Lie Groups and Algebras

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1 Intro

Until now a couple of Lie groups, such as $SU(2)$, $SU(3)$ and $SO(4)$, have been covered. Their representations have been used to extract their algebra, then their irreducible representations have been found by the use of different methods - lowering and raising operators, use of characters, explicit tensor representations and Young tableaux.

For the general treatment of Lie algebras an extension of the already treated methods by the names of Cartan and Weyl will be used. The generators will be split, into a set of $\{H\}$ which is commutating among its elements and therefore can be assigned eigenvalues and a rest $\{E\}$ containing raising and lowering operators. Now the Lie groups may be classified by split and commutation relations which will total up to the determination of simple roots with special lengths and scalar products. The properties can be presented with either the Cartan matrix or the Dynkin diagram in the end.

In the end there are four regular infinite series of Lie algebras and five irregular algebras that are not part of either series. All of them can be treated with the method of lowering and raising operators. These methods can as well be applied to more complicated Lie algebras.

2 The Adjoint Representation and the Killing Form

Definition 2.1. A Lie algebra of dimension d is specified by a set of d generators T_i closed under commutation:

$$[T_\alpha, T_\beta] = i f^{\gamma}_{\alpha\beta} T_\gamma \tag{1}$$

The Lie product $[A, B]$ does not have to be a commutator of operators or matrices but a skew-symmetric mapping from $L \times L$ onto L with the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{2}$$

Above is trivially satisfied by $[A, B]$ being a commutator.
The Poisson bracket of Hamiltonian classical mechanics is an example of a Lie product which is not defined as a commutator:

$$[A, B]_{PB} := \sum_{\sigma} \left(\frac{\partial A}{\partial q_\alpha}\frac{\partial B}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha}\frac{\delta B}{\partial q_\alpha}\right) \tag{3}$$

The algebra being dealt with here is a vector space and the Lie product has to be consistent with its addition:

$$[\lambda_1 A + \lambda_2 B, C] = \lambda_1 [A, C] + \lambda_2 [B, C] \tag{4}$$

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Definition 2.2. The Lie algebras commutation with a fixed generator maps the generators onto themselves, creating a new representation of dimension d. This is called the adjoint representation $\mathcal{A}$.

By setting a fixed T_α for formula (1) its second element is mapped onto a linear combination of the generators, giving a representation whose structure constants are the matrix representatives of the generators:

$$(D_{\mathcal{A}}(T_\alpha))^\gamma_\beta = if^\gamma_{\alpha\beta} \tag{5}$$

Now f is obviously anti-symmetric in alpha and beta but lacks any symmetries for gamma.

This is where the Killing form comes into play. It can be used to lower the unwelcome index and create a completely anti-symmetric f.

Definition 2.3. The Killing form, $Tr(AB)$, is the trace of the product of the matrices representing A and B in the adjoint representation:

$$(A, B) := Tr(D_{\mathcal{A}}(A)D_{\mathcal{A}}(B)) \tag{6}$$

Definition 2.4. By applying the Killing form to the generators one gets the cartan metric:

$$g_{\alpha\beta} := Tr_{\mathcal{A}}(T_\alpha T_\beta) = -f^\delta_{\alpha\gamma} f^\gamma_{\beta\delta} \tag{7}$$

Using $g_{\delta\gamma}$ to lower the last index on $f^\delta_{\alpha\beta}$ one receives an $f_{\alpha\beta\gamma} = f^\delta_{\alpha\beta} g_{\delta\gamma}$ that is completely antisymmetric in α , β and γ.

With $Tr_{\mathcal{A}}([T_\alpha, T_\beta], T_\gamma])$ one finds that $f_{\alpha\beta\gamma} = f_{\gamma\alpha\beta} = -f_{\alpha\gamma\beta}$ etc. is true:

$$Tr_{\mathcal{A}}([T_\alpha, T_\beta], T_\gamma]) = if^\delta_{\alpha\beta} Tr_{\mathcal{A}}(T_\delta T_\gamma) = if_{\alpha\beta\gamma} \tag{8}$$

And by taking the cyclic property of the trace into account:

$$Tr([A, B]C) = Tr([B.C]A) = Tr([C, A]B) \tag{9}$$

There are Lie algebras that contain subalgebras with elements that may generate a subgroup of the group generated by L. To get a subgroup there either has to be an invariant subalgebra or an ideal.

Definition 2.5. An ideal is a subspace I such that all commutators involving I lie within it.

$$[I, L] \subset I \tag{10}$$

Lie algebras can be expressed as a direct sum of simple Lie algebras. A simple Lie algebra is one that does not contain any proper ideals. (A semi-simple one doesn't contain any Abelian ideals.)

The Cartan metric can tell whether a Lie algebra is semi-simple, which is true if $det(g) \neq 0$

or equivalently $(A, X) = 0 \forall X \in L \rightarrow A = 0$.

If L contains an ideal, to get to a direct sum, the following is done and repeated until only a direct sum of simple Lie algebras is left:

First take P, defined as the orthogonal complement to I with respect to the Killing form:

$$(I, P) = 0 \tag{11}$$

and then, since this is a subalgebra, I is an ideal and the trace owns cyclic properties:

$$([P, P]I) = ([P, I]P) = (IP) = 0 \tag{12}$$

$([P, I]I) = ([I, I]P] = [IP] = 0$ proves to be orthogonal to every element within the given algebra and the Killing form is non-degenerate and therefore $[P, I]$ is zero.

Obviously L can be written as $L = I \oplus P$. Now the process will be repeated with P if it is not already simple.

3 The Cartan Basis of a Lie Algebra

A Lie algebra is determined by its structure constants but this doesn't work the other way, since one may change the basis vectors to a set created from linear combinations of old ones which wwould result in a change of the commutation relations.

One may take for example the $SU(2)$ algebra:

$$[J_i, J_j] = i\eta_{ijk} J_k \tag{13}$$

By use of J_3 and $J_\pm = J_1 \pm iJ_2$ one receives:

$$[J_3, J_\pm] = \pm J_\pm \ and \ [J_+, J_-] = 2J_3 \tag{14}$$

Definition 3.1. Now the generalized method behind this is the Cartan presentation of commutation relations. One looks for a maximal set of commutating generators. This set $\{H_i\}, i = 1...r$ will be the basis for the Cartan subalgebra.

Definition 3.2. The number of generators is called rank r of an algebra.

Definition 3.3. The weight of the H_i are their eigenvalues, which are used for labeling states of representations.

One part of the commutation relations now may look like this:

$$[H_i, H_j] = 0 \tag{15}$$

For the remaining $d - r$ generators one wants to take linear combinations $\{E_\alpha\}$ so that:

$$[H_i, E_\alpha] \propto E_\alpha \tag{16}$$

For that, eigenvalues need to be found:

$$det(C_{\alpha\beta} - \lambda g_{\alpha\beta}) = 0 \tag{17}$$

Here $C_{\alpha\beta}$ equals $if_{l\alpha\beta}$ for particular fixed i. Assuming that the algebra was cast in such a way that all fs are real, C is purely imaginary and antisymmetric. Thus it is Hermitian with real eigenvalues, with r being zero and $d - r$ non-zero.
Cartans theorem says that those non-zero eigenvalues are non-degenerate:

$$[H_i, E_\alpha] = (\alpha)_i E_\alpha \tag{18}$$

In the following we'll have a look at all possible H_i. For a start with use of the Jacobi identity there is:

$$[H_i, [H_j, E_\alpha]] = [H_j, [H_i, E_\alpha]] - [E_\alpha, [H_i, H_j]] \tag{19}$$

From the above one may conclude that:

$$[H_i, [H_j, E_\alpha]] = (\alpha)_i [H_j, E_\alpha] \tag{20}$$

Thus $[H_j, E_\alpha]$ is eigenvector of the adjoint action of H_i and the respective eigenvalue is $(\alpha)_i$. A proportionality with E_α is obvious due to the eigenvectors being degenerate:

$$[H_j, E_\alpha] = (\alpha)_j E_\alpha \tag{21}$$

Definition 3.4. The r-dimensional vectors this results in are called roots of the Lie algebra while the eigenvectors are called root vectors (or step operators).

4 Properties of the Roots and Root Vectors

The commutation relations between the E_α will now be examined.
With use of the Jacobi identity and (21):

$$[H_i, [E_\alpha, E_\beta]] = -[E_\alpha, [E_\beta, H_i]] - [E_\beta, [H_i, E_\alpha]] = (\alpha + \beta)_i [E_\alpha, E_\beta] \tag{22}$$

So $[E_\alpha, E_\beta]$ is a root vector too, with root $\alpha + \beta$ unless $\alpha + \beta = 0$ or $[E_\alpha, E_\beta] = 0$. If $\alpha + \beta = 0$ one can see that $[E_\alpha, E_\beta] = 0$ commutes with each H_i and therefore belongs to the Cartan subalgebra:

$$[E_\alpha, E_{-\alpha}] = \lambda_i H_i and [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} for (\beta \neq -\alpha \tag{23}$$

with $N_{\alpha\beta} = 0$ if $\alpha + \beta$ is not a root.

We assumed that if α is a root, $-\alpha$ is a root as well which can easily be shown.

Because of the form of the commutators in the Cartan basis, scalar products of the $\{H_i\}$ and the products of have simple properties:

$$Tr_{\mathcal{A}}([[H_j, E_\alpha]H_i]) = -Tr_{\mathcal{A}}([[H_j, H_i]E_\alpha]) = 0 \tag{24}$$

because H_j and H_i commute, but:

$$(H_i, E_\alpha) = 0 \tag{25}$$

For $\alpha + \beta \neq 0$ one can see that (E_α, E_β) in the same way. As contrast $(E_\alpha, E_{-\alpha})$ now must be non-zero for a simple Lie algebra with a non-degenerate metric. Normalized one may get:

$$(E_\alpha, E_{-\alpha}) = 1 \tag{26}$$

With the Killing form being non-degenerate and considering that $(H_i, E_\alpha) = 0$ also is block diagonal in the basis of H_i and E_α the sub-matrix $[H_i, H_j]$ must also be non-degenerate and consequently with a convenient transformation of the Cartan subalgebra may be regarded as:

$$(H_i, H_j) = \delta_{ij} \tag{27}$$

Moving forward to the scalar product of this with H_j and keeping H_i's orthogonally in mind the result is:

$$([E_\alpha, E_{-\alpha}], H_j) = \lambda_j \tag{28}$$

When using the cyclic properties of the trace one can see that $\lambda_j = (\alpha)_j$:

$$([E_\alpha, E_{-\alpha}], H_j) = ([H_j, E_\alpha], E_{-\alpha}) = (\alpha)_j (E_\alpha, E_{-\alpha}) = (\alpha)_j \tag{29}$$

So the commutators and scalar products of the generators in Cartan-Weyl basis are:

$$\begin{aligned}
&[H_i, H_j] = 0 \\
&[H_i, E_\alpha] = (\alpha)_i E_\alpha \\
&[E_\alpha, E_{-\alpha}] = (\alpha)_i H_i \\
&[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad \alpha + \beta \neq 0 \\
&(H_i, H_j) = \delta_{ij} \\
&(H_i, E_\alpha) = 0 \\
&(E_\alpha, E_{-\alpha}) = 1 \\
&(E_\alpha, E_\beta) = 0
\end{aligned} \tag{30}$$

5 Quantization of Roots

Roots are r-dimensional vectors with components $(\alpha)_i$ as defined before. Thus scalar products may be defined:

$$\begin{aligned}
\alpha \times \beta &:= (\alpha)_i (\beta)_i \\
and \alpha H &:= (\alpha) H_i
\end{aligned} \tag{31}$$

Here an application of our topic will be presented: For a start define:

$$H_\alpha := \frac{2}{\alpha^2}\alpha H \qquad (32)$$

The H_α are linear combinations of the H_i which means that they commute:

$$[H_\alpha, H_\beta] = 0 \qquad (33)$$

Looking at the commutators in the Cartan-Weyl basis one can see that

$$[H_\alpha, E_\beta] = \frac{2\alpha\beta}{\alpha^2}E_\beta \qquad (34)$$

while

$$[E_\alpha, E_{-\alpha}] = \frac{1}{2}\alpha^2 H_\alpha \qquad (35)$$

For the first of those we have the special cases $[H_\alpha, 2E_{-\alpha}] = \pm 2E_{\pm\alpha}$ By comparing this to the commutation relations of the $SU(2)$ generators one will find that each α will give an $SU(2)$ algebra which will be called S_α here and their identifications being:

$$J_3 = \frac{1}{2}H_\alpha, J_\pm = (2/\alpha^2)^{\frac{1}{2}}E_{\pm\alpha} \qquad (36)$$

These equations explains the step operators involved. The eigenvalues of J_3 are half-integral, hence the ones belonging to H_α will be integral. Other step operators, for example E_β on H_α will result in a step of $2\alpha\beta/\alpha^2$ which leads us to:

$$\frac{2\alpha\beta}{\alpha^2} = n = integer \qquad (37)$$

with a limitation that can be shown by use of the Schwartz inequality:

$$(\frac{2\alpha\beta}{\alpha^2})(\frac{2\beta\alpha}{\beta^2}) \leq 4 \qquad (38)$$

Now the angle between two vectors α and β shall be defined as:

$$cos\theta = \frac{\alpha\beta}{|\alpha||\beta|} \qquad (39)$$

which leads to another limitation: $n_1 n_2 \leq 4$ and $cos\theta = \frac{1}{2}(n_1 n_2)^{\frac{1}{2}}$.
Since these n must be relatively positive one needs a condition for positive definiteness: $0 \leq n_2 \leq n_1$ and in the same way $0 \leq \theta \leq 2/\pi$ and in the same way . n_1 may not be equal to 4 as that would lead to a contradiction. The existing possibilities are:

$$\begin{array}{llll} \text{a)} & n_1 = 0 \Rightarrow n_2 = 0 & \text{corresponding to} & \theta = \pi/2 \\ b) & n_1 = n_2 = 1 & \text{corresponding to} & \theta = \pi/3 \text{ and } |\alpha| = |\alpha| \\ c) & n_1 = 2, n_2 = 1 & \text{corresponding to} & \theta = \pi/4 \text{ and } |\beta| = \sqrt{2}|\alpha| \\ d) & n_1 = 3, n_2 = 1 & \text{corresponding to} & \theta = \pi/6 \text{ and } |\beta| = \sqrt{3}|\alpha| \end{array} \qquad (40)$$

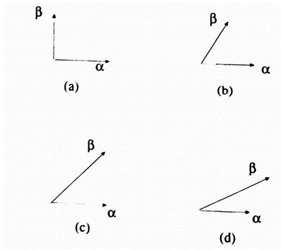

Figure 1: First roots

The roots may be generated into complete sets.$E_{\pm\beta}$ interacting with E_α will generate a beta-root-string through α which will extend from $\alpha + p\beta$ to $\alpha - q\beta$ in general.
One wants root vectors from the basis of an irreducible representation of S_β of dimension $2j + 1$ that correlate with the given string. At the ends of the string one has eigenvalues of $\frac{1}{2}H_\beta$ that are $\pm j$. So:

$$\begin{aligned} jE_{\alpha+p\beta} &= [\tfrac{1}{2}H_\beta, E_{\alpha+p\beta}] = \tfrac{(\alpha+p\beta)\beta}{\beta^2}E_{\alpha+p\beta} \\ and \quad j &= \tfrac{(\alpha+p\beta)\beta}{\beta^2}, -j = \tfrac{(\alpha-q\beta)\beta}{\beta^2} \end{aligned} \quad (41)$$

with the end result:

$$\begin{aligned} &\text{q+p=2j} \\ &q - p = 2\alpha\beta/\beta^2 = n_2 \end{aligned} \quad (42)$$

One sees that: if $p = 0$ the multiplicity is $2j+1 = q+1 = n_2+1$ and same with the second string. With these roots one can extend the set of roots to a full one, also including the negative ones.

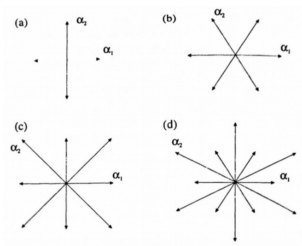

Figure 2: Extended roots

The symmetry in the complete diagrams is obvious. It is a consequence of the Weyl reflections which may be explained through the $SU(2)$ groups related to each root. The S_α group is given with the raising and lowering operators $J_\pm = E_{\pm\alpha}/|\alpha|$ which may be used to define the operator $J_2 := (J_+ - J_-)/2i$. One of its properties is:

$$e^{i\pi J_2}J_3e^{-i_2} = -J_3 \quad (43)$$

or:

$$e^{i\pi J_2}\alpha HJ_3e^{-i_2} = -\alpha H \quad (44)$$

On vH this lacks any effect when the vector v is orthogonal to α since

$$\begin{aligned}&[\text{vH,E}_{\pm\alpha}] = vH \text{ and}\\ &e^{i\pi J_2}\alpha H J_3 e^{-i_2} = vH\end{aligned} \tag{45}$$

So one finds that any general linear combination xH of the H's is transformed to $\sigma_\alpha(x)H$ while

$$\sigma_\alpha(x) = x - \frac{2x\alpha}{\alpha^2}\alpha \tag{46}$$

One sees the decomposition of x into its components perpendicular and parallel to . The Weyl reflection of a root is also a root:

$$\sigma_\alpha(\beta) = \beta - \frac{2\beta\alpha}{\alpha^2}\alpha \tag{47}$$

One defines:

$$[xH, E_\beta] = x\beta E_\beta \tag{48}$$

which is for arbitrary x and its transform is:

$$[\sigma(x)H, \tilde{E}_\beta] = x\beta\tilde{E}_\beta \tag{49}$$

while $\tilde{E}_\beta = exp(i\pi J_2)E_\beta exp(-i\pi J_2)$. The scalar product $x\beta$ is invariant under Weyl reflections:

$$x\beta = \sigma_\alpha(x)\sigma_\alpha(\beta) \tag{50}$$

One defines:

$$y := \sigma_\alpha(x) \tag{51}$$

Now above equation now says:

$$[yH, \tilde{E}_\beta] = y\sigma_\alpha(\beta)\tilde{E}_\beta \tag{52}$$

Therefore $\sigma_\alpha(\beta)$ must be a root with the root vector $\tilde{E}_\beta$
The final consequence is that for two distinct roots alpha and beta:

$$\begin{cases}\alpha\beta < 0 \Rightarrow \alpha + \beta \text{ is a root}\\ \alpha\beta > 0\alpha - \beta \text{ is a root}\end{cases} \tag{53}$$

References

[1] [H.F. Jones; Groups, Representations and Physics; Second Edition]