic terms of the chiral multiplet is found to be

$$\mathcal{L}_{\rm kin} = -\partial_\mu\phi\partial^\mu\bar{\phi} - i\bar{\chi}\sigma^\mu\partial_\mu\chi + F\bar{F} \ . \tag{2.18}$$

One can check $\delta_\xi \mathcal{L}_{\rm kin} = \partial_\mu j^\mu$ such that the action is invariant for appropriate boundary conditions of the fields. The equations of motion derived from $\mathcal{L}_{\rm kin}$ read

$$\Box\phi = 0\ , \qquad \bar\sigma^\mu\partial_\mu\chi = 0\ , \qquad F = 0\ . \tag{2.19}$$

We see that the equations of motion is purely algebraic which is the characteristic feature of auxiliary fields in supersymmetric theories.

One can add mass terms as

$$\mathcal{L}_{\rm m} = -\tfrac{1}{2}m\big(\chi\chi + \bar\chi\bar\chi + 2\phi F + 2\bar\phi\bar F\big)\ . \tag{2.20}$$

$\mathcal{L}_{\rm kin} + \mathcal{L}_{\rm m}$ now have the equations of motion

$$\Box\phi + m\bar F = 0\ , \qquad \bar\sigma^\mu\partial_\mu\chi + m\bar\chi = 0\ , \qquad F + m\bar\phi = 0\ . \tag{2.21}$$

Again the equation of motion for F is algebraic and thus can be inserted into the first equation yielding the familiar Klein-Gordon equation $(\Box - m^2)\phi = 0$.

Finally, the most general renormalizable Lagrangian for n_c chiral multiplets reads

$$\begin{aligned}\mathcal{L} = &- \partial_\mu\phi^i\partial^\mu\bar\phi^i - i\bar\chi^i\bar\sigma^\mu\partial_\mu\chi^i + F^i\bar F^i \\ &- \tfrac{1}{2}W_{ij}\chi^i\chi^j - \tfrac{1}{2}\bar W_{ij}\bar\chi^i\bar\chi^j + F^iW_i + \bar F^i\bar W_i\ ,\end{aligned} \tag{2.22}$$

where $i,j = 1,\ldots,n_c$. W_i and W_{ij} in (2.22) are the first and second derivatives of the superpotential $W(\phi)$, which is a holomorphic function of the fields ϕ^i, and in renormalizable theories constrained to be at most cubic

$$\begin{aligned} W(\phi) &= \tfrac{1}{2}m_{ij}\phi^i\phi^j + \tfrac{1}{3}Y_{ijk}\phi^i\phi^j\phi^k\,, \\ W_i &\equiv \frac{\partial W}{\partial\phi^i} = m_{ij}\phi^j + Y_{ijk}\phi^j\phi^k\,, \\ W_{ij} &\equiv \frac{\partial^2 W}{\partial\phi^i\partial\phi^j} = m_{ij} + 2Y_{ijk}\phi^k\,. \end{aligned} \tag{2.23}$$

m_{ij} is the mass matrix while Y_{ijk} are the Yukawa couplings.[4] Eliminating the auxiliary fields F^i by

$$\frac{\delta\mathcal{L}}{\delta\bar F^i} = F^i + \bar W^i = 0\,, \tag{2.24}$$

and inserted back into (2.22) yields

$$\mathcal{L} = -\,\partial_\mu\phi^i\partial^\mu\bar\phi^i - i\bar\chi^i\bar\sigma^\mu\partial_\mu\chi^i - \tfrac{1}{2}W_{ij}\chi^i\chi^j - \tfrac{1}{2}\bar W_{ij}\bar\chi^i\bar\chi^j - V(\phi,\bar\phi)\ , \tag{2.25}$$

where V is the scalar potential given by

$$V(\phi,\bar\phi) = F_i\bar F_i = W_i\bar W_i\,. \tag{2.26}$$

[4] Of course both couplings are constrained by any symmetry (e.g. gauge symmetry) the theory under consideration might have.

3 Superspace and the Chiral Multiplet

3.1 Basic set-up

The coordinates of superspace are $(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})$ where $\theta^\alpha, \bar\theta^{\dot\alpha}$ are Grassmann coordinates which satisfy

$$\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta^2 \ , \qquad \theta^\alpha\theta^\beta\theta^\gamma = 0 \ . \tag{3.1}$$

Superfields are functions on superspace and due to (3.1) have an expansion

$$\begin{aligned} f(x,\theta,\bar\theta) =& f(x) + \theta^\alpha\chi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\rho^{\dot\alpha}(x) + \theta^2 m(x) + \bar\theta^2 n(x) + \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}A_\mu \\ &+ \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \bar\theta^2\theta^\alpha\psi_\alpha(x) + \theta^2\bar\theta^2 d(x) \ . \end{aligned} \tag{3.2}$$

We see that the following ordinary complex fields are combined into a superfield

$$\begin{aligned} &s=0: \quad f(x), m(x), n(x), d(x) \ , && n_B = 8 \\ &s=\tfrac{1}{2}: \quad \chi_\alpha(x), \bar\rho^{\dot\alpha}(x), \bar\lambda^{\dot\alpha}(x), \psi_\alpha(x) \ , && n_F = 16 \\ &s=1: \quad A_\mu \ , && n_B = 8 \end{aligned} \tag{3.3}$$

Note that due to (3.1) sums and products of superfields are again superfields

$$f_1(x,\theta,\bar\theta) + f_2(x,\theta,\bar\theta) = f_3(x,\theta,\bar\theta) \ , \qquad f_1(x,\theta,\bar\theta)f_2(x,\theta,\bar\theta) = f_4(x,\theta,\bar\theta) \ . \tag{3.4}$$

In this formalism supersymmetry transformations are translations in superspace. Recall that a finite translation in Minkowski space is generated by

$$G(a) := e^{i(-a^\mu P_\mu)} \ . \tag{3.5}$$

The generalization in superspace is defined to be

$$G(a,\bar\eta,\eta) := e^{i(-a^\mu P_\mu + \eta Q + \bar\eta\bar Q)} \ . \tag{3.6}$$

The product of two transformations can be computed with help of the Hausdorff-formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots}$

$$G(b,\xi,\bar\xi)\, G(a,\eta,\bar\eta) = G\big(a+b-i(\xi\sigma\bar\eta - \eta\sigma\bar\xi), \xi+\eta, \bar\xi+\bar\eta\big) \tag{3.7}$$

By acting infinitesimally on a superfield one determines $Q, \bar Q$ as differential operators

$$\begin{aligned} G(0,\xi,\bar\xi)f(x,\theta,\bar\theta) &= (1 + i\xi Q + \bar\xi\bar Q)f + \mathcal{O}(\xi^2) = f(x - i(\xi\sigma\bar\theta - \theta\sigma\bar\xi), \theta+\xi, \bar\theta+\bar\xi) \\ &= f(x,\theta,\bar\theta) - i(\xi\sigma^\mu\bar\theta - \theta\sigma^\mu\bar\xi)\partial_\mu f + +\xi^\alpha\partial_\alpha f + \bar\xi_{\dot\alpha}\partial^{\dot\alpha} f + \mathcal{O}(\xi^2) \ , \end{aligned} \tag{3.8}$$

where

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} = -\epsilon_{\alpha\beta}\partial^\beta \ . \tag{3.9}$$

From (3.8) one finds a representation for $Q, \bar Q$ in terms of differential operators

$$Q_\alpha = \partial_\alpha - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , \qquad \bar Q_{\dot\alpha} = -\partial_{\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \ , \tag{3.10}$$

and checks

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \ , \qquad \{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} \, . \tag{3.11}$$

Note that for left multiplication that we used above the sign of P_μ changed. For right multiplication one finds the representation

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , \qquad \bar{D}_{\dot\alpha} = -\partial_{\dot\alpha} - i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \ , \tag{3.12}$$

which satisfy

$$\{D_\alpha, \bar{D}_{\dot\beta}\} = -2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \ , \qquad \{D_\alpha, D_\beta\} = 0 = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} \, . \tag{3.13}$$

Whichever representation one uses, the respective "other" differential operators represent covariant derivatives on superspace as they satisfy

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_\beta\} = \{\bar{D}_{\dot\alpha}, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \ . \tag{3.14}$$

Supersymmetry transformations can be systematically computed by

$$\begin{aligned} \delta_\xi f(x,\theta,\bar\theta) =& \delta_\xi f(x) + \theta^\alpha\delta_\xi\chi_\alpha(x) + \bar\theta_{\dot\alpha}\delta_\xi\bar\rho^{\dot\alpha}(x) + \ldots + \theta^2\bar\theta^2\delta_\xi d(x) \\ =& (\xi Q + \bar\xi\bar{Q}) f(x,\theta,\bar\theta) \end{aligned} \tag{3.15}$$

In particular one finds that the highest component $d(x)$ of any superfield always transforms as a total divergence.

3.2 Chiral Multiplet

We already observed that a general superfield $f(x,\theta,\bar\theta)$ has $n_B = n_F = 16$ which is too large for the multiplets we have constructed in the previous section. One can reduce the number of degrees of freedom by imposing algebraic supersymmetric constraints. For a chiral multiplet this constraint reads

$$\bar{D}_{\dot\alpha}\Phi = 0 = D_\alpha\bar\Phi \ . \tag{3.16}$$

They are supersymmetric since D anticommutes with Q. Furthermore, the solution of this constraint in terms of the components of $f(x,\theta,\bar\theta)$ are the algebraic equations

$$\rho = \psi = n = 0 \ , \qquad A_\mu = i\partial_\mu f \ , \qquad \lambda_{\dot\alpha} = -\tfrac{i}{2}\partial_\mu\chi^\beta\sigma^\mu_{\beta\dot\alpha} \ , \qquad d = \tfrac{1}{4}\Box f \ . \tag{3.17}$$

Or if one renames $f = \phi, \chi \to \sqrt{2}\chi, m = F$

$$\begin{aligned} \Phi(x,\theta,\bar\theta) \ &= \phi(x) + \sqrt{2}\theta\chi(x) + \theta^2 F(x) + \theta\sigma^\mu\bar\theta\partial_\mu\phi(x) \\ &\quad - \tfrac{i}{\sqrt{2}}\theta^2\partial_\mu\chi(x)\sigma^\mu\bar\theta + \tfrac{1}{4}\theta^2\bar\theta^2\Box\phi(x) \ . \end{aligned} \tag{3.18}$$

The field redefinition $y^\mu := x^\mu + i\theta\sigma^\mu\bar\theta$ removes the $\bar\theta$ dependence and yields

$$\Phi(y,\theta) \ = \phi(y) + \sqrt{2}\theta\chi(y) + \theta^2 F(y) \ . \tag{3.19}$$

Now one can work out the supersymmetry transformation law

$$\begin{aligned}\delta\phi &= (\xi Q+\bar\xi\bar Q)\Phi\big|_{\theta=\bar\theta=0} = \ldots = \sqrt{2}\xi\chi\\ \delta\chi &= \tfrac{1}{\sqrt{2}}\,(\xi Q+\bar\xi\bar Q)\Phi\big|_{\theta} = \ldots = \sqrt{2}\xi F + i\sqrt{2}\sigma^\mu\bar\xi\partial_\mu\phi\ ,\\ \delta_\xi F &= (\xi Q+\bar\xi\bar Q)\Phi\big|_{\theta^2} = \ldots = i\sqrt{2}\bar\xi\bar\sigma^\mu\partial_\mu\chi\ ,\end{aligned} \tag{3.20}$$

which indeed coincides with (2.16).

The supersymmetric action is constructed by choosing appropriate highest components of superfields or rather products of superfields. Note that due to (3.16)

$$\bar D_{\dot\alpha}\Phi^n = n\Phi^{n-1}\bar D_{\dot\alpha}\Phi = 0\ ,\qquad \bar D_{\dot\alpha}W(\Phi) = \frac{\partial W}{\partial\Phi}\,\bar D_{\dot\alpha}\Phi = 0\ . \tag{3.21}$$

Thus the θ^2 component of W transforms as a total divergence. One finds

$$W(\phi+\sqrt{2}\theta\chi+\theta^2F)\Big|_{\theta^2} = \partial W|_{\theta=\bar\theta=0}\,F + \tfrac12\,\partial^2W\big|_{\theta=\bar\theta=0}\,\chi\chi \tag{3.22}$$

or for n_c chiral multiplets $\Phi^i, i=1,\ldots,n_c$

$$W(\Phi^i)\big|_{\theta^2} = W_i(\phi)F^i + \tfrac12 W_{ij}(\phi)\chi^i\chi^j\ , \tag{3.23}$$

where $W_i(\phi), W_{ij}(\phi)$ are defined in (2.23).

The kinetic terms arise from $\Phi\bar\Phi$ which is not chiral and thus one has to take the $\theta^2\bar\theta^2$ component

$$\Phi\bar\Phi\big|_{\theta^2\bar\theta^2} = -\partial_\mu\phi\partial^\mu\bar\phi + F\bar F - i\bar\chi\partial\!\!\!/\chi\ . \tag{3.24}$$

Thus altogether we have

$$\mathcal{L} = \Phi\bar\Phi\big|_{\theta^2\bar\theta^2} + W(\Phi^i)\big|_{\theta^2} + \bar W(\bar\Phi^i)\big|_{\bar\theta^2}\ . \tag{3.25}$$

3.3 Berezin integration

There is an alternative way to display this result. One defines an integral for Grassmann variables by

$$\int d\theta = 0\ ,\qquad \int \theta d\theta = 1\ , \tag{3.26}$$

such that for $f(\theta) = f(A+\theta\chi)$ one finds

$$\int f(\theta)d\theta = \chi\ ,\qquad \int f(\theta)\theta d\theta = A\ . \tag{3.27}$$

This can be generalized to $\theta^\alpha, \bar\theta^{\dot\alpha}$ by defining the measures

$$d^2\theta := -\tfrac14 d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta}\ ,\qquad d^2\bar\theta := -\tfrac14 d\theta_{\dot\alpha}d\theta_{\dot\beta}\epsilon^{\dot\alpha\dot\beta}\ ,\qquad d^4\theta := d^2\theta d^2\bar\theta\ . \tag{3.28}$$

with

$$\int \theta^2d^2\theta = 1 = \int \bar\theta^2 d^2\bar\theta = \int \theta^2\bar\theta^2 d^2\theta d^2\bar\theta\ . \tag{3.29}$$

In this notation the Lagrangian (3.25) reads

$$\mathcal{L} = \int \Phi\bar\Phi d^4\theta + \int\ W(\Phi^i)d^2\theta + \int \bar W(\bar\Phi^i)d^2\bar\theta\ . \tag{3.30}$$

3.4 R-symmetry

The supersymmetry algebra (1.17) has an $U(1)$ auter automorphism (called R-symmetry) which transforms Q as

$$Q \to Q' = e^{-i\alpha} Q \ , \qquad \bar{Q} \to \bar{Q}' = e^{i\alpha} \bar{Q} \ , \qquad \alpha \in \mathbb{R} \ . \tag{3.31}$$

This implies that the members of supermultiplet transform differently and one has the R-charges for the chiral multiplet

$$R(\Phi^i) = R(\phi^i) = q_i \ , \qquad R(\chi^i) = q_i - 1 \ , \qquad R(F^i) = q_i - 2 \ , \qquad R(\theta) = 1 \ . \tag{3.32}$$

The kinetic terms are automatically invariant but the interactions might break this symmetry. $R(\theta) = 1$ implies $R(d^2\theta) = -2, R(d^4\theta) = 0$ and thus one needs $R(W) = 2$ which indeed constrains the interactions.

4 Super Yang-Mills Theories

In Table 2.2 we saw that the massless vector multiplet contains the states $|\lambda = \pm 1\rangle$ and $|\lambda = \pm 1\rangle$. In a QFT they correspond to a gauge boson $A_\mu(x)$ and a Weyl fermion $\lambda_\alpha(x)$ termed gaugino. Off-shell the gauge boson and the gaugino have $n_B = n_F = 4$.

A massless A_μ has a gauge invariance which removes one degree of freedom

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \ . \tag{4.1}$$

Therefore we expect a real scalar auxiliary field $D(x)$ to complete the off-shell massless vector multiplet. The gauge invariant field strengh is defined by

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{4.2}$$

4.1 The Vector Multiplet in Superspace

In the previous lecture we discussed the general superfield $f(x, \theta, \bar\theta)$ in (3.2) which has $n_B = n_F = 16$. The vector multiplet V satisfies the constraint $V = V^\dagger$, has $n_B = n_F = 8$ and a θ-expansion

$$\begin{aligned} V(x,\theta,\bar\theta) =& f(x) + i\theta^\alpha \chi_\alpha(x) - i\bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x) + \tfrac{i}{2}\theta^2 m(x) - \tfrac{i}{2}\bar\theta^2 \bar m(x) - \theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} A_\mu \\ &+ i\theta^2 \bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) - i\bar\theta^2\theta^\alpha \lambda_\alpha(x) + \tfrac{1}{2}\theta^2\bar\theta^2 d(x) \ , \end{aligned} \tag{4.3}$$

where the convention compared to (3.2) was slightly changed for later convenience. The bosonic fields of V are the real f, d, A_μ and the complex m while the fermions are χ, λ.

Since the massless vector has a gauge invariance we need to implement this at the level of superfields. We will see that the right transformation (for the Abelian case) is

$$V \to V' = V + \Lambda + \bar\Lambda \ , \tag{4.4}$$

where Λ is a chiral multiplet (i.e. $\bar D_{\dot\alpha}\Lambda = 0 = D_\alpha \bar\Lambda$) . Let us denote the component of Λ by (Λ, ψ, F) and one computes

$$\begin{aligned} V + \Lambda + \bar\Lambda =& f + (\Lambda + \bar\Lambda) + \theta(i\chi + \sqrt{2}\psi) - \bar\theta(i\bar\chi - \sqrt{2}\bar\psi) \\ &+ \tfrac{1}{2}\theta^2(im + 2F) + \tfrac{1}{2}\bar\theta^2(-\bar m + 2\bar F) - \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}(A_\mu - i\partial_\mu(\Lambda - \bar\Lambda) \\ &+ i\theta^2\bar\theta(\bar\lambda + \tfrac{1}{\sqrt{2}}\bar\sigma^\mu\partial_\mu\psi - i\bar\theta^2\bar\theta(\lambda - \tfrac{1}{\sqrt{2}}\sigma^\mu\partial_\mu\bar\psi + \tfrac{1}{2}\theta^2\bar\theta^2(d + \tfrac{1}{4}\Box(\Lambda + \bar\Lambda)) \ . \end{aligned} \tag{4.5}$$

This shows that f, χ, m and the longitudinal component of A_μ are $n_B = n_F = 4$ gauge degrees of freedom. Finally one performs the field redefinition

$$\lambda \to \lambda + \tfrac{i}{2}\sigma^\mu\partial_\mu\bar\chi \ , \qquad d \to D + \tfrac{1}{2}\Box f \ , \tag{4.6}$$

such that the gauge transformation of the physical components (A_μ, λ, D) become

$$\delta A_\mu = -i\partial_\mu(\Lambda - \bar\Lambda) \ , \qquad \delta\lambda = 0 \ , \qquad \delta D = 0 \ . \tag{4.7}$$

With the help of the gauge invariance (4.5) one can gauge fix to the (non-supersymmetric) Wess-Zumino (WZ) gauge and set $f = \chi = m = 0$. Then one has

$$\begin{aligned} V =& -\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} A_\mu + i\theta^2\bar\theta\bar\lambda - i\theta^2\bar\theta\lambda + \tfrac{1}{2}\theta^2\bar\theta^2 D \ , \\ V^2 =& -\tfrac{1}{2}\theta^2\bar\theta^2 A_\mu A^\mu \ , \\ V^3 =& 0 \ . \end{aligned} \tag{4.8}$$

The gauge invariant field strength is defined as

$$W_\alpha := -\tfrac{1}{4}\bar D^2 D_\alpha V \ , \qquad \bar W_{\dot\alpha} := -\tfrac{1}{4} D^2 \bar D_{\dot\alpha} V \ , \tag{4.9}$$

and one checks that under the gauge transformation (4.4)

$$W'_\alpha = W_\alpha - \tfrac{1}{4}\bar D^2 D_\alpha(\Lambda + \bar\Lambda) - \tfrac{1}{4} D^2 \bar D_{\dot\alpha}(\Lambda + \bar\Lambda) = W_\alpha \tag{4.10}$$

due to the chiral property of Λ. One also has

$$\bar D_{\dot\alpha} W_\beta = 0 = D_\beta \bar W_{\dot\alpha} \ , \tag{4.11}$$

and an expansion

$$W_\alpha = -i\lambda_\alpha + (\delta_\alpha^\beta D - \tfrac{i}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha^{\ \beta} F_{\mu\nu})\theta_\beta + \theta^2\sigma^\mu\partial_\mu\bar\lambda \ . \tag{4.12}$$

The supersymmetry transformations can be found from the generic transformation rules given in section 3 to be

$$\begin{aligned} \delta_\xi A_\mu &= -i\bar\lambda\bar\sigma^\mu\xi + i\bar\xi\bar\sigma^\mu\lambda \, , \\ \delta_\xi\lambda &= i\xi D + \sigma^{\mu\nu}\xi F_{\mu\nu} \, , \\ \delta_\xi D &= -\xi\sigma^\mu\partial_\mu\bar\lambda - (\partial_\mu\lambda)\,\bar\sigma^\mu\bar\xi \, . \end{aligned} \tag{4.13}$$

The Lagrangian in terms of superfields is

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{4}\ W_\alpha W^\alpha|_{\theta^2} + \tfrac{1}{4}\ \bar W^{\dot\alpha}\bar W_{\dot\alpha}\big|_{\bar\theta^2} = \tfrac{1}{4}\int d^2\theta\, W_\alpha W^\alpha + \tfrac{1}{4}\int d^2\bar\theta\, \bar W^{\dot\alpha}\bar W_{\dot\alpha} \\ &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar\lambda\bar\partial\!\!\!/\lambda + \tfrac{1}{2}D^2 \ . \end{aligned} \tag{4.14}$$

It is possible to add a supersymmetric and gauge invariant Fayet-Iliopoulos (FI) term

$$\mathcal{L}_{FI} = \xi_{FI}\int d^4\theta V = \xi_{FI} D \ . \tag{4.15}$$

Due to $\int d^4\theta(\Lambda + \bar\Lambda) = 0$ it is gauge invariant. The equation of motion for D in this case becomes $D = -\xi_{FI}$.

4.2 Non-Abelian vector multiplets

In non-Abelian gauge theories A_μ carries the adjoint representation of the gauge group G, i.e., $A_\mu = A_\mu^a T^a$ where T^a are the generators of G obeying

$$[T^a, T^b] = i f^{abc} T^c \ , \quad \mathrm{Tr}\,(T^a T^b) = k\,\delta^{ab} \ , \ k > 0 \ , \quad a,b,c = 1,\ldots,n_v = \dim(ad(G)) \ . \tag{4.16}$$

The generators of G commute with the supersymmetry generators, i.e. $[T^a, Q] = 0$, so that all members of any supermultiplet carry the same representation of G. Therefore the entire superfield V carries the adjoint representation, i.e. $V = V^a T^a$.

The non-Abelian field strength is defined by

$$\begin{aligned} F_{\mu\nu} &:= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \tfrac{i}{2}[A_\mu, A_\nu] \ \equiv \ F_{\mu\nu}^a T^a \ , \\ F_{\mu\nu}^a &:= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - \tfrac{1}{2} f^{abc} A_\mu^b A_\nu^c \ . \end{aligned} \tag{4.17}$$

The (unitary) gauge transformation read

$$\begin{aligned} A_\mu \to A'_\mu &= U^\dagger A_\mu U - U^\dagger \partial_\mu U \ , \\ F_{\mu\nu} \to F'_{\mu\nu} &= U^\dagger F_{\mu\nu} U \ , \end{aligned} \tag{4.18}$$

for $U^\dagger U = \mathbf{1}$.

The superspace generalization is

$$W_\alpha := -\tfrac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V = W_\alpha^a T^a \ , \tag{4.19}$$

with a gauge transformation

$$e^V \to e^{V'} = e^{-i\bar{\Lambda}} e^V e^{i\Lambda} \ , \qquad W_\alpha \to W'_\alpha = e^{-i\Lambda} W_\alpha e^{i\Lambda} \ , \tag{4.20}$$

where $\Lambda = \Lambda^a T^a$.

The non-Abelian Lagrangian then reads

$$\begin{aligned} \mathcal{L} &= \frac{\tau}{16k} \int d^2\theta\, \mathrm{Tr}\, W_\alpha W^\alpha + \frac{\bar{\tau}}{16k} \int d^2\bar{\theta}\, \mathrm{Tr}\, \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \\ &= -\tfrac{1}{4g^2} F_{\mu\nu}^a F^{\mu\nu\, a} - \tfrac{\theta}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{\mu\nu\, a} - \tfrac{i}{g^2} \bar{\lambda}^a \not{D} \lambda^a + \tfrac{1}{2g^2} D^a D^a \ . \end{aligned} \tag{4.21}$$

where

$$D_\mu \lambda^a = \partial_\mu \lambda^a - f^{abc} A_\mu^b \lambda^c \ , \qquad \tilde{F}^{\mu\nu\, a} := \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a \ , \tag{4.22}$$

and we introduced the complex coupling constant

$$\tau := g^{-2} + i\tfrac{\theta}{8\pi^2} \ . \tag{4.23}$$

The equation of motion for D reads

$$D^a = 0 \ . \tag{4.24}$$

For a non-Abelian gauge group a FI-term cannot be added as it is not gauge invariant.

5 Super YM Theories coupled to matter and the MSSM

5.1 Coupling to matter

Chiral multiplets can carry a reprensentation $\mathbf{r}$ of the gauge group G. In this case one has n_c chiral multiplets $\Phi^i = (\phi^i, \chi^i, F^i), i = 1, \ldots, n_c = \dim(\mathbf{r})$ transforming as

$$\Phi \to \Phi' = e^{-i\Lambda}\Phi \ , \qquad \bar{\Phi} \to \bar{\Phi}' = \bar{\Phi}e^{i\bar{\Lambda}} \ , \qquad \Lambda = \Lambda^a T^a_{\mathbf{r}} \ , \tag{5.1}$$

where $T^a_{\mathbf{r}}$ are the generators of G in representation $\mathbf{r}$. Gauge invariance then requires changing the kinetic term of the chiral multiplets as follows

$$\bar{\Phi}\Phi \to \bar{\Phi}e^V\Phi \ , \tag{5.2}$$

which indeed is consistent with (4.20). The gauge invariant non-Abelian Lagrangian then reads

$$\begin{aligned} \mathcal{L} \ &= \frac{\tau}{16k}\int d^2\theta \, \mathrm{Tr}\, W_\alpha W^\alpha + \frac{\bar{\tau}}{16k}\int d^2\bar{\theta}\, \mathrm{Tr}\, \bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}} \\ &+ \int d^4\theta \, \bar{\Phi}e^V\Phi + \int d^2\theta \, W(\Phi) + \int d^2\bar{\theta}\, \bar{W}(\bar{\Phi}) \ . \end{aligned} \tag{5.3}$$

After the rescaling $V \to 2gV$ it reads in components

$$\begin{aligned} \mathcal{L} = &- \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu\, a} - i\bar{\lambda}^a \bar{D\!\!\!/}\lambda^a + \tfrac{1}{2}D^aD^a - D_\mu\phi^i D^\mu\bar{\phi}^i - i\bar{\chi}^i \bar{D\!\!\!/}\chi^i + F^i\bar{F}^i \\ &+ i\sqrt{2}\, g\left(\bar{\phi}^i T^a_{ij}\chi^j\lambda^a - \phi^i T^a_{ij}\bar{\lambda}^a\bar{\chi}^j\right) + gD^a\bar{\phi}^i T^a_{ij}\phi^j \\ &- \tfrac{1}{2}W_{ij}\chi^i\chi^j - \tfrac{1}{2}\bar{W}_{ij}\bar{\chi}^i\bar{\chi}^j + F^iW_i + \bar{F}^i\bar{W}_i \ , \end{aligned} \tag{5.4}$$

where W_i and W_{ij} are defined in (2.23) and the covariant derivatives are defined in (4.22) and as

$$D_\mu\phi^i = \partial_\mu\phi^i + igA^a_\mu T^{ai}{}_j\phi^j \ , \qquad D_\mu\chi^i = \partial_\mu\chi^i + ig\phi^a_\mu T^{ai}{}_j\chi^j \, . \tag{5.5}$$

$\mathcal{L}$ is invariant under the combined supersymmetry transformations

$$\begin{aligned} \delta_\xi\phi^i &= \sqrt{2}\xi\chi^i \ , \\ \delta_\xi\chi^i &= \sqrt{2}\xi F^i + i\sqrt{2}\sigma^\mu\bar{\xi}D_\mu\phi^i \ , \\ \delta_\xi F^i &= i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu D_\mu\chi^i \ , \\ \delta_\xi v^a_\mu &= -i\bar{\lambda}^a\bar{\sigma}^\mu\xi + i\bar{\xi}\bar{\sigma}^\mu\lambda^a \, , \\ \delta_\xi\lambda^a &= i\xi D^a + \sigma^{\mu\nu}\xi F^a_{\mu\nu} \, , \\ \delta_\xi D^a &= -\xi\sigma^\mu D_\mu\bar{\lambda}^a - (D_\mu\lambda^a)\,\bar{\sigma}^\mu\bar{\xi} \, . \end{aligned} \tag{5.6}$$

The additional terms compared to (2.16) and (2.15) are enforced by gauge invariance.

The auxiliary fields F^i, D^a can be eliminated by their algebraic equations of motions

$$\frac{\delta\mathcal{L}}{\delta D^a} = D^a + g\bar{\phi}^i T^a_{ij}\phi^j = 0 \ , \qquad \frac{\delta\mathcal{L}}{\delta\bar{F}^i} = F^i + \bar{W}^i = 0 \, . \tag{5.7}$$

Inserted into the Lagrangian (5.4) then yields

$$
\begin{aligned}
\mathcal{L} = & -\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} - \tfrac{i}{2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} - i\bar{\lambda}^a \bar{\not{D}} \lambda^a - D_\mu 1^i D^\mu \bar{\phi}^i - i\bar{\chi}^i \bar{\not{D}} \chi^i \\
& + i\sqrt{2}\, g \left(\bar{\phi}^i T^a_{ij} \chi^j \lambda^a - \phi^i T^a_{ij} \bar{\lambda}^a \bar{\chi}^j \right) - \tfrac{1}{2} W_{ij} \chi^i \chi^j - \tfrac{1}{2} \bar{W}_{ij} \bar{\chi}^i \bar{\chi}^j - V(\phi, \bar{\phi})\,,
\end{aligned}
\tag{5.8}
$$

where V is the scalar potential given by

$$
V(\phi, \bar{\phi}) = W_i \bar{W}_i + \tfrac{1}{2} g^2 \left(\bar{\phi}^i T^a_{ij} \phi^j \right) \left(\bar{\phi}^k T^a_{kl} \phi^l \right) = F_i \bar{F}_i + \tfrac{1}{2} D^a D^a\,. \tag{5.9}
$$

Before we continue let us make the following remarks:

- V is positive semi-definite $V \geq 0$.
- V is not the most general scalar potential, i.e. there is no independent $\lambda(\phi\bar{\phi})^2$ coupling. Instead the quartic scalar couplings arise from Y^2 in the F-term or g^2 in the D-term. In the SSM this properties leads to a light Higgs boson.
- V depends only on g, m_{ij}, Y_{ijk} with no additional new parameters being introduced.
- There is a "new" Yukawa coupling proportional to $g\bar{\phi}\chi\lambda$.

5.2 The minimal supersymmetric Standard Model (MSSM)

The basic idea of the supersymmetric Standard Model (MSSM) is to promote each field of the Standard Model (SM) to an appropriate supermultiplet. In particular the quarks, leptons and Higgs reside in chiral multiplets while the gauge bosons are members of vector multiplets. Since the gauge generators commute with the Q's, the supermultiplets have to carry the same representations as their SM-components. The gauge group of the SM is $G = SU(3) \times SU(2) \times U(1)_Y$ which is spontaneously broken to $G = SU(3) \times U(1)_{\mathrm{em}}$.

5.2.1 The Spectrum

The spectrum of the MSSM is summarized in Table 5.1.

Before we turn to the Lagrangian let us note that two Higgs doublets (i.e. an extended Higgs sector) are necessary. This is imposed on the theory by supersymmetry as gauge invariance of the superpotential otherwise cannot be achieved. Alternatively, the absence of a gauge anomaly leads to the same conclusion as the Higgs multiplets contain two new chiral fermions which have to be in vector-like representations of the gauge group.

Let us also summarize the new fields in the spectrum. For $s = 0$ these are the squarks $\tilde{q}, \tilde{u}, \tilde{d}$ and the sleptons $\tilde{l}, \tilde{e}, \tilde{\nu}$. For $s = 1/2$ these are the Higgsinos $\tilde{h}_u, \tilde{h}_d$ and the gauginos $\tilde{G}, \tilde{W}, \tilde{B}$. They will often be regrouped into the four neutralinos $\tilde{h}^0_u, \tilde{h}^0_d, \tilde{\gamma}^0, \tilde{Z}$ (where $\tilde{\gamma}^0, \tilde{Z}$ are called photino and Zino) and the four charginos $\tilde{h}^+_u, \tilde{h}^-_d, \tilde{W}^\pm$.

	SM fields	$SU(3)\times SU(2)\times U(1)_Y$	$U(1)_{\rm em}$	supermultiplet	F	B
quarks	$q_L^I = \begin{pmatrix} u_L^I \\ d_L^I \end{pmatrix}$	$(3,2,\frac{1}{6})$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$	$Q_L^I = \begin{pmatrix} U_L^I \\ D_L^I \end{pmatrix}$	q_L^I	$\tilde{q}_L^I$
	u_R^I	$(\bar{3},1,-\frac{2}{3})$	$-\frac{2}{3}$	U_R^I	u_R^I	$\tilde{u}_R^I$
	d_R^I	$(\bar{3},1,-\frac{1}{3})$	$-\frac{1}{3}$	D_R^I	d_R^I	$\tilde{d}_R^I$
leptons	$l_L^I = \begin{pmatrix} \nu_L^I \\ e_L^I \end{pmatrix}$	$(1,2,-\frac{1}{2})$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$L_L^I = \begin{pmatrix} N_L^I \\ E_L^I \end{pmatrix}$	l_L^I	$\tilde{l}_L^I$
	e_R^I	$(1,1,1)$	1	E_R^I	e_R^I	$\tilde{e}_R^I$
	ν_R^I	$(1,1,0)$	0	N_R^I	ν_R^I	$\tilde{\nu}_R^I$
Higgs	$\begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix}$	$(1,2,\frac{1}{2})$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}$	$\begin{pmatrix} \tilde{h}_u^+ \\ \tilde{h}_u^0 \end{pmatrix}$	$\begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix}$
		$(1,2,-\frac{1}{2})$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}$	$\begin{pmatrix} \tilde{h}_d^0 \\ \tilde{h}_d^- \end{pmatrix}$	$\begin{pmatrix} h_d^0 \\ h_d^- \end{pmatrix}$
gauge	G	$(8,1,0)$	0	G	$\tilde{G}$	G
bosons	W	$(1,3,0)$	$(0,\pm 1)$	W	$\tilde{W}$	W
	B	$(1,1,0)$	0	B	$\tilde{B}$	B

Table 5.1: Particle content of the supersymmetric Standard Model. The column below 'F' ('B') denotes the fermionic (bosonic) content of the model. The index $I = 1, 2, 3$ labels the three families of the SM.

5.2.2 The Lagrangian

The Lagrangian for the supersymmetric Standard Model has to be of the form (5.8) with gauge group $G = SU(3) \times SU(2) \times U(1)_Y$. This specifies the covariant derivatives in (5.5) appropriately. The superpotential (2.23) has to be chosen such that the Lagrangian of the non-supersymmetric Standard Model is contained. This is achieved by

$$W = \sum_{I,J=1}^{3} \big((Y_u)_{IJ} H_u Q_L^I U_R^J + (Y_d)_{IJ} H_d Q_L^I D_R^J + (Y_l)_{IJ} H_d L_L^I E_R^J + m_{IJ} N_R^I N_R^J\big) + \mu\, H_u H_d \ , \tag{5.10}$$

where $(Y_u)_{IJ}, (Y_d)_{IJ}, (Y_l)_{IJ}$ are the measured Yukawa couplings of the SM, μ a Higgs-mass parameter and m_{IJ} a possible mixing matrix of the right handed neutrinos. Now we see more explicitly that a $h_u \bar{h}_u$ Higgs mass term as in the SM is incompatible with the holomorphicity of W. This forces the presence of a second Higgs doublet h_d in the complex conjugate representation of $SU(2) \times U(1)$.

From Table 5.1 we see that in terms of quantum numbers there is no distinction between the chiral superfields L_L and H_d. This in turn leads to additional gauge invariant couplings which are possible in W. These are

$$\Delta W = aH_uL_L + b\,L_LQ_LD_R + c\,D_RD_RU_R + d\,L_LL_LE_R\,, \tag{5.11}$$

which, however, violate baryon or lepton number conservation and thus easily lead to unacceptable physical consequences (for example fast proton decay). Such couplings can be excluded by imposing a discrete R-parity. Particles of the Standard Model (including both Higgs doublets) are assigned R-charge 1 while all new supersymmetric particles are assigned R-charge -1. This eliminates all terms in (5.11) while the superpotential given in (5.10) is left invariant. An immediate consequence of this additional symmetry is the fact that the lightest supersymmetric particle (often denoted by the 'LSP') is necessarily stable and thus a candidate for WIMP dark matter. However, one should stress that R-parity is not a phenomenological necessity. Viable models with broken R-parity can be constructed and they also can have some phenomenological appeal.

Another extension of the SSM (often called the NMSSM) adds an additional singlet chiral multiplet S with couplings

$$W_{NMSSM} = \tfrac{1}{2}\mu_S S^2 + \tfrac{1}{6}Y_S S^3 + \lambda_S S H_u H_d + W_{SSM}\ . \tag{5.12}$$

6 Spontaneous Supersymmetry Breaking

6.1 Order parameters of supersymmetry breaking

Recall that in a theory with a spontaneously broken symmetry the action of the theory is invariant under the symmetry transformation but its ground state or background is not. Here we consider backgrounds which preserve four-dimensional Lorentz invariance and minimize the potential V. In supersymmetric theories we have generically

$$\langle \delta\text{fermion} \rangle \sim \langle \text{boson} \rangle \ , \qquad\qquad \langle \delta\text{boson} \rangle \sim \langle \text{fermion} \rangle = 0 \ , \tag{6.1}$$

where the second transformation always vanishes in a Lorentz-invariant background. Therefore we see that the Lorentz-scalar part of $\langle \delta\text{fermion} \rangle$ is the order parameter of supersymmetry breaking. For super Yang-Mills theories we have

$$\langle \delta\chi^i_\alpha \rangle = \sqrt{2}\xi_\alpha \langle F^i \rangle \ , \qquad \langle \delta\lambda^a_\alpha \rangle = i\xi_\alpha \langle D^a \rangle \ , \tag{6.2}$$

where all additional terms vanish in a Lorentz-invariant background. We see that we can have spontaneous supersymmetry breaking if and only if

$$\langle F^i \rangle \neq 0 \quad (F\text{−term breaking}) \ , \quad \text{and/or} \quad \langle D^a \rangle \neq 0 \quad (D\text{−term breaking}) \ , \tag{6.3}$$

i.e. $\langle F^i \rangle$ and $\langle D^a \rangle$ are the order parameters of supersymmetry breaking in that non-vanishing F- or D-terms signal spontaneous supersymmetry breaking.

Let us determine the minimum of the scalar potential (2.26)

$$V = F_i \bar{F}_i + \tfrac{1}{2} D^a D^a \geq 0 \ . \tag{6.4}$$

Its first derivative reads

$$\partial_j V = F_i \partial_j \bar{F}_i + (\partial_j D^a) D^a = \bar{W}_i W_{ij} + (\partial_j D^a) D^a = 0 \ . \tag{6.5}$$

We immediately see that the minimum of V is at

$$\langle F_i \rangle = \langle \bar{F}_i \rangle = \langle D^a \rangle = \langle V \rangle = 0 \ . \tag{6.6}$$

Conversely, $\langle V \rangle = 0$ implies that supersymmetry is unbroken while $\langle V \rangle \neq 0$ implies that supersymmetry is broken.

6.2 Models for spontaneous supersymmetry breaking

Let us now discuss models for spontaneous supersymmetry breaking. The idea is to add fields to the spectrum with couplings such that supersymmetry is spontaneously broken. Concretely one needs to forbid solutions with $\langle F_i \rangle = \langle D^a \rangle = 0$ which is surprisingly difficult to arrange. Let us start with F-term breaking.

6.2.1 F-term breaking

In the O'Raifeartaigh model [?] one introduces three chiral superfields Φ_0, Φ_1, Φ_2 and the following superpotential:

$$W = \lambda\phi_0 + m\phi_1\phi_2 + Y\phi_0\phi_1^2 \,, \qquad m^2 > 2\lambda Y \,. \tag{6.7}$$

The algebraic equations for the F-terms are:

$$\begin{aligned} F_0 &= \frac{\partial W}{\partial\phi^0} = \lambda + Y\phi_1^2 \,, \\ F_1 &= \frac{\partial W}{\partial\phi^1} = m\phi_2 + 2Y\phi_0\phi_1 \,, \\ F_2 &= \frac{\partial W}{\partial\phi^2} = m\phi_1 \,. \end{aligned} \tag{6.8}$$

$\langle F_0\rangle = 0 = \langle F_2\rangle$ has no solution and thus supersymmetry must be broken.

The scalar potential reads

$$V = \left|\lambda + Y\phi_1^2\right|^2 + |m\phi_2 + 2Y\phi_0\phi_1|^2 + |m\phi_1|^2 \,. \tag{6.9}$$

It is minimized by $\langle\phi_1\rangle = 0 = \langle\phi_2\rangle$, $\langle\phi_0\rangle$ arbitrary, such that $\langle F_1\rangle = 0 = \langle F_2\rangle$ and $\langle F_0\rangle \neq 0$. The mass spectrum of the 6 real bosons and the 3 Weyl fermions is found to be

$$\begin{aligned} \text{bosons}: &\quad (0, 0, m^2, m^2, m^2 \pm 2Y\lambda) \,, \\ \text{fermions}: &\quad (0, m, m) \,. \end{aligned} \tag{6.10}$$

We observe a mass splitting of the boson-fermion mass degeneracy but a sum rule still holds

$$\text{Str}M^2 := \sum_s (-)^{2s}(2s+1)\text{Tr}\, M_s^2 = \text{Tr}\, M_0^2 - 2\text{Tr}\, M_{1/2}^2 = 4m^2 - 4m^2 = 0 \,. \tag{6.11}$$

(In section 6.3.2 we will derive the general form of the sum rule and show its validity.)

Phenomenologically the sum rule (6.11) is problematic for the supersymmetric Standard Model. Since none of the supersymmetric partners has been observed yet, they must be heavier than the particles of the Standard Model. Close inspection of (6.11) shows that this cannot be arranged within a spontaneously broken supersymmetric Standard Model. Nevertheless let us continue and discuss D-term breaking.

6.2.2 D-term breaking

We already discussed the possibility of adding a Fayet-Iliopoulos term to the supersymmetry Lagrangian for any $U(1)$ factor in the gauge group. Let us therefore consider a $U(1)$ vector multiplet and one charged chiral multiplet with vanishing $W = 0$ but the additional FI coupling (4.15). In this case the D-term and the potential read

$$D = -(g\bar{\phi}\phi + \xi_{FI}) \,, \qquad V = \tfrac{1}{2}D^2 = \tfrac{1}{2}\left(g\bar{\phi}\phi + \xi_{FI}\right)^2 \,. \tag{6.12}$$

We need to distinguish the cases $g\xi_{FI} < 0$ and $g\xi_{FI} > 0$. For $g\xi_{FI} < 0$ the minimum is at $\langle \bar{\phi}\phi \rangle = -\xi_{FI}/g$ with $\langle D \rangle = 0 = \langle V \rangle$. Thus the $U(1)$ gauge symmetry is spontaneously broken but supersymmetry is intact. For $g\xi_{FI} > 0$ the condition $\langle D \rangle = 0$ has no solution. The minimum is at $\langle \phi \rangle = 0$ with $\langle V \rangle = \xi_{FI}^2/2$, $\langle D \rangle = -\xi_{FI}$. In this case the $U(1)$ is unbroken but supersymmetry is broken. Thus the vector multiplet remains massless, the chiral fermion remains massless as $W = 0$ and only ϕ receives a mass

$$m_\phi^2 = \langle \partial_\phi \partial_{\bar{\phi}} V \rangle = -2\xi_{FI} g \ . \tag{6.13}$$

In this case we have the sum rule

$$\mathrm{Str} M^2 = m_\phi^2 = -2gD \ . \tag{6.14}$$

6.3 General considerations

6.3.1 Fermion mass matrix and Goldstone's theorem for supersymmetry

Let us start by computing the generic fermion mass matrix including the case where the scalar fields ϕ^i have a non-trivial background value $\langle \phi^i \rangle \neq 0$. It arises from the following terms of the Lagrangian (5.8)

$$\mathcal{L}_{M_{1/2}} = -\tfrac{1}{2} W_{ij} \chi^i \chi^j + \tfrac{1}{2} \bar{W}_{ij} \bar{\chi}^i \bar{\chi}^j + i\sqrt{2} g \left(\bar{\phi}^i T^a_{ij} \chi^j \lambda^a - \bar{\lambda}^a T^a_{ij} \phi^i \bar{\chi}^j \right) \ . \tag{6.15}$$

These terms can be arranged in matrix form

$$\mathcal{L}_{M_{1/2}} = -\tfrac{1}{2} \left(\chi^i, \lambda^a \right) M_{1/2} \begin{pmatrix} \chi^j \\ \lambda^b \end{pmatrix} + \text{h.c.} \,, \tag{6.16}$$

for

$$M_{1/2} = \left. \begin{pmatrix} W_{ij} & i\sqrt{2}\partial_i D^a \\ i\sqrt{2}\partial_j D^b & 0 \end{pmatrix} \right|_{\min(V)} , \tag{6.17}$$

where $\partial_i D^a = -g\bar{\phi}^j T^a_{ji}$. Similarly

$$\bar{M}_{1/2} = \left. \begin{pmatrix} \bar{W}_{ij} & -i\sqrt{2}\bar{\partial}_i D^a \\ -i\sqrt{2}\bar{\partial}_j D^b & 0 \end{pmatrix} \right|_{\min(V)} . \tag{6.18}$$

Note that for $\langle \phi^i \rangle = 0$ only $W_{ij} = m_{ij}$ survives in $M_{1/2}$. For later use we compute

$$\mathrm{Tr}\, M_{1/2} \bar{M}_{1/2} = \left. \left(W_{ij} \bar{W}_{ji} + 4 \partial_i D^a \bar{\partial}_i D^a \right) \right|_{\min(V)} \ . \tag{6.19}$$

Goldstone's theorem implies that any spontaneously broken global symmetry leads to a massless state in the spectrum. This also holds for supersymmetry where the broken generator is a Weyl spinor and thus there has to be an massless Goldstone fermion. Indeed for arbitrary background values, $M_{1/2}$ always has a zero eigenvalue corresponding to the Goldstone fermion. This can be seen by identifying the corresponding null vector. Consider

$$M_{1/2} \begin{pmatrix} \bar{W}_j \\ \frac{-i}{\sqrt{2}} D^a \end{pmatrix} = \begin{pmatrix} W_{ij} \bar{W}_j + (\partial_j D^a) D^a \\ i\sqrt{2} (\partial_j D^b) \bar{W}_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ , \tag{6.20}$$

where in the first equation we used (6.18). In the second equation the upper component vanishes due to (6.5) while the lower component vanishes due to gauge invariance of W. Gauge invariance indeed implies

$$\delta W = W_i \delta\phi^i = i\alpha^a W_i \left(T^a\right)^i_j \phi^j = i\alpha^a W_i \partial_{\bar{i}} D^a = 0\,. \tag{6.21}$$

This proves Goldstones theorem for supersymmetry. Phenomenologically, however, the presence of a massless Goldstone fermion poses a problem for the SSM as no massless fermion has been observed yet. This already hints at the super Higgs effect where the Goldstone fermion is "eaten" by the gauge field of local supersymmetry, the gravitino.

6.3.2 Mass sum rules and the supertrace

In order to determine the scalar mass matrix we need to consider the second derivatives of V. From (2.26) we find

$$\begin{aligned}
\partial_j V &= W_{ij}\bar{W}_i + (\partial_j D^a)D^a\,,\\
\partial_j\partial_k V &= W_{ijk}\bar{W}_i + (\partial_j D^a)(\partial_k D^a)\\
\partial_j\bar{\partial}_k V &= W_{ij}\bar{W}_{ik} + (\partial_j\bar{\partial}_k D^a)D^a + (\partial_j D^a)(\bar{\partial}_k D^a)\,,
\end{aligned} \tag{6.22}$$

where

$$\begin{aligned}
D^a = -g\bar{\phi}^i T^a_{ij}\phi^j - \xi_{FI}\delta^{aU(1)}\,, \qquad \partial_j D^a = -g\bar{\phi}^i T^a_{ij}\,,\\
\bar{\partial}_i D^a = -gT^a_{ij}\phi^j\,, \qquad \partial_j\bar{\partial}_k D^a = -gT^a_{kj}\,.
\end{aligned} \tag{6.23}$$

The scalar masses can also be written in matrix form

$$V = \tfrac{1}{2}\left(\bar{\phi}^i, \phi^j\right) M_0^2 \begin{pmatrix}\phi^k\\ \bar{\phi}^l\end{pmatrix} \tag{6.24}$$

for

$$M_0^2 = \left.\begin{pmatrix}\bar{\partial}_i\partial_k V & \bar{\partial}_i\bar{\partial}_l V\\ \partial_j\partial_k V & \partial_j\bar{\partial}_l V\end{pmatrix}\right|_{\min(V)}\,. \tag{6.25}$$

Note that for $\langle\phi^i\rangle = 0$, M_0^2 is block diagonal with m^2_{ij} appearing in the diagonal. The trace is

$$\operatorname{Tr} M_0^2 = 2\left.\left(W_{ij}\bar{W}_{ji} + (\partial_i\bar{\partial}_i D^a)D^a + (\partial_i D^a)(\bar{\partial}_i D^a)\right)\right|_{\min}\,. \tag{6.26}$$

Finally, the mass matrix of the gauge bosons arises from

$$\mathcal{L}_{M_1} = -D_\mu\bar{\phi}^i D^\mu\phi^i = -\tfrac{1}{2}M^2_{Ib}A^I_\mu A^{b\,\mu} + \ldots\,, \tag{6.27}$$

with

$$M^2_{Ib} = 2g^2\bar{\phi}^j T^I_{jk}T^b_{kl}\phi^l = 2(\partial_k D^a)(\bar{\partial}_k D^b)\,, \tag{6.28}$$

where we used

$$D_\mu\phi^i = \partial_\mu\phi^i + igA^a_\mu T^a_{ij}\phi^j\,, \qquad D_\mu\bar{\phi}^i = \partial_\mu\phi^i - igA^a_\mu T^a_{ij}\bar{\phi}^j\,. \tag{6.29}$$

Note that for $\langle\phi^i\rangle = 0$ all gauge bosons are massless.

One defines the supertrace of the mass matrices by

$$\mathrm{Str} M^2 := \sum_{s=0}^{1} (-)^{2s}(2s+1)\mathrm{Tr}\, M_s^2 \ . \tag{6.30}$$

For the case at hand we find from (6.19), (6.26), (6.28)

$$\begin{aligned} \mathrm{Str} M^2 =& \mathrm{Tr}\, M_0^2 - 2\mathrm{Tr}\, M_{1/2} + 3\mathrm{Tr}\, M_1^2 \\ =& 2(W_{ij}\bar{W}_{ji} + (\partial_i\bar{\partial}_i D^a)D^a + (\partial_i D^a)(\bar{\partial}_i D^a)) \\ & - 2(W_{ij}\bar{W}_{ji} + 4\partial_i D^a \bar{\partial}_i D^a) + 6(\partial_i D^a)(\bar{\partial}_i D^a) \\ =& 2(\partial_i\bar{\partial}_i D^a)D^a = -2g\, (\mathrm{Tr}\, T^a)\, D^a \, . \end{aligned} \tag{6.31}$$

For a non-Abelian gauge group the generators are traceless while for an Abelian ($U(1)$) gauge group the trace is proportional to the sum of the $U(1)$ charges q. Thus we have altogether

$$\mathrm{Str} M^2 = -2g\, (\mathrm{Tr}\, T^a)\, D^a = \begin{cases} 0 & \text{for non-Abelian } G \\ -2g\, (\sum q)\, D^{U(1)} & \text{for } G = U(1) \end{cases} \ . \tag{6.32}$$

However, for $\sum q \neq 0$ the theory has a gravitational anomaly and thus cannot be coupled to gravity.

7 Non-renormalizable couplings

In this lecture we consider supersymmetric theories as effective theories below some (cut-off) scale M. Such effective theories are non-renormalizable and therefore we have to generalize the considerations so far. In particular the following three generalizations will be important:

1. The superpotential $W(\phi)$ will be an arbitrary holomorphic function and no longer contraint to be cubic.

2. The gauge couplings will be field dependent $\tau \to \tau(\phi)$.

3. The kinetic term of the scalar fields will have the form of an interacting σ-model.

Let us start with the latter.

7.1 Non-linear σ-models

The renormalizable kinetic term for n scalar fields given by

$$\mathcal{L} = -\delta_{ij}\, \partial_\mu \phi^i \partial^\mu \phi^j \,, \quad i, j = 1, \ldots, n \,, \tag{7.1}$$

can be generalized as

$$\mathcal{L} = -G_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j \,, \tag{7.2}$$

where $G_{ij}(\phi)$ is a symmetric, positive and invertible matrix depending on ϕ^i. A theory with the Lagrangian (7.2) is called non-linear σ-model which, due to the ϕ dependence of $G_{i\bar{j}}$, is non-renormalizable.

The scalar fields ϕ^i can be interpreted as coordinate of an n-dimensional Riemannian target space $\mathcal{M}$ and $G_{i\bar{j}}$ as its metric. Indeed an arbitrary field redefinition $\phi^i \to \phi^{i\,\prime}(\phi^j)$ implies

$$\partial_\mu \phi^i \to \partial_\mu \phi^{i\,\prime} = \frac{\partial \phi^{i\,\prime}}{\partial \phi^j}\, \partial_\mu \phi^j \,. \tag{7.3}$$

$\mathcal{L}$ is invariant if $G_{i\bar{j}}$ transforms inversely, i.e.,

$$G_{ij} \to G'_{ij} \;=\; \frac{\partial \phi^k}{\partial \phi^{i\,\prime}} \frac{\partial \phi^l}{\partial \phi^{j\,\prime}}\, G_{kl} \,, \tag{7.4}$$

which is precisely the transformation of the metric on $\mathcal{M}$. The scalar fields can thus be viewed as the map

$$\phi^i(x): \quad M_4 \to \mathcal{M} \,, \tag{7.5}$$

where M_4 is the Minkowski space and $\mathcal{M}$ a Riemannian target space.

Let us also recall that the metric has an expansion in Riemann normal coordinates. For $\phi^i = \phi^i_0 + \delta\phi^i$ one has

$$G_{ij} = \delta_{ij} + M^{-2} R_{ijkl}(\phi_0) \delta\phi^k \delta\phi^l + \mathcal{O}((\delta\phi)^3) \,, \tag{7.6}$$

where we have chosen $G_{ij}(\phi_0) = \delta_{ij}$ and R_{ijkl} is the curvature tensor on $\mathcal{M}$. We also included the cut-off scale M which is necessary due to the mass dimensions $[\phi^i] = 1, [G_{ij}] = 0$. This M-dependence is another way to see the non-renormalizability of the non-linear σ-model. For complex scalar fields $\mathcal{M}$ is a complex manifold.

7.2 Couplings of neutral chiral multiplet

Let return to supersymmetric theories and first discuss the couplings of chiral multiplets. As we already said, W is no longer constrained to be cubic and the kinetic term $\phi^i\bar{\phi}^{\bar{j}}$ is replaced by an arbitrary real function $K(\phi^i, \bar{\phi}^{\bar{j}})$ where now one conventionally also puts a "bar" over the index of the anti-holomorphic scalar $\bar{\phi}^{\bar{j}}$.

The couplings are determined most easily in superspace where the non-renormalizable Lagrangian replacing (3.30) reads

$$\mathcal{L} = \int d^4\theta\, K(\Phi^i, \bar{\Phi}^i) + \int d^2\theta\, W(\Phi^i) + \int d^2\bar{\theta}\, \bar{W}(\bar{\Phi}^i) \ . \tag{7.7}$$

Note that K is not uniquely defined but only up to so called Kähler transformations as for $K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + f(\Phi^i) + \bar{f}(\bar{\Phi}^i)$ one has

$$\int K(\Phi^i, \bar{\Phi}^i) d^4\theta \to \int K(\Phi^i, \bar{\Phi}^i) + \int f(\Phi^i) d^4\theta + \int \bar{f}(\bar{\Phi}^i) d^4\bar{\theta} = \int K(\Phi^i, \bar{\Phi}^i) \ , \tag{7.8}$$

where we used $\int f(\Phi^i) d^4\theta = 0 = \int \bar{f}(\bar{\Phi}^i) d^4\theta = 0$.

In components the Lagrangian (7.7) is found to be (generalizing (2.25))

$$\begin{aligned}\mathcal{L} = & -G_{i\bar{j}}(\phi, \bar{\phi})\, \partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{j}} - iG_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu D_\mu\chi^i - V(\phi, \bar{\phi}) \\ & - \tfrac{1}{2}(\nabla_i\partial_j W)\, \chi^i\chi^j - \tfrac{1}{2}(\nabla_{\bar{i}}\partial_{\bar{j}} W)\, \bar{\chi}^{\bar{i}}\bar{\chi}^{\bar{j}} + R_{i\bar{j}k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}} \ .\end{aligned} \tag{7.9}$$

The σ-model metric $G_{i\bar{j}}(\phi, \bar{\phi})$ is related to K by

$$G_{i\bar{j}} = \frac{\partial}{\partial\phi^i}\frac{\partial}{\partial\bar{\phi}^{\bar{j}}} K(\phi, \bar{\phi}) \, , \tag{7.10}$$

which is the metric on a Kähler manifold with K being the Kähler potential. Supersymmetry forces the fermions χ^i to transform as vectors under the coordinate change (7.3) and thus its covariant derivative reads

$$D_\mu\chi^i := \partial_\mu\chi^i + \Gamma^i_{jk}\, \partial_\mu\phi^j\chi^k \ , \tag{7.11}$$

where the Christoffel symbols on a Kähler manifold read

$$\Gamma^i_{jk} = G^{i\bar{l}}\partial_j G_{\bar{l}k} \ , \tag{7.12}$$

with $\Gamma^{\bar{k}}_{\bar{i}\bar{j}}$ being the only other non-vanishing one. $R_{i\bar{j}k\bar{l}}$ is the Riemann curvature tensor which on a Kähler manifold reads

$$R_{i\bar{j}k\bar{l}} = G_{m\bar{l}}\partial_{\bar{j}}\Gamma^m_{ik} \ . \tag{7.13}$$

Furthermore, the generalized Yukawa couplings are given by

$$\nabla_i\partial_j W = \partial_i\partial_j W - \Gamma^k_{ij}\partial_k W \ . \tag{7.14}$$

Finally, the scalar potential is given by

$$V = G^{i\bar{j}}\partial_i W\partial_{\bar{j}}\bar{W} \ . \tag{7.15}$$

7.3 Couplings of vector multiplets – gauged σ-models

For vector multiplets the non-renormalizable generalization of (4.21) is

$$
\begin{aligned}
\mathcal{L} &= \tfrac{1}{16}\int d^2\theta\, \tau_{ab}(\phi) W^a_\alpha W^{b\,\alpha} + \text{h.c.} \\
&= -\tfrac{1}{4}\text{Re}\tau_{ab} F^a_{\mu\nu} F^{\mu\nu\, b} - \tfrac{1}{4}\text{Im}\tau_{ab} F^a_{\mu\nu} \tilde{F}^{\mu\nu\, b} - i\text{Re}\tau_{ab}\bar{\lambda}^a \bar{\not{D}}\lambda^b + \tfrac{1}{2}\text{Re}\tau_{ab} D^a D^b \ ,
\end{aligned} \tag{7.16}
$$

where $\tau_{ab}(\phi)$ is holomorphic and called the gauge kinetic function. We thus see that the gauge couplings and the θ-angles became field dependent

$$
g^{-2}_{ab} = \text{Re}\tau_{ab} \ , \qquad \frac{\theta_{ab}}{8\pi^2} = \text{Im}\tau_{ab} \ . \tag{7.17}
$$

Note that for a simple gauge group one has $\tau_{ab} = \delta_{ab}\tau(\phi)$.

Coupling chiral multiplets is not straightforward. Gauge invariance of the non-linear σ-model (7.2) requires the metric $G_{i\bar{j}}$ to admit (non-Abelian) isometries. In particular one needs that the metric is invariant (i.e. $\delta_\Lambda G_{i\bar{j}} = 0$) under the gauge transformation

$$
\delta_\Lambda \phi^i = \Lambda^a(x)\, k^{ai}(\phi)\,, \qquad \delta_\Lambda \bar{\phi}^{\bar{i}} = \Lambda^a(x)\, \bar{k}^{ai}(\bar{\phi})\,, \tag{7.18}
$$

where $\Lambda^a(x)$ is the gauge parameter and $k^{ai}(\phi)$ are Killing vector fields.[5]

Demanding $\delta_\Lambda G_{i\bar{j}} = 0$ results in the Killing equations

$$
\nabla_i \bar{k}^a_j + \nabla_j \bar{k}^a_i = 0 \ , \qquad \nabla_i k^a_{\bar{j}} + \bar{\nabla}_{\bar{j}} \bar{k}^a_i = 0 \ , \tag{7.19}
$$

where

$$
\bar{k}^a_j(\phi,\bar{\phi}) := G_{j\bar{i}}(\phi,\bar{\phi})\, \bar{k}^{a\bar{i}}(\bar{\phi}) \ , \qquad k^a_{\bar{j}}(\phi,\bar{\phi}) := G_{\bar{j}i}(\phi,\bar{\phi})\, k^{ai}(\phi) \ . \tag{7.20}
$$

The solution of the first equation in (7.19) is an (anti-) holomorphic Killing vector field $\bar{k}^{\bar{i}} = \bar{k}^{\bar{i}}(\bar{\phi})$ as promised. The solution of the second equation locally reads

$$
k^a_{\bar{j}} = G_{\bar{j}i}\, k^{ai} = -i\frac{\partial P^a}{\partial \bar{\phi}^{\bar{j}}} \ , \qquad \bar{k}^a_i = G_{i\bar{j}}\, \bar{k}^{a\bar{j}} = -i\frac{\partial P^a}{\partial \phi^i} \ , \tag{7.21}
$$

where the P^a are real and called moment maps or Killing prepotentials. They are unique up to holomorphic terms which in particular include the Fayet-Iliopoulos terms. The relation (7.21) can also be inverted leading to

$$
P^a = i\big(\bar{k}^{a\bar{j}}\partial_{\bar{j}}K - \bar{r}^a(\bar{\phi})\big) = -i\big(k^{ai}\partial_i K - r^a(\phi)\big) \ , \tag{7.22}
$$

where $r^a(\phi)$ contains the FI-terms $r^a = \delta^{a\,U(1)}\xi_{FI}$.

In order to construct a gauge invariant Lagrangian one also needs the covariant derivatives. They are given by

$$
D_\mu\phi^i = \partial_\mu\phi^i - A^a_\mu k^{ai}(\phi) \ , \qquad D_\mu\chi^i = \partial_\mu\chi^i + \Gamma^i_{jk}\,\partial_\mu\phi^j\chi^k - A^a_\mu\partial_k k^{ai}\chi^k \ . \tag{7.23}
$$

[5] Consistency requires that the $k^{a\,i}(\phi)$ are holomorphic functions of the ϕ^i and we will see shortly that this also results from the solution of the Killing equation.

Together with (7.18) gauge invariance requires that $k^{ai}(\phi)$ carry a representation of the gauge group G. One defines

$$k^a := k^{ai}\frac{\partial}{\partial\phi^i}\ , \qquad \bar{k}^a = \bar{k}^{a\bar{i}}\frac{\partial}{\partial\bar{\phi}^{\bar{i}}}\ , \tag{7.24}$$

and shows

$$\left[k^a, k^b\right] = -f^{abc}\, k^c\ , \qquad \left[\bar{k}^a, \bar{k}^b\right] = -f^{abc}\, \bar{k}^c\ , \qquad \left[k^a, \bar{k}^b\right] = 0\ , \tag{7.25}$$

where f^{abc} are the structure constants of G.

Let us check the renormalizable limit. Using again dimensional analysis reveals that $[\phi] = [A_\mu] = 1$ implies $[k^{ai}] = 1, [K] = 2$. Thus expanding k^{ai}, K for small ϕ yields

$$k^{ai} = iT^{a\,i}{}_{j}\phi^{\bar{j}} + \mathcal{O}(\phi^3)\ , \qquad K = \delta_{i\bar{j}}\phi^i\bar{\phi}^{\bar{j}} + \mathcal{O}(\phi^3)\ . \tag{7.26}$$

Inserted into (7.22) then yields

$$P^a = -\bar{\phi}^{\bar{i}}T^a_{\bar{i}j}\phi^j + \mathcal{O}(\phi^3)\ , \tag{7.27}$$

which indeed shows that the Killing prepotentials P^a are related to the D-terms at lowest order.

Let us now give the final Lagrangian

$$\begin{aligned}
\mathcal{L} = {} & \tfrac{1}{4}\mathrm{Re}\tau_{ab}F^a_{\mu\nu}F^{\mu\nu\, b} - \tfrac{1}{4}\mathrm{Im}\tau_{ab}F^a_{\mu\nu}F^{\mu\nu\, b} - i\mathrm{Re}\tau_{ab}\bar{\lambda}^a \not{D}\lambda^b \\
& - G_{i\bar{j}}(\phi,\bar{\phi})\, D_\mu\phi^i D^\mu\bar{\phi}^{\bar{j}} - iG_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu D_\mu\chi^i \\
& + \sqrt{2}\, k^a_i\chi^i\lambda^a + \sqrt{2}\, \bar{k}^a_{\bar{i}}\bar{\chi}^{\bar{i}}\bar{\lambda}^a \\
& - \tfrac{1}{2}(\nabla_i\partial_j W)\, \chi^i\chi^j - \tfrac{1}{2}(\nabla_{\bar{i}}\partial_{\bar{j}} \bar{W})\, \bar{\chi}^{\bar{i}}\bar{\chi}^{\bar{j}} \\
& + R_{i\bar{j}k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}} - V(\phi,\bar{\phi})\ .
\end{aligned} \tag{7.28}$$

where

$$V = G^{i\bar{j}}\partial_i W\partial_{\bar{j}}\bar{W} + \tfrac{1}{2}\,\mathrm{Re}\tau^{-1}_{ab}P^aP^b\ , \tag{7.29}$$

and $P^a = \mathrm{Re}\tau^{ab}D^b$. We see that altogether $\mathcal{L}$ is determined by the couplings $K(\phi,\bar{\phi})$, $W(\phi)$, $\tau(\phi)$ and the Killing prepotentials $P^a(\phi,\bar{\phi})$.

8 $N = 1$ Supergravity

8.1 General Relativity and the vierbein formalism

Let us first recall a few facts about General Relativity. It can be viewed as a (semi-) classical field theory for a spin 2 field, the metric $g_{\mu\nu}(x)$ which is a symmetric tensor field on an arbitrary (pseudo-) Riemannian manifold. Its Lagrangian is given by

$$\mathcal{L}_{\rm EH} = -\tfrac{1}{2\kappa^2}\sqrt{-g}\big(R+\Lambda\big) + \mathcal{L}_{\rm mat}\,, \tag{8.1}$$

where $\kappa^2 = 8\pi M_{Pl}^{-2}$, $g = \det g_{\mu\nu}$, R is the Ricci-scalar, Λ is the cosmological constant and $\mathcal{L}_{\rm mat}$ contains the couplings to matter and gauge fields. The equations of motion derived from the action are the Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\big(R+\Lambda\big) = \kappa T_{\mu\nu}\ , \tag{8.2}$$

where $R_{\mu\nu}$ is the Ricci tensor while $T_{\mu\nu}$ is the energy-momentum tensor defined as

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial\mathcal{L}_{\rm matter}}{\partial g_{\mu\nu}}\ . \tag{8.3}$$

The matter couplings summarized in $\mathcal{L}_{\rm mat}$ are obtained from the corresponding flat-space version by replacing $\eta^{\mu\nu}\to g^{\mu\nu}$ and multiplication by $\sqrt{-g}$. For a scalar field it reads

$$\mathcal{L}_{\rm mat} = -\sqrt{-g}\, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\ , \tag{8.4}$$

for a gauge field one has

$$\mathcal{L}_{\rm mat} = -\tfrac{1}{4}\sqrt{-g}\, g^{\mu\nu}g^{\kappa\rho}F_{\mu\kappa}F_{\nu\rho} \tag{8.5}$$

In order to couple fermions on needs the vierbein formalism where one defines the 4×4 matrix, the vierbein, $e^a_\mu(x)$ by

$$g_{\mu\nu}(x) = e^m_\mu(x)\,\eta_{mn}\,e^n_\nu(x)\,,\quad \mu,\nu = 0,\ldots,3\ ,\quad m,n = 0,\ldots,3\ . \tag{8.6}$$

At each space-time point x^μ it erects a local Lorentz-frame. Note that (8.6) is invaraint under the local Lorentz transformation

$$e^m_\mu \to e^{m\,\prime}_\mu = \Lambda^m_n\, e^n_\mu\ , \tag{8.7}$$

where Λ is the rotation matrix defined in (1.2). Finally, the inverse vierbeins are defined by

$$e^m_\mu e^\nu_m = \delta^\nu_\mu\ ,\qquad e^\mu_n e^m_\mu = \delta^m_n\ . \tag{8.8}$$

With the help of the vierbein one can give the Weyl action for a spin-1/2 fermion χ as

$$\mathcal{L} = -ie\,\bar\chi\bar\sigma^m e^\mu_m D_\mu\chi\,, \tag{8.9}$$

where $e = \det(e^m_\mu) = \sqrt{-g}$ and σ^m are the Pauli matrices as defined in (1.11). The covariant derivative is given by

$$D_\mu\chi = \partial_\mu\chi + \omega_{\mu mn}\sigma^{mn}\chi\ , \tag{8.10}$$

where $\omega = \omega(e, \partial e)$ is the spin connection and σ^{mn} is defined in (1.12).

Demanding metric compatibilty, i.e.

$$
\begin{aligned}
D_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\kappa_{\mu\nu} g_{\kappa\rho} - \Gamma^\kappa_{\mu\rho} g_{\kappa\nu} = 0 \ , \\
D_\mu e^m_\nu &= \partial_\mu e^m_\nu - \Gamma^\rho_{\mu\nu} e^m_\rho + e^n_\nu \omega^m_{\mu n} = 0 \ ,
\end{aligned}
\tag{8.11}
$$

expresses $\Gamma = \Gamma(g, \partial g)$ and

$$
\omega_{\mu\nu\rho} = \omega^m_{\mu n} e^n_\nu e^\kappa_m g_{\kappa\rho} = -\tfrac{1}{2}\big(e_{\rho m}(\partial_\mu e^m_\nu - \partial_\nu e^m_\mu) + e_{\nu m}(\partial_\rho e^m_\mu - \partial_\mu e^m_\rho) - e_{\mu m}(\partial_\nu e^m_\rho - \partial_\rho e^m_\nu)\big) \ . \tag{8.12}
$$

One also has the relation

$$
\Gamma^\rho_{\mu\nu} e^m_\rho = \partial_\mu e^m_\nu + e^n_\nu \omega_{\mu n}{}^m \ . \tag{8.13}
$$

With the help of the vierbein one defines for m vector field v_μ

$$
v_m := e^\mu_m v_\mu \ , \tag{8.14}
$$

and the covariant derivatives

$$
\begin{aligned}
D_\mu v_\nu &= \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu} v_\rho \ , \qquad & D_\mu v^\nu &= \partial_\mu v^\nu + v^\rho \Gamma^\nu_{\mu\rho} \ , \\
D_\mu v_m &= \partial_\mu v_m - \omega_{\mu m}{}^n v_n \ , \qquad & D_\mu v^m &= \partial_\mu v^m + v^n \omega_{\mu n}{}^m \ ,
\end{aligned}
\tag{8.15}
$$

The mction (8.1) hms two sets of invmrimnces. Firstly there are the general coordinate transformations

$$
x^\mu \to x'^\mu = x^\mu - a^\mu(x) \ , \tag{8.16}
$$

which leads to the infinitesimal transformations of vector fields v_μ

$$
\delta v_\mu = -a^\rho \partial_\rho v_\mu - (\partial_\mu a^\rho)\, v_\rho \ . \tag{8.17}
$$

In addition there are local Lorentz transformations for vector fields which are defined in the local tangent space and which carry m-type indices

$$
\delta v^m = v^n L_n{}^m(x) \ , \qquad \delta v_m = -L_m{}^n(x)\, v_n \ . \tag{8.18}
$$

(Note that in the notation of (1.4) we abbreviated $L_n{}^m \equiv -\frac{i}{2}\omega_{[\mu\nu]}(L^{[\mu\nu]})_n{}^m$.) Thus the vierbein itself transforms accordingly as

$$
\delta e^m_\mu = -a^\rho \partial_\rho e^m_\mu - (\partial_\mu a^\rho)\, e^m_\rho + e^n_\mu L_n{}^m \ , \tag{8.19}
$$

while ω transforms as

$$
\delta \omega_{\mu m}{}^n = -a^\rho \partial_\rho \omega_{\mu m}{}^n - (\partial_\mu a^\rho)\, \omega_{\rho m}{}^n + \omega_{\mu m}{}^c L_c{}^n - L_m{}^c \omega_{\mu c}{}^n - \partial_\mu L_m{}^n \ . \tag{8.20}
$$

Note that ω transforms as a connection of local Lorentz transformations.

8.2 Pure supergravity

The goal now is to supersymmetrize General Relativity. From lecture 2 (Table 2.2) we know that we need a spin – or rather helicity-3/2 field, the gravitino $\psi_{\mu\alpha}$ in the gravitational multiplet.

Before the invention of supersymmetry there was a no-go theorem stating that a massless spin-3/2 field cannot be consistently coupled in an interacting QFT. Let us briefly review this argument. In fact any massless field with $s \geq 1$ has to couple to a conserved current since as a consequence there is a local gauge invariance which removes possible ghost-like excitations and renders the Hilbert space positive definite. Concretely for $s = 1$ one has

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu \ , \tag{8.21}$$

with the equation of motion

$$\partial_\mu F^{\mu\nu} = j^\mu \ . \tag{8.22}$$

Taking the derivative one obtains $\partial_\mu j^\mu = 0$ which implies that $\mathcal{L}$ is gauge invariant and the ghost-like excitations of A_μ can be removed. Similarly, taking the derivative of (8.2) implies $D_\mu T^{\mu\nu} = 0$ or in flat space $\partial_\mu T^{\mu\nu} = 0$. Here the local symmetry is reparametrization invariance. For $s = 3/2$ no appropriate local symmetry was known before supersymmetry and in fact Coleman and Mandula showed that it cannot exist and at the same time be compatible with the axioms of a QFT [10]. Of course there was the hidden assumption in their argument that all symmetry generators are bosonic charges which is not the case for supersymmetry. Thus ψ_μ has to couple to the supercurrent, i.e. the generator of the supersymmetry transformation and it has to be a local invariance.

The kinetic term for ψ_μ (also known as the Rarita-Schwinger field strength) is given by

$$\mathcal{L}_\psi = \tfrac{e}{\kappa^2}\, \bar\psi_\kappa \bar\sigma^{[\kappa}\sigma^\mu\bar\sigma^{\nu]} D_\mu \psi_\nu \ , \tag{8.23}$$

where the covariant derivative includes appropriate connections and we defined $\sigma_\rho := e_\rho^m \sigma_m$. Together $\mathcal{L}_{\rm EH}(\Lambda = 0) + \mathcal{L}_\psi$ are invariant under the local supersymmetry transformations[6]

$$\delta_\xi e_\mu^m = i\big(\psi_\mu \sigma^m \bar\xi(x) - \xi(x)\sigma^m \bar\psi_\mu\big)\,, \qquad \delta_\xi \psi_\mu = -D_\mu \xi(x)\,. \tag{8.24}$$

Note that $\xi(x)$ is a local parameter and ψ_μ transforms like a gauge field or a connection that is inhomogenously.

Let us close this lecture by counting degrees of freedom. Off-shell e_μ^a has $6 = 16 - 4 - 6$ degrees of freedom (4 are removed by coordinate invariance and 6 by local Lorentz invariance) while ψ_μ has $12 = 16 - 4$ degrees of freedom (4 are removed by local supersymmetry). There are various off-shell multiplets with different auxiliary field structure. In the following we concentrate on the old minimal multiplet with six auxiliary fields, a real b_μ and a complex M. Altogether the off-shell multiplet is

$$(g_{\mu\nu}, \psi_{\mu\,\alpha}, b_\mu, M) \ , \quad \text{with d.o.f. :} \quad (6, 12, 4, 2) \ . \tag{8.25}$$

Including the auxiliary fields the Lagrangin reads

$$\mathcal{L} = -\tfrac{e}{2\kappa^2}\big(R + \bar\psi_\kappa \bar\sigma^{[\kappa}\sigma^\mu\bar\sigma^{\nu]} D_\mu\psi_\nu + M\bar M + b^\mu b_\mu\big) \ . \tag{8.26}$$

[6] He we only discuss pure supergravity in Minkowskian background and return to the issue of the cosmological constant in the next section.

In this case the equation of motion for the auxiliary fields is

$$b_\mu = 0 \ , \qquad M = 0 \ . \tag{8.27}$$

9 Coupling of $N = 1$ Supergravity to super Yang-Mills with matter

In this lecture we discuss the couplings of chiral and vector multiplets to $N = 1$ supergravity. Since theories including gravity are non-renormalizability we can use the results of Section 7 with the cut-off scale being the Planck scale, i.e. $M = M_{\rm Pl}$.

There are basically three approaches to construct the action:

1. via superspace [5].

   In this case one promotes the vierbein $e_\mu^n(x)$ to a full superfield $E_\Pi^N(x, \theta, \bar\theta), N = (n, \alpha, \dot\alpha), \Pi = \mu, \underline{\alpha}, \underline{\dot\alpha}$, where the underline variables contain a vielbein. This step introduces too many d.o.f. and it is necessary to impose covariant constraint on torsion and curvature.

2. One constructs the couplings of superconformal supergravity and then gauge fixes to Poincare supergravity [3].

3. One constructs the action for linearized gravity and then systematically adds higher order terms to Lagrangian and transformation laws [4].

The resulting action is given in Appendix G of [5]. In the following we focus on selected terms of this action.

9.1 The bosonic Lagrangian

Let us first focus on the bosonic terms as they also fix all fermionic couplings by supersymmetry. They read

$$\begin{aligned}\mathcal{L} =& \tfrac{e}{2\kappa^2}\, R - \tfrac{1}{4}\,{\rm Re}\tau_{ab}(\phi) F^a_{\mu\nu} F^{b\,\mu\nu} - \tfrac{1}{4}\,{\rm Im}\tau_{ab}(\phi) F^a \widetilde{F}^b \\ & - G_{i\bar{j}} D_\mu\phi^i D^\mu\bar\phi^{\bar{j}} - V(\phi, \bar\phi) + \text{ fermionic terms}\,,\end{aligned} \tag{9.1}$$

where the covariant derivatives are exactly as in (7.23) and the metric continues to be Kähler and given by (7.10). The scalar potential V is given by

$$V = e^{\kappa^2 K}\left(D_i W G^{i\bar{j}} D_{\bar{j}}\bar{W} - 3\kappa^2|W|^2\right) + \tfrac{1}{2}{\rm Re}\tau_{ab} P^a P^b \tag{9.2}$$

where

$$\begin{aligned} D_i W :=& \frac{\partial W}{\partial\phi^i} + \kappa^2\left(\frac{\partial K}{\partial\phi^i}\right) W\ , \\ P^a =& -\tfrac{i}{2}(k^{ai}\partial_i K - \bar{k}^{a\bar{j}}\partial_{\bar{j}} K) - {\rm Im} r^a\ .\end{aligned} \tag{9.3}$$

As in Section 7 the couplings in $\mathcal{L}$ of (9.1) are determined by the three functions $K(\phi, \bar\phi), W(\phi), f(\phi)$ and the choice of P^a. In the flat limit $\kappa \to 0$ the potential reduces to the V given in (5.9).

The Lagrangian (9.1) has a modified Kähler invariance under which the couplings transform accordingly

$$K \to K + f(\phi) + f(\bar{\phi}) \ , \quad W \to W e^{-\kappa^2 f} \ , \tag{9.4}$$

which leave the metric $G_{i\bar{j}}$ and the potential V invariant.[7]

One can combine K and W into an invariant combination $G = \kappa^2 K + \ln \kappa^6 |W|^2$ and in terms of G the potential takes the form

$$V = \kappa^{-4} e^G \big(G_i G^{i\bar{j}} G_{\bar{j}} - 3 \big) \ . \tag{9.5}$$

In this formulation it appears that W can be redefined into K. However, the definition of G is problematic for $\langle W \rangle = 0$. In fact the physical meaning of W are its zeros and poles.

The fermions also transform under Kähler transformations as can be seen from their covariant derivatives

$$\begin{aligned}
D_\mu \chi^i &= \partial_\mu \chi + \chi^i \omega_\mu + \Gamma^i_{jk} D_\mu \phi^j \chi^k - g A^a_\mu \partial_j k^{ai} \chi^j - K_\mu \chi^i - \tfrac{i}{2} g A^a_\mu P^a \chi^i \ , \\
D_\mu \lambda^a &= \partial_\mu \lambda^a + \lambda^a \omega_\mu + - g f^{abc} A^b_\mu \lambda^c + K_\mu \lambda^a + \tfrac{i}{2} g A^b_\mu P^b \lambda^a \ , \\
D_\mu \psi_\nu &= \partial_\mu \psi_\nu + \psi_\nu \omega_\mu + + K_\mu \psi_\nu + \tfrac{i}{2} g A^b_\mu P^b \psi_n u \ ,
\end{aligned} \tag{9.6}$$

where

$$K_\mu := \tfrac{1}{4} (K_i \partial_\mu \phi^i - K_{\bar{i}} \partial_\mu \bar{\phi}^{\bar{i}}) \ . \tag{9.7}$$

K_μ is called the composite Kähler connection as it transforms under Kähler transformations as

$$K_\mu \to K_\mu + \tfrac{i}{2} \partial_\mu \mathrm{Im} f \ . \tag{9.8}$$

Accordingly the fermions transform as

$$\psi_\mu \to \psi_\mu \, e^{-\frac{i}{2} \mathrm{Im} f} \ , \qquad \lambda \to \lambda \, e^{-\frac{i}{2} \mathrm{Im} f} \ , \qquad \chi \to \chi \, e^{\frac{i}{2} \mathrm{Im} f} \ . \tag{9.9}$$

So far the Kähler transformations (9.4) were a mere redundancy of the couplings but not a symmetry. However, it can happen that under a gauge transformation K does transform as in (9.4). In this case one has

$$\begin{aligned}
\delta K &= \Lambda^a (r^a + \bar{r}^a) \ , \\
\delta W &= - \Lambda^a r^a W \ , \\
\delta \chi^i &= \Lambda^a \partial_j k^{ai} \chi^j + \tfrac{i}{2} \Lambda^a \mathrm{Im} r^a \chi \ , \\
\delta \lambda^a &= f^{abc} \Lambda^b \lambda^c - \tfrac{i}{2} \Lambda^b \mathrm{Im} r^b \lambda^a \ , \\
\delta \psi_\mu &= - \tfrac{i}{2} \Lambda^a \mathrm{Im} r^b \psi_\mu \ .
\end{aligned} \tag{9.10}$$

We see that in this case the fermions have an additional coupling to the gauge field which is often referred to as gauging the R-symmetry.

[7] P^a is invariant after an appropriate shift of r^a.

10 Spontaneous supersymmetry breaking in supergravity

As in section 6 the order parameters of spontaneous supersymmetry breaking are the scalar parts of the fermionic supersymmetry transformations. In supergravity they are given by

$$\delta_\xi \chi^i \sim F^i \xi \ , \qquad \delta_\xi \lambda^a \sim g D^a \xi \ , \qquad \delta_\xi \psi_\mu \sim D_\mu \xi + i e^{\frac{1}{2}\kappa^2 K} W \sigma_\mu \bar{\xi} \ , \tag{10.1}$$

where

$$F^i = e^{\frac{1}{2}\kappa^2 K} G^{i\bar{j}} \bar{D}_{\bar{j}} \bar{W} = e^{G/2} G^{i\bar{j}} G_{\bar{j}} \ . \tag{10.2}$$

We see that, as before, $\langle F^i \rangle$ and $\langle D^a \rangle$ are the order parameters of supersymmetry breaking.[8] For $\langle F^i \rangle = \langle D^a \rangle = 0$, the potential evaluated at the minimum is

$$\langle V \rangle = -3\kappa^2 \langle e^{\kappa^2 K} |W|^2 \rangle \leq 0 \ . \tag{10.3}$$

$\langle V \rangle$ plays the role of a cosmological constant and for $\langle W \rangle = \langle V \rangle = 0$ one has a Minkowski background M_4. For $\langle W \rangle \neq 0$ follows $\langle V \rangle < 0$, i.e. one has an AdS_4-background. Note that a dS-background is incompatible with unbroken supersymmetry.

10.1 F-term breaking

Let us focus on F-term breaking in a Minkowski background M_4 (and set $\kappa = 1$ most of the time). In this case we have

$$\langle e^{G/2} G^{i\bar{j}} G_{\bar{j}} \rangle \neq 0 \ , \qquad \langle G^{i\bar{j}} G_i G_{\bar{j}} \rangle = 3 \ . \tag{10.4}$$

The non-derivative couplings of the gravitino read

$$-e^{G/2} \big(\psi_\mu \sigma^{\mu\nu} \psi_\nu + \tfrac{i}{\sqrt{2}} G_i \chi^i \sigma^\mu \bar{\psi}_\mu + \tfrac{1}{2} (\nabla_i G_j + G_i G_j) \chi^i \chi^j + h.c. \big) \ . \tag{10.5}$$

These terms can be diagonalized by the redefinition

$$\tilde{\psi}_\mu = \psi_\mu + \tfrac{\sqrt{2}}{3 m_{3/2}} \, \partial_\mu \eta + \tfrac{i\sqrt{2}}{6} \, \sigma_\mu \bar{\eta} \ , \tag{10.6}$$

for $\eta := G_i \chi^i$. Inserted into (10.5) one obtains

$$m_{3/2} (\tilde{\psi}_\mu \sigma^{\mu\nu} \tilde{\psi}_\nu) + \tfrac{1}{2} m_{ij} \chi^i \chi^j + h.c. \ , \tag{10.7}$$

where

$$m_{3/2}^2 = \kappa^{-2} e^{\langle G \rangle} = \kappa^4 \langle e^{\kappa^2 K} |W|^2 \rangle \ , \qquad m_{ij} = \langle \nabla_i G_j + \tfrac{1}{3} G_i G_j \rangle m_{3/2} \ . \tag{10.8}$$

One can show that the mass matrix m_{ij} of the chiral fermions always has a zero eigenvalue corresponding to the Goldstone fermion (GF). Indeed, using the minimization condition of the potential and (10.4) one shows $m_{ij} G^j = 0$ and thus η is indeed the GF. One can

[8] For $\langle F^i \rangle = \langle D^a \rangle = 0$ one can always find $\langle \delta_\xi \psi_\mu \rangle = 0$ which determines a Minkowski or AdS-background.

further show that η also disappears from the kinetic terms and thus η is "eaten" by the massive gravitino and becomes its "longitudinal" d.o.f..

Let us turn to the bosonic mass matrices. In terms of $G = K + \ln|W|^2$ they read

$$\begin{aligned} M^2_{i\bar{\jmath}} &= \langle (D_i G_k \bar{D}_{\bar{\jmath}} G^k - R_{i\bar{\jmath}k\bar{l}} G^k G^{\bar{l}} + G_{i\bar{\jmath}}) e^G \rangle \ , \\ M^2_{ij} &= \langle (G^k D_i D_j G_k + D_i G_j + D_j G_i) e^G \rangle \ . \end{aligned} \tag{10.9}$$

The sum rule (6.32) is modified and now reads

$$\mathrm{Str} M^2 = \sum_{s=0}^{3/2} (-)^{2s}(2s+1)\,\mathrm{Tr}\, M_J^2 = 2(n_c - 1)\, m^2_{3/2} - 2\langle R_{i\bar{\jmath}} G^i G^{\bar{\jmath}} \rangle m^2_{3/2} \ . \tag{10.10}$$

The new terms arise due to the massive gravitino and as a consequence it becomes possible to have all scalar partners of the SM fermions heavy.

In AdS_4 similar formulas exist but they are more complicated as the cosmological explicitly contributes.

10.2 The Polonyi model

After these generalities let us come to a concrete realization of supersymmetry breaking. As in global supersymmetry the basic idea is to add a "hidden sector" which is responsible for the supersymmetry breaking. That is one adds

$$W = W_{\mathrm{MSSM}} + W_{\mathrm{hidden}} \tag{10.11}$$

such that

$$\langle \bar{F}^{\bar{\imath}} \rangle = \langle e^{\kappa^2 K/2} G^{\bar{\imath}j} D_j W_{\mathrm{hidden}} \rangle \neq 0 \ . \tag{10.12}$$

The simplest concrete W is the Polonyi model where one singlet ϕ is added with the following couplings

$$\begin{aligned} W &= M_s^2\,(\phi + \beta) \ , \qquad M_s, \beta \in \mathbb{R} \ , \\ K &= \phi\bar{\phi} + K_{\mathrm{MSSM}} \ , \quad G_{\phi\bar{\phi}} = \partial_\phi \partial_{\bar{\phi}} K = 1 \ . \end{aligned} \tag{10.13}$$

Computing

$$D_\phi W = \partial_\phi W + \kappa^2 K_\phi W = m^2 + \kappa^2 \bar{\phi} m_s^2\,(\phi + \beta) \ , \tag{10.14}$$

one see that $D_\phi W = 0$ has no solution for $\kappa\beta < 2$. Minimizing V and tuning $\langle V \rangle = 0$ by choosing β appropriately one finds

$$\begin{aligned} \kappa\beta &= \pm(2 - \sqrt{3}) \ , \qquad \langle \phi \rangle = \pm(\sqrt{3} - 1) \ , \\ \langle D_\phi W \rangle &= \sqrt{3} m_s^2 e^{\left(2-\sqrt{3}\right)} \ , \qquad \langle W \rangle = \pm \kappa^{-1} m_s^2 \ . \end{aligned} \tag{10.15}$$

10.3 Generic gravity mediation

In this section we want to identify the effect of supersymmetry breaking in the observable (MSSM) sector following [11]. We distinguish the observable charged matter fields Q^I from neutral (hidden) scalars T^i and assume $\langle Q^I \rangle = 0$. Then we expand their Kähler potential in a power series in Q^I as

$$K \;=\; \kappa^{-2}\hat{K}(T,\bar{T}) \;+\; Z_{\bar{I}J}(T,\bar{T})\,\bar{Q}^{\bar{I}}Q^{\bar{J}} \;+\; \left(\tfrac{1}{2}H_{IJ}(T,\bar{T})\,Q^I Q^{\bar{J}} \;+\; \text{c.c.}\right) + \;\cdots\,, \quad (10.16)$$

where we neglect terms of order $\mathcal{O}(Q^3)$. In this notation the superpotential is given by

$$W(T,Q) = W_{\text{obs}}(T,Q) + W_{\text{hidden}}(T)\;, \quad (10.17)$$

with

$$W_{\text{obs}}(T,Q) = \tfrac{1}{2}m_{IJ}(T)\,Q^I Q^{\bar{J}} + \tfrac{1}{3}Y_{IJL}(T)\,Q^I Q^J Q^K \;+\; \cdots \quad (10.18)$$

For $W_{\text{hidden}}(T)$ we make the following assumption:

1. some $\langle F^i \rangle \neq 0$,
2. all $\langle T^i \rangle$ fixed,
3. $\langle V \rangle = 0$,
4. $m_{3/2} \ll M_{Pl}$.

With these assumption one can compute the leading order effect in the limit $M_{\text{Pl}} \to \infty$ with $m_{3/2}$ fixed. One finds that the (canonically normalized) gaugino masses are given by

$$\tilde{m} \;=\; \tfrac{1}{2}F^i\partial_i \log g^{-2} + \tfrac{1}{16\pi^2}\; b\, m_{3/2}\;, \quad (10.19)$$

where b is the one-loop coefficient of the β-function and the second term is know as a contribution from anomaly mediation [12]. The potential reads

$$\begin{aligned} V \;&=\tfrac{1}{4}g^2\left(\bar{Q}^{\bar{I}}Z_{\bar{I}J}T^a Q^J\right)^2 \;+\; \partial_I\hat{W}Z^{I\bar{J}}\bar{\partial}_{\bar{J}}\hat{\bar{W}} \\ &\quad+\; m^2_{I\bar{J}}Q^I\bar{Q}^{\bar{J}} \;+\; \left(\tfrac{1}{3}A_{IJL}Q^IQ^JQ^L \;+\; \tfrac{1}{2}B_{IJ}Q^IQ^J \;+\; \text{c.c.}\right). \end{aligned} \quad (10.20)$$

The first line is the scalar potential of an effective theory with unbroken rigid supersymmetry while the second line is comprised of the soft supersymmetry-breaking terms. $\hat{W}$ is given by

$$\hat{W}(Q) \;=\; \tfrac{1}{2}\hat{\mu}_{IJ}Q^IQ^J \;+\; \tfrac{1}{3}\hat{Y}_{IJL}Q^IQ^JQ^L, \quad (10.21)$$

where

$$\begin{aligned} \hat{\mu}_{IJ} &:=\; e^{\hat{K}/2}m_{IJ} \;+ m_{3/2}H_{IJ} \;-\; \bar{F}^{\bar{j}}\bar{\partial}_{\bar{j}}H_{IJ}\,, \\ \hat{Y}_{IJL} &:=\; e^{\hat{K}/2}\,Y_{IJL}\,. \end{aligned} \quad (10.22)$$

The coefficients of the soft terms in the second line of (10.20) are as follows:

$$\begin{aligned} m^2_{I\bar{J}} \;&=\; m^2_{3/2}Z_{I\bar{J}} \;-\; F^i\bar{F}^{\bar{j}}R_{i\bar{j}I\bar{J}}\,, \\ A_{IJL} \;&=\; F^iD_i\hat{Y}_{IJL}\,, \\ B_{IJ} \;&=\; F^iD_i\hat{\mu}_{IJ} \;-\; m_{3/2}\hat{\mu}_{IJ}\,, \end{aligned} \quad (10.23)$$

where

$$\begin{aligned}
R_{i\bar{j}I\bar{J}} &= \partial_i \bar{\partial}_{\bar{j}} Z_{I\bar{J}} - \Gamma^N_{iI} Z_{N\bar{L}} \bar{\Gamma}^{\bar{L}}_{\bar{j}\bar{J}}\,, \qquad \Gamma^N_{iI} = Z^{N\bar{J}} \partial_i Z_{I\bar{J}}\,, \\
D_i \hat{Y}_{IJL} &= \partial_i \hat{Y}_{IJL} + \tfrac{1}{2} \hat{K}_i \hat{Y}_{IJL} - \Gamma^N_{i(I} \hat{Y}_{JL)N}\,, \\
D_i \hat{\mu}_{IJ} &= \partial_i \hat{\mu}_{IJ} + \tfrac{1}{2} \hat{K}_i\, \hat{\mu}_{IJ} - \Gamma^N_{i(I}\, \hat{\mu}_{J)N}\,.
\end{aligned} \tag{10.24}$$

Notice that all quantities appearing in eqs. (10.19), (10.22) and (10.23) are covariant with respect to the supersymmetric reparametrization of matter and moduli fields as well as covariant under Kähler transformations.

According to eq. (10.23), $m^2_{\bar{I}J} \sim m^2_{3/2}$, $A_{IJL} \sim m_{3/2} \hat{Y}_{IJL}$, and $B_{IJ} \sim m_{3/2} \hat{m}_{IJ}$; nevertheless, the soft terms are generally *not universal*, i.e. $A_{IJL} \neq \text{const} \cdot m_{3/2} \hat{Y}_{IJL}$ and $m^2_{I\bar{J}} \neq \text{const} \cdot m^2_{3/2} Z_{I\bar{J}}$, even at the tree level. In the context of the MSSM, this non-universality means that the absence of flavor-changing neutral currents is not an automatic feature of supergravity but a non-trivial constraint that has to be satisfied by a fully realistic theory.

Phenomenological viability of the MSSM imposes yet another requirement: The supersymmetric mass term μ for the two Higgs doublets should be comparable in magnitude with the non-supersymmetric mass terms. Equation (10.22) displays m_{IJ} and H_{IJ} as two independent sources of $\hat{m}_{IJ}$. The contribution of a non-vanishing H_{IJ} to $\hat{m}$ is automatically of order $m_{3/2}$, without any fine-tuning. This fact is known as the Giudice-Masiero mechanism [13].

10.4 Soft Breaking of Supersymmetry

As we have seen in section 6 models with spontaneously broken supersymmetry are phenomenologically not acceptable. For example the mass formula (6.32), generally valid in such cases, forbids that all supersymmetric particles acquire masses large enough to make them invisible in present experiments. One way to overcome those difficulties is to allow explicit supersymmetry breaking.

A special case of explicit breaking is so called soft supersymmetry breaking where the added terms do not introduce quadratic divergences into the theory and thus stabilizes the Higgs mass and the weak scale.

One possibility to identify the soft breaking terms is to investigate the divergence structure of the effective potential [14]. Consider a quantum field theory of a scalar field ϕ in the presence of an external source J. The generating functional for the Green's functions is given by

$$e^{-iE[J]} = \int \mathcal{D}\phi \exp\left[i \int d^4x (\mathcal{L}[\phi(x)] + J(x)\phi(x)) \right] . \tag{10.25}$$

The effective action $\Gamma(\phi_{cl})$ is defined by the Legendre transformation

$$\Gamma(\phi_{cl}) = -E[J] - \int d^4x J(x) \phi_{cl}(x) \ , \tag{10.26}$$

where $\phi_{cl} = -\frac{\delta E[J]}{\delta J(x)}$. $\Gamma(\phi_{cl})$ can be expanded in powers of momentum; in position space this expansion takes the form

$$\Gamma(\phi_{cl}) = \int d^4x[-V_{eff}(\phi_{cl}) - \frac{1}{2}(\partial_m\phi_{cl})(\partial^m\phi_{cl})Z(\phi_{cl}) + \dots\ ]\ . \tag{10.27}$$

The term without derivatives is called the effective potential $V_{eff}(\phi_{cl})$. It can be calculated in a perturbation theory of $\hbar$:

$$V_{eff}(\phi_{cl}) = V^{(0)}(\phi_{cl}) + \hbar V^{(1)}(\phi_{cl}) + \dots \tag{10.28}$$

where $V^{(0)}(\phi_{cl})$ is the tree level and $V^{(1)}(\phi_{cl})$ the one-loop contribution. In a theory with scalars, fermions and vector bosons the one-loop contribution takes the form [15]

$$V^{(1)} \sim \int d^4k\,\mathrm{Str}\ln(k^2+M^2) = \sum_s(-1)^{2s}(2s+1)\,\mathrm{Tr}\int d^4k\ \ln(k^2+M_s^2)\ , \tag{10.29}$$

where M_s^2 is the matrix of second derivatives of $\mathcal{L}|_{k=0}$ at zero momentum for scalars ($s=0$), fermions ($s=1/2$) and vector bosons ($s=1$).[9] The UV divergences of (10.29) can be displayed by expanding the integrand in powers of large k. This leads to

$$V^{(1)} \sim \mathrm{Str}\mathbf{1}\int\frac{d^4k}{(2\pi)^4}\ln k^2 + \mathrm{Str}M^2\int\frac{d^4k}{(2\pi)^4}k^{-2} + \dots\ . \tag{10.30}$$

If a UV-cutoff Λ is introduced the first term in (10.30) is $\mathcal{O}(\Lambda^4\ln\Lambda)$. Its coefficient $\mathrm{Str}\mathbf{1} = n_B - n_F$ vanishes in theories with a supersymmetric spectrum of particles. The second term in (10.30) is $\mathcal{O}(\Lambda^2)$ and determines the presence of quadratic divergences at one-loop level. Therefore quadratic divergences are absent if

$$\mathrm{Str}M^2 = 0\ . \tag{10.31}$$

More precisely, one can also tolerate a constant $\mathrm{Str}M^2$ since this would correspond to a shift of the zero point energy which, without coupling to gravity, is undetermined. In theories with exact or spontaneously broken supersymmetry (10.31) is fulfilled whenever the trace-anomaly vanishes as we learned in (6.32).[10]

The soft supersymmetry breaking terms are defined as those non-supersymmetric terms that can be added to a supersymmetric Lagrangian without spoiling $\mathrm{Str}M^2 = \text{const.}$. One finds the following possibilities [14]:

- Holomorphic terms of the scalars proportional to ϕ^2, ϕ^3 and the corresponding complex conjugates.[11]
- Mass terms for the scalars proportional to $\bar\phi\phi$. (They only contribute a constant, field independent piece in $\mathrm{Str}M^2$).

[9] M_s^2 is not necessarily evaluated at the minimum of V_{eff}. Rather it is a function of the scalar fields in the theory. The mass matrix is obtained from M_s^2 by inserting the vacuum expectation values of the scalar fields.

[10] Indeed, theories with a non-vanishing D-term have been shown to produce a quadratic divergence at one-loop [16].

[11] Higher powers of ϕ are forbidden since they generate quadratic divergences at the 2-loop level [14].

- Gaugino mass terms.

Thus the most general Lagrangian with softly broken supersymmetry takes the form

$$\mathcal{L} = \mathcal{L}_{\text{susy}} + \mathcal{L}_{\text{soft}} \ , \tag{10.32}$$

where $\mathcal{L}_{\text{susy}}$ is of the form (5.4) and

$$\mathcal{L}_{\text{soft}} = -m_{ij}^2 \phi^i \bar{\phi}^j - (B_{ij}\phi^i\phi^j + A_{ijk}\phi^i\phi^j\phi^k + \text{h.c.}) - \tfrac{1}{2}\,\tilde{m}_{ab}\lambda^a\lambda^b + \text{h.c.} \ . \tag{10.33}$$

m_{ij}^2 and B_{ij} are mass matrices for the scalars, A_{ijk} are trilinear couplings (often called 'A-terms') and $\tilde{m}_{ab}$ is a mass matrix for the gauginos. As we have seen in the previous section these terms precisely appear in spontaneously broken supergravity (c.f. (10.20)).

We see that many new parameters are introduced which are only constrained by gauge invariance. For the SSM (with R-parity) one has

$$\begin{aligned}
\mathcal{L}_{\text{soft}} \ &= \ -\Big((A_u)_{IJ} h_u \tilde{q}_L^I \tilde{u}_R^J + (A_d)_{IJ} h_d \tilde{q}_L^I \tilde{d}_R^J + (A_e)_{IJ} h_d \tilde{l}_L^I \tilde{e}_R^J + B h_u h_d + \ \text{h.c.} \ \Big) \\
&\quad - \sum_{\text{all scalars}} m_{ij}^2 \phi^i \bar{\phi}^j - \Big(\tfrac{1}{2}\sum_{(a)=1}^{3} \tilde{m}_{(a)} (\lambda\lambda)_{(a)} + \ \text{h.c.} \ \Big) \ ,
\end{aligned} \tag{10.34}$$

where the index (a) runs over the three factors in the SM gauge group. Obviously a huge number of new parameters is introduced via $\mathcal{L}_{\text{soft}}$. The parameters of $\mathcal{L}_{\text{susy}}$ are the Yukawa couplings Y and the parameter μ in the Higgs potential. The Yukawa couplings are determined experimentally already in the non-supersymmetric Standard Model. In the softly broken supersymmetric Standard Model the parameter space is enlarged by

$$\big(\mu, (a_u)_{IJ}, (a_d)_{IJ}, (a_e)_{IJ}, b, m_{ij}^2, \tilde{m}_{(a)}\big) \ . \tag{10.35}$$

Not all of these parameters can be arbitrary but quite a number of them are experimentally constrained.

Within this much larger parameter space it is possible to overcome several of the problems encountered in the supersymmetric Standard Model. For example, the supersymmetric particles can now easily be heavy (due to the arbitrariness of the mass terms m_{ij}^2) and therefore out of reach of present experiments. Furthermore, the Higgs potential is changed and vacua with spontaneous electroweak symmetry breaking can be arranged.

However, the soft breaking terms introduce their own set of difficulties. For generic values of the parameters (10.35) the contribution to flavor-changing neutral currents is unacceptably large, additional (and forbidden) sources of CP-violation occur and finally the absence of vacua which break the $U(1)_{\text{em}}$ and/or $SU(3)$ is no longer automatic. It is beyond the scope of these lectures to review all of these aspects in detail.

11 N-extended Supersymmetries

11.1 Supersymmetry Algebra

Let us consider the generalized situation of N supercharges $Q^I_\alpha,\ I = 1, \ldots, N$. In this case the superalgebra reads

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}\, P_\mu \delta^{IJ}\,, \quad \{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta} Z^{IJ}\,, \quad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 2\epsilon_{\dot\alpha\dot\beta}\bar{Z}^{IJ}\,, \tag{11.1}$$

where the commutations relations for each of the Q^I with the generators of the Poincare group $L_{\mu\nu}, P_\mu$ are as in (1.17). The Jacobi-identity requires that Z commutes with all generators $[Z, Q] = [Z, P] = [Z, L] = 0$ and thus these are (Lorentz-invariant) central charges of the algebra.

Furthermore (11.1) is left invariant by an $U(N)$ automorphism which rotates the charges according to

$$\begin{gathered} Q^I \to Q'^I = Q^J\, U_J{}^I\,, \qquad \bar{Q}^I \to \bar{Q}'^I = U^{\dagger I}{}_J\, \bar{Q}^{\ J}\,, \\ Z^{IJ} \to Z'^{IJ} = U^I{}_K\, Z^{KL}\, U_L{}^J\,, \end{gathered} \tag{11.2}$$

where $UU^\dagger = \mathbb{1}$. One can use this freedom to bring Z into "normal-form", i.e., into 2×2 antisymmetric block-matrices leaving $N/2$ physical real central charges:[12]

$$Z^{IJ} = \begin{pmatrix} 0 & -Z_1 & & & \\ Z_1 & 0 & & & \\ & & 0 & -Z_2 & \\ & & Z_2 & 0 & \\ & & & & \ddots \end{pmatrix}\,. \tag{11.3}$$

11.2 Representations of extended supersymmetry

For massive representation with $P_\mu = (-m, 0, 0, 0)$, the superalgebra becomes

$$\{Q^\Sigma_\alpha, \bar{Q}^\Pi_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta}\delta^{\Sigma\Pi}\,, \qquad \{Q^\Sigma_\alpha, Q^\Pi_\beta\} = 2\epsilon_{\alpha\beta}Z^{\Sigma\Pi}\,, \qquad \{\bar{Q}^\Sigma_{\dot\alpha}, \bar{Q}^\Pi_{\dot\beta}\} = 2\epsilon_{\dot\alpha\dot\beta}\bar{Z}^{\Sigma\Pi}\,. \tag{11.4}$$

Let us start with $N = 2$ and define

$$a_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger)\,, \qquad b_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger)\,. \tag{11.5}$$

Inserted into (11.4) one obtains

$$\{a_\alpha, a^\dagger_\beta\} = 2\delta_{\alpha\beta}(m + Z)\,, \qquad \{b_\alpha, b^\dagger_\beta\} = 2\delta_{\alpha\beta}(m - Z)\,, \tag{11.6}$$

with all other anticommutators vanishing. Positivity of the quantum mechanical Hilbert space requires

$$m \geq Z\,. \tag{11.7}$$

This constraint is known as the Bogolmoni-Prasad-Sommerfield (BPS) bound.

[12] For N odd there is a single zero in the bottom right corner.

11.2.1 $N = 2, m > Z$

From (11.6) we see that for $m > Z$ there are $4\,(= 2N)$ fermionic creation operators $a^\dagger_\alpha, b^\dagger_\beta$. Let us combine these four operators as $A^\dagger := (a^\dagger_\alpha, b^\dagger_\beta)$ and construct the representations by acting with $A^\dagger$ on the spin-s Clifford vacuum $|s\rangle$ which is annihilated by A, i.e.

$$A|s\rangle = 0\ . \tag{11.8}$$

The states with there multiplicities are given in Table 11.1. We see that the total number

states	spin	multiplicity
$\|s\rangle$	s	$2s+1$
$A\|s\rangle$	$s \pm \frac{1}{2}$	$4(2s+1)$
$AA\|s\rangle$	$s, s\pm 1$	$6(2s+1)$
$AAA\|s\rangle$	$s \pm \frac{1}{2}$	$4(2s+1)$
$AAAA\|s\rangle$	s	$2s+1$

Table 11.1: Massive $N = 2$ states.

of states is $16(2s+1)$ while the spins range from $s+1, \ldots, s-1$.

The different multiplets are summarized in Table 11.2.

Spin	$\|0\rangle$	$\|\frac{1}{2}\rangle$	$\|1\rangle$
0	5	4	1
$\frac{1}{2}$	4	6	4
1	1	4	6
$\frac{3}{2}$		1	4
2			1
$n_B = n_F$	8	16	32
	long vector multiplet	spin $\frac{3}{2}$ multiplet	spin 2 multiplet

Table 11.2: Massive $N = 2$ multiplets.

11.2.2 N even, $m > Z_r$

In this case one follows the same construction as for $N = 2$. The total number of states of a multiplet is given by

$$n = (2s+1)\sum_{k=0}^{2N}\binom{2N}{k} = 2^{2N}(2s+1)\,, \tag{11.9}$$

where $(2s+1)$ is the multiplicity of $|s\rangle$. The number of bosonic and fermionic states therefore is

$$n_{\mathrm{B}} = 2^{2N-1}(2s+1) = n_{\mathrm{F}}\,, \tag{11.10}$$

and the different spins occurring in the multiplet are $(s+\frac{N}{2},\ldots,s-\frac{N}{2})$. The $N=4$ massive graviton multiplet is given in Table 11.3.

Spin	$\|0\rangle$
0	42
$\frac{1}{2}$	48
1	27
$\frac{3}{2}$	8
2	1
$n_B=n_F$	$128=2^7$

Table 11.3: Massive $N=4$ graviton multiplet.

11.2.3 $N=2, m=Z$

Let us now turn to the situation where the mass m saturates the BPS bound in (11.6), i.e., $m=Z$. In this case the $N=2$ fermionic creation operators $b^\dagger_\alpha$ decouple and we are left only with the $a^\dagger_\alpha$ or in other words with an "$N/2$ situation". The number of states in a multiplet is only half, i.e., $n=2^N(2s+1)$. The $N=2$ BPS multiplets coincide with the massive $N=1$ multiplets given in Table 2.1. We repeat them in adapted form in Table 11.4. Note that in $N=2$ supersymmetry there are two distinct vector multiplets. A "short" BPS multiplet with a total of 8 states and a "long" non-BPS multiplet with a total of 16 states.[13]

Spin	$\|0\rangle$	$\|\frac{1}{2}\rangle$	$\|1\rangle$	$\|\frac{3}{2}\rangle$
0	2	1		
$\frac{1}{2}$	1	2	1	
1		1	2	1
$\frac{3}{2}$			1	2
2				1
$n_B=n_F$	2	4	6	8
	half-hyper multiplet	short vector multiplet	spin $\frac{3}{2}$ BPS multiplet	spin 2 BPS multiplet

Table 11.4: $N=2$ BPS multiplets.

11.2.4 N arbitrary, $m=Z_i$

For $N>2$ there can be $N/2$ distinct central charges $Z_i, i=1,\ldots,N/2$ and the multiplet structure depends on how many BPS bounds are saturated. In the generic case one has $m>Z_i, \forall\, i$ and we discussed this in 11.2.1 and 11.2.2. Then one can have the situation that $r<N/2$ BPS bounds are saturated, i.e., $m=Z_i, \forall\, i=1,\ldots,r$. In

[13]Half-hypermultiplets in complex representation of the gauge group G are inconsistent. However, they appear to be possible in pseudo-real representations of G [17, 18].

this case the representation theory coincides with a theory with $N-r$ supercharges. Finally, all BPS charges might be saturated $m = Z_i, \forall\, i$ and then one encounters the representations of $N/2$ supersymmetry. The importance of the BPS bound comes the fact that it only depends on the algebra and therefore is expected to hold after including quantum corrections.

11.3 massless representation

Finally, let us turn to massless representations where again a light-like frame with momentum $P_\mu = (-E, 0, 0, E)$ is chosen. The superalgebra becomes

$$\begin{aligned}
\{Q^\Sigma_\alpha, \bar{Q}^\Pi_{\dot\beta}\} &= 2E(-\sigma^0 + \sigma^1)_{\alpha\dot\beta}\,\delta^{\Sigma\Pi} = 2E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \delta^{I\Pi}\,, \\
\{Q^\Sigma_\alpha, Q^\Pi_\beta\} &= 0 = \{\bar{Q}^\Sigma_{\dot\alpha}, \bar{Q}^\Pi_{\dot\beta}\}\ .
\end{aligned} \tag{11.11}$$

Thus we have the same situation as in the BPS case with all charges saturated, namely N fermionic creation operators $(Q^\Pi_1)^\dagger$. For multiplets which are in accord with the CPT theorem we thus have for the number of states in a multiplet

$$n = 2^N \times \begin{cases} 1 & \text{if the multiplet is CPT complete} \\ 2 & \text{if the CPT conjugate has to be added} \end{cases} \tag{11.12}$$

The massless multiplets for $N = 2, 4, 8$ are given in Tables 11.5,11.6,11.7, respectively.

λ	$\|-\frac{1}{2}\rangle$	$\|-1\rangle$	$\|0\rangle$	$-\|\frac{3}{2}\rangle$	$\|\frac{1}{2}\rangle$	$\|-2\rangle$	$\|1\rangle$
-2						1	
$-\frac{3}{2}$				1		2	
-1		1		2		1	
$-\frac{1}{2}$	1	2		1			
0	2	1	1				
$+\frac{1}{2}$	1		2		1		
$+1$			1		2		1
$+\frac{3}{2}$					1		2
$+2$							1
$n_B = n_F$	2	4		4		4	
	half-hyper multiplet	vector multiplet		gravitino multiplet		graviton multiplet	

Table 11.5: The massless multiplets for $N = 2$.

We see that for $N \geq 4$ no matter multiplets exists and for $N = 8$ there is a unique massless multiplet incorporating all helicities $\lambda = 0, \ldots, \pm 2$. For $N > 8$ one necessarily has states with $|\lambda| > 2$ in the spectrum which is believed to be inconsistent in a Minkowski background. Therefore one confines the attention to $N \leq 8$.

λ	$\|-1\rangle$	$\|-\frac{3}{2}\rangle$	$\|-\frac{1}{2}\rangle$	$\|-2\rangle$	$\|0\rangle$
-2				1	
$-\frac{3}{2}$		1		4	
-1	1	4		6	
$-\frac{1}{2}$	4	6	1	4	
0	6	4	4	1	1
$+\frac{1}{2}$	4	1	6		4
$+1$	1		4		6
$+\frac{3}{2}$			1		4
$+2$					1
$n_B = n_F$	8	16		16	
	vector multiplet	gravitino multiplet		graviton multiplet	

Table 11.6: The massless multiplets for $N = 4$.

λ	$\|-2\rangle$
-2	1
-$\frac{3}{2}$	8
-1	28
-$\frac{1}{2}$	56
0	70
$+\frac{1}{2}$	56
$+1$	28
$+\frac{3}{2}$	8
$+2$	1

Table 11.7: Massless multiplet for $N = 8$

12 QFTs with global $N = 2$ supersymmetry

12.1 The $N = 2$ action for vector multiplets

The massless vector multiplet consists in one vector A_μ, two fermions λ_α^Σ $(\Sigma = 1, 2)$ and a complex scalars z. For n_v vector multiplets we use the notation $\left(A_\mu^a, \lambda_\alpha^{\Sigma\, a}, z^a\right)$ with $a = 1, \ldots, n_v$. All members of the multiplet transform in the adjoint representation of some gauge group G. In terms of $N = 1$ multiplets, we have the decomposition:

$$\left(A_\mu^a, \lambda_\alpha^{\Sigma\, a}, z^a\right) \to \left(A_\mu^a, \lambda_\mu^{1a}\right) \oplus \left(\lambda_\alpha^{2a}, z^a\right) \ , \tag{12.1}$$

where the first multiplet is the $N = 1$ vector multiplet while the second is a $N = 1$ chiral multiplet. The bosonic Lagrangian is

$$\begin{aligned} \mathcal{L} = & -\tfrac{1}{4}(\mathrm{Im}F)_{ab}(z, \bar{z})\, F_{\mu\nu}^a F^{b\mu\nu} - \tfrac{1}{4}(\mathrm{Re}F)_{ab}(z, \bar{z})\, F_{\mu\nu}^a F^{\mu\nu b} \\ & - G_{a\bar{b}}(z, \bar{z}) D_\mu z^a D^\mu \bar{z}^b - V(z, \bar{z}) \ , \end{aligned} \tag{12.2}$$

where due to supersymmetry the couplings are now interrelated. In particular the gauge kinetic function F_{ab} and the σ-model metric $G_{a\bar{b}}$ are both expressed in terms of one holomorphic prepotential $F(z)$.[14] Concretely, $G_{a\bar{b}}$ is again Kähler but with a specific Kähler potential

$$K = i\left(\bar{F}_a z^a - F_a \bar{z}^a\right) \ , \qquad F_a = \partial_a F\left(z\right) \tag{12.3}$$

such that

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K = 2\mathrm{Im} F_{ab} \ . \tag{12.4}$$

Manifolds with this property have been termed rigid special Kähler manifolds.

The gauge kinetic functions are also determined by the second derivative of F according to

$$F_{ab} = \partial_a \partial_b F(z) \ . \tag{12.5}$$

Note that the physical requirement of properly propagating fields imposes $\mathrm{Im}\left(F_{ab}\right) > 0$.

The scalars z^a transform in the adjoint representation of G and thus the covariant derivatives are given by

$$D_\mu z^a = \partial_\mu z^a - v_\mu^b k^{ba}\left(z\right) \ . \tag{12.6}$$

As in $N = 1$ the Killing vectors k^{ba} are expressed in terms of Killing prepotentials P^a according to

$$k^b_{\bar{a}}(z, \bar{z}) = G_{\bar{a}c} k^{bc}\left(z\right) = -i\partial_{\bar{a}} P^b(z, \bar{z}) \ , \qquad \bar{k}^b_a(z, \bar{z}) = G_{a\bar{c}} \bar{k}^{b\bar{c}}\left(\bar{z}\right) = i\partial_a P^b(z, \bar{z}) \ . \tag{12.7}$$

In $N = 2$ no superpotential is possible and the potential is entirely determined by the Killing vectors

$$V = G_{a\bar{b}} k^a_{\bar{c}} \bar{k}^{\bar{b}}_d \bar{z}^{\bar{c}} z^d \ . \tag{12.8}$$

For renormalizable theories one finds

$$F = \tfrac{i}{4} z^a z^a \ , \qquad K = \delta_{a\bar{b}} z^a \bar{z}^{\bar{b}} \ , \qquad G_{a\bar{b}} = \delta_{a\bar{b}} \ , \qquad k^{ab} = i f^{abc} z^c \ . \tag{12.9}$$

Thus the potential is quartic and reads

$$V \ \sim \ \delta_{a\bar{d}} f^{abc} f^{def} z^b \bar{z}^c z^e \bar{z}^f \ \sim \ \mathrm{Tr}\left[z, \bar{z}\right]^2 \geq 0 \ , \tag{12.10}$$

where in the second step we defined $z = z^a T^a$, $\bar{z} = \bar{z}^a T^a$.

Due to the semi-positivity of V its minimum is at $\langle V \rangle = 0$ which holds for example at the origin of field space $\langle z^a \rangle = 0$. However, there is a moduli space of solutions spanned by the directions which point along the Cartan subalgebra of G. For $z = z^{\hat{a}} T^{\hat{a}}$ where $T^{\hat{a}}$ are the generators of the Cartan subalgebra which obey $[T^{\hat{a}}, T^{\hat{b}}] = 0$, the potential remains zero but the gauge group is broken $G \to U(1)^{\mathrm{rank}G}$. This moduli space is called the Coulomb branch of the theory.

12.2 Hypermultiplets

A hypermultiplet consists of two half-hypermultiplets and thus contains four real scalars q^u and two Weyl fermions χ^i. It can also be viewed as the product of two chiral multiplets. The ungauged bosonic Lagrangian reads

$$\mathcal{L} = -\tfrac{1}{2} G_{uv} \partial_\mu q^u \partial^\mu q^v \ , \tag{12.11}$$

[14]Note that compared to $N = 1$ the notation changed as the gauge kinetic function is now called F_{ab} and the role of real and imaginary part have been interchanged.

where G_{uv} is a metric on a Hyperkähler manifold. This is related to the fact that the four scalars can be combinded into complex fields in three different ways. This corresponds to the fact that three complex structures $J^x, x = 1, 2, 3$ exist. Therefore let us make a brief excursion into (almost) complex structures and Hyperkähler manifolds.

12.2.1 Almost complex structures

An almost complex structure J maps the tangent space $T(M)$ of a manifold M to itself

$$J: \ T(M) \to T(M) \ , \qquad q^u \to J^u_v q^v \ , \quad u = 1, \ldots, \mathrm{dim} M \ , \tag{12.12}$$

where q^u are tangent vectors. J obeys

$$J^u_v J^v_w = -\delta^u_w \quad \Leftrightarrow \quad J^2 = -\mathbf{1} \ . \tag{12.13}$$

Therefore J has eigenvalues $\pm i$ and $T(M)$ splits into two subspaces $T = T_+ \oplus T_-$. This in turn can be used to define complex vectors $\phi^i = q^i_+ + iq^i_-$ where $q_\pm \in T_\pm$.

If J exits complex vectors can be defined locally. However, there is an obstruction to do this globally on a manifold which is called the Nijenhuis-tensor N. It is defined by

$$N^w_{uv} := J^r_u(\partial_r J^w_v - \partial_v J^w_r) - (u \leftrightarrow v) \ . \tag{12.14}$$

If $N^w_{uv} = 0$ global complex coordinates exist, J is called a complex structure and the manifold M is called complex. If $N^w_{uv} \neq 0$ global complex coordinates do not exist, J is called an almost complex structure and the manifold M is called almost complex.

If the metric G in M satisfies

$$J^u_v G_{vw} J^w_r = G_{ur} \quad \Leftrightarrow JGJ^T = G \ , \tag{12.15}$$

it is called a hermitian metric. One defines the fundamental two-form K

$$K := K_{uv} dq^u dq^v \ , \qquad \text{where} \qquad K_{uv} := G_{uw} J^w_v = -K_{vu} \tag{12.16}$$

If J is covariantly constant w.r.t. a hermitian metric, i.e.

$$\nabla_w J^u_v = 0 \tag{12.17}$$

then M is complex and Kähler. In this case K is closed, i.e. $dK = 0$.

On a Hyperkähler manifold three covariantly constant complex structures $J^x, x = 1, 2, 3$ exist and they obey

$$J^x J^y = -\delta^{xy}\mathbf{1} + i\epsilon^{xyz} J^z \ , \tag{12.18}$$

The metric is hermitian w.r.t. all three complex structures and thus we have

$$\nabla J^x = 0 \qquad \Leftrightarrow \qquad dK^x = 0 \quad \forall x \ . \tag{12.19}$$

Hyperkähler manifolds are Ricci-flat.

12.2.2 Isometries on Hyperkähler manifolds

Under an isometry the scalars transform as

$$\delta q^u = \Lambda^a k^{au} \ . \tag{12.20}$$

As they are isometries on a Kähler manifold they obey $k^a_u = \partial_u P^a$. However they should leave all three Kählerforms invariant which in turn implies that are tri-holomorphic in that they obey

$$K^x_{uv} k^{av} = \partial_u P^{ax} \ , \tag{12.21}$$

i.e. we have three Killing prepotentials P^x. In addition the Killing vectors obey the Lie algebra (7.25). For Kähler manifolds with Killing vectors that obey (7.21) this implies the equivariance condition

$$k^i_a G_{i\bar{j}} \bar{k}^{\bar{j}}_b - k^i_b G_{i\bar{j}} \bar{k}^{\bar{j}}_a = i f_{abc} P_c \ . \tag{12.22}$$

Analogously, for Hyperkähler manifolds one has

$$k^u_a K^x_{uv} k^v_b = f_{abc} P^x_c \ . \tag{12.23}$$

The bosonic Lagrangian now reads

$$\mathcal{L} = -\tfrac{1}{2} G_{uv} D_\mu q^u D^\mu q^v - V \ , \tag{12.24}$$

where

$$\begin{aligned} D_\mu q^u &= \partial_\mu q^u - A^a_\mu \, k^{au}(q) \ , \\ V &= (G_{a\bar{b}} k^a_{\bar{c}} \bar{k}^{\bar{b}}_d + 4 G_{uv} k^u_d k^v_c) z^d \bar{z}^{\bar{c}} + G^{a\bar{b}} \sum_x P^x_a P^x_b \geq 0 \end{aligned} \tag{12.25}$$

13 $N = 2$ Supergravity coupled to SYM and charged matter

The massless multiplets for $N = 2$ are given in Table 11.5. The gravitational multiplet contains the metric $g_{\mu\nu}$, two gravitini $\psi_\mu^{\Sigma=1,2}$ and a vector A_μ^0 called the graviphoton, i.e. together $(g_{\mu\nu}, \psi_\mu^\Sigma, A_\mu^0)$. The vector multiplet $\left(A_\mu^a, \lambda^{\Sigma a}, z^a\right)$ with $a = 1, \ldots, n_v$ and the hypermultiplet (χ^i, q^u), $i = 1, \ldots, 2n_H$, $u = 1, \ldots, 4n_H$ were already discussed in the previous section.

The bosonic Lagrangian in reads

$$\begin{aligned}\mathcal{L} = &- \tfrac{1}{2}R - \tfrac{1}{4}\mathrm{Im}\,(\mathcal{N})_{AB}\, F_{\mu\nu}^A F^{B\mu\nu} - \tfrac{1}{4}\mathrm{Re}\,(\mathcal{N})_{AB}\, F_{\mu\nu}^A F_{\rho\sigma}^B \epsilon^{\mu\nu\rho\sigma} \\ &- G_{a\bar{b}}(z,\bar{z}) D_\mu z^a D^\mu \bar{z}^b - G_{uv}(q) D_\mu q^u D^\mu q^v - V\left(z,\bar{z},q\right) \ ,\end{aligned} \tag{13.1}$$

where $A = 0, \ldots, n_v$ also labels the graviphoton. The scalar field space is locally the product

$$\mathcal{M} = \mathcal{M}_{v,SK}^{2n_v} \times \mathcal{M}_{h,QK}^{4n_h} \ , \tag{13.2}$$

where $\mathcal{M}_{v,SK}^{2n_v}$ is a $2n_v$-dimensional special Kähler manifold while $\mathcal{M}_{h,QK}^{4n_h}$ is a $4n_h$-dimensional quaternionic-Kähler manifold. Let us discuss both geometries in turn [**?**, 3].

13.1 Special Kähler geometry

A special Kähler manifold is a Kähler manifold where the Kähler potential is of the specific form

$$K = -\ln i\left(\bar{Z}^A F_A(Z) - Z^A \bar{F}_A(\bar{Z})\right) \ , \tag{13.3}$$

with

$$F_A := \frac{\partial F}{\partial Z^A} \qquad \text{and} \qquad Z^A F_A = 2F \ , \tag{13.4}$$

i.e. $F(Z)$ is homogeneous of degree 2 in the coordinates Z^A. The physical scalar fields z^a are defined as the projective coordinates $z^a = \frac{Z^a}{Z^0}$. Using the homogeneity of $F(Z)$ we can define a $\mathcal{F}(z^a)$ via $F(Z^A) = i(Z^0)^2\mathcal{F}(z^a)$. In terms of $\mathcal{F}$ the Kähler potential reads

$$K = -\ln\left(2(\mathcal{F} + \bar{\mathcal{F}}) - \left(\mathcal{F}_a - \bar{\mathcal{F}}_a\right)(z^a - \bar{z}^a)\right) - \ln|Z^0|^2 \ , \tag{13.5}$$

where the last terms can be removed by a Kähler transformations. Indeed, the rescalings $Z^A \to Z^A e^{-f(z)}, F_A \to F_A e^{-f(z)}$ induce a Kähler transformation of the form $K \to K + f(z) + \bar{f}(z)$ and can be used to set $Z^0 = 1$. The choice of coordinates $Z^A = (1, z^a)$ are called special coordinates.

There also is again an invariant symplectic form of K given by

$$K = -\ln i\left(V^\dagger \Omega V\right) \ , \qquad \text{with} \qquad \Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \ , \qquad V = \begin{pmatrix} F_A \\ Z^B \end{pmatrix} \ . \tag{13.6}$$

The symplectic section V transforms according to

$$V \to V' = SV \ , \tag{13.7}$$

with S being an element of $Sp(2n_v+2,\mathbb{R})$ obeying

$$S^{\mathrm{T}}\Omega S = \Omega \ . \tag{13.8}$$

The gauge kinetic matrix $\mathcal{N}$ is given by:

$$\mathcal{N}_{AB} = \bar{F}_{AB} - \frac{(\mathrm{Im}F)_{AC}\, Z^C\, (\mathrm{Im}F)_{BD} Z^D}{Z^C\, (\mathrm{Im})_{CD}\, Z^D} \ , \tag{13.9}$$

where the second term is due to the graviphoton. Finally, the covariant derivatives read

$$D_\mu z^a = \partial_\mu z^a - A_\mu^B\, k^{Ba}(z) \ , \tag{13.10}$$

where the holomorphic Killing vectors $k^{Ba}(z)$ can again be expressed in terms of Killing prepotential P_0^B by

$$k_{\bar{a}}^B = G_{\bar{a}b} k^{Bb} = i\partial_{\bar{a}} P_0^B \ . \tag{13.11}$$

Finally, if one decouples gravity the geometry reduces to the geometry discussed in Section 12.

13.2 Quaternionic-Kähler geometry

For n_h hypermultiplets one has $4n_h$ real scalars $q^u, u = 1,\dots,4n_h$ which span the $4n_h$-dimensional target space $\mathcal{M}_{h,QK}^{4n_h}$. In supergravity it is not a Hyperkähler manifold bur rather a quaternionic-Kähler manifold. This means that again three almost complex structures $(J^x)_u^v, x = 1,2,3$ which satisfy (12.18) exist. The metric G_{uv} is again Hermitian with respect to all three of them

$$(J^x)_u^v G_{vw} (J^x)_s^w = G_{us} \ . \tag{13.12}$$

Other than for Hyperkähler manifolds they are covariantly constant with respect to an $SU(2)$ connection ω in that they obey

$$\nabla_w (J^x)_u^v + \epsilon^{xyz}\omega_w^y (J^z)_u^v = 0 \ , \tag{13.13}$$

instead of (12.19) For each J^x there is again an associated two-form K^x with coefficients $K_{uv}^x = G_{uw}(J^x)_v^w$ which now obey

$$dK^x + \epsilon^{xyz} w^y \wedge K^z = 0 \ . \tag{13.14}$$

Note that Hyper-Kähler manifold are Ricci-flat while quaternionic-Kähler manifold are Einstein manifolds.

The q^u can be charged with respect to an Abelian or non-Abelian gauge group. This requires the couplings to vector multiplet via the covariant derivatives

$$D_\mu q^u = \partial_\mu q^u - A_\mu^A k_A^u(q) \ , \tag{13.15}$$

where the Killing vectors $k_A^u(q)$ can be expressed in terms of Killing prepotential P_A^x by

$$k_A^u K_{uv}^x = -D_v P_A^x = -(\partial_\nu P_A^x + \epsilon^{xyz} w_v^y P_A^z) \ . \tag{13.16}$$

The equivariance condition now reads

$$k_A^u K_{uv}^x k_B^v = f_{ABC} P_C^x + \epsilon^{xyz} P_A^y P_B^z \ . \tag{13.17}$$

Fianlly, the potential is given by

$$V = e^K \Big(G_{a\bar{b}} k_A^a \bar{k}_B^{\bar{b}} Z^A \bar{Z}^B + 4 G_{uv} k_A^u k_B^v Z^A \bar{Z}^B + \big(G^{a\bar{b}} (\partial_a Z^A)(\bar{\partial}_{\bar{b}} \bar{Z}^B) - 3 Z^A \bar{Z}^B \big) P_A^x P_B^x \Big) \ . \tag{13.18}$$

Before we continue let us mention one caveat. The situation discussed here only features multiplets which are charged with respect to electric gauge bosons but not their magnetic duals. In string theory it is sometimes convenient to go to a different symplectic basis and includes magnetic charges. This can be done via the embedding tensor formalism [**?**].

14 Seiberg-Witten theory

14.1 Preliminaries

In section 12.1 we discussed the couplings of the vector multiplets in global $N = 2$ field theories. We noted that their Lagrangian is entirely determined by a holomorphic prepotential $F(z)$. The scalar fields pointing in the direction of the Cartan subalgebra $z = z^{\hat{a}}T^{\hat{a}}$ where $T^{\hat{a}}$ are the generators of the Cartan subalgebra are flat directions of the potential and span what is called the moduli space of the theory.

So far we did not discuss any quantum corrections. The one loop corrections to the gauge coupling reads

$$g^{-2}(\mu) = g_0^{-2}(\Lambda_{UV}) + \frac{b}{8\pi^2} \ln \frac{\Lambda_{UV}}{\mu} \ , \tag{14.1}$$

where g_0 is the bare coupling defined at some UV-scale Λ_{UV} and b is the one-loop coefficient of the β-function. If G is asymptotically free there also is an IR-scale Λ_{IR} where the gauge coupling becomes infinite. For $SU(2)$ and $\langle z^3 \rangle > \Lambda_{IR}$ the logarithmic running stops and the gauge coupling stays constant below $\langle z^3 \rangle$. For large $\langle z^3 \rangle$ one has a classical $U(1)$ theory at all scales while for small $\langle z^3 \rangle$ classically a gauge enhancement to $SU(2)$ occurs. The question Seiberg and Witten addressed is to what extent this perturbative picture holds non-perturbatively [19].[15]

Concretely they determined the prepotential F exactly, i.e. including all non-perturbative corrections. The generic form of F was known to be of the form [21]

$$F(a) = \tfrac{1}{2}\tau_0 a^2 + \tfrac{i}{\pi}\, a^2 \ln \frac{a^2}{\Lambda_{UV}^2} + \frac{a}{2\pi i} \sum_{l=1}^{\infty} c_l \left(\frac{\Lambda_{UV}}{a} \right)^{4l} \ , \tag{14.2}$$

where for simplicity one denotes $a = z^3$. The first two terms are perturbative but due to the non-renormalization theorem there is no further perturbative correction and only a sum of non-perturbative contributions. Note that the perturbative part of F is not single valued due to the $\ln a^2$ term.

The holomorphic gauge coupling is defined as

$$\tau(a) = \tfrac{1}{\pi}\theta(a) + 8\pi i g^{-2}(a) = \frac{\partial^2 F}{\partial a^2} = \tau_0 + \tfrac{2i}{\pi} \ln \frac{a^2}{\Lambda^2} + \ldots \ . \tag{14.3}$$

Since $\mathrm{Im}\tau$ also determines the σ-model metric we need to have $\mathrm{Im}\tau > 0$. However since $\mathrm{Im}\tau$ is harmonic it can have no minimum unless it is constant and thus turns negative somewhere on the moduli space. This in turn implies that $\tau(a)$ is only locally and for large a well defined but the global description of the moduli space should be different. On the other hand the physics properties of a theory should not depend on a specific parameterization.

The resolution of this apparent paradox is that only the equation of motion have to be well defined while the action might not be. For the case at hand it turns out that for small a the theory is better described in terms of a dual gauge theory. Let us therefore pause and discuss the electric-magnetic duality.

[15] For a review of Seiberg-Witten theory see, for example, [20].

14.2 Electric-magnetic duality

For a $U(1)$ gauge theory the e.o.m. and the Bianchi identity reads

$$\partial^\mu F_{\mu\nu} = 0 \ , \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \ . \tag{14.4}$$

In terms of the dual field strength $\tilde{F}^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ one has

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu \tilde{F}_{\rho\sigma} = 0 \ , \qquad \partial^\mu \tilde{F}_{\mu\nu} = 0 \ , \tag{14.5}$$

i.e. e.o.m. and B.I. are interchanged. For field dependent gauge couplings one has

$$\partial^\mu(g^{-2}(a)F_{\mu\nu} + \tfrac{i}{8\pi^2}\theta(a)\tilde{F}_{\mu\nu}) = 0 \ , \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \ . \tag{14.6}$$

It is convenient to define the self-dual and anti self-dual combinations

$$F^{\pm}_{\mu\nu} := \tfrac{1}{2}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}) \ , \tag{14.7}$$

and

$$G^{-}_{\mu\nu} := \tau F^{-}_{\mu\nu} \ , \qquad G^{+}_{\mu\nu} := \bar{\tau} F^{+}_{\mu\nu} \ . \tag{14.8}$$

In terms of these quantities e.o.m. and B.I. are equivalent to

$$\partial^\mu \mathrm{Im} F^{-}_{\mu\nu} = 0 \ , \qquad \partial^\mu \mathrm{Im} G^{-}_{\mu\nu} = 0 \ . \tag{14.9}$$

In terms of these quantities the electromagnetic duality can be expressed as a $SL(2,\mathbb{R})$ transformation

$$\begin{pmatrix} G^{-}_{\mu\nu} \\ F^{-}_{\mu\nu} \end{pmatrix} \to \begin{pmatrix} G'^{-}_{\mu\nu} \\ F'^{-}_{\mu\nu} \end{pmatrix} = S \begin{pmatrix} G^{-}_{\mu\nu} \\ F^{-}_{\mu\nu} \end{pmatrix} \ , \qquad \tau \to \tau' = \frac{a\tau + b}{c\tau + d} \ , \tag{14.10}$$

where

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \qquad ad - bc = 1 \ , \quad a, b, c, d \in \mathbb{R} \ . \tag{14.11}$$

At the same time one needs to transform

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a'_D \\ a' \end{pmatrix} = S \begin{pmatrix} a_D \\ a \end{pmatrix} \ , \qquad \text{where} \qquad a_D := \frac{\partial F}{\partial a} \ . \tag{14.12}$$

14.3 The Seiberg-Witten solution

For the case at hand we have for the perturbative terms

$$\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} \frac{2i}{\pi}\sqrt{u}\ln\frac{u}{\Lambda^2} \\ \sqrt{u} \end{pmatrix} \ , \tag{14.13}$$

where $u = a^2$. The transformation $u \to u' = e^{2\pi i}u$ induces

$$\begin{pmatrix} a'_D \\ a' \end{pmatrix} = M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} \ , \qquad \text{where} \qquad M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \ . \tag{14.14}$$

Seiberg and Witten suggested that a global description of the moduli space exists with two singularities at $u = \pm\Lambda^2$ where magnetically charged states (a monopole and a dyon) become massless and a perturbative description in terms of the dual gauge theory exits.[16] Consistency requires that the mondromy matrices M obey

$$M_{+\Lambda^2} M_{-\Lambda^2} = M_\infty \ . \tag{14.15}$$

For a dyon of magnetic charge g and electric charge q the monodromy matrix is

$$M^{(g,q)} = \begin{pmatrix} 1+qg & q^2 \\ -g^2 & 1-gq \end{pmatrix} \ . \tag{14.16}$$

One can check that (14.15) is satisfied for a monopole of charge $(1,0)$ and a dyon of charge $(1,-2)$.

The next step is find $a(u), a_D(u)$ that display the required monodomies. This is a version of the Riemann-Hilbert problem and there are two basic strategies:

1. determine $a(u), a_D(u)$ as a solution of a singular so called Picard-Fuchs differential equation.
2. express $a(u), a_D(u)$ as period integrals of an auxiliary spectral surface.

Seiberg and Witten chose the second route and due to the $SL(2)$ considered a torus as the auxiliary spectral surface. One finds

$$\tau(u) = \frac{\omega_D}{\omega} \ , \qquad \omega_D = \frac{\partial a_D}{\partial u} = \oint_\beta \omega \ , \qquad \omega = \frac{\partial a}{\partial u} = \oint_\alpha \omega \ , \tag{14.17}$$

where ω is a certain one-form on the torus and (α, β) are the two cycles of the torus. Further discussions about the solution are beyond the scope of these lecture and we refer to the literature [19, 20].

[16] The Seiberg-Witten proposal was later on verfied by explicit instanton computations. See [20] for a list of references.

15 $N = 4$ and $N = 8$ Supergravity

For supergravities with $N \geq 4$ the scalar field space $\mathcal{M}$ is a coset space of the form $\mathcal{M} = G/H$ where G is a non-compact Lie group and H its maximal compact subgroup. Therefore let us first discuss such σ-models in general [22].

15.1 σ-models on coset spaces G/H

In order to describe such σ-models we introduce an $n \times n$ G-valued matrix $\mathcal{V}$ and assign the transformation law

$$\delta_G \mathcal{V} = \Lambda \mathcal{V} \ , \qquad \delta_G \mathcal{V}^{-1} = -\mathcal{V}^{-1}\Lambda \ , \tag{15.1}$$

where Λ is the constant parameter of the transformation with values in the Lie algebra of G which in the following we denote as Lie(G). H is a subgroup of G and $\mathcal{V}$ transforms under H according to

$$\delta_H \mathcal{V} = -\mathcal{V} h(x) \ , \qquad \delta_H \mathcal{V}^{-1} = h(x)\mathcal{V}^{-1} \ , \tag{15.2}$$

where $h(x)$ is a space-time dependent parameter (local parameter) with values in Lie(H). It is convenient to define the current

$$J_\mu := \mathcal{V}^{-1}\partial_\mu \mathcal{V} \ \in \mathrm{Lie}(G) \ . \tag{15.3}$$

One readily determines its transformation law to be

$$\delta_G J_\mu = 0 \ , \qquad \delta_H J_\mu = -\partial_\mu h + [h, J_\mu] \ . \tag{15.4}$$

Next one decomposes J_μ as

$$J_\mu := Q_\mu + P_\mu \ , \quad \text{with} \quad Q_\mu \in \mathrm{Lie}(H) \ , \quad P_\mu \in (\mathrm{Lie}(H))^T \ , \tag{15.5}$$

where $(\mathrm{Lie}(H))^T$ is the orthogonal compliment of H in G. With this assignment one can read off their transformation properties from (15.4) to be

$$\delta_H Q_\mu = -\partial_\mu h + [h, Q_\mu] \ , \qquad \delta_H P_\mu = [h, P_\mu] \ . \tag{15.6}$$

We see that Q_μ transforms as a composite H-connection while P_μ transforms as a tensor of H.

If G is non-compact (as it is in extended supergravity) it is not straightforward to construct a Lagrangian. $\mathcal{L} \sim \mathrm{Tr}\, J_\mu J^\mu$ is not postive as the trace involves the indefinite Cartan metric. $\mathcal{L} \sim \mathrm{Tr}\, Q_\mu Q^\mu$ is positive but not gauge invariant. Instead

$$\mathcal{L} = \mathrm{Tr}\, P_\mu P^\mu = \mathrm{Tr}\, \mathcal{V}^{-1}(D_\mu \mathcal{V})\mathcal{V}^{-1} D^\mu \mathcal{V} \ , \tag{15.7}$$

where $D_\mu \mathcal{V} = \partial_\mu \mathcal{V} - \mathcal{V}Q$ is positive and gauge invariant if H is the maximal compact subgroup of G.

An important quantity is the positive, H-invariant matrix

$$\mathcal{N} := \mathcal{V}\mathcal{V}^T \ , \tag{15.8}$$

with transformation law

$$\delta_G \mathcal{N} = \Lambda \mathcal{N} + \mathcal{N} \Lambda^T \ , \qquad \delta_H \mathcal{N} = 0 \ . \tag{15.9}$$

As we will see shortly $\mathcal{N}$ parametrizes the inverse gauge couplings but also the Lagrangian $\mathcal{L}$ can be expressed in terms of the $\mathcal{N}$ as

$$\mathcal{L} = \mathrm{Tr}\, \partial_\mu \mathcal{N} \partial^\mu \mathcal{N}^{-1}, \tag{15.10}$$

The physical scalars parametrize the coset, i.e. there is a "unitray gauge" where

$$\mathcal{V} = e^{\phi^{\hat{a}} T^{\hat{a}}} \ , \qquad \hat{a} = 1, \ldots, \dim(\mathrm{Lie}(G)) - \dim(\mathrm{Lie}(H)) \ , \tag{15.11}$$

with $T^{\hat{a}}$ being the non-compact generators.

15.2 $N = 4$ global supersymmetry

The massless multiplets in $N = 4$ supergravity are recorded in Table 11.6. The only multiplet with possibly renormalizable interactions is the vector multiplet. Its field content is $(A^a_\mu, \lambda^{a\Sigma}, \phi^{a[\Sigma\Pi]})$ with $\Sigma, \Pi = 1, \ldots, 4$. Apart from the vector we have four gaugini in the **4** of the R-symmetry group $U(4)$ and six scalars in the **6** of $U(4)$. All reside in the adjoint representation of some gauge group G_0 and their bosonic Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} - D_\mu \phi^{ia} D^\mu \phi^{ia} - V(\phi) \ , \tag{15.12}$$

where $a = 1, \ldots, n_v$ and for simplicity we label the six scalars by the index $i = 1, \ldots, 6$. The potential takes the form

$$V \sim \sum_{ij} \mathrm{Tr}\, [\phi^i, \phi^j]^2 \ , \quad \text{for} \quad \phi^i \equiv \phi^{ia} T^a \ . \tag{15.13}$$

Remarks:

- The σ-model metric for the scalars is flat and the gauge kinetic function is constant.
- It is a conformal theory with a vanishing β function to all orders.
- It is a finite theory.

15.3 $N = 4$ Supergravity

The $N = 4$ gravitational multiplet is given in Table 11.6 and has the field content $(g_{\mu\nu}, \psi^\Sigma_\mu, A^{[\Sigma\Pi]}_\mu, \chi^\Sigma, \tau)$. Apart from the graviton it contains a **4** of gravitini, a **6** of graviphotons , a **4** of spin-1/2 fermions and a complex scalar τ.

The Lagrangian is of the form

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} R - \tfrac{1}{4} \mathrm{Im}\tau \mathcal{N}_{AB} F^A_{\mu\nu} F^{\mu\nu A} - \tfrac{1}{4} \mathrm{Re}\tau \eta_{AB} F^A_{\mu\nu} \tilde{F}^{\mu\nu A} \\ & - \mathrm{Tr}\, \partial_\mu \mathcal{N} \partial^\mu \mathcal{N}^{-1} - \tfrac{1}{4} (\mathrm{Im}\tau)^{-2} \partial_\mu \tau \partial^\mu \bar{\tau} \ , \end{aligned} \tag{15.14}$$

where $A, B = 1, \ldots, n_v + 6$, η_{AB} is the flat metric of $SO(6, n_v)$, and $\mathcal{N}$ is an $SO(6, n_v)$ valued matrix and the coset representative of the scalar field space

$$\mathcal{M} = \frac{SO\,(6, n_v)}{SO\,(6) \times SO\,(n_v)} \times \frac{SU(1,1)}{U(1)} \ . \tag{15.15}$$

The first component of the product is spanned by the scalars of the vector multiplet and represented by $\mathcal{N}$ as in (15.8) while the second component is spanned by τ.

This Lagrangian can be gauged for a gauge group G_0 which is a compact subgroup of $G = SO(6, n_v) \times SU(1, 1)$. In this case a potential appears which is of the form

$$V(\phi) = \tfrac{1}{9}|A_2(\phi,\tau)|^2 + \tfrac{1}{2}|A_{2a}(\phi,\tau)|^2 - \tfrac{1}{3}|A_1(\phi,\tau)|^2 \ , \tag{15.16}$$

where the matrices $A_{1,2,2a}$ are again the scalar parts of the fermion variations

$$\begin{aligned} \delta\psi_\mu^\Sigma &\sim A_{1\Pi}^{\Sigma}\epsilon^\Pi + \ldots \ , \qquad \Sigma, \Pi = 1, \ldots, 4 \ , \\ \delta\chi^\Sigma &\sim A_{2\Pi}^{\Sigma}\epsilon^\Pi + \ldots \ , \\ \delta\lambda_a^\Sigma &\sim A_{2a\Pi}^{\Sigma}\epsilon^\Pi + \ldots \ . \end{aligned} \tag{15.17}$$

The explicit form of the A's together with the entire $N = 4$ Lagrangian can be found, for example, in [23]. As in $N = 2$, the A's depend on the Killing vectors of $\mathcal{M}$ and vanish in the ungauged theory.

15.4 $N = 8$ Supergravity

For $N = 8$ there only is the gravitational multiplet which we listed in Table 11.7. It has the field content

$$(g_{\mu\nu}, \psi_\mu^\Sigma, A_\mu^{[\Sigma\Pi]}, \chi^{[\Sigma\Pi\Lambda]}, \phi^{[\Sigma\Pi\Lambda\Omega]}) \ , \tag{15.18}$$

residing in the $U(8)$ representations $(\mathbf{1}, \mathbf{8}, \mathbf{2}8, \mathbf{5}6, \mathbf{7}0)$. The scalar field space turns out to be

$$\mathcal{M} = \frac{E_{7(7)}}{SU(8)} \ , \tag{15.19}$$

where $E_{7(7)}$ is a non-compact version of E_7 where the number of non-compact generators minus the number of compact generators (which is also called the character of the algebra) is equal to the rank (= 7) of E_7 [3]. For this version of E_7, $SU(8)$ is the maximal compact subgroup and the dimension of $\mathcal{M}$ is indeed $\dim \mathcal{M} = \dim(E_{7,7}) - \dim(SU(8)) = 133 - 63 = 70$.

The Lagrangian is of the form

$$\mathcal{L} = -\tfrac{1}{2}R - \tfrac{1}{4}\mathcal{N}_{AB}F_{\mu\nu}^A F^{\mu\nu A} - \mathrm{Tr}\,\partial_\mu\mathcal{N}\partial^\mu\mathcal{N}^{-1} + \ldots \tag{15.20}$$

where $A, B = 1, \ldots, 28$, and $\mathcal{N}$ is an $E_{7(7)}$ valued matrix and the coset representative of the scalar field space $\mathcal{M}$ given in (15.19).

It has been a questions which subgroups of $E_{7(7)}$ can be gauged. For a partial answer see, for example, [24].

16 Supersymmetry in arbitrary dimensions

16.1 Spinor representations of $SO(1, D-1)$

The spinor representations of $SO(1, D-1)$ are constructed from Dirac matrices γ^M satisfying the Clifford/Dirac algebra

$$\{\gamma^M, \gamma^N\} = 2\eta^{MN} , \qquad M, N = 0, \ldots, D-1 . \tag{16.1}$$

Then the operators

$$\Sigma^{MN} := \tfrac{1}{4} [\gamma^M, \gamma^N] \tag{16.2}$$

satisfy the $SO(1, D-1)$ algebra and thus are generator of (the spinor representations of) $SO(1, D-1)$.

Concretely let us consider $SO(1, D-1)$ for D even.[17] We choose $D = 2k+2$, $k = 0, 1, 2, \ldots$ and define

$$\begin{aligned} \gamma^{0\pm} &:= \tfrac{1}{2} \left(\pm\gamma^0 + \gamma^1 \right) , \\ \gamma^{a\pm} &:= \tfrac{1}{2} \left(\gamma^{2a} \pm i\gamma^{2a+1} \right) , \quad a = 1, \ldots, k , \\ \gamma^{A\pm} &:= \left(\gamma^{0\pm}, \gamma^{a\pm} \right) , \qquad\quad A = 0, \ldots, k . \end{aligned} \tag{16.3}$$

Inserting these definitions into (17.18), one obtains the relations

$$\{\gamma^{A+}, \gamma^{B-}\} = \delta^{AB} , \qquad \{\gamma^{A\pm}, \gamma^{B\pm}\} = 0 . \tag{16.4}$$

This corresponds to the algebra of $k+1$ fermionic creation and annihilation operators (oscillators). One can construct the Dirac representation from the a Clifford vacuum $|\Omega\rangle$ defined by

$$\gamma^{A-}|\Omega\rangle = 0 , \qquad \forall A . \tag{16.5}$$

The states are constructed by acting with γ^{A+} in all possible ways on $|\Omega\rangle$ using $\left(\gamma^{A+}\right)^2 = 0$. The (complex) dimension of the Dirac representation thus is

$$n = \dim_{\mathbb{C}} (\text{Dirac rep.}) = \sum_{i=0}^{k+1} \binom{k+1}{i} = 2^{k+1} . \tag{16.6}$$

For $D = 4$ we have $k = 1$ and thus $n = 2^2 = 4$. For $D = 2$ we have $k = 0$ and thus $n = 2$. Let us exemplary construct the matrix representation for $D = 2$ explicitly. The only non-zero matrices are γ^{0+} and γ^{0-} with

$$\gamma^{0+}|\Omega\rangle = |1\rangle , \qquad \gamma^{0-}|1\rangle = |\Omega\rangle . \tag{16.7}$$

Therefore we can read off the matrix representation

$$\gamma^{0+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad \gamma^{0-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \tag{16.8}$$

[17] Here we follow Appendix B of Vol II of [25].

and thus according to (16.3)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{16.9}$$

The construction for arbitrary k can be obtained similarly [25].

It is possible to define a 'generalized γ_5' by

$$\gamma_{D+1} := i^k \gamma^0 \gamma^1 \dots \gamma^{D-1}, \tag{16.10}$$

satisfying

$$\{\gamma_{D+1}, \gamma^M\} = 0, \qquad \left[\gamma_{D+1}, \Sigma^{MN}\right] = 0, \qquad (\gamma_{D+1})^2 = 1. \tag{16.11}$$

Then one can define two projection operators, $1 \pm \gamma_{D+1}$, that split the Dirac representation into two Weyl representations with eigenvalues ± 1. The dimension of the Weyl representation thus is

$$\dim_{\mathbb{C}}(\text{Weyl rep.}) = 2^k. \tag{16.12}$$

One can check that $\left(\gamma^M\right)^*$ and $\left(-\gamma^M\right)^*$ both satisfy the Dirac algebra (17.18). Since the previous construction was unique both have to be similar to γ^M itself. Indeed one defines

$$B_1 := \gamma^3 \cdots \gamma^{D-1}, \qquad B_2 := \gamma_{D+1} B_1, \tag{16.13}$$

and shows

$$B_1 \gamma^M B_1^{-1} = (-1)^k \left(\gamma^M\right)^*, \qquad B_2 \gamma^M B_2^{-1} = (-1)^{k+1} \left(\gamma^M\right)^*. \tag{16.14}$$

i.e., for any k a similarity transformation exists. Furthermore

$$B_{1,2} \gamma_{D+1} B_{1,2}^{-1} = (-1)^k (\gamma_{D+1})^*, \tag{16.15}$$

so that for k even, i.e., $D = 2, 6, 10, \dots$, the Weyl representation is its own conjugate (s.c.), while for k odd, i.e., $D = 4, 8, \dots$, the Weyl representations are conjugate to each other (c.c.). From

$$B_{1,2} \Sigma^{MN} B_{1,2}^{-1} = -\left(\Sigma^{MN}\right)^* \tag{16.16}$$

it follows that both ψ and $B^{-1}\psi^*$ obey the same Lorentz transformation law, i.e.,

$$\delta\psi = i\omega_{MN} \Sigma^{MN} \psi, \qquad \delta B^{-1}\psi^* = i\omega_{MN} \Sigma^{MN} B^{-1} \psi^*. \tag{16.17}$$

Thus one can impose a Majorana condition and define the Majorana Spinor ψ being a Dirac spinor but with the additional requirement (reality condition)

$$\psi^* = B\psi. \tag{16.18}$$

Thus the dimension of the Majorana representation is

$$\dim_{\mathbb{C}}(\text{Majorana rep.}) = 2^k, \qquad \text{or} \qquad \dim_{\mathbb{R}}(\text{Majorana rep.}) = 2^{k+1}. \tag{16.19}$$

From (16.18) we find

$$\psi = B^* \psi^* = B^* B \psi, \tag{16.20}$$

and thus

$$BB^* = 1\,. \tag{16.21}$$

From the definition (16.13) one computes

$$B_1 B_1^* = (-1)^{\frac{k}{2}(k+1)} \;\Rightarrow\; k = 0, 3, 7, \ldots \;\; (D = 2, 8, \ldots)\,, \tag{16.22}$$

$$B_2 B_2^* = (-1)^{\frac{k}{2}(k-1)} \;\Rightarrow\; k = 1, 4, 8, \ldots \;\; (D = 4, 10, \ldots)\,. \tag{16.23}$$

A Majorana-Weyl (MW) representation is only possible if the Weyl representation is self-conjugated, i.e., k is even, and hence, for $k = 0, 4, 8, \ldots$ $(D = 2, 10, \ldots)$. Its dimension is

$$\dim_{\mathbb{R}}(\mathrm{MW}) = 2^k\,. \tag{16.24}$$

For D odd and $D = 2k+1$ there are no Weyl representation and a Majorana representation is possible only in $D = 1, 3, 9, 11, \ldots$. Its dimension is

$$\dim_{\mathbb{R}}(\text{Majorana rep.}) = 2^k\,. \tag{16.25}$$

In this case the dimension of the Dirac representation is

$$\dim_{\mathbb{R}}(\text{Dirac rep.}) = 2^{k+1}\,. \tag{16.26}$$

All the possible representations are summarized in Table 16.1.

D	k	Majorana	Weyl	M-W	$\dim_R$
2	0	✓	s.c.	✓	1
3	1	✓	-	-	2
4	1	✓	c.c.	-	4
5	2	-	-	-	8
6	2	-	s.c.	-	8
7	3	-	-	-	16
8	3	✓	c.c.	-	16
9	4	✓	-	-	16
10	4	✓	s.c.	✓	16
11	5	✓	-	-	32
12	5	✓	c.c.	-	64

Table 16.1: Spinor representations for $2 \leq D \leq 12$.

16.2 Supersymmetry algebra

The supersymmetry algebra is an extension of the Poincare algebra. In arbitrary space-time dimensions D it depends on the spinor representations of $SO(1, D-1)$. Schematically it reads

$$\begin{aligned} \{Q^I, \bar{Q}^J\} &\sim \gamma^M P_M \delta^{IJ}\,, \qquad \{Q^I, Q^J\} \sim Z^{IJ}\,, \\ \left[L_{MN}, Q^I\right] &\sim \Sigma_{MN} Q^I\,, \qquad \left[P_M, Q^I\right] = 0\,, \end{aligned} \tag{16.27}$$

where $M = 0, \dots, D-1$. Q^I is a spinor in the smallest spinor representation listed in Table 16.1. The Jacobi-identity requires that Z^{IJ} commutes with all generators and this is a central element of the algebra. Positivity requires the BPS-bound

$$M \geq |Z| \ . \tag{16.28}$$

For arbitrary D it is more convenient to counts real supercharges (which we denote by q) instead of the number of spinor representations. For example, $N = 1$ in $D = 4$ has $q = 4$ real supercharges, or in general $q = 4N$ for arbitrary N in $D = 4$. For this notation the various supersymmetric theories for $4 \leq D \leq 12$ and $4 \leq q \leq 32$ are displayed in Table 16.2.[18]

D \ q	4	8	...	16	...	24	...	32	64
4	$\times$ (N=1)	$\circ$ (N=2)	$\circ$	$\circ$ (N=4)	$\circ$	$\circ$	$\circ$	$\circ$ (N=8)	
5		$\times$		$\circ$		$\circ$		$\circ$	
6		$\times$ (1,0)		$\circ$ $\circ$ (1,1) (2,0)		$\circ$ $\circ$		$\circ$ (2,2)	
7				$\times$				$\circ$	
8				$\times$				$\circ$	
9				$\times$				$\circ$	
10				$\times$ I				$\circ$ $\circ$ IIA IIB	
11								$\times$	
12									$\times$

Table 16.2: Table of supersymmetric theories. "$\times$" denotes the theories with the minimal number of supersymmtries.

Most of the entries in Table 16.2 are self-explanatory. However note that in $D = 6$ the supercharge Q is self-conjugate and two independent Weyl representations of opposite chirality, denoted $\mathbf{8}$ and $\mathbf{8}'$, of $SO(1,5)$ exist. For the theory denoted by $(1,1)$ the two supercharges transform as $Q_1 \in \mathbf{8}$, $Q_2 \in \mathbf{8}'$ and thus the theory is non-chiral while the (2,0) theory has $Q_1 \in \mathbf{8}$, $Q_2 \in \mathbf{8}$ and therefore is chiral.

In $D = 10$ also two Majorana-Weyl representations of opposite chirality $\mathbf{16}, \mathbf{16}'$ exist. Type IIA is non-chiral with $Q_1 \in \mathbf{16}$, $Q_2 \in \mathbf{16}'$ while type IIB is chiral with $Q_1 \in \mathbf{16}$, $Q_2 \in \mathbf{16}$.

In $D = 2$ the Lorentz group is $SO(1,1)$ and the supercharges Q are real one-dimensional Majorana-Weyl spinors. The type (p,q) superalgebra in two dimensions reads

$$\begin{aligned} \{Q_L^{I_L}, Q_L^{J_L}\} &= \delta^{I_L J_L}\, P^- \,, \qquad I_L, J_L = 1, \dots, p \,, \\ \{Q_R^{I_R}, Q_R^{J_R}\} &= \delta^{I_R J_R}\, P^+ \,, \qquad I_R, J_R = 1, \dots, q \,, \\ \{Q_L^{I_L}, Q_R^{I_R}\} &= Z^{I_L I_R} \,. \end{aligned} \tag{16.29}$$

[18]For $q = 64$ one goes beyond $N = 8$ and thus has higher spin fields in the massless multiplet. For these theories one does not have a consistent interacting quantum field theory in a Minkowski background.

17 Kaluza-Klein Compactification

17.1 S^1-compactification

The basic idea of Kaluza-Klein theory is to formulate gauge symmetries as space-time symmetries of a higher-dimensional space-time. The simplest example is the compactification on a circle S^1, i.e. the space-time background is the five-dimensional space-time

$$\mathbb{R}_{1,3} \times S_1 \ , \tag{17.1}$$

with coordinates

$$x^M = (x^\mu, y) \ , \qquad M = 0, ..., 4 \ , \quad \mu = 0, ..., 3 \ . \tag{17.2}$$

y is the periodic coordinate of the circle, i.e. $y = y + 2\pi R$ with R being the radius of the circle. Thus a scalar field $\Phi(x^M)$ in this background satisfies

$$\Phi(x^\mu, y + 2\pi R) = \Phi(x^\mu, y) \ , \tag{17.3}$$

and therefore can be expanded in terms of Eigenfunctions of S^1 as

$$\Phi(x^M) = \sum_{n=-\infty}^{\infty} \phi^{(n)}(x^\mu)\, e^{iny/R} \ . \tag{17.4}$$

If Φ satisfies the massless Klein-Gordon equation one has

$$\Box_5 \Phi := \eta^{MN} \partial_M \partial_N \Phi = (\eta^{\mu\nu} \partial_\mu \partial_\nu + \partial_y{}^2) \Phi = \sum_n \big(\Box_4 \phi^{(n)} - m^2_{(n)} \phi^{(n)}\big) e^{iny/R} = 0 \ , \tag{17.5}$$

so that each Fourier mode $\phi^{(n)}$ satisfies

$$\Box_4 \phi^{(n)} - m_n^2 \phi^{(n)} = 0 \ , \qquad m^2_{(n)} = n^2/R^2 \ . \tag{17.6}$$

From an $\mathbb{R}_{1,3}$ perspective the $\phi^{(n)}$ form an infinite tower of massive scalar fields called the Kaluza-Klein tower.

One can estimate the size of S_1 by experimentally testing the validity of the two $\frac{1}{r}$ potentials:

$$\begin{aligned} V_{\text{Coulomb}} &\sim \frac{e}{r} \ , \qquad \text{confirmed for} \quad r > 10^{-18} m \ , \\ V_{\text{Newton}} &\sim \frac{m}{r} \ , \qquad \text{confirmed for} \quad r > 10^{-4} m \ . \end{aligned} \tag{17.7}$$

Therefore, generically we need to choose $R < 10^{-18} m$ so that the extra dimensions are not visible.[19]

In any case the SM modes should be found among the massless (zero) modes, i.e. for $n = 0$. In this case Φ has no y dependence $\Phi(x^\mu, y) \to \phi^{(0)}(x^\mu)$.

[19] However there is an exception pioneered by Rubakov and Shaposhnikov [26]. Whenever the gauge theory is localized on a three-dimensional hyperplane inside a higher dimensional space-time then a large R with $R < 10^{-4} m$ is allowed. In string theory this scenario is realized by D-branes [25].

One includes gravity by considering and reducing the higher-dimensional Einstein-Hilbert action

$$S = \frac{-1}{2\kappa_5^2}\int d^5x\sqrt{-g^{(5)}}\mathcal{R}^{(5)} \ , \tag{17.8}$$

where $\mathcal{R}^{(5)}$ is the five-dimensional Ricci scalar. One expands the five-dimensional metric around an $SO(1,3)$ invariant background with some fluctuation δg_{MN}. A convenient parametrization is

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + r^2 A_\mu A_\nu & r^2 A_\mu \\ r^2 A_\nu & r^2 \end{pmatrix} = \langle g_{MN}\rangle + \delta g_{MN} \ , \qquad \langle g_{MN}\rangle = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & < r^2 > \end{pmatrix} \tag{17.9}$$

and one obtains after Weyl rescaling

$$S = \int d^4x\sqrt{\tilde{g}^{(4)}}\left(\frac{-1}{2\kappa_4^2}\tilde{\mathcal{R}}^{(4)} - \tfrac{1}{4}r^2\tilde{F}_\mu\tilde{F}^{\mu\nu} - \frac{\partial_\mu r\partial^\mu r}{\kappa_4^2 r^2}\right), \tag{17.10}$$

where

$$\kappa_4^{-2} = \frac{2\pi R}{\kappa_5^2} \ , \qquad \tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu \ , \qquad \tilde{A}_\mu = \kappa_4 A_\mu \ . \tag{17.11}$$

This shows that the five-dimensional Einstein-Hilbert action decomposes into the four-dimensional Einstein-Hilbert action, a $U(1)$ gauge theory plus a neutral scalar without potential. The local $U(1)$ gauge transformation arises from general coordinate invariance of the five-dimensional theory $y \to y - \xi(x^\mu)$.

17.2 Generalization

As a first generalization consider a spacetime of the form:

$$M_4 \times T^d \tag{17.12}$$

where T^d is a d-dimensional torus. As in (17.9) the metric is parametrized as

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + g_{mn}A_\mu^m A_\nu^n & g_{np}A_\mu^p \\ g_{np}A_\nu^p & g_{mn} \end{pmatrix} \ , \qquad m, n = 1, ..., d \quad . \tag{17.13}$$

For a 4-dimensional perspective we have

zero modes	spin	multiplicity
$g_{\mu\nu}$	2	1
A_μ^p	1	d
g_{mn}	0	$\frac{1}{2}d(d+1)$

.

The isometries of T^d read $y^m \to y^m - \xi^m(x)$ and induce

$$A_\mu^m \to A_\mu^m - \partial_\mu\xi^m \ , \tag{17.14}$$

corresponding to a $[U(1)]^d$ gauge theory. The action of the zero modes after the KK reduction and Weyl rescaling takes the form

$$S = \int d^4x\sqrt{-\tilde{g}^{(4)}}\Big(\frac{1}{2\kappa_4^2}\tilde{\mathcal{R}}^{(4)} + g_{mn}\tilde{F}_{\mu\nu}^m\tilde{F}^{n\mu\nu} - (\partial_\mu g_{mn})(\partial^\mu g^{mn}) + (\partial_\mu \ln\det g)(\partial^\mu \ln\det g\Big) \ . \tag{17.15}$$

Note that again there is no potential and the σ-model target space is the coset

$$\mathcal{M} = \frac{GL(d)}{SO(d)} \ . \tag{17.16}$$

One can consider more general Kaluza-Klein theories in that one takes a space-time background of arbitrary dimension and includes a compact d-dimensional manifold X^d as follows

$$\mathbb{R}^{(1,D-1)} \times X^d \ , \tag{17.17}$$

where X^d is a d-dimensional compact Riemannian manifold. Following the same procedue as just outlined one derives an effective theory in $\mathbb{R}^{(1,D-1)}$ for the massless modes. This includes the metric, a set of gauge bosons and a set of scalars. Any isometry of X^d then appears in $\mathbb{R}^{(1,D-1)}$ as a gauge symmetry. It can be non-Abelian if X^d admits non-Abelian isometries which, for example, a group manifold $X^d = G/H$ has.[20] The problem with such compactifications is that generically they are not flat and the Ricci-scalar of X^d induces a large cosmological constant.

17.3 Supersymmetry in Kaluza Klein theories

Let us first include fermions into Kaluza-Klein theories and consider the massless Dirac equation in the background (17.17)

$$i\gamma^M D_M \Psi(x,y) = (i\gamma^\mu D_\mu + i\gamma^m D_m)\Psi = 0 \ , \tag{17.18}$$

where γ denotes Dirac matrices and $x^\mu, \mu = 0, ..., D-1$ are coordinates on $\mathbb{R}^{(1,D-1)}$ while $y^m, m = 1, ..., d$ are coordinates on X^d. One expands

$$\Psi(x,y) = \sum_n \psi_n(x)\,\eta_n(y) \ , \qquad \text{with} \qquad i\gamma^m D_m \eta_n = m_n \eta_n \ , \tag{17.19}$$

(no sum on n in the last term). The η_n are called Killing spinors and at level n there can be more than one of them. Inserted into (17.18) implies

$$(i\gamma^\mu D_\mu + m_n)\,\psi_n(x) = 0 \ , \tag{17.20}$$

with $m_n = 0$ corresponding to the massless (zero) modes.

As an aside let us remark that the Atiah-Hirzebruch index theorem states that if X^d has continuous isometries, the (character-valued) index of the Dirac operators vanishes

$$\text{index}(\not{D}(X)) = n_L - n_R = 0 \ , \tag{17.21}$$

where $n_{L/R}$ counts the left-handed/right-handed zero modes of D. This has an immediate application to Kaluza-Klein theories as gauge theories which arise from higher-dimensional backgrounds (17.17) necessarily need an X^d with continuous isometries. Therefore, a chiral spectrum in $\mathbb{R}^{(1,D-1)}$ cannot be generated by compactifications on a smooth manifold unless it was already chiral before the compactification [27].

[20] One can also replace $\mathbb{R}^{(1,D-1)}$ by an anti de Sitter or de Sitter background.

For the supercharges one makes the Ansatz

$$Q(x,y) = \sum_{\Sigma} Q^{\Sigma}(x)\,\eta^{\Sigma}(y)\ , \tag{17.22}$$

where $Q^{\Sigma}(x)$ are a spinors on $\mathbb{R}^{1,D-1}$ and $\eta^{\Sigma}(y)$ are spinors on X^d. One further requires:

1. The η^{Σ} are normalizable, i.e. $||\eta^{\Sigma}|| = 1$ which implies that the η^{Σ} are nowhere vanishing on X^d.

2. The η^{Σ} should be globally well-defined which implies that they are singlets of the structure group.[21]

The existence of globally well-defined η^{Σ} imply a reduction of the structure group and X^d is called a manifold with the G-structure [28]. G is a subgroup of $GL(d,\mathbb{R})$, i.e. $G \subset GL(d,\mathbb{R})$ which leaves η invariant.

As an example let us choose $D = 4, d = 6$ in which case the Lorentz group $SO(1,9)$ decomposes as

$$SO(1,9) \to SO(1,3) \times SO(6)\ . \tag{17.23}$$

The spinor representation **16** of $SO(1,9)$ decomposes accordingly

$$\mathbf{16} \to (\mathbf{2},\mathbf{4}) + (\bar{\mathbf{2}},\bar{\mathbf{4}})\ . \tag{17.24}$$

If one demands a specific number of supercharges in $\mathbb{R}^{(1,3)}$ the structure group is fixed and the class of manifolds determined.

Let us consider the following examples [28]:

1. $X^6 = T^6$ or a torus fibration. In this case one has an identity structure with four $\eta^{\Sigma} \in \mathbf{4}$ of $SO(6)$ which correspond to four supercharges.

2. X^6 with $SU(2)$ structure. In this case two η^{Σ} exist corresponding to two supercharges.

3. X^6 with $SU(3)$ structure. In this case one η exists corresponding to one supercharge.

An additional requirement one can impose is that the η is covariantly constant (w.r.t. the Levi-Civita or spin connection)

$$\not{D}\eta = 0\ . \tag{17.25}$$

In this case the following properties hold:

1. $Hol(X^d) \subset SU(\frac{d}{2})$,

2. $R_{nm}(X^d) = 0$,

3. $c_1(X^d) = 0$.

[21] This is the group acting on the frame bundle.

This implies that X^d is a Calabi-Yau manifold.

Let us briefly discuss the implications for supersymmetry. The supersymmetry transformation of the gravitino Ψ_M is schematically given by

$$\delta\Psi_M = D_M\xi + \sum_p a_p \gamma \cdot F_p\,\xi + \ldots \tag{17.26}$$

where ξ is the parameter of the supersymmetry transformation, a_p are some constants and $\gamma \cdot F_p$ denotes appropriate contractions of the p-form field strength present in the theory. Expanding $\xi(x,y) = \xi(x)\eta(y)$ as in (17.22) we find in the background (17.17)

$$\langle\delta\Psi_\mu\rangle = 0\ , \qquad \langle\delta\Psi_m\rangle \sim D_m\eta + \sum_p a_p\gamma \cdot \langle F_p\rangle\eta\ . \tag{17.27}$$

We see that depending on the $\langle F_p\rangle$ and $D_m\eta$, supersymmetry can be intact or spontaneously broken. In standard Calabi-Yau compactifications one has $\langle F_p\rangle = 0 = D_m\eta$ and supersymmetry is preserved.

18 Supergravities with $q = 32$ supercharges

18.1 Counting degrees of freedom in D dimernsions

The most economic way to count on-shell degrees of freedom for massless fields is to go a light-like frame $p^M = (-E, E, 0, \ldots)$. In D space-time dimensions this choice is left invariant by the little group $SO(D-2)$ and the massles modes fall into representations of this group.[22] The on-shell degrees of freedom for the various fields are recorded in Table 18.1.[23]

fields	on-shell d.o.f.
scalar ϕ	1
spin-1/2 fermion χ	$\frac{1}{2}$ dim(spinor rep.)
gauge boson A_M	$D-2$
gravitino ψ_M	$\frac{1}{2}(D-3)$ dim(spinor rep.)
graviton g_{MN}	$\frac{1}{2}(D-2)(D-1)-1$
p-form $A_{[M_1 M_2 \ldots M_p]}$	$\binom{D-2}{p}$

Table 18.1: Counting degrees of freedom (d.o.f.).

18.2 $D = 11$ Supergravity

In $D = 11$ the massless multiplet contains the metric g_{MN} with 44 d.o.f., an antisymmetric 3-index tensor A_{MNP} with 84 d.o.f. and a gravitino Ψ_M with 128 d.o.f.. The action is

$$S = \tfrac{1}{2}\int d^{11}x\sqrt{-g}\big(R - \tfrac{1}{2}|F_4|^2\big) - \tfrac{1}{6}\int A_3 \wedge F_4 \wedge F_4 + \text{(fermionic interactions)}\,, \quad (18.1)$$

where $F_4 = dA_3$ is the field strength of the three from and we abbreviate

$$|F_p|^2 = \frac{1}{p!} F_{M_1,\ldots M_p}\, F^{M_1,\ldots,M_p}\,. \quad (18.2)$$

$\mathcal{L}$ has diffeomorphism invariance and local supersymmetry by construction. In addition there is gauge invariance related to the three-form

$$\delta A_3 = \mathrm{d}\Lambda_2\,, \qquad \delta F_4 = 0\,, \qquad \delta\Psi = 0\,, \quad (18.3)$$

where Λ_2 is a 2-form and $\mathrm{d}F_4 = 0$.

[22]For massive representations one goes to the rest frame $p^M = (-m, 0, 0, \ldots)$ and in this case the little group is $SO(D-1)$.

[23]A counting using gauge invariance and the equation of motions can be found for example in [29].

18.3 Compactification on S^1: Type II A supergravity in $D = 10$

Compactifying $D = 11$ supergravity on an S^1 yields the spectrum

$$\begin{aligned} g_{MN} &\to g_{\mu\nu}\ , \quad g_{\mu 10} \sim A_\mu\ , \quad g_{10,10} \sim \phi\ , \\ A_{MNP} &\to A_{\mu\nu\rho}\ , \quad A_{\mu\nu 10} \sim B_{\mu\nu}\ , \\ \Psi_M &\to (\Psi_{\mu,\alpha}, \Psi_{\mu\,\dot\alpha})\ , \quad (\Psi_{10\,\alpha} \sim \lambda_\alpha, \Psi_{10\,\dot\alpha} \sim \lambda_{\dot\alpha})\ . \end{aligned} \tag{18.4}$$

Altogether the $D = 10$ gravitational multiplet thus contains

$$\underbrace{\underset{(35)}{G_{\mu\nu}}\ ,\ \underset{(28)}{B_{\mu\nu}}\ ,\ \underset{(1)}{\phi}\ ,\ \underset{(8)}{A_\mu}\ ,\ \underset{(56)}{A_{[\mu\nu\rho]}}}_{(128)}\ ,\ \underbrace{\underset{(56)}{\Psi_{\mu\alpha}}\ ,\ \underset{(56)}{\Psi_{\mu\dot\alpha}}\ ,\ \underset{(8)}{\lambda_\alpha}\ ,\ \underset{(8)}{\lambda_{\dot\alpha}}\ ,}_{(128)} \tag{18.5}$$

where we indicated the number of d.o.f. in brackets.

Performing the KK-reduction one obtains

$$S_{11} = \int_{S'} \mathrm{d}y \int \mathrm{d}^{10}x\, \mathcal{L}_{11} = \int \mathrm{d}^{10}x\, \mathcal{L}_{\mathrm{II\,A}}\,, \tag{18.6}$$

where one only keep the lowest KK modes after the compactification (no y-dependence). The type IIA actions is

$$\begin{aligned} S_{\mathrm{IIA}} = \tfrac{1}{2} \int \mathrm{d}^{10}x \sqrt{-g} \left(e^{-\phi} \left(R + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}|H_3|^2 \right) - \tfrac{1}{4}(|F_2|^2 + |\hat{F}_4|^2) \right) \\ - \tfrac{1}{4} \int \mathrm{d}^{10}x\, B_2 \wedge F_4 \wedge F_4 + (\text{fermionic interactions})\,, \end{aligned} \tag{18.7}$$

where $F_2 = \mathrm{d}A_1$, $H_3 = \mathrm{d}B_2$, $F_4 = \mathrm{d}A_3$ and $\hat{F}_4 = F_4 - A_1 \wedge H_3$.

The IIA theory has two local supersymmetries of opposite chirality. In addition there three independent gauge symmetries related to the various p-forms present. They are

$$(i) \qquad \delta A_1 = \mathrm{d}\Lambda_0\,, \qquad \delta A_3 = \Lambda_0 H_3\ , \tag{18.8}$$

$$(ii) \qquad \delta B_2 = \mathrm{d}\Lambda_1\,, \qquad \delta H_3 = 0\,, \tag{18.9}$$

$$(iii) \qquad \delta A_3 = \mathrm{d}\Lambda_2\,, \tag{18.10}$$

with parameters $\Lambda_0, \Lambda_1, \Lambda_2$. Note that the theory contains no charged fermions.

18.4 Kaluza-Klein reduction on T^d

Reducing eleven-dimensional supergravity on T^d one has a space-time background $M_{D=11-d} \times T^d$. The bosonic spectrum of the gravitational multiplet is

$$\begin{aligned} g_{MN} &\to \begin{cases} g_{\mu\nu} & \\ g_{\mu i}\ , i = 1, \ldots, d & \to d \text{ graviphotons} \\ g_{ij} & \to \tfrac{1}{2}d(d+1) \text{ scalars} \end{cases} \\ A_{MNP} &\to \begin{cases} A_{\mu\nu\rho} & \\ A_{\mu\nu i} & \to d \text{ two-forms } B_{\mu\nu} \\ A_{\mu ij} & \to \tfrac{1}{2}d(d-1) \text{ graviphotons} \\ A_{ijk} & \to \binom{d}{3} \text{ scalars} \end{cases} \end{aligned} \tag{18.11}$$

Altogether the gravitational multiplet in $11-d$ dimensions contains the bosonic components

$$g_{\mu\nu},\ A_{[\mu\nu\rho]},\ d\,B_{\mu\nu},\ \tfrac{1}{2}d(d+1)\,A_\mu,\ \tfrac{1}{6}d(d^2+5)\,\phi\ . \tag{18.12}$$

In order to understand the scalar field space $\mathcal{M}$ we need to pause and discuss Poincare-duality in D space-time dimensions. Consider a p-form $B_p = B_{M_1\ldots M_p}dx^{M_1}\wedge\ldots dx^{M_p}$. Its field strength is $H_{p+1} = dB_p$ and one has a gauge invariance $B_p \to B_p + d\Lambda_{p-1}$. Poincare duality related

$$H_{p+1} \simeq \tilde{H}_{D-p-1}\ . \tag{18.13}$$

For example in $D=4$ a two form B_2 is dual to a scalar while in $D=3$ a vector A_1 is dual to a scalar. In even dimension for can impose a self-duality condition on a $p+1 = D/2$-form. For example, in $D=4$ a self-dualty condition can be imposed on a a one-form A_1 or rather its field strength $F_2 = \tilde{F}_2$.

The scalar fields arise from the metric g_{MN} and from the three-Form $A_{[MNP]}$. Together they form the scalar manifold

$$\mathcal{M} = \frac{E_{d,d}}{H}\ , \tag{18.14}$$

where the groups $E_{d,d}, H$ are given in Table 18.2.

As we already said the scalars arise from the metric g_{MN} and the three-form $A_{[MNP]}$. For $3 \le d \le 6$ $\dim(\mathcal{M})$ is the sum number of scalar fields $\frac{1}{6}d(d^2+5)$ given in (18.12). For $D=4, d=7$ however, there are 7 additional scalars from the duality $B_2 \sim \phi$ so that the total number is 70. For $D=3, d=8$ there are 36 additional scalars from the duality $A_1 \sim \phi$ so that the total number is 128.

d	$E_{d,d}$	H	$\dim(G/H)$
1	$SO(1,1)$	1	1
2	$GL(2)$	$SO(2)$	3
3	$SL(2)\times SL(3)$	$SO(2)\times SO(3)$	7
4	$SL(5)$	$SO(5)$	14
5	$SO(5,5)$	$SO(5)\times SO(5)$	25
6	$E_{6,6}$	$USp(8)$	42
7	$E_{7,7}$	$SU(8)$	70
8	$E_{8,8}$	$SO(16)$	128

Table 18.2: Scalar coset for $q=32$.

19 Supergravities with $q = 16$ supercharges

19.1 Type I supergravity in $D = 10$

In this supergravity there are two multiplets. The gravitational multiplet contains

$$\underbrace{\underset{(35)}{g_{MN}}\ ,\ \underset{(28)}{B_{MN}}\ ,\ \underset{(1)}{\phi}}_{(64)}\ ,\ \underbrace{\underset{(56)}{\Psi_{M\alpha}}\ ,\ \underset{(8)}{\lambda_{\dot\alpha}}}_{(64)}\ , \tag{19.1}$$

while the vector multiplets features

$$\underset{(8)}{A_M}\ ,\ \underset{(8)}{\chi_\alpha}\ . \tag{19.2}$$

Now non-Abelian gauge symmetries are possible in that the vector multiplet can transform in the adjoint representation of some non-Abelian gauge group G. The Lagrangian is

$$S = \tfrac{1}{2}\int \mathrm{d}^2x\,\sqrt{-g}e^{-2\phi}\left(R + 4\partial_M\phi\partial^M\phi - |\hat{H}_3|^2\right) + \mathrm{Tr}\,\left(F_{MN}F^{MN}\right) + \dots\,, \tag{19.3}$$

where

$$\hat{H}_3 := \mathrm{d}B_2 - \omega_3\ , \qquad \omega_3 := \mathrm{Tr}\,\left(A\wedge\mathrm{d}A + \tfrac{2}{3}A\wedge A\wedge A\right)\ . \tag{19.4}$$

ω_3 is the Yang-Mills Chern-Simons term which obeys $\mathrm{d}\omega_3 = F_2\wedge F_2$.

The theory is general coordinate invariant and has one local supersymmetry and associated with B_2 there is a two-form gauge invariance

$$\delta B_2 = \mathrm{d}\Lambda_1\ , \tag{19.5}$$

where Λ_1 is a one-form parameter.

The non-Abelian gauge invariance transforms $\delta A = \mathrm{d}\Lambda + [A,\Lambda]$ but from (19.4) we infer that also ω_3 transforms and as a consequence B_2 has to transform in order to keep $\hat{H}_3$ invariant. Altogether one finds

$$\delta A = \mathrm{d}\Lambda + [A,\Lambda]\ , \qquad \delta\omega_3 = \mathrm{Tr}\ \mathrm{d}(\Lambda dA)\,, \qquad \delta B_2 = \mathrm{Tr}\,(\Lambda dA)\,. \tag{19.6}$$

Type I supergravity is chiral in and generically anomalous [**?**]. The anomaly can only be cancelled for two gauge groups $E_8\times E_8$ and $SO(32)$.

19.2 Compactification of type I on T^d

Compactifying $D = 10$ Type I supergravity on T^d yields the bosonic spectrum

$$\begin{aligned}
&g_{MN} \to \begin{cases} g_{\mu\nu} & \\ g_{\mu i}\ , i = 1,\dots,d & \to d \text{ graviphotons} \\ g_{ij} & \to \tfrac{1}{2}d(d+1) \text{ scalars}\end{cases}\\
&B_{MN} \to \begin{cases} B_{\mu\nu} & \\ B_{\mu i} & \to d \text{ graviphotons} \\ B_{ij} & \to \tfrac{1}{2}d(d-1) \text{ scalars}\end{cases}\\
&A_M^a \to \begin{cases} A_\mu^a & \text{gauge boson in adjoint representation} \\ A_i^a & \to d \text{ scalars in adjoint representation}\end{cases}
\end{aligned} \tag{19.7}$$

Altogether the gravitational and vector multiplet in $10-d$ dimensions contains the bosonic components

$$\begin{aligned} \text{gravitational multiplet :} &\quad (g_{\mu\nu},\, B_{\mu\nu},\, d\, A_\mu,\, \phi)\ , \\ \text{vector multiplet :} &\quad (A_\mu, d\,\phi)\ . \end{aligned} \tag{19.8}$$

Thus in $10-d$ dimensions there is one gravitational multiplet and $(d+n_v)$ vector multiplets. The $d(d+n_v)+1$ scalars parameterize the coset

$$\mathcal{M} = \begin{cases} \frac{SO(d,d+n_v)}{SO(d)\times SO(d+n_v)} \times \mathbb{R}^+ & \text{for } d \leq 5 \\ \frac{SO(6,6+n_v)}{SO(6)\times SO(6+n_v)} \times \frac{SU(1,1)}{U(1)} & \text{for } d = 6 \\ \frac{SO(8,8+n_v)}{SO(8)\times SO(8+n_v)} & \text{for } d = 7 \end{cases}\ , \tag{19.9}$$

where the second factor is the dilaton for $d \leq 5$ and the dilaton plus the dual of $B_{\mu\nu}$ for $d=6$. For $d=7$ also all $14+n_v$ vector fields are dualized to scalars.

20 Chiral supergravities and supergravities with $q = 8$ supercharges

20.1 Type II B supergravity

In type IIB supergravity there is again only one massless multiplet, the gravitational multiplet, which contains the fields

$$\underbrace{\underset{(35)}{g_{MN}}\,,\ \underset{(28)}{B^1_{MN}}\,,\ \underset{(28)}{B^2_{MN}}\,,\ \underset{(1)}{\phi}\,,\ \underset{(1)}{l}\,,\ \underset{(1)}{A_M}\,,\ \underset{(35)}{A^*_{MNPQ}}}_{(128)}\,,\ \underbrace{\underset{(56)}{\Psi^1_{M\alpha}}\,,\ \underset{(56)}{\Psi^2_{M\alpha}}\,,\ \underset{(8)}{\lambda^1_{\dot\alpha}}\,,\ \underset{(8)}{\lambda^2_{\dot\alpha}}}_{(128)}\,, \tag{20.1}$$

where the four-form has a self-dual field strength

$$F_5 = \mathrm{d}A^*_4 = \tilde F_5\,, \qquad \text{with} \qquad \tilde F_{M_1,\ldots,M_5} = \epsilon_{M_1,\ldots,M_{10}} F^{M_6,\ldots,M_{10}}\,. \tag{20.2}$$

This theory has no Lorentz invariant action but only field equations due to the self-duality constraint. One can give the action without imposing the constraint and include it on the field equation by hand. In this case the action reads

$$\begin{aligned} S = {}& \tfrac12 \int \mathrm{d}^{10}x\,\sqrt{-g}\left(R + \tfrac12 \frac{\partial_M\tau\partial^M\tau}{(\mathrm{Im}\,\tau)^2} - \tfrac12 M_{ij}F^i_{MNP}F^{j\,MNP} - \tfrac12|\hat F_5|^2\right) \\ & - \tfrac14\epsilon_{ij}\int \mathrm{d}^{10}x\, A_4\wedge F^i_3\wedge F^j_3\,, \end{aligned} \tag{20.3}$$

where

$$\begin{aligned} &\tau = l + ie^{-\phi}\,, \qquad M_{ij} = \frac{1}{\mathrm{Im}\,\tau}\begin{pmatrix} |\tau|^2 & -\mathrm{Re}\,\tau \\ -\mathrm{Re}\,\tau & 1\end{pmatrix}\,, \\ &F^i_3 = \begin{pmatrix}\mathrm{d}B^1_2 \\ \mathrm{d}B^2_2\end{pmatrix}\,, \qquad \hat F_5 = \mathrm{d}A_4 + \tfrac12\epsilon_{ij}B^i_2\mathrm{d}B^j_2\,. \end{aligned} \tag{20.4}$$

The type IIB theory has two supersymmetries of the same chirality. There are two p-form gauge symmetries

$$\begin{aligned} (i)\quad & \delta A_4 = \mathrm{d}\Lambda_3\,, \\ (ii)\quad & \delta B^i_2 = \mathrm{d}\Lambda^i_1\,, \qquad \delta A_4 = -\tfrac12\epsilon_{ij}\Lambda^i_1\mathrm{d}B^j_2\,, \end{aligned} \tag{20.5}$$

where $\Lambda_3\,(\Lambda_1)$ is a parameter three-form (one-form). Finally there also is an $SL(L,R)$ symmetry acting as

$$\begin{aligned} &\tau \to \tau' = \frac{a\tau+b}{c\tau+d}\,, \qquad a,b,c,d\in\mathbb{R}\,, \quad ad-bc=1\,, \\ &M \to M' = (\Lambda^{-1})^T M\Lambda^{-1}\,, \qquad \Lambda = \begin{pmatrix} d & c \\ b & a\end{pmatrix}\,, \\ &F^i_3 \to F'^{\,i}_3 = \Lambda^i{}_jF^j_3\,. \end{aligned} \tag{20.6}$$

20.2 $(2,0)$ supergravity in $D = 6$

This is a supergravity in $D = 6$ with two supercharges of the same chirality. The gravitational multiplet contains

$$\underset{(9)}{g_{MN}},\ \underset{(5\times 3=15)}{B^{+\,a=1,\ldots 5}_{MN}}\ ,\ \ \underset{(2\times 12=24)}{\Psi^{i=1,2}_{M\alpha}}\ . \tag{20.7}$$

The only other massless multiplet is a tensor multiplet with field content

$$\underset{(3)}{B^{-}_{MN}},\ \underset{(5)}{\phi^{a=1,\ldots 5}}\ ,\ \ \underset{(2\times 4=8)}{\chi^{i=1,2}}\ . \tag{20.8}$$

Recall that in $D = 6$ a two-form is Poincare dual to a two-form $B_2 \simeq \tilde{B}_2$ and thus one can impose a self- or anti-self duality condition on B_2. Five self-dual B_2^+ are part of the gravitional multiplet while the tensor multiplets contain the anti-selfdual B_2^-. As a consequencet there is again no Lorentz-invariant action and the theory is defined via its field equations. The scalar field space turns out to be

$$\mathcal{M} = \frac{SO(5, n_T)}{SO(5) \times SO(n_T)}\ , \tag{20.9}$$

where n_T counts the number of tensor multiplets. The theory is anomaly free only for $n_T = 21$. Note that the theory contains no gauge fields at all.

20.3 $(1,0)$ supergravity in $D = 6$

This is also a chiral supergravity in $D = 6$ as it has one chiral supercharge. The massless multiplets are

$$\begin{aligned}
\text{gravitational multiplet} :&\quad \underset{(9)}{G_{MN}},\ \underset{(3)}{B^{+}_{MN}}\ ,\ \underset{(12)}{\Psi_{M\alpha}}\ ,\\
\text{tensor multiplet} :&\quad \underset{(3)}{B^{-}_{MN}},\ \underset{(1)}{\phi}\ ,\ \underset{(4)}{\chi}\ ,\\
\text{vector multiplet} :&\quad \underset{(4)}{A^{a}_{M}},\ \underset{(4)}{\lambda^{a}}\ ,\\
\text{hypermultiplet} :&\quad \underset{(4)}{q^{u=1,\ldots,4}},\ \underset{(4)}{\chi}\ ,
\end{aligned} \tag{20.10}$$

where the vector multiplet is in the adjoint representation of some gauge group G while the hypermultiplets can be in any representation.

The scalar field space turns out to be

$$\mathcal{M} = \frac{SO(1, n_T)}{SO(n_T)} \times \begin{cases} \mathcal{M}_{\text{HK}} & \text{for global supersymmetry} \\ \mathcal{M}_{\text{QK}} & \text{for local supersymmetry} \end{cases}\ , \tag{20.11}$$

where the second component is spanned by the scalars in the hypermultiplets.

Remarks:

1. A Lorentz-invariant action only exits for $n_T = 1$.

2. The hypermultiplets are exactly as in $D = 4, N = 2$ (and in $D = 5, N = 2$).

3. As in $D = 4, N = 1$ the vector multiplets have no scalars.

4. The theory is anomaly free only for $n_h - n_v + 29n_T = 273$.

20.4 $q = 8$ ($N = 2$) supergravity in $D = 5$

This theory can be constructed from the $(1,0)$ theory in $D = 6$ by KK-reduction. In $D = 4$ a two-form is Poincare dual to a vector $B_2 \simeq A_1$ and thus one can dualize an entire tensor multiplet to a vector multiplet.[24]

The massless multiplets are

$$\begin{aligned} \text{gravitational multiplet}: &\quad \underset{(5)}{g_{\mu\nu}},\ \underset{(3)}{A^0_\mu},\ \underset{(8)}{\Psi_\mu}\ , \\ \text{tensor multiplet}: &\quad \underset{(3)}{B^-_{\mu\nu}},\ \underset{(1)}{\phi}\ ,\ \underset{(4)}{\chi}\ , \\ \text{vector multiplet}: &\quad \underset{(3)}{A^a_\mu},\ \underset{(1)}{\phi^a},\ \underset{(4)}{\lambda^a}\ , \\ \text{hypermultiplet}: &\quad \underset{(4)}{q^{u=1,\ldots,4}},\ \underset{(4)}{\chi}\ , \end{aligned} \tag{20.12}$$

where the vector multiplet is again in the adjoint representation of some gauge group G while tensor- and hypermultiplets can be in any representation.

The scalar field space turns out to be

$$\mathcal{M} = \mathcal{M}_{\mathrm{SR}} \times \begin{cases} \mathcal{M}_{\mathrm{HK}} & \text{for global supersymmetry} \\ \mathcal{M}_{\mathrm{QK}} & \text{for local supersymmetry} \end{cases}\ , \tag{20.13}$$

where $\mathcal{M}_{\mathrm{SR}}$ denotes a "very special" real manifold spanned by the scalars in tensor- and vector multiplets while the second component is again spanned by the scalars in the hypermultiplets.

Let us briefly describe the geometry of the very special real manifold $\mathcal{M}_{\mathrm{SR}}$. Its coordinates are the real scalar fields ϕ^a, $a = 1, \ldots n_v + n_T$ from both tensor- and vector multiplets. They are the soltion of a cubic constraint

$$c_{ABC} h^A(\phi) h^B(\phi) h^C(\phi) = 1\ , \qquad A, B, C = 0, \ldots, n_v + n_T\ , \tag{20.14}$$

where the constant, symmetric tensor c_{ABC} specifies the theory. The metric is obtained by

$$G_{ab} = -3(\partial_a h^A)(\partial_b h^B) c_{ABC} h^C\ . \tag{20.15}$$

More details can be found in [3].

[24] There is a caveat in that the vector multiplet has to be in the adjoint representation of G while the tensor multiplet can be in any representation $\mathbf{r}$ of G.

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