The Homogenous Lorentz Group

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February 3, 2016

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1 Proper Lorentz Transforms

Before we get started let us revise the Lorentz transformation between two equally oriented inertial systems moving with velocity v along the x^1-axis. Here $x' = (x'^0, x'^1, x'^2, x'^3)$, $x = (x^0, x^1, x^2, x^3)$ are the space-time coordinates of our inertial systems. With $x^0 := ct$ can write such transformation in coordinates:

$$x'^0 = \gamma x^0 - \beta\gamma x^1, \tag{1}$$
$$x'^1 = -\beta\gamma x^0 + \gamma x^1, \tag{2}$$
$$x'^2 = x^2,$$
$$x'^3 = x^3,$$

where $\beta := v/c$ and $\gamma := 1/\sqrt{1-\beta^2}$.

Easily, we see that this translates perfectly into matrix operations:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{3}$$

We go a step further and define the 'angle'

$$\Psi := \cosh^{-1}\gamma. \tag{4}$$

Then we have $\gamma = \cosh\Psi$ and $\beta = \tanh\Psi$, which turns our transformation matrix into

$$\begin{pmatrix} \cosh\Psi & -\sinh\Psi & 0 & 0 \\ -\sinh\Psi & \cosh\Psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

This reminds us very much of a rotation in $\mathbb{R}^3$ through an angle θ about the x^3-axis:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

There are a few differences however. First of all, there is a difference in signs. Moreover, the rotation matrix contains *sin*, *cos*, while the matrix of our Lorentz transformation contains *sinh* and *cosh*. Yet, the different signs cause their determinants to be the same. For the rotation matrix it is 1; for our Lorentz transformation matrix it is also $(cosh\Psi)^2 - (sinh\Psi)^2 = 1$.

2 Four Vectors

In special relativity we work in the 4-dimensional *Minkwoski space* denoted as $\mathbb{M}$, which is an $\mathbb{R}$-vector space with a pseudo scalar product $\langle\cdot,\cdot\rangle_{\mathbb{M}}$. The latter is different from the Euclidean scalar product. For convenience we will simply write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{\mathbb{M}}$. Its elements are called *four vectors* $x^\mu = (x^0, x^1, x^2, x^3)$. In physics we choose $x^0 = ct$ with the speed of light c and the spatial coordinates $\vec{x} = (x^1, x^2, x^3)$. While we might be tempted to identify Minkowski space with the 4-dimensional $\mathbb{R}^4$, its structure is inherently different from that of Euklidean space.

Convention: In this presentation we are using the *Einstein sum convention.* Over the same indices appearing multiple times, each as a sub- *and* a superscript, a summation is carried out, e.g. $a^i_{j,k} b_j := \sum_i a^i_{j,k} b_i$.

We can define a *pseudo scalar product* in $\mathbb{M}$:

$$\langle x, y\rangle = x^0 y^0 - (x^1 y^1 + x^2 y^2 + x^3 y^3). \tag{7}$$

Moreover, we distinct between *covariant* and *controvariant* vectors. If $x^\mu = (x^0, x^1, x^2, x^3)$ is a covariant vector, then $x_\mu = (x^0, -x^1, -x^2, -x^3)$ is its corresponding controvariant vector. The only difference is the different sign in the spatial coordinates. The time coordinate however is left unchanged. Now we can introduce the *pseudo metric tensor* $g^{\mu\nu}$, $g_{\mu\nu}$ (from now on simply 'metric tensor'). We define its covariant form as

$$g^{\mu\nu} = g_{\mu\nu} := \begin{cases} 1 & ,\mu = \nu = 0 \\ -1 & ,1 \leq \mu = \nu \leq 3 \\ 0 & ,\mu \neq \nu \end{cases} \tag{8}$$

If not otherwise noted μ, ν run from 0 to 3. Furthermore, the metric tensor has following important properties:

- $g^{\mu\nu} x_\mu = x^\nu$,
- $g_{\mu\nu} x^\mu = x_\nu$.

So basically, the metric tensor switches indices and turns a covariant into a controvariant, or a controvariant into a covariant vector. We will be using these properties later on.

In fact, we can identify vectors and even scalars with tensors. We call a tensor with n distinct indices a *tensor of n-th order*. Thus, a scalar is a 0^{th}-order tensor, a vector a 1^{st}-order tensor. The highest order we will deal with in this presentation are 2^{nd}-order tensors. There are 3 types of them, classified by the position of their indices:

- covariant tensors $\Lambda^{\mu\nu}$,
- controvariant tensors $\Lambda_{\mu\nu}$,
- and mixed tensors $\Lambda^\mu{}_\nu, \Lambda_\nu{}^\mu$.

With the aid of this definition we can redefine the scalar product in a more useful manner:

$$\langle x, y \rangle := x^\mu g_{\mu\nu} y^\nu = x^\mu y_\mu. \tag{9}$$

If we identify the diagonal matrix $g := diag(1, -1, -1, -1)$ with the metric tensor, the corresponding vector notation becomes $\langle x, y \rangle = x^T gy$. Now it becomes clear why this is not a real scalar product: The matrix g corresponding to this bilinear form is indefinite.

3 Basic Properties of the Transformations

The general homogenous Lorentz transformations are mappings of $\mathbb{M}$ onto itself. They are linear mappings preserving the Minkowski norm $\|x\| := \sqrt{\langle x, x \rangle}$ of four vectors. Like all linear mappings they can be represented by a quadratic matrix Λ. Thus, the transformation in coordinate form is defined by

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{10}$$

The homogenous Lorentz transformations conserve the pseudo norm induced by the scalar product. Let $x' := \Lambda x$, then

$$
\begin{aligned}
x'^2 = x'^T g x = (\Lambda x)^T g (\Lambda x) && (11)\\
= x^T \Lambda^T g \Lambda x && (12)\\
\stackrel{!}{=} x^T g x && (13)\\
= x^2. && (14)
\end{aligned}
$$

Hence, we can conclude that $\Lambda^T g \Lambda = g$, which gives us

$$
\Lambda^T g \Lambda = g \Rightarrow \det(\Lambda^T \Lambda) = (\det \Lambda)^2 = 1. \tag{15}
$$

So we find $\det \Lambda = \pm 1$. This means that $\Lambda \in Gl_4(\mathbb{R})$ is an invertible 4x4-matrix.

Now, let $\mathcal{L} := \{\Lambda \in Gl_4(\mathbb{R} \,|\, \Lambda^T g \Lambda = g\}$ be the set of all homogenous Lorentz transformations. Then $\mathcal{L}$ is a group.

Proof
Let Λ_1, Λ_2 be in $\mathcal{L}$. We define $\Lambda_3 := \Lambda_1 \Lambda_2$. Then Λ_3 is another homogenous Lorentz transformation since

$$
\Lambda_3^T g \Lambda_3 = \Lambda_2^T \Lambda_1^T g \Lambda_1 \Lambda_2 = \Lambda_2^T g \Lambda_2 = g. \tag{16}
$$

Moreover, the product of any $\Lambda_i \in \mathcal{L}$ is associative since matrix multiplication is associative. The identity element is the unit matrix $\mathbb{E}_4$. Since $\det \Lambda \neq 0$ there is an inverse element $\Lambda^{-1} \in \mathcal{L}$ for each $\Lambda \in \mathcal{L}$. $\square$

Let us go back to the scalar product in $\mathbb{M}$. First we contemplate the component with $\sigma = \rho = 0$ in ${\Lambda^\mu}_\sigma g_{\mu\nu} {\Lambda^\nu}_\rho = g_{\rho\sigma}$.

We find, that

$$
\begin{aligned}
g_{\mu\nu} {\Lambda^\mu}_0 {\Lambda^\nu}_0 = g_{00} \left({\Lambda^0}_0\right)^2 + \sum_{\mu,\nu=1}^{3} g_{\mu\nu} {\Lambda^\mu}_0 {\Lambda^\nu}_0 && (17)\\
= g_{00} \left({\Lambda^0}_0\right)^2 - \sum_{\nu=1}^{3} \left({\Lambda^\nu}_0\right)^2. && (18)
\end{aligned}
$$

This yields $1 \leq \left({\Lambda^0}_0\right)^2 \Leftrightarrow {\Lambda^0}_0 \geq 1 \vee {\Lambda^0}_0 \leq -1$. Combined with the fact that $\det \Lambda = \pm 1$, this implies that $\mathcal{L}$ is not a connected group. It is made up of four disjoint sets which we call *components*. Moreover, there are three discrete transformations in $\mathcal{L}$:

- the identity Id,

- space inversion I_S,
- time inversion I_T,
- and space-time inversion $I_{ST} = I_S \circ I_T = I_T \circ I_S = -Id$.

symbol	Λ^0_0	$\det \Lambda$	**discrete transformation**
$\mathcal{L}^\uparrow_+$	$\geq +1$	$+1$	Id
$\mathcal{L}^\uparrow_-$	$\geq +1$	-1	I_S
$\mathcal{L}^\downarrow_+$	≤ -1	$+1$	I_T
$\mathcal{L}^\downarrow_-$	≤ -1	-1	$-Id$

Table 1: The different components of $\mathcal{L}$ and their properties

The space-time inversion I_{ST} is really just a composition of the first two inversions. The set $\{Id, I_S, I_T I_{ST}\} \subset \mathcal{L}$ is an Abelian subgroup of the Lorentz group. Table 1 gives us an oversight over the four components of $\mathcal{L}$.

Of all the components only the *proper orthochronous Lorentz group* $\mathcal{L}^\uparrow_+$ is a subgroup of $\mathcal{L}$. The other components can be identified with left cosets of the corresponding discrete transformation and $\mathcal{L}^\uparrow_+$. Moreover, the union of $\mathcal{L}^\uparrow_+$ with any of the other components forms a subgroup of $\mathcal{L}$. From now on, we will restrict our discussion to the proper orthochronous Lorentz group $\mathcal{L}^\uparrow_+$. Because its parameter space is not bound.

4 Connection to $SL(2, \mathbb{C})$

We want to show the relation of $\mathcal{L}^\uparrow_+$ to $SL(2, \mathbb{C})$. In doing so, we use the Pauli matrices:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With the 2x2-identity matrix $E_2 =: \sigma_0$ we define a controvariant 4-tupel

$$\sigma_\mu := (\sigma_0, \sigma_1, \sigma_2, \sigma_3,) = (\sigma_0, \vec{\sigma}).$$

It is important to keep in mind that this is now not a 4-tupel of numbers, but a 4-tupel of *matrices*. We define another such *contro*variant 4-tupel

$$\underline{\sigma}_\mu := \sigma^\mu = (\sigma_0, -\vec{\sigma}). \tag{19}$$

With these definitions, we associate to each x^μ a hermitian 2x2-matrix

$$X = \sigma_\mu x^\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^3 & x^0 - x^3 \end{pmatrix}. \tag{20}$$

This means that $X \in Mat_2(\mathbb{C})$ is self-adjoined, i.e. $\overline{X} = X^T$. First we show that

$$Tr(\underline{\sigma}_\mu \sigma_\nu) = 2g_{\mu\nu}. \tag{21}$$

Proof
We need following property of the Pauli matrices:

$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu}\sigma_0 + i \cdot \epsilon_{\mu\nu\rho}\sigma_\rho. \tag{22}$$

With the definitions of the controvariant 4-tupels and the properties of the Pauli matrices we now find, that

$$\begin{aligned}
\mu = \nu = 0 \qquad & \Rightarrow Tr(\sigma_0\sigma_0) = 2, & (23)\\
\mu = 0, 1 \leq \nu \leq 3 \qquad & \Rightarrow Tr(\underline{\sigma}_0\sigma_\nu) = Tr(\sigma_\nu) = 0, & (24)\\
1 \leq \mu \leq 3, \nu = 0 \qquad & \Rightarrow Tr(\underline{\sigma}_0\sigma_\nu) = -Tr(\sigma_\mu) = 0, & (25)\\
1 \leq \mu = \nu \leq 3 \qquad & \Rightarrow Tr(\underline{\sigma}_0\sigma_\nu) = -Tr(\delta_{\mu\mu}\sigma_0) = -2, & (26)\\
1 \leq \mu, \nu \leq 3, \mu \neq \nu \qquad & \Rightarrow Tr(\underline{\sigma}_0\sigma_\nu) = -i \cdot \epsilon_{\mu\nu\rho} \cdot Tr(\sigma_\rho) = 0. & (27)
\end{aligned}$$

Hence, this yields $Tr(\underline{\sigma}_\mu \sigma_\nu) = 2g_{\mu\nu}$. Using this, we find the equation

$$\begin{aligned}
Tr(\underline{\sigma}^\mu X) &= Tr(\underline{\sigma}^\mu \sigma_\nu x^\nu)\\
&= Tr(g^{\mu\rho}\underline{\sigma}_\rho \sigma_\nu x^\nu)\\
&= g^{\mu\rho} x^\nu \cdot Tr(\underline{\sigma}_\rho \sigma_\nu\\
&= g^{\mu\rho} 2 \cdot g_{\rho\nu} x^\nu\\
&= 2 g^{\mu\rho} x_\rho\\
&= 2x^\mu.
\end{aligned}$$

Moreover, the determinant of X is

$$\begin{aligned}
\det X &= \begin{vmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^3 & x^0 - x^3 \end{vmatrix}\\
&= (x^0)^2 - (x^3)^2 - \big((x^1 - ix^2)(x^1 + ix2)\big)\\
&= (x^0)^2 - \vec{x}^2 = x^2.
\end{aligned} \tag{28}$$

Given a unimodular complex matrix $T \in \{A \in Gl_2(\mathbb{C}) | \det A = 1\}$ we transform the matrix X:

$$X' = TXT^\dagger. \tag{29}$$

Clearly, since the determinants of $T, T^\dagger$ equal 1, the Minkowski norm is conserved under such transformation. Hence, X corresponds to a Lorentz transformation Λ.

5 Generators of $\mathcal{L}_+^\uparrow$

Since pure rotations do not affect the x^0-coordinate, one can extend the generators of $SO(3)$ by a null row and column for it:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

. At last, we have to find the corresponding generator for the *Lorentz boost*

$$B_1 = \begin{pmatrix} \cosh\Psi & -\sinh\Psi & 0 & 0 \\ -\sinh\Psi & \cosh\Psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{31}$$

It is obtained by taking the derivative of B_1 at $\Psi = 0$:

$$i\frac{dB_1(\Psi)}{d\Psi}\bigg|_{\Psi=0} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: Y_1. \tag{32}$$

The Y_2, Y_3 for Lorentz boosts along the x^2-,x^3-axes are found in an analogous fashion. The commutators of these generators are

- $[X_i, X_j] = i\epsilon_{ijk}X_k$,
- $[X_i, Y_j] = i\epsilon_{ijk}Y_k$,
- $[Y_i, Y_j] = -i\epsilon_{ijk}Y_k$.

The first equation shows clearly that the spatial rotations$\{X_1, X_2, X_3\}$ form the subalgebra $SO(3)$. The second equation shows us that the tripel (Y_1, Y_2, Y_3) transforms just like a vector under spatial rotations. The last equation shows that the Lorentz boosts $\{Y_1, Y_2, Y_3\}$ do *not* form a subalgebra, i.e. they do not close in on themselves.

The commutation relations are further simplified by defining the linear combinations

$$\vec{X}^{\pm} := \frac{1}{2}\left(\vec{X} \pm i\vec{Y}\right), \tag{33}$$

where $\vec{X} := (X_1, X_2, X_3)$, and $\vec{Y} := (Y_1, Y_2, Y_3)$. Using this, we acquire the commutations

- $[X_i^+, X_j^+] = i\epsilon_{jik}X_k^+$,

- $[X_i^-, X_j^-] = i\epsilon_{jik} X_k^-$,
- $[X_i^+, X_j^-] = 0$.

We see that the first two equations close in on themselves. They form two independent $SU(2)$ algebras. Further, we note that there is a big similarity to the $SO(4)$ in this respect. Howoever, these equations here are complex equations. The commutation relations for $SO(4)$ on the other hand are real equations. Both are *locally isomorphic* to one another.

The irreducible representations j_1, j_2 of the first and second $SO(2)$ respectively can now be enumerated. Each has $2j_{1,2} + 1$ degrees of freedom. $\mathcal{L}_+^\uparrow$ is not compact. Hence, its irreducible representations cannot be unitary. We want to discuss some examples.

- The trivial representation is (0,0).
- The *Weyl* representations $(0, \frac{1}{2}), (\frac{1}{2}, 0)$ are the lowest dimensional irreducible representations with two degrees of freedom.
- The *Dirac* representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ is used in quantum electrodynamics to describe the electron. It acts on 4-component spinors.
- The representation $(\frac{1}{2}, \frac{1}{2})$ corresponds to the defining representation of $\mathcal{L}_+^\uparrow$. Of course, it has four degrees of freedom.
- The representation $(1, 0) \oplus (0, 1)$ with 6 degrees of freedom is carried by antisymmetric tensors $F_{\mu\nu}$. In electrodynamics the six independent components can be identified with the electrical and magnetic field components.

6 Summary

Let us resume the core aspects of this presentation.

- The Minkowski space $\mathbb{M}$ is a 4-dimensional $\mathbb{R}$-vector space with the pseudo scalar product $\langle x, y \rangle := x^\mu g_{\mu\nu} y^\nu$.
- The Lorentz transformations are all endomorphisms of $\mathbb{M}$ onto itself that leave the Minkowski norm $\|x\| := \sqrt{\langle x, x \rangle}$ invariant, their absolute determinant is 1.
- The Lorentz group $\mathcal{L}$ is the set of all such transformations. It is not connected and consists of 4 disjoint sets called components.
- The proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ is the only component that is also a subgroup of $\mathcal{L}$. Its elements Λ have the properties $\det \Lambda = +1$, ${\Lambda^0}_0 \leq +1$ and its discrete transformation is $Id = g^{\mu\nu}$.

- In $SL(2,\mathbb{C})$ we can define a hermitian 2x2-matrix $X := \sigma_\mu x^\mu$, which is invariant under transformation $X' = TXT^\dagger$ with a unimodular matrix T. It corresponds to a Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$.
- The generators of $\mathcal{L}_+^\uparrow$ are the generators $X_{1,2,3}$ of spatial rotations with an additional null row and column for the x^0-coordinate. The generators of the Lorentz boosts Y_j along the x^j-axis (j=1,2,3) are obtained by taking the derivative $Y_j := i\, dB_j(\Psi)/d\Psi|_{\Psi=0}$ at $\Psi = 0$.
- The Lie algebra of $\mathcal{L}_+^\uparrow$ is

  $$[X_i^+, X_j^+] = i\epsilon_{ijk}X_k^+, \quad [X_i^-, X_j^-] = i\epsilon_{ijk}X_k^-, \quad [X_i^+, X_j^-] = 0,$$

  where $\vec{X}^\pm := \frac{1}{2}\left(\vec{X} \pm i\vec{Y}\right)$.

Literatur

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