Problem 1.1

a) Show that for infinitesimal rotations with angle θ_i and infinitesimal Lorentz boosts with velocity v_i one has

$$\delta x^i = -\tfrac{i}{2}\theta_k \epsilon^{kmn} L^{mn} x^i \ , \qquad \delta x^\mu = -i v_k L^{0k} x^\mu \ , \quad \text{with} \quad L^{\mu\nu} = -i\big(x^\mu \partial^\nu - x^\nu \partial^\mu\big) \ .$$

b) Show

$$[L^{\mu\nu}, L^{\rho\sigma}] = -i\left(\eta^{\nu\rho}L^{\mu\sigma} - \eta^{\nu\sigma}L^{\mu\rho} - \eta^{\mu\rho}L^{\nu\sigma} + \eta^{\mu\sigma}L^{\nu\rho}\right)$$

c) Define

$$L^i := \tfrac{1}{2}\,\epsilon^{ijk}L^{jk} \ , \qquad K^i := L^{0i} \ ,$$

and show

$$[K^i, K^j] = -i\epsilon^{ijk}L^k \ , \qquad [K^i, L^j] = i\epsilon^{ijk}K^k \ , \qquad [L^i, L^j] = i\epsilon^{ijk}L^k \ .$$

d) Define $L^i_\pm := \frac{1}{2}\big(L^i \pm iK^i\big)$ and show

$$[L^i_\pm, L^j_\pm] = i\epsilon^{ijk}L^k_\pm \ , \qquad [L^i_+, L^j_-] = 0 \ .$$

What does this show?

Problem 1.2

a) Show the following properties of σ^μ

$$\sigma^2(\bar\sigma^\mu)^*\sigma^2 = \sigma^\mu \ , \qquad \bar\sigma^2(\sigma^\mu)^*\bar\sigma^2 = \bar\sigma^\mu \ , \qquad \bar\sigma^2(\sigma^{\mu\nu})^*\sigma^2 = -\bar\sigma^{\mu\nu} \ ,$$

where

$$\sigma^\mu = (-\mathbf{1}, \sigma^i) \ , \quad \bar\sigma^\mu = (-\mathbf{1}, -\sigma^i) \ , \quad \sigma^{\mu\nu} = \tfrac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu) \ , \quad \bar\sigma^{\mu\nu} = \tfrac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu) \ .$$

Problem 1.2

b) Using a) compute the Lorentz transformation of $-\bar{\sigma}^2\chi^*$ where χ_α is a Weyl spinor transforming as $\delta\chi = \frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\chi$. What does the result show?

c) The Lagrangian for a massive Dirac field Ψ_D reads

$$\mathcal{L} = \bar{\Psi}_D(i\gamma^\mu\partial_\mu + m)\Psi_D \ ,$$

where $\bar{\Psi}_D = \Psi_D^\dagger\gamma^0$. Decompose $\mathcal{L}$ in terms of Weyl spinors with $\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$.

Problem 1.3

In the lectures we defined

$$\delta_\xi := \xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} \ ,$$

where ξ is an anticommuting Grassmann variable.

a) Show

$$[\delta_\eta, \delta_\xi] = -2i(\eta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\eta})\partial_\mu \ . \tag{1}$$

b) Show that (1) is fulfilled for $[\delta_\eta, \delta_\xi]A(x)$ and

$$\delta_\xi A = \sqrt{2}\xi\chi \ , \quad \delta_\xi\chi = \sqrt{2}\xi F + i\sqrt{2}\sigma^\mu\bar{\xi}\partial_\mu A \ , \quad \delta_\xi F = i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\chi \ .$$