Preliminary lecture notes

Introduction to String Phenomenology

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ABSTRACT

After a brief introduction/review of string theory the course aims at developing the connection between string theory, particle physics and cosmology.

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Contents

1 Introduction to string theory

1.1 Basic assumptions

The basic idea of string theory is to replace a point-like particle by an extended object – a string which can be open or closed (Fig. 1.1). One then develops a quantum theory of strings. In order to do so one needs to define time t and energy H. Therefore one assumes that the strings move in a D-dimensional space-time $\mathbb{R}_{1,D-1}$ with Minkowskian signature $(1, D-1)$ (Fig. 1.2). The symmetry of this space-time is the Poincare group and thus t, H, mass m and spin s are defined by the representation theory. The drawback however is that the space-time background has to be assumed from the beginning. With this preliminaries one can define (perturbative) string theory as the quantum theory of extended objects (strings).

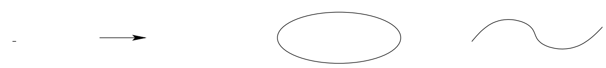


Figure 1.1: point-like particles are replaced by strings.

Figure 1.2: String moving in space-time background.

1.2 The string action

Let us denote the coordinates of the string by X^M. It is a map from the worldsheet Σ (with coordinates (τ, σ)) into the target space $\mathbb{R}_{1,D-1}$

$$X^M(\sigma^\alpha) : \ \Sigma \to \mathbb{R}_{1,D-1} \ , \quad M = 0, \dots D-1, \quad \sigma^\alpha = (\tau, \sigma), \ \alpha = 0, 1 \ , \quad 0 \leq \sigma < l \ . \tag{1.1}$$

The Nambu-Goto action is

$$S_{\text{NG}} = -T \int_\Sigma dA \ , \tag{1.2}$$

where A denotes the area of Σ (measured in coordinates of $\mathbb{R}_{1,D-1}$). T is the tension of the string with units of energy/unit volume.

The line element of $\mathbb{R}_{1,D-1}$ is

$$ds^2 = -\eta_{MN} dx^M(\sigma^\alpha) dx^M(\sigma^\alpha) = -G_{\alpha\beta} d\sigma^\alpha d\sigma^\beta \ , \tag{1.3}$$

where $G_{\alpha\beta}$ is the induced metric on Σ given by

$$G_{\alpha\beta} = \eta_{MN} \frac{\partial X^M}{\partial \sigma^\alpha} \frac{\partial X^N}{\partial \sigma^\beta} \ . \tag{1.4}$$

In terms of the metric the area A is given by

$$A = \sqrt{-\det G_{\alpha\beta}}\, d\sigma d\tau \ . \tag{1.5}$$

X^M and σ^α have dimension of length or inverse mass of $\mathbb{R}_{1,D-1}$. As a consequence $G_{\alpha\beta}$ is dimensionless and the tension T has dimension $(\text{length})^{-2} = (\text{mass})^2$. One defines

$$T \equiv \frac{1}{2\pi\alpha'} \ , \qquad l_s \equiv 2\pi\sqrt{\alpha'} \ , \qquad M_s \equiv \frac{1}{\sqrt{\alpha'}} \ . \tag{1.6}$$

α' is called the Regge slope, l_s the string length and M_s the string (mass) scale.

In addition to $G_{\alpha\beta}$ one defines the intrinsic metric $h_{\alpha\beta}(\tau, \sigma)$ on Σ. In terms of h one can rewrite the Nambu-Goto action as the Polyakov action

$$S_{\mathrm{P}} = -\frac{T}{2} \int_\Sigma d^2\sigma \sqrt{-\det h}\, h^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \eta_{MN} \ . \tag{1.7}$$

$h^{\alpha\beta}$ acts here as a Lagrange multiplier as its kinetic term is topological

$$\tfrac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{-\det h}\, R(h) = \chi(\Sigma) = 2 - 2g \ , \tag{1.8}$$

where $R(h)$ is the Riemann scalar and g the genus of Σ. The equation of motion

$$\frac{\delta S_{\mathrm{P}}}{\delta h^{\alpha\beta}} = 0 \tag{1.9}$$

yields S_{NG}. The advantage of using S_{P} instead of S_{NG} is that it corresponds to the standard action of D scalar fields in a two-dimensional (2d) field theory.

S_{P} has the following symmetries:

1. D-dimensional Poincare invariance

$$X^M \to X^{M\prime} = \Lambda^M_N X^N + a^M \ , \tag{1.10}$$

where $\Lambda \in SO(1, D-1)$ and a^M parameterizes translations. As a consequence energy, momentum and angular momentum E, P^M, L^{MN} are conserved.

2. Reparametrizations of Σ

$$\sigma^\alpha \to \sigma^{\alpha\prime}(\sigma^\alpha) \ . \tag{1.11}$$

As a consequence the energy-momentum tensor $T^{\alpha\beta}$ of the 2d field theory is (covariantly) conserved $D_\alpha T^{\alpha\beta} = 0$.

3. Local Weyl invariance

$$h_{\alpha\beta} \to e^{w(\sigma^\alpha)} h_{\alpha\beta}. \tag{1.12}$$

As a consequence $T_\alpha^\alpha = 0$.

The symmetries 2. and 3. have three local parameters and as a consequence $h_{\alpha\beta}$ has no degrees of freedom (dof). Thus S_P is a conformal field theory (CFT) on Σ. Its Weyl anomaly corresponds to the Liouville mode.

The equation of motion in the gauge $h_{\alpha\beta} = \mathrm{diag}(-1, 1)$ reads

$$\Box X^M = h^{+-}\partial_+\partial_- X^M = 0 \ , \tag{1.13}$$

where

$$\partial_\pm = \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma) \ , \qquad \sigma^\pm = \tau \pm \sigma \ . \tag{1.14}$$

The solution reads

$$X^M = X_L^M(\sigma^+) + X_R^M(\sigma^-) \ . \tag{1.15}$$

The boundary conditions of the <u>closed</u> string are

$$X^M(\tau, \sigma) = X^M(\tau, \sigma + l) \tag{1.16}$$

so that X^M can be expanded in Eigenfunctions of a circle

$$X_{L,R}^M = \tfrac{1}{2} x_0^M + \tfrac{\pi\alpha'}{l} p_0^M + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n\neq 0} \tfrac{1}{n} \alpha_{L,R\,n}^M e^{-i\frac{2\pi}{l} n\sigma^\pm} \ . \tag{1.17}$$

1.3 Quantization and excitation spectrum

The next step is to canonically quantize the string by replacing

$$X^M \to \hat{X}^M \ , \qquad \Pi_M = \frac{\partial \mathcal{L}}{\partial \dot{X}^M} \to \hat{\Pi}_M \tag{1.18}$$

and imposing

$$[\hat{\Pi}^M(\tau, \sigma), \hat{X}^N(\tau, \sigma')] = -i\delta(\sigma - \sigma')\eta^{MN} \ , \qquad [\hat{\Pi}_M, \hat{\Pi}_N] = 0 = [\hat{X}^M, \hat{X}^N] \ . \tag{1.19}$$

Due to the signature of η the construction of a positive definite Fock space is problematic. One finds that it requires $D = 26$ which coincides with an anomaly-free Weyl invariance.[1]

[1] More precisely, one need 26 scalar fields which, however, do not all have to be interpreted as space-time coordinates.

By applying creation opertors on the Fock vacuum one finds an infite tower of states with masses

$$M^2 = nM_s \ , \qquad n \in \{-1, 0, 1, 1, \ldots\} \ . \tag{1.20}$$

There is a unique state for $n = -1$ called the tachyon and a graviton $G_{(MN)}$, an anti-symmetric tensor $B_{[MN]}$ and a dilaton ϕ for $n = 0$. This situation can be improved by requiring supersymmetry in the 2d field theory on Σ. Redoing the Fock-space analysis on finds $D = 10$ and for a specific projection (GSO-projection) no tachyon.

In two space-time dimensions the superalgebra splits on the light cone into what is called (p, q)-supersymmetry where p denotes the left-moving supercharges and q the right-moving supercharges (see Appendix A for more details). For $D = 10$ and $(1, 1)$ super-symmetry on Σ one has two inequivalent theories termed type IIA and type IIB. Both are $N = 2$ space-time supersymmetric, type IIA is non-chiral while type IIB is chiral. For $D = 10$ and $(0, 1)$ supersymmetry on Σ there are three inequivalent theories termed type I, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. Type I includes closed and open strings and all three are $N = 1$ space-time supersymmetric.

In Table 1.1 we list the massless spectrum of type II string theories in $\mathbb{R}_{1,9}$ in the Neveu-Schwarz-Neveu-Schwarz (NS-NS), the Ramond-Ramond (R-R) and Neveu-Schwarz-Ramond (NS-R) sector while in Table 1.2 we display it for type I and heterotic strings. The C_p are antisymmetric tensors in p indices or equivalently the coeffcents of a p-form and A_M denotes a gauge boson. $\Psi_{M\pm}$ is the gravitino where $\pm$ indicates the 10d chirality and λ is the dilatino.

	Type IIA	Type IIB
NS-NS	$G_{(MN)},\, B_{[MN]},\, \phi$	
R-R	$C_1,\, C_3$	$l,\, C_2,\, C_4^*$
NS-R	$\Psi_{M+}, \Psi_{M-}, \lambda_+, \lambda_-$	$\Psi_{M+}^{1,2},\ \lambda_-^{1,2}$

Table 1.1: Massless spectrum of type II strings.

	Type I, Heterotic
NS	$G_{MN},\, B_{[MN]},\, \phi,\, A_M^a \in G = SO(32), E_8 \times E_8$
NS-R	$\Psi_{M+},\ \lambda_-,\ \lambda_-^a$

Table 1.2: Massless spectrum of heterotic and type I strings.

1.4 Interactions

The fundamental string interaction is depicted in Fig. 1.3. The strength of this inter-

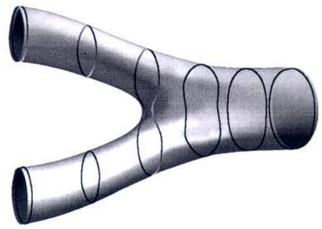


Figure 1.3: Fundamental string vertex.

action is measured by the dimensionless string coupling g_s which is proportional to the background value of the dilaton ϕ via $g_s = e^{\langle\phi\rangle}$.

From the fundamental vertex one constructs all scattering amplitudes. As an example the four-point amplitude is depicted in in Fig. 1.4.

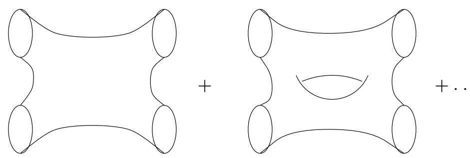


Figure 1.4: Four-point amplitude.

The g_s-dependence of the amplitude is

$$A = \sum_{n=0}^{\infty} A^{(n)} g_s^{2+2n} + \mathcal{O}(e^{-g_s^{-2}}) \ . \tag{1.21}$$

Remarks:

1. Interactions are introduced via "Feynman-diagrams" and corresponds to a sum over all worldsheet topologies. However, the object which leads to the expansion (1.21) is not known or in other words there is no analog of the action functional/path integral known. As a consequence even a formal definition of the theory is not available.

2. The graphs are "smeared" versions of the standard Feynman-diagrams in a quantum field theory which is the origin of the UV-finiteness of A.

3. For $g_s < 1$ a perturbative evaluation of A is sensible.

4. In the limit $l_s \to 0$ one obtains the amplitudes of a QFT coupled to classical relativity.

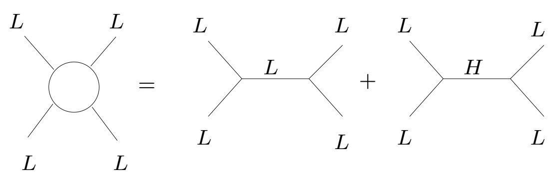
2 The low energy effective action of string theory

2.1 The S-matrix approach

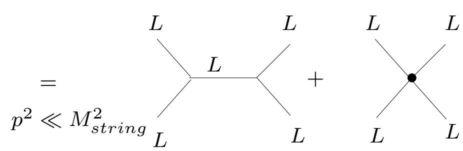
In field theories with light (L) and heavy (H) fields, i.e. with $m_L \ll m_H$, one defines for $p \ll m_H$ a low energy effective action formally by

$$e^{-i\int \mathcal{L}_{\text{eff}}(L)} = \int DH\, e^{-i\int \mathcal{L}(L,h)} \ . \tag{2.1}$$

In string theory there is no analog of the path integral but one can do the same procedure at the level of the S-matrix as depicted in Fig. 2.1



+ t and u channels



+ t and u channels

For $p^2 \ll M^2_{string}$ one obtains the amplitudes of an effective field theory.

The method (called the S-matrix approach) can be systematically used to construct

$$\mathcal{L}_{\text{eff}} = \sum_{n=0}^{\infty} \left(\frac{p^2}{M_s^2}\right)^n \mathcal{L}_{\text{eff}}^{(n)} \ . \tag{2.2}$$

$n = 0$ corresponds to the potential and Yukawa-interactions while $n = 1$ give the standard kinetic terms. In pratice one uses symmetries to simplify the analysis.

2.2 Type II A supergravity in $D = 10$

We consider now type II A supergravity in $D = 10$. The multiplet contains

$$\underbrace{\underset{(35)}{G_{MN}}\,,\ \underset{(28)}{B_{MN}}\,,\ \underset{(1)}{\phi}\,,\ \underset{(8)}{C_M}\,,\ \underset{(56)}{C_{[MNP]}}}_{(128)}\,,\ \underbrace{\underset{(56)}{\Psi_{M\alpha}}\,,\ \underset{(56)}{\Psi_{M\dot\alpha}}\,,\ \underset{(8)}{\lambda_\alpha}\,,\ \underset{(8)}{\lambda_{\dot\alpha}}}_{(128)}\,, \tag{2.3}$$

where we indicated the number of d.o.f. in brackets. The bosonic Lagrangian has the form

$$\mathcal{L}_{\rm IIA} = \mathcal{L}_{\rm NS} + \mathcal{L}_{\rm RR} + \mathcal{L}_{\rm CS} \ , \tag{2.4}$$

where in the string frame

$$\begin{aligned} \mathcal{L}_{\rm NS} =& \tfrac{1}{2\tilde{\kappa}^2}\, e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}|H_3|^2\right) \ , \qquad \tilde{\kappa}^2 = \tfrac{(4\pi^2\alpha')^4}{4\pi} = \tfrac{l_s^8}{4\pi} \\ \mathcal{L}_{\rm RR} =& -\tfrac{1}{8\tilde{\kappa}^2}\left(\tfrac{1}{2}|F_2|^2 + \tfrac{1}{4!}|\hat{F}_4|^2\right) \\ \mathcal{L}_{\rm CS} =& -\tfrac{1}{4\tilde{\kappa}^2}\, B_2 \wedge F_4 \wedge F_4 \,, \end{aligned} \tag{2.5}$$

and $F_2 = dC_1$, $H_3 = dB_2$, $F_4 = dC_3$, $\hat{F}_4 = F_4 - C_1 \wedge H_3$.

The IIA theory has two local supersymmetries of opposite chirality. In addition there three independent gauge symmetries related to the various p-forms present. They are

$$(i) \qquad \delta C_1 = d\Lambda_0 \,, \qquad \delta C_3 = \Lambda_0 H_3 \,, \qquad \delta F_2 = \delta\hat{F}_4 = 0 \ , \tag{2.6}$$

$$(ii) \qquad \delta B_2 = d\Lambda_1 \,, \qquad \delta H_3 = 0 \,, \tag{2.7}$$

$$(iii) \qquad \delta C_3 = d\Lambda_2 \,, \qquad \delta\hat{F}_4 = 0 \ , \tag{2.8}$$

with parameters $\Lambda_0, \Lambda_1, \Lambda_2$. Note that the theory contains no charged fermions.

2.3 Type II B supergravity

The multiplet contains

$$\underbrace{\underset{(35)}{G_{MN}}\,,\ \underset{(28)}{B^1_{MN}}\,,\ \underset{(28)}{B^2_{MN}}\,,\ \underset{(1)}{\phi}\,,\ \underset{(1)}{l}\,,\ \underset{(1)}{C_M}\,,\ \underset{(35)}{C^*_{MNPQ}}}_{(128)}\,,\ \underbrace{\underset{(56)}{\Psi^1_{M\alpha}}\,,\ \underset{(56)}{\Psi^2_{M\alpha}}\,,\ \underset{(8)}{\lambda^1_{\dot\alpha}}\,,\ \underset{(8)}{\lambda^2_{\dot\alpha}}}_{(128)}\,, \tag{2.9}$$

where the four-form has a self-dual field strength

$$F_5 = dC_4^* = \tilde{F}_5 \,, \qquad \text{where} \qquad \tilde{F}_{M_1,\ldots,M_5} = \epsilon_{M_1,\ldots,M_{10}} F^{M_6,\ldots,M_{10}} \ . \tag{2.10}$$

This theory has no Lorentz invariant action but only field equations due to the self-duality constraint. One can give the action without imposing the constraint and include it on the field equation by hand. One has

$$\mathcal{L}_{\rm IIB} = \mathcal{L}_{\rm NS} + \mathcal{L}_{\rm RR} + \mathcal{L}_{\rm CS} \ , \tag{2.11}$$

where $\mathcal{L}_{\rm NS}$ is as in (2.5) while

$$\begin{aligned} \mathcal{L}_{\rm RR} =& -\tfrac{1}{4\tilde{\kappa}^2}\left(\tfrac{1}{2}|F_1|^2 - \tfrac{1}{3!}|\hat{F}_3|^2 + \tfrac{1}{2\cdot 5!}|\hat{F}_5|^2\right) \ , \\ \mathcal{L}_{\rm CS} =& -\tfrac{1}{4\tilde{\kappa}^2}\, C_4 \wedge H_3 \wedge F_3 \ , \end{aligned} \tag{2.12}$$

with

$$\hat{F}_3 = dC_2 - lH_3 \ , \qquad \hat{F}_5 = dC_4 + \tfrac{1}{2}B_2 \wedge F_3 - \tfrac{1}{2}C_2 \wedge H_3 \, . \tag{2.13}$$

The type IIB theory has two supersymmetries of the same chirality. The p-form gauge symmetries are

$$\delta C_4 = d\Lambda_3 \, , \qquad \delta B_2 = d\Lambda_1^B \, , \qquad \delta C_2 = d\Lambda_1^C \, , \qquad \delta C_4 = -\tfrac{1}{2}\Lambda_1^B \wedge F_3 + \tfrac{1}{2}\Lambda_1^C \wedge H_3 \, . \tag{2.14}$$

The type IIB theory also has $SL(2,\mathbb{R})$ symmetry which is visible in the Einstein frame. Defining

$$\begin{aligned} G^E_{MN} &= e^{-\phi} G_{MN} \ , \qquad \tau = l + ie^{-\phi} \, , \\ M_{ij} &= \frac{1}{\operatorname{Im}\tau} \begin{pmatrix} |\tau|^2 & -\operatorname{Re}\tau \\ -\operatorname{Re}\tau & 1 \end{pmatrix} \ , \qquad F_3^i = \begin{pmatrix} H_3 \\ F_3 \end{pmatrix} , \quad i = 1, 2 \end{aligned} \tag{2.15}$$

the Einstein frame Lagrangian reads

$$\mathcal{L}_{\rm IIB} = \frac{1}{2\kappa_{10}^2} \left(R_E - \tfrac{1}{2} \frac{\partial_M \tau \partial^M \tau}{(\operatorname{Im}\tau)^2} - \tfrac{1}{2} M_{ij} \left(F^i_{MN} F^{j\,MN} \right) - \tfrac{1}{2\cdot 5!} |\hat{F}_5|^2 - \tfrac{1}{4}\epsilon_{ij} C_4 \wedge F_3^i \wedge F_3^j \right) . \tag{2.16}$$

In the Einstein frame one can check the $SL(2,\mathbb{R})$ symmetry acting as

$$\begin{aligned} &\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \, , \qquad a, b, c, d \in \mathbb{R} \, , \quad ad - bc = 1 \, , \\ &M \to M' = (\Lambda^{-1})^T M \Lambda^{-1} \, , \qquad \Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \, , \\ &F_3^i \to F_3'^{\,i} = \Lambda^i{}_j F_3^j \, . \end{aligned} \tag{2.17}$$

2.4 Heterotic and type I

The effective actions of these two string theories are very similar. The massless multiplets are the gravitational multiplet containing

$$\underbrace{\underset{(35)}{G_{MN}} \, , \ \underset{(28)}{B_{MN}} \, , \ \underset{(1)}{\phi}}_{(64)} \, , \ \underbrace{\underset{(56)}{\Psi_{M\alpha}} \, , \ \underset{(8)}{\lambda_{\dot\alpha}}}_{(64)} \, , \tag{2.18}$$

and the vector multiplet featureing

$$\underset{(8)}{A_M} \, , \ \underset{(8)}{\chi_\alpha} \, , \tag{2.19}$$

Now non-Abelian gauge symmetries are possible. The Lagrangian is

$$\mathcal{L}_{\rm het/I} = \mathcal{L}_{\rm NS} + \mathcal{L}_{\rm v\,het/I} \ , \tag{2.20}$$

where $\mathcal{L}_{\rm NS}$ is again given by

$$\mathcal{L}_{\rm NS} = \tfrac{1}{2\tilde\kappa^2} e^{-2\phi} \left(R + 4\partial_M\phi\partial^M\phi - |\hat H_3|^2 \right) \ , \tag{2.21}$$

but now with

$$\begin{aligned} \hat H_3 &= dB_2 - \tfrac14 \alpha'(\Omega_{\rm YM} - \Omega_{\rm YM}) \ , \\ \Omega_{\rm YM} &= {\rm Tr}\left(A_1 \wedge dA_1 - \tfrac{2i}{3} A_1 \wedge A_1 \wedge A_1 \right) \ , \\ \Omega_{\rm L} &= {\rm Tr}\left(\omega_1 \wedge d\omega_1 + \tfrac{2}{3}\omega_1 \wedge \omega_1 \wedge \omega_1 \right) \ . \end{aligned} \tag{2.22}$$

$\Omega_{\rm YM}$ is the Yang-Mills Chern-Simons term which obeys $d\Omega_{\rm YM} = F_2 \wedge F_2$ while $\Omega_{\rm L}$ is the Lorentz Chern-Simons term which obeys $d\Omega_L = R_2 \wedge R_2$. This implies

$$d\hat H_3 = -\tfrac14 \alpha'({\rm Tr} F\wedge F - {\rm Tr} R \wedge R) \ . \tag{2.23}$$

The kinetic term for the vector multiplet $\mathcal{L}_{\rm vhet/I}$ reads

$$\begin{aligned} \text{heterotic} : \qquad \mathcal{L}_{\rm vhet/I} &= -\tfrac{1}{2\tilde g_{10}^2} e^{-2\phi} {\rm Tr}\left(F_{MN}F^{MN}\right) \ , \\ \text{type I} : \qquad \mathcal{L}_{\rm vhet/I} &= -\tfrac{1}{2\tilde g_{10}^2} e^{-\phi} {\rm Tr}\left(F_{MN}F^{MN}\right) \ , \end{aligned} \tag{2.24}$$

where

$$\tfrac{\tilde\kappa^2}{\tilde g_{10}^2} = \tfrac14 \alpha' \ . \tag{2.25}$$

The theory has one local supersymmetry and the two-form gauge symmetries $\delta B_2 = d\Lambda_1$ together with the gauge invariance

$$\delta A_1 = d\Lambda + i[A_1, \Lambda] \, , \qquad \delta\Omega_{\rm YM} = d{\rm Tr}\,(\Lambda \wedge F) \, , \qquad \delta B_2 = \tfrac14 \alpha' \, {\rm Tr}(\Lambda\wedge F) \, , \tag{2.26}$$

and an analogous symmetry for $\Omega_{\rm L}$. The theory is anomaly free for $E_8 \times E_8$, $SO(32)$.

3 Calabi-Yau compactifications

In the way we discussed the quantization of string theory in section 1 we need ten scalar fields $X^M, M = 0, \ldots, 9$ on the worldsheet Σ. These scalar fields are interpreted as the coordinates of a target space which is identified as our space-time. However, the choice of the global symmetry in the target space is not fixed by the consistency of the quantization. In sections 1 and 2 we discussed $\mathbb{R}_{1,9}$ with a Lorentz symmetry $SO(1,9)$ as an instructive example but this is by no means necessary. Instead we can have a target space $\mathbb{R}_{1,d-1} \times Y_{10-d}$ with symmetry group $SO(1,d-1) \times SO(10-d)$ where Y_{10-d} is a compact $(10-d)$-dimensional manifold. Such backgrounds are commonly referred to as compactifications of $\mathbb{R}_{1,9}$ and have been prominently discussed in Kaluza-Klein theories [9]. In string theory there is an additional consistency condition in that the background has to be a SCFT on the worldsheet. This can be satisfied by choosing Y_{10-d} to be Ricci-flat or by turning on appropriate background values (background flux) of other fields such that

$$\text{Ric}(Y_{10-d}) = 0 \ , \qquad \text{or} \qquad \text{Ric}(Y_{10-d}) + \text{background flux} = 0 \ . \tag{3.1}$$

In fact it is possible to abandon the concept of a geometrical background altogether and have instead $\mathbb{R}_{1,d-1} \times$ SCFT where SCFT denotes an appropriate two-dimensional SCFT which plays the role of the $(10-d)$ compact dimension but does not admit any geometricl interpretation in terms of some target manifold.[2] This state of affairs is another manifestation of the fact that currently we do not understand how in string theory the space-time background the string moves in is choosen.

The Ricci-flat compact manifolds have been studied in mathematics. They consist of:

- Tori $T^{(10-d)}$ which are even flat in that also the Riemann-tensor vanishes,
- four-dimensional K3-surfaces (they correspond to $d = 6$),
- Calabi-Yau threefolds which are complex three-dimensional manifolds corresponding to $d = 4$.[3]

In the following we will concentrate on $d = 4$ and thus Calabi-Yau threefolds which we denote by Y_3.

3.1 Calabi-Yau manifolds

There are different equivalent definitions of Calabi-Yau manifolds. From [2] we take:

[2] Fermionic construction or asymmetric orbifold are prominent examples of this situation.

[3] Calabi-Yau n-folds exist for any n but for the application discussed here only $n = 3, 4$ will be relevant.

Definition: A Calabi-Yau n-fold is a complex n-dimensional compact Kähler manifold with a metric of holonomy $H = SU(n)$ (or $H \subset SU(n)$).

This implies the following properties:

1. The metric is Ricci-flat.
2. The first Cern class vanishes $c_1(Y_n) = 0$.
3. Precisely one covariantly constant spinor η exist for $H = SU(n)$ or at least one for $H \subset SU(n)$.
4. Y_n has a unique holomorphic nowhere vanishing and covariantly constant $(n, 0)$-form Ω.

(For more details see Appendix B.)

3.2 Supersymmetry in Calabi-Yau compactifications

If we consider a background $\mathbb{R}_{1,d-1} \times Y_{10-d}$ instead of $\mathbb{R}_{1,9}$ the Lorentz group $SO(1, 9)$ decomposes as

$$SO(1,9) \to SO(1,d-1) \times SO(10-d) \ . \tag{3.2}$$

The spinor representation $\mathbf{16}$ of $SO(1, 9)$ decomposes accordingly

$$\begin{aligned} \mathbf{16} &\to (\mathbf{2^{\frac{1}{2}(d-2)}}, \mathbf{2^{(4-\frac{d}{2})}}) + (\mathbf{2^{\frac{1}{2}(d-2)\prime}}, \mathbf{2^{(4-\frac{d}{2})\prime}}) \ , \\ \mathbf{16}' &\to (\mathbf{2^{\frac{1}{2}(d-2)}}, \mathbf{2^{(4-\frac{d}{2})\prime}}) + (\mathbf{2^{\frac{1}{2}(d-2)\prime}}, \mathbf{2^{(4-\frac{d}{2})}}) \ , \end{aligned} \tag{3.3}$$

where $'$ denotes the inequivalent Weyl representation. For $d = 4$ one has $SO(1, 9) \to SO(1, 3) \times SO(6)$ and

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \ . \tag{3.4}$$

In particular the supercharge $Q \in \mathbf{16}$ decomposes into

$$Q \to Q^I_\alpha \ , \ \bar{Q}^I_{\dot\alpha} \ , \quad \alpha, \dot\alpha = 1, 2, \quad I = 1, \ldots, 4 \ . \tag{3.5}$$

On a flat background T^6 all supercharges exist and thus one obtains $N = 4$ supercharges in $d = 4$ from one supercharge in $d = 10$. On curved Calabi-Yau backgrounds one has to make sure that the supercharges are globally defined spinors. On $K3$ there are two such spinors corresponding to eight well defined supercharges while on Y_3 there is one such spinor corresponding to four supercharges or $N = 1$. The situation is depicted in Fig. 3.1.

Constructing the effective low energy action one can use two different approaches. It is again possible to compute the massless spectrum of the theory directly in string theory and then use the S-matrix approach in $\mathbb{R}_{1,d-1}$ to compute $\mathcal{L}_{\text{eff}}$. Alternatively one can perform a Kaluza-Klein reduction which we turn to now.

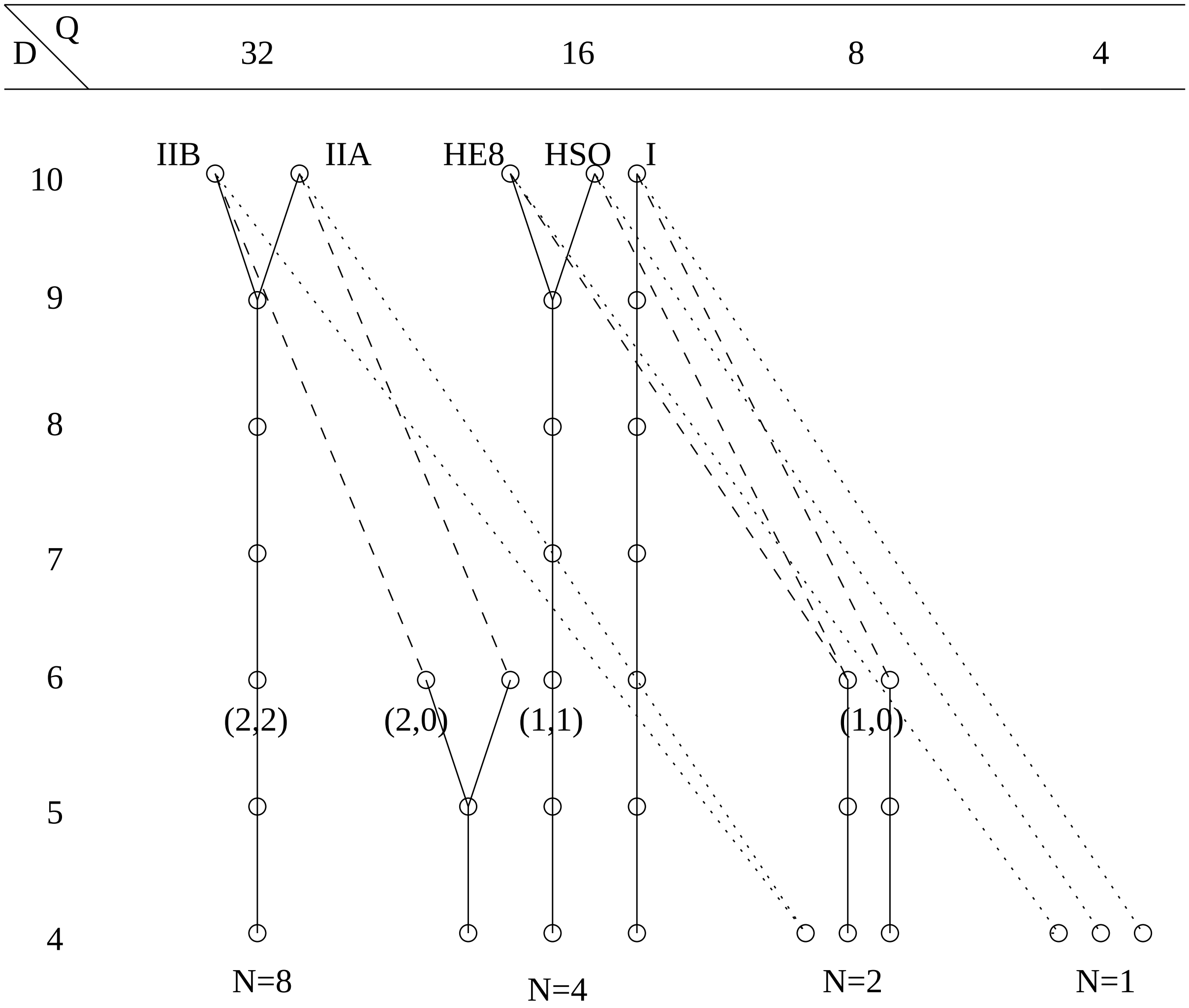


Figure 3.1: Calabi-Yau compactifications of the 10-dimensional string theories. The solid line (−) denotes toroidal compactification, the dashed line (−−) denotes $K3$ compactifications and the dotted line ($\cdots$) denotes Y_3 compactifications. Whenever two compactifications (two lines) terminate in the same point, the two string theories are related by a perturbative duality. (A line crossing a circle is purely accidental and has no physical significance.)

3.3 Kaluza-Klein formalism

The massless wave equations in $\mathbb{R}_{1,9}$ read

$$\Box\phi = 0 \ , \qquad i\gamma^M D_M \psi = 0 \ . \tag{3.6}$$

The corresponding wave equations in $\mathbb{R}_{1,d-1} \times Y_{10-d}$ then read

$$(\Box_{1,d-1} + \Delta_y)\phi(x,y) = 0 \ , \qquad i(\gamma^\mu D_\mu + \gamma^m D_m)\psi(x,y) = 0 \ , \tag{3.7}$$

where x^μ are the coordinates of $\mathbb{R}_{1,d-1}$ while y^m are the coordinates of Y_{10-d}. Both fields can be expanded in terms of Eigenfunctions of the wave-operators on Y_{10-d}

$$\phi = \sum_n \phi^{(n)}(x)\, \theta^{(n)}(y) \ , \qquad \psi = \sum_n \psi^{(n)}(x)\, \chi^{(n)}(y) \ , \tag{3.8}$$

where

$$\Delta_y \theta^{(n)} = m_\phi^{2(n)} \theta^{(n)} \ , \qquad i\gamma^m D_m \chi^{(n)} = m_\psi^{(n)} \chi^{(n)} \tag{3.9}$$

Inserted into (3.7) yields

$$(\Box_{1,d-1} + m^{2(n)})\, \phi^{(n)}(x) = 0 \ , \qquad i(\gamma^\mu D_\mu + m^{(n)})\, \psi^{(n)}(x) = 0 \ . \tag{3.10}$$

The scale of m is related to the Kaluza-Klein scale $l_{\rm KK}$ via

$$m \sim l_{\rm KK}^{-1} \ , \qquad V_{10-d} \sim l_{\rm KK}^{10-d} \ , \tag{3.11}$$

where V denotes the volume of Y_{10-d}. From (3.10) we see that the massless modes correspond to the zero modes of the wave operator on Y_{10-d}. For Calabi-Yau manifolds these zero modes are in one-to-one correspondence with the harmonic forms on Y which in turn are in one-to-one correspondence with elements of the Dolbeault cohomology groups $H^{(p,q)}(Y)$ defined as

$$H^{(p,q)}(Y) := \frac{\text{closed } (p,q)\text{-forms}}{\text{exact } (p,q)\text{-forms}} \ . \tag{3.12}$$

Here (p,q) denotes the number of holomorphic and anti-holomorphic differentials of the harmonic forms. The dimensions of $H^{(p,q)}(Y)$ are called Hodge numbers and denoted as $h^{p,q} = \dim H^{p,q}(Y)$. They are conventionally arranged in a Hodge diamond which on a Calabi-Yau manifold simplifies as follows

$$\begin{array}{ccccccc}
 & & & h^{(0,0)} & & & \\
 & & h^{(1,0)} & & h^{(0,1)} & & \\
 & h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} & \\
h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\
 & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\
 & & h^{(3,2)} & & h^{(2,3)} & & \\
 & & & h^{(3,3)} & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{(1,1)} & & 0 & \\
1 & & h^{(1,2)} & & h^{(1,2)} & & 1 \\
 & 0 & & h^{(1,1)} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \ . \tag{3.13}$$

Or in other words the $h^{(p,q)}$ satisfy

$$h^{(1,0)} = h^{(0,1)} = h^{(2,0)} = h^{(0,2)} = h^{(3,1)} = h^{(1,3)} = h^{(3,2)} = h^{(2,3)} = 0 \ , \qquad (3.14)$$
$$h^{(0,0)} = h^{(3,0)} = h^{(0,3)} = h^{(3,3)} = 1 \ , \qquad h^{(2,1)} = h^{(1,2)} \ , \qquad h^{(1,1)} = h^{(2,2)} \ .$$

We see that $h^{(1,1)}$ and $h^{(1,2)}$ are the only non-trivial, i.e. arbitrary Hodge numbers on a Calabi-Yau threefold.

The deformations of the Calabi-Yau metric $g_{i\bar{j}}, i, \bar{j} = 1, \ldots, 3$ which do not disturb the Calabi-Yau condition correspond to moduli scalars in the low energy effective action. They naturally split into deformations of the complex structure δg_{ij} and deformations of the Kähler form $\delta g_{i\bar{j}}$. The latter are in one to one correspondence with the harmonic $(1,1)$-forms and thus can be expanded as

$$\delta g_{i\bar{j}} = i v^{\alpha}(x)\, \omega^{\alpha}_{i\bar{j}} \ , \qquad \alpha = 1, \ldots, h^{(1,1)} \ , \qquad (3.15)$$

where ω_a are harmonic $(1,1)$-forms on Y which form a basis of $H^{(1,1)}(Y)$. The v^a denote $h^{(1,1)}$ moduli which in the effective action appear as scalar fields. Similarly the deformations of the complex structure are parameterized by complex moduli z^k which are in one-to-one correspondence with harmonic $(1,2)$-forms via

$$\delta g_{ij} = \frac{i}{||\Omega||^2}\, \bar{z}^a(x) \bar{\chi}^a_{i\bar{i}\bar{j}}\, \Omega^{\bar{i}\bar{j}}{}_j \ , \quad a = 1, \ldots, h^{(1,2)} \ , \qquad (3.16)$$

where Ω is the holomorphic (3,0)-form, $\bar{\chi}_k$ denotes a basis of $H^{(1,2)}$ and we abbreviate $||\Omega||^2 \equiv \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk}$.

4 Calabi-Yau compactifications of the heterotic string

Let us recall that the massless spectrum of the heterotic string in $\mathbb{R}_{1,9}$ contains a gravitational multiplet consisting of the ten-dimensional metric $G_{MN}, M, N = 0, \dots 9$, an antisymmetric two-tensor B_{MN}, the dilaton ϕ, a left-handed Majorana-Weyl gravitino ψ_M and a right handed Majorana-Weyl fermion, the dilatino λ. Additionally, we have a Yang-Mills vector multiplet which features a gauge boson A_M^a and a gaugino χ^a, both transforming in the adjoint representation of either $E_8 \times E_8$ or SO(32). The corresponding action was discussed in Section 2.4.

4.1 The four-dimensional spectrum

Let us first discuss the massless spectrum of the compactified theory in the background $\mathbb{R}_{1,3} \times Y_3$ where Y_3 is a Calabi-Yau manifold. The metric G_{MN} decomposes into the metric $g_{\mu\nu}, \mu, \nu = 0, \dots, 3$ of $\mathbb{R}_{1,3}$ and the $h^{(1,1)} + 2h^{(1,2)}$ geometric moduli v^α, z^a discussed in the previous section (cf. in (3.15),(3.16)). The component $g_{\mu i}$ has no zero modes as there are no harmonic one-forms on Y_3 (cf. (3.13)). Similarly, B_{MN} decomposes into $B_{\mu\nu}$ and $h^{(1,1)}$ scalar moduli b^α (cf. (B.21)).

For the fermions let us recall the decomposition of the **16** spinor representation discussed in Section 3.2. For the group decomposition

$$SO(1,9)) \to SO(1,3) \times SO(6) \to SO(1,3) \times SU(3) \times U(1) \tag{4.1}$$

one has

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \to (\mathbf{2}, \mathbf{1}) + (\mathbf{2}, \mathbf{3}) + (\bar{\mathbf{2}}, \mathbf{1}) + (\bar{\mathbf{2}}, \bar{\mathbf{3}}) \ . \tag{4.2}$$

Therefore the 10-dimensional gravitino ψ_M decomposes into $\psi_\mu \in (\mathbf{2}, \mathbf{1})$ and $\psi_m \in (\mathbf{2}, \mathbf{3})$ The latter is a spin-1/2 fermion in the $(\mathbf{3} + \bar{\mathbf{3}}) \times \mathbf{3} \sim \mathbf{6} + \bar{\mathbf{3}} + \mathbf{8} + \mathbf{1}$. Since there is non zero mode corresponding to the $\bar{\mathbf{3}}$ we are left with the **6** and the $(\mathbf{8} + \mathbf{1})$. Finally the dilatino $\lambda \in \mathbf{16}$ decomposes into $(\mathbf{2}, \mathbf{1}) + (\bar{\mathbf{2}}, \mathbf{1})$.

These bosons and fermions combine into the following 4d $N = 1$ multiplets:

gravity:	$(g_{\mu\nu}, \psi_\mu)$
dilaton:	$(\phi, B_{\mu\nu}, \lambda)$
$h^{1,1}$ Kähler structure moduli:	$(t^\alpha, \psi^\alpha) \in (\mathbf{8} + \mathbf{1})$ of $SU(3)$
$h^{1,2}$ complex structure moduli:	$(z^a, \psi^a) \in \mathbf{6}$ of $SU(3)$

For the vector multiplets the identification of the zero modes is more subtle due to (2.22). On a Calabi-Yau one has

$$\int_{Y_3} d\hat{H}_3 = -\tfrac{1}{4}\alpha' \int_{Y_3} (\mathrm{Tr} F \wedge F - \mathrm{Tr} R \wedge R) = 0 \ . \tag{4.3}$$

Since $\int_{Y_3} \text{Tr} R \wedge R \neq 0$ one needs a non-trivial gauge bundle on Y_3. The simplest solution (called the standard embedding) is to impose

$$\text{Tr} F \wedge F = \text{Tr} R \wedge R \ , \qquad \hat{H}_3 = 0 \ . \tag{4.4}$$

In terms of the gauge fields is says $A = \omega$ or in other words the gauge connection is equal to the spin connection. The latter is an element of $SU(3) \subset SO(6)$ and thus one has to break the ten-dimensional heterotic gauge groups as

$$E_8 \times E_8 \to E_8 \times E_6 \times SU(3) \ , \qquad SO(32) \to SO(26) \times U(1) \times SU(3) \ , \tag{4.5}$$

and identify the $SU(3)$ factor with the spin connection. Let us focus on $E_8 \times E_8$ where the adjoint representation of E_8 decomposes under $E_6 \times SU(3)$ as

$$\mathbf{248} \to (\mathbf{27}, \mathbf{3}) + (\mathbf{\bar{27}}, \mathbf{\bar{3}}) + (\mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}) \ . \tag{4.6}$$

Correspondingly the gauge field decomposes $A^a_M \to (A^a_\mu, A^a_m)$ with each field possibly in the representation (4.6). However, for A^a_μ only the $(\mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{8})$ survive as zero mode on Y_3 as there are no one-forms. The $(\mathbf{78}, \mathbf{1})$ is identified with the E_6 gauge field while the $(\mathbf{1}, \mathbf{8})$ is identified with the spin connection and therefore does not contribute to the low energy spectrum.[4] For A^a_m on the other hand the $(\mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{8})$ cannot appear, again because there are no one-forms on Y_3. In this case the $(\mathbf{27}, \mathbf{3}) + (\mathbf{\bar{27}}, \mathbf{\bar{3}})$ can appear. Using again $(\mathbf{3} + \mathbf{\bar{3}}) \times \mathbf{3} \sim \mathbf{6} + \mathbf{\bar{3}} + \mathbf{8} + \mathbf{1}$ one finds $2h^{1,2}\mathbf{27} + 2h^{1,1}\mathbf{\bar{27}}$ scalar fields. λ^a has a similar decomposition and thus ten-dimensional vector multiplet decomposes into 4d $N = 1$ multipletsshown in Table 4.1.[5]

vector:	$(A^a_\mu, \lambda^a) \in (\mathbf{78}, \mathbf{248})$ of $E_6 \times E_8$
chiral matter:	$h^{1,2}$ families $(A^a, \psi^a) \in \mathbf{27}$
	$h^{1,1}$ families $(A^\alpha, \psi^\alpha) \in \mathbf{\bar{27}}$

Table 4.1: Massless heterotic spectrum

4.2 The low energy effective action

The low energy effective action of the compactification is a 4d $N = 1$ supergravity whose bosonic Lagrangian reads

$$\begin{aligned} \mathcal{L} = & - \tfrac{1}{2\kappa^2} R - \tfrac{1}{4} g^{-2}_{ab} F^a_{\mu\nu} F^{\mu\nu b} + \tfrac{1}{32\pi^2} \, \Theta_{ab} \, \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^b_{\rho\sigma} \\ & - G_{A\bar{B}} \left(\Phi, \bar{\Phi}\right) D_\mu \Phi^A D^\mu \bar{\Phi}^{\bar{B}} - V\left(\Phi, \bar{\Phi}\right) \ , \end{aligned} \tag{4.7}$$

[4]If one considers deformation which do not preserve $A = \omega$ but do respect supersymmetry one finds that these defomations do give rise to chiral multiplets termed bundle moduli.

[5]Note that there is a slight clash in the notation. The index a is used to denote the adjoint representation of the gauge group and also counts the number of $(1, 2)$-forms.

where $\kappa^2 = 8\pi M_{Pl}^{-2}$, R is the Einstein-Hilbert-term and we have collectively denotes all scalar fields as Φ^A. The Lagrangian (D.1) is characterized by four functions $K(\Phi, \bar{\Phi})$, $f(\Phi)$, $W(\Phi)$ and D^a. K is the Kähler potential which determines the sigma-model metric by

$$G_{A\bar{B}} = \partial_A \partial_{\bar{B}} K \ . \tag{4.8}$$

The (inverse) gauge couplings and the Θ-angle combine into the holomorphic gauge kinetic function

$$f_{ab} = g_{ab}^{-2} + \tfrac{i}{8\pi^2}\,\Theta_{ab} \ . \tag{4.9}$$

The potential is given by

$$V = e^{\kappa^2 K}\left[(D_A W)\, G^{-1A\bar{B}}\left(D_{\bar{B}}\bar{W}\right) - 3\kappa^2|W|^2\right] + \tfrac{1}{2} g_{ab}^{-1} D^a D^b \ , \tag{4.10}$$

where W is the holomorphic superpotential and

$$D_A W := \frac{\partial W}{\partial \Phi^A} + \kappa^2 \left(\frac{\partial K}{\partial \Phi^A}\right) W \ . \tag{4.11}$$

The D-term is given by

$$D^a = -i(\partial_A K) k^{aA} + \xi \delta^{aU(1)} \ , \tag{4.12}$$

where k^{aA} denotes a Killing vector and ξ is the Fayet-Illiopoulos (FI) parameter. (For further details, see Appendix C.) Finally the covariant derivatives are

$$D_\mu \Phi^A = \partial_\mu \Phi^A - A_\mu^a\, k_a^A(\Phi) \ . \tag{4.13}$$

Inserting the KK-expansion discussed in section 4.1 into (2.21) and (2.24), performing appropriate field redefinitions one computes the function K, W, f to be

$$K = -\ln(S+\bar{S}) + K_{\rm ks}(t,\bar{t}) + K_{\rm cs}(z,\bar{z}) + K_{\rm m}(A,\bar{A},t,\bar{t},z,\bar{z}) \ , \tag{4.14}$$

where $S = e^{-2\phi} + ia$ is the complexified dilaton with a being the dual of $B_{\mu\nu}$. For $K_{\rm ks}, K_{\rm cs}$ one finds (cf. (B.27), (B.30))

$$\begin{aligned} K_{\rm ks}(t,\bar{t}) &= -\ln i d_{\alpha\beta\gamma}(t-\bar{t})^\alpha (t-\bar{t})^\beta (t-\bar{t})^\gamma \ , \\ K_{\rm cs}(z,\bar{z}) &= -\ln\Big[-i\int_Y \Omega\wedge\bar{\Omega}\Big] \ , \end{aligned} \tag{4.15}$$

while $K_{\rm m}(A,\bar{A},t,\bar{t},z,\bar{z})$ is only known at leading order in A. The moduli dependence of the matter metric reads

$$\begin{aligned} G_{{\rm m}\,\alpha\bar{\beta}}(A,\bar{A},t,\bar{t},z,\bar{z})|_{A=\bar{A}=0} &= \partial_{A^\alpha}\partial_{\bar{A}^{\bar{\beta}}} K_{\rm m}(A,\bar{A},t,\bar{t},z,\bar{z})|_{A=\bar{A}=0} = e^{\frac{1}{3}(K_{\rm cs}-K_{\rm ks})} G_{\alpha\bar{\beta}} \ , \\ G_{{\rm m}\,a\bar{b}}(A,\bar{A},t,\bar{t},z,\bar{z})|_{A=\bar{A}=0} &= \partial_{A^a}\partial_{\bar{A}^{\bar{b}}} K_{\rm m}(A,\bar{A},t,\bar{t},z,\bar{z})|_{A=\bar{A}=0} = e^{-\frac{1}{3}(K_{\rm cs}-K_{\rm ks})} G_{a\bar{b}} \ , \end{aligned} \tag{4.16}$$

where $G_{\alpha\bar\beta}$ and $G_{a\bar b}$ denote the metrics derived via (4.8) from $K_{\rm ks}, K_{\rm cs}$ in (4.15).

The K given in (4.14)–(4.16) is only its tree level contribution. K is corrected at any order in perturbation theory and also non-perturbatively. Generically, little is known about these corrections.

The gauge kinetic function turns out be universal at the tree level and given by the complexified dilaton[6]

$$f_{ab} = S\delta_{ab}\ . \tag{4.17}$$

f receives perturbative correction at one-loop but not beyond and non-perturbative correction.

The superpotential reads

$$W = Y_{abc}(z)A^aA^bA^c + Y_{\alpha\beta\gamma}(t)A^\alpha A^\beta A^\gamma + \mathcal{O}(A^4)\ , \tag{4.18}$$

where we suppress the gauge indices. The Yukawa couplings are the third derivative of the holomorphic prepotential $\mathcal{F}$ defined in Appendix B

$$Y_{abc} = \partial_{z^a}\partial_{z^b}\partial_{z^c}\mathcal{F}_{\rm cs}(z)\ , \qquad Y_{\alpha\beta\gamma} = \partial_{t^\alpha}\partial_{t^\beta}\partial_{t^\gamma}\mathcal{F}_{\rm ks}(t)\ . \tag{4.19}$$

Note that W does not depend on S at all and the Yukawa couplings do not depend on both types of moduli. The superpotential does not receive any perturbative correction and is only corrected non-perturbatively.

The (supersymmetric) minima of the potential V are the solutions of

$$D_AW = 0 = D^a\ . \tag{4.20}$$

For the case at hand the minimum is degenerate with $\langle A\rangle = 0$ and $\langle S\rangle, \langle t^\alpha\rangle, \langle z^a\rangle$ undetermined. This is consistent with the notion that they are moduli of Calabi-Yau manifolds. The Yukawa couplings are field dependent and thus could be dynamically determined. However, as they depend only on moduli fields they remain free parameters at least in perturbation theory.

[6]Strictly speaking different factors of the gauge group can have different normalizations labelled by an integer k called the Kac-Moody level of the SCFT [7].

5 Supersymmtry breaking and gaugino condensation

5.1 Supersymmetry breaking in supergravity

In any supersymmetric theory bosons (B) transform into fermions (F)

$$\delta B \sim F \ , \qquad \delta F \sim B \ . \tag{5.1}$$

If the vacuum (the background) is maximally symmetric (ie. preserves the Lorentz-group $SO(1, d-1)$) one needs $\langle F \rangle = 0$ while scalar fields can have a non-trivial background value $\langle B_{s=0} \rangle \neq 0$. Therefore $\langle \delta B \rangle = 0$ has to holds while $\langle \delta F \rangle|_{s=0}$ can be non-zero. In this case it signals spontaneous supersymmetry breaking or in other words $\langle \delta F \rangle|_{s=0}$ is the order parameter of supersymmetry breaking.

In $d = 4, N = 1$ theories the supersymmetry transformations of the fermions read

$$\begin{aligned} \text{chiral fermions}: &\quad \delta\chi^A \sim F^A \epsilon \ , \\ \text{gauginos}: &\quad \delta\lambda^a \sim g D^a \epsilon \ , \\ \text{gravitino}: &\quad \delta\psi_\mu \sim D_\mu \epsilon + i e^{\frac{1}{2}\kappa^2 K} W \sigma_\mu \bar{\epsilon} \ , \end{aligned} \tag{5.2}$$

where $F^A \sim e^{\frac{1}{2}\kappa^2 K} G^{A\bar{B}} \bar{D}_{\bar{B}} \bar{W}$ with W being the superpotential. Thus $\langle F^A \rangle$ and $\langle D^a \rangle$ are the order parameters of supersymmetry breaking.[7]

Unbroken supersymmetry thus corresponds to $\langle F^A \rangle = \langle D^a \rangle = 0$ which when inserted into (4.10) yields

$$\langle V \rangle = -3\kappa^2 \langle e^{\kappa^2 K} |W|^2 \rangle \leq 0 \ . \tag{5.3}$$

$\langle V \rangle$ plays the role of a cosmological constant and for $\langle W \rangle = \langle V \rangle = 0$ one has a Minkowski background. For $\langle W \rangle \neq 0$ follows $\langle V \rangle < 0$, i.e. one has an AdS-background. Note that a dS-background is incompatible with unbroken supersymmetry.

Broken supersymmetry corresponds to $\langle F^A \rangle \neq 0$ and/or $\langle D^a \rangle \neq 0$. In the following we concentrate on F-term breaking (ie. $\langle D^a \rangle = 0$). If in addition the cosmological constant vanishes, ie. $\langle V \rangle = 0$, one needs (cf. (4.10))

$$\langle |DW|^2 \rangle = 3\kappa^2 \langle |W|^2 \rangle \ . \tag{5.4}$$

In this case one defines the gravitino mass

$$m_{3/2}^2 := \kappa^4 \langle e^{\kappa^2 K} |W|^2 \rangle \tag{5.5}$$

as the scale of supersymmetry breaking.

[7] For $\langle F^A \rangle = \langle D^a \rangle = 0$ one can always find $\langle \delta\psi_\mu \rangle = 0$ which determines a Minkowski or AdS-background.

5.2 Supersymmetry breaking in String Theory

At the string tree-level supersymmetry is unbroken by construction and the cosmological constant vanishes. Indeed, the superpotential given in (4.18) obeys $\langle DW\rangle = \langle W\rangle = 0$. Thus supersymmetry can only be broken by quantum corrections.

As we recalled in the previous section the Lagrangian is characterized by the couplings K, W and f which do receive perturbative and non-perturbative quantum corrections. K is corrected at all orders while the holomorphicity of $W(\Phi)$ and $f(\Phi)$ lead to two perturbative non-renormalization theorems: $W(\Phi)$ receives no perturbative corrections [?] while $f(\Phi)$ is only corrected at one-loop order but has no further perturbative corrections [?]. Altogether one has

$$\begin{aligned} K &= \sum_{n=0}^{\infty} K^{(n)} + K^{(\mathrm{np})} \ , \\ W &= W^{(0)} + W^{(\mathrm{np})} \ , \\ f &= f^{(0)} + f^{(1)} + f^{(\mathrm{np})} \ , \end{aligned} \tag{5.6}$$

where the superscript (np) indicates possible non-perturbative corrections. These correction are in general non-universal and depend on the background under consideration. What is universal is the dilaton dependence of the couplings. As we discussed in the previous section $W^{(0)}$ is independent on the dilaton, $f^{(0)} = S$ and $K^{(0)} = -\ln(S + \bar{S}) + K^{(0)}(\Phi, \bar{\Phi})$ where Φ collectively denotes all other fields. $f^{(1)}$ is independet of the dilaton but can depend on Φ. The perturbative expansion in K is in fact an expansion in $(S + \bar{S})^{-1}$. Finally the non-perturbative corrections generically behave as e^{-S}.[8] So altogether we have in the heterotic string

$$\begin{aligned} K &= -\ln(S+\bar{S}) + K^{(0)}(\Phi,\bar{\Phi}) + \sum_{n=1}^{\infty} \frac{\hat{K}^{(n)}(\Phi,\bar{\Phi})}{(S+\bar{S})^n} + K^{(\mathrm{np})}(e^{-S},\Phi,\bar{\Phi}) \ , \\ W &= W^{(0)}(\Phi) + W^{(\mathrm{np})}(e^{-S},\Phi) \ , \\ f &= S + f^{(1)}(\Phi) + f^{(\mathrm{np})}(e^{-S},\Phi) \ , \end{aligned} \tag{5.7}$$

It is not possible to induce supersymmetry breaking perturbatively. This can be seen as follows

$$\langle D_A W\rangle = \langle \partial_A W + (\partial_A K) W\rangle = \langle D_A W^{(0)}\rangle + \langle (\partial_A K^{\mathrm{corr}}) W^{(0)}\rangle = 0 \ , \tag{5.8}$$

where in the last step we used that the first term vanish as supersymmetry is unbroken at the tree level while the second vanishes due to $\langle W^{(0)}\rangle = 0$. Thus supersymmetry can only be broken by non-perturbative effects which has the additional advantage that it might generate a hierarchy $m_{3/2} \ll M_{\mathrm{Pl}}$.

[8] Exceptions to this rule will be discussed in section **??**.

5.3 Non-perturbative effects in string theory

So far we only constructed string theories perturbatively as an expansion in topologies of the worldsheet. Therefore it is difficult to say something about the non-perturbative properties of the theory.[9] What has been done is to study the non-perturbative effects of effective field theory which certainly also are part of string theory and then assume that they dominate over 'stringy' non-perturbative contributions.

The prime example of a field-theoretic non-perturbative effect in supersymmetric theories is gaugino condensation. One considers a "hidden sector" with an asymptotically free supersymmetric gauge theory which is weakly coupled at $M_{\rm Pl}$.[10] The E_8 pure gauge theory of the Standard Embedding is a perfect example of this situation. The gauge couplings are scale dependent and in any QFT evolve according to

$$g^{-2}(\mu) = g^{-2}(M_{\rm Pl}) - \frac{b}{8\pi^2}\ln\frac{M_{\rm Pl}}{\mu} + \Delta\ , \quad \mu < M_{\rm Pl}\ , \tag{5.9}$$

where b is the one-loop coefficient of the β-function given by

$$\begin{aligned} b &= \tfrac{11}{3}T(G) - \tfrac{2}{3}\sum_{\mathbf{r}} n_{\mathbf{r}}^{\rm WF}T(\mathbf{r})\ + \tfrac{1}{6}\sum_{\mathbf{r}} n_{\mathbf{r}}^{\rm S}T(\mathbf{r}) \\ b_{N=1} &= 3T(G) - \sum_{\mathbf{r}} n_{\mathbf{r}}^{\rm C}T(\mathbf{r}) \end{aligned} \tag{5.10}$$

where $n_{\mathbf{r}}^{\rm WF}$ ($n_{\mathbf{r}}^{\rm S}$) counts the number of Weyl fermions (real scalars) in the representation $\mathbf{r}$ and we defined the indices of the gauge group according to

$$T(\mathbf{r})\ \delta^{ab} \equiv\ {\rm Tr}_{\mathbf{r}}(T^aT^b)\ , \qquad T(G)\ \equiv\ T(\text{adjoint of } G)\ . \tag{5.11}$$

In the second line of (5.10) we gave b for an $N = 1$ supersymmetric theory with $n_{\mathbf{r}}^{\rm C}$ counting the number of chiral multiplets. Δ in (5.9) denotes the IR-finite threshold corrections which arise from integrating out heavy states with masses $\mathcal{O}(M_{\rm Pl})$.

An asymptotically free gauge theory has $b > 0$ and becomes strong at the scale Λ where $g^{-2}(\mu = \Lambda) = 0$. Inserted into (5.9) this determines Λ to be

$$\Lambda\ =\ M_{\rm Pl}\, e^{-\frac{8\pi^2}{b}(g^{-2}(M_{\rm Pl})+\Delta)} < M_{\rm Pl}\ . \tag{5.12}$$

Thus a hierarchy $\frac{\Lambda}{M_{\rm Pl}} \ll 1$ is generated if g and/or b are small.

An effective theory below Λ in terms of gauge singlet has been derived in refs. [?, ?, ?]. One finds a superpotential

$$W(\Phi) \sim \Lambda_s^3(\Phi) \qquad \text{with} \qquad \Lambda_s(\Phi) = M_{\rm Pl}\, e^{-\frac{8\pi^2}{b}f(\Phi)}\ , \tag{5.13}$$

[9] We will discuss them in later in the context of string dualities.

[10] A hidden sector is defined by the absence of renormalizable couplings with the observable sector.

where $f(\Phi)$ is the gauge kinetic function. For the heterotic string one has

$$f = S + f^{(1)}(\Phi) + f^{(\mathrm{np})}(\Phi) \ , \tag{5.14}$$

where $f^{(1)}(\Phi)$ is independent of the dilaton. Comparing with the notation in (5.9) we identify

$$g^{-2}(M_{\mathrm{Pl}}) = \mathrm{Re}\, f^{(0)} = \mathrm{Re}\, S \ , \qquad \Delta = \mathrm{Re}\, f^{(1)} \ . \tag{5.15}$$

The potential derived from (5.13) reads

$$V \sim \frac{|\Lambda|^6}{M_{\mathrm{Pl}}^2} \ . \tag{5.16}$$

Its S-dependent part is depicted in Fig. 5.1 and shows the "dilaton problem" [**?**]. It is a "run-away" potential with a minimum at $\langle \mathrm{Re}\, S \rangle \to \infty$.

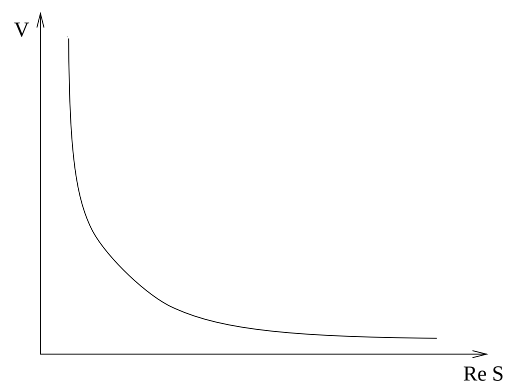


Figure 5.1: The "run-away" of the dilaton potential

This generic problem of all heterotic string vacua is surprisingly difficult to get around. One suggestion are the so called <u>race-track</u> scenarios [**?**, ?]. One considers two (or more) hidden asymptotically free gauge theories with gauge groups $G_{\mathrm{hidden}} = \times_a G_a$.[11] Each G_a has a different one-loop corrections $f^{(1)}$ so that the condensation scale for each group factor reads

$$\Lambda_a = \ M_{\mathrm{Pl}}\, e^{-\frac{8\pi^2}{b_a}(S + f_a^{(1)})}. \tag{5.17}$$

The resulting potential at leading order is

$$V \approx \frac{1}{M_{\mathrm{Pl}}^2} \left|\Lambda_1^3 + \Lambda_2^3\right|^2 \tag{5.18}$$

with a minimum at $|\Lambda_1| = |\Lambda_2|$. Inserting (5.17) one obtains

$$\langle \mathrm{Re}\, S \rangle \approx \frac{b_1 b_2}{b_1 - b_2} \left(\frac{f_1^{(1)}}{b_1} - \frac{f_2^{(2)}}{b_2} \right) \ , \tag{5.19}$$

[11]This cannot occur in the Standard Embedding but in generalizations one can break the hidden E_8.

where we need $b_1 > b_2$, $\frac{f_1^{(1)}}{b_1} > \frac{f_2^{(2)}}{b_2}$ for consistency.

An additional constraint arises from the fact that the tree level gauge coupling of the heterotic string is universal (cf. (4.17)) in that $\langle \mathrm{Re}\, S\rangle$ determines its value for the hidden and observable sector simultaneously. In the observable sector a reasonable estimate for the the size of $\langle \mathrm{Re}\, S\rangle$ is given by the GUT value

$$\langle \mathrm{Re} S\rangle \sim \frac{\alpha_{\mathrm{GUT}}}{4\pi} \sim \frac{23}{4\pi} \qquad \Rightarrow \qquad \langle \mathrm{Re} S\rangle \sim \mathcal{O}(2)\ . \tag{5.20}$$

We can also estimate the size of $\Delta = \mathrm{Re}\, f^{(1)}$. It arises from integrating out heavy modes with masses m_H of order $\mathcal{O}(M_{\mathrm{GUT}})$ or $\mathcal{O}(M_{\mathrm{Pl}})$ and thus can be estimated as $\Delta \sim \mathcal{O}(\frac{b}{8\pi} \ln \frac{m_h}{M_{\mathrm{Pl}}}) \sim \mathcal{O}(\frac{b}{8\pi})$. Thus $\frac{f}{b} \sim \mathcal{O}(\frac{1}{8\pi^2}) \sim \frac{1}{100}$. Therefore, for generic b we have

$$\langle \mathrm{Re} S\rangle \sim \frac{b}{8\pi^2}\ , \tag{5.21}$$

and thus need $b \sim \mathcal{O}(100)$ to achieve (5.20). (Note $b_{E_8} = 90$.)

Let us now estimate the scale of the possible supersymmetry breaking. Inserting (5.13) into (5.5) we have

$$m_{3/2} \approx \frac{\Lambda^3}{M_{\mathrm{Pl}}^2}\ , \tag{5.22}$$

so that for $\Lambda \sim 10^{13} - 10^{14} GeV$ one obtains $m_{3/2} \sim 10^1 - 10^3 GeV$ which is the 'desired' mass scale for low energy supersymmetry. For a Λ in that range we need $b \approx 22$, ie. a small b. We see that there is a tension between low energy supersymmetry and a phenomenological preferable gauge couplings.

One way out is to fine-tune the denominator in (5.19) such that the prefactor is large. This however, cannot be done at will as the rank of the hidden gauge group is bounded. For the Standard Embedding we have $\mathrm{rk}(E_8) = 8$ while for non-standard heterotic compactification one has the bound[12]

$$\mathrm{rk}(G_{\mathrm{hid}}) \leq 22\ . \tag{5.23}$$

For a hidden gauge group $G_{\mathrm{hid}} = SU(8) \times SU(9)$ one finds that both constraints are satisfied, ie. $\Lambda \sim 10^{14} GeV$ and $\langle \mathrm{Re} S\rangle \sim 2$. It is possible to further improve on this by having matter in the hidden sector. In this case the prefactor in (5.19) can be fine-tuned more easily.

Nevertheless, the racetrack scenarios have two remaining problems:

1. a negative cosmological constant, and
2. $\langle F^S\rangle = 0$, ie. supersymmetry is unbroken.

[12] The right moving central charge is $c_R = 26$ and the rank of the Standard Model gauge group subtracts $c_R(SM) = 4$.

This can be further improved by noting that in string theory $f^{(1)}$ is in general moduli-dependent and thus one has

$$W(S,\Phi) = M_{\rm Pl}^3\, e^{-\frac{24\pi^2}{b}(S+f^{(1)}(\Phi))}. \tag{5.24}$$

In addition this opens up the possibility of stabilizing the moduli at the same time. The computation of $f^{(1)}(\Phi)$ can be done via an appropriate string loop diagram where heavy states with moduli dependent masses $m = m(\Phi)$ contribute to $f^{(1)}(\Phi)$). Alternatively one can use the holomorphic anomaly (cf. Appendix C) to infer $f^{(1)}(\Phi)$). However the results depend on the background under consideration and no generic analysis or statement is possible. For specific background (orbifolds) the dependence on the untwistd moduli t is known. Minimization of the potential leads to

$$\langle F_S\rangle = 0\ , \quad \langle S\rangle \text{ fixed}\ , \quad \langle F_t\rangle \neq 0\ , \quad \langle t\rangle \text{ fixed}\ , \quad \langle V\rangle < 0\ . \tag{5.25}$$

Let us briefly summarize the lessons of this section:

- Gaugino condensation does induce a non-perturbative potential $V(S,\Phi)$ for the dilaton S and the moduli Φ.
- The perturbatively flat directions can be lifted and vacuum expectation values $\langle S\rangle > 1$ and $\langle\Phi\rangle$ can be generated.
- Supersymmetry can be broken by an F-term in the moduli direction $\langle F_\Phi\rangle \neq 0$.
- The cosmological constant is generically negative $\langle V\rangle < 0$

6 D-branes in type II Calabi-Yau compactifications

6.1 D-branes

If one includes open strings into string theory one needs to specify their boundary conditions (BC). One can have:

- Neumann BC

$$\partial_\sigma X^\mu(\sigma,\tau)|_{\sigma=0,l} = 0 \ , \qquad \mu = 0,\dots,p \ . \tag{6.1}$$

- Dirichlet BC

$$X^i(\sigma,\tau)|_{\sigma=0,l} = X_0^i \ , \qquad i = p+1,\dots,q \ . \tag{6.2}$$

- mixed DN-BC

$$X^i(\sigma,\tau)|_{\sigma=0,} = X_0^i \ , \qquad \partial_\sigma X^1(\sigma,\tau)|_{\sigma=l} = 0 \ . \tag{6.3}$$

This implies that that Dirichlet BC define a hyperplane where the string ends (see fig. 6.1).

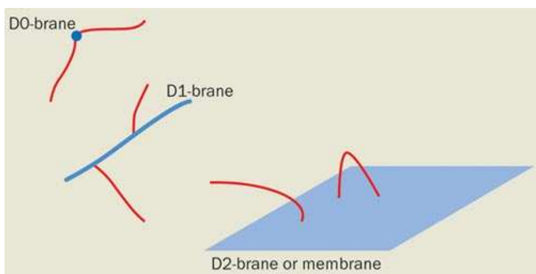


Figure 6.1: D-branes

The quantization proceeds as for the closed string with the BC taken into account. The D-Branes can be viewed as dynamical objects of string theory with excitations related to the attached open strings. In $D = 10$ the massless open string excitations are $N = 1$ vector multiplets in the adjoint of $SO(32)$. On a D_p-brane one has a $U(1)$ vector multiplet while on a stack of N D_p-branes one has a vector multiplet in the adjoint of $U(N)$. Note that the gauge theory is localized on the D-brane.

The D-brane action contains two pieces

$$S = S_{\text{DBI}} + S_{\text{CS}} \ . \tag{6.4}$$

S_{DBI} is a generalization of the Nambu-Goto action termed Dirac-Born-Infeld (DBI) action and is given by

$$S_{\text{DBI}} = -\mu_p \int_{W_{p+1}} dx^{p+1} e^{-\phi} \sqrt{-\det(P[G+B] - 2\pi\alpha' F)} \tag{6.5}$$

with a tension $\mu_p = (2\pi)^{-p}(\alpha')^{-\frac{1}{2}(p+1)}$. (Note that the physical tension includes the background value of the dilaton and thus is given by $\mu_{\rm phys} = g_s^{-1}\mu_p$.) W_{p+1} is the worldsheet of the brane and P denotes the pullback

$$P[G]_{\mu\nu} = G_{\mu\nu} + G_{\mu i}\partial_\nu x^i + \partial_\mu x^i G_{i\nu} + \partial_\mu x^i \partial_\nu x^j G_{ij} \ . \tag{6.6}$$

Finally F is the field strength of the $U(1)$ gauge boson.

The second term in (6.4) $S_{\rm CS}$ is the Chern-Simons action given by

$$S_{\rm CS} = \mu_p \int_{W_{p+1}} \left(P\big[\sum_q C_q\big] \wedge e^{(2\pi\alpha' F - B_2)} \wedge \hat{A}(R) \right)_{p+1} \ , \tag{6.7}$$

where C_q are the RR gauge potential and the A-roof polynomial reads

$$\hat{A}(R) = 1 - \frac{1}{24(8\pi^2)} {\rm Tr} R^2 + \ldots \ . \tag{6.8}$$

Expanding $S_{\rm CS}$ for $B_2 = 0$ one obtains

$$\begin{aligned} S_{\rm CS} = \mu_p \int_{W_{p+1}} C_{p+1} + 2\pi\alpha' \int_{W_{p+1}} C_{p-1} \wedge {\rm Tr} F \\ + \tfrac{1}{2} \int_{W_{p+1}} C_{p-3} \wedge {\rm Tr} F^2 - \frac{1}{24(8\pi^2)} \int_{W_{p+1}} C_{p-3} \wedge {\rm Tr} R^2 + \ldots \ . \end{aligned} \tag{6.9}$$

Remarks:

- D-branes carry RR-charge

$$\begin{aligned} Q_e &= \int_{S_{8-p}} *F_{p+2} = \ldots = \mu_p \ , \\ Q_m &= \int_{S_{p+2}} F_{p+2} = \ldots = \mu_{6-p} \ , \end{aligned} \tag{6.10}$$

  which satisfy a Dirac quantization condition

$$Q_e Q_m = 2\pi n \ , \quad n \in \mathbb{Z} \ . \tag{6.11}$$

- D-branes are BPS states and preserve half of the supercharges.
- D-branes are non-perturbative states in the sense that $\mu_{\rm phys} \sim g_s^{-1}$
- type IIA has $p =$ even Dbranes, type IIA has $p =$ odd Dbranes, type I has $p = 1, 5$ and the heterotic string has no D-branes.

D-branes also arise as (static) solitonic (ie. non-perturbative) solutions of the low energy effective supergravity. The solutions reads

$$
\begin{aligned}
ds^2 &= (Z(r))^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + (Z(r))^{1/2}dx^m dx^m \ , \\
e^{2\phi} &= (Z(r))^{\frac{1}{2}(3-p)} \ , \\
Z(r) &= 1 + \frac{\rho^{7-p}}{r^{7-p}} \ , \qquad r = \sum_m (x^m)^2 \ , \quad \rho^{7-p} \sim g_s N \ , \\
F_{8-p} &= \frac{N}{r^{8-p}} d(\mathrm{vol})_{S_{8-p}} = *F_{p+2} \ , \qquad \int_{S_{8-p}} F_{8-p} = N \ .
\end{aligned}
\tag{6.12}
$$

Here N is the number of D-branes and $x^0, \ldots, x^p$ are the coordinates along the brane while $x^{p+1}, \ldots, x^9$ are the coordinates transverse to the brane.

The solution has the following properties:

- It is Lorentz-invariant on the brane.
- It is rotationally invariant in the transverse space.
- For $r \to \infty$ one obtains flat $\mathbb{R}_{1,9}$.
- For small r one has a throat of size $g_s N$.
- It can be shown that these p-brane solutions are the supergravity approximation of string-theoretic D-branes.

6.2 Orientfolds

Since D-branes have RR-charge it seems at first problematic to have them on a compact manifold. One way out are orientifolds. The string background is modded out by an isometry ΩG which includes worldsheet parity Ω which acts as $\Omega : \sigma \to l - \sigma$. An example is that IIB/Ω = type I. We see that the projection removes half of the supercharges.

It is also possible to project within the same theory by including a space-time isometry G which includes an involution σ^*. For example, on S^1 $\sigma^* : X^9 \to -X^9$. σ^* has two fix-points at $X^9 = 0, \pi R$. They define eight-dimensional O_8-planes.

Calabi-Yau manifolds can only have discrete isometries which then act on the coordinates. Consistency requires

$$
\begin{aligned}
\text{IIA}: &\quad \sigma^*(\Omega_3) = \bar{\Omega}_3 \ , \qquad \sigma^*(J) = -J \ . \\
\text{IIB}: &\quad \sigma^*(\Omega_3) = \pm\Omega_3 \ , \qquad \sigma^*(J) = J \ .
\end{aligned}
\tag{6.13}
$$

By using a local representation

$$\Omega_3 \sim dz^1 \wedge dz^2 \wedge dz^3 \ , \qquad J \sim \sum_i dz^i \wedge d\bar{z}^i \ , \tag{6.14}$$

we can infer that for IIA and $z^i = x^i + iy^i$

$$\sigma^*(y^i) = -y^i \ , \qquad \sigma^*(x^i) = x^i \ . \tag{6.15}$$

This fixes three coordinates and defines O_6 orientifold planes. For type IIB the plus-sign in (6.13) fixes no or two coordinates corresponding to O_9 and O_5 orientifold planes, respectively, while the minus-sign in (6.13) fixes one or three coordinates corresponding to O_7 and O_3 orientifold planes, respectively. If we assume that the orientifold planes fill space-time $\mathbb{R}_{1,3}$ we have

$$\begin{array}{ll} O_3 \rightarrow & \text{point in } Y_3 \ , \\ O_5 \rightarrow & \text{wraps 2-cycle in } Y_3 \ , \\ O_5 \rightarrow & \text{wraps 3-cycle in } Y_3 \ , \\ O_5 \rightarrow & \text{wraps 4-cycle in } Y_3 \ . \end{array}$$

If a Calabi-Yau manifold admits an involution the cohomology groups split according to

$$H^{(p,q)} = H^{(p,q)}_+ \oplus H^{(p,q)}_- \ , \qquad h^{(p,q)} = h^{(p,q)}_+ + h^{(p,q)}_- \ , \tag{6.16}$$

where $H^{(p,q)}_+$ contains even forms under σ^* and $H^{(p,q)}_-$ contains odd forms. As a consequence of (6.13) one has

$$\begin{aligned} \text{IIB}: \quad & h^{(3,0)}_\pm = h^{(0,3)}_\pm = 1 \ , \quad h^{(3,0)}_\mp = h^{(0,3)}_\mp = 0 \ , \\ & h^{(0,0)}_+ = h^{(3,3)}_+ = 1 \ , \quad h^{(0,0)}_- = h^{(3,3)}_- = 0 \ , \\ \text{IIA}: \quad & h^{(1,1)}_\pm = h^{(2,2)}_\mp \ , \quad h^{(0,0)}_- = h^{(3,3)}_+ = 0 \ , \quad h^{(0,0)}_+ = h^{(3,3)}_- = 1 \ , \\ & h^3_+ = h^3_- = h^{(2,1)} + 1 \ , \end{aligned} \tag{6.17}$$

From worldsheet description of the uncompactified 10d theory one also finds

$$\begin{array}{llll} \text{IIB}: & \sigma^*\phi = \phi \ , & & \sigma^* l = \mp l \ , \\ & \sigma^* g = g \ , & & \sigma^* C_2 = \pm C_2 \ , \\ & \sigma^* B_2 = -B_2 \ , & & \sigma^* C_4 = \mp C_4 \ . \\ \text{IIA}: & \sigma^*\phi = \phi \ , & & \\ & \sigma^* g = g \ , & & \sigma^* C_1 = -C_1 \ , \\ & \sigma^* B_2 = -B_2 \ , & & \sigma^* C_3 = C_3 \ . \end{array} \tag{6.18}$$

Performing a KK-reduction keeping only the even (invariant) modes one obtains in type IIB the expansions

$$
\begin{array}{llll}
\text{IIB}: & & O3/O7 & O5/O9 \\
& J = & v\cdot\omega_+^{(1,1)} & v\cdot\omega_+^{(1,1)} \\
& \delta g_{ij} = & z\cdot\chi_-^{(1,2)} & z\cdot\chi_+^{(1,2)} \\
& B_2 = & b\cdot\omega_-^{(1,1)} & b\cdot\omega_-^{(1,1)} \\
& C_2 = & c\cdot\omega_-^{(1,1)} & b\cdot\omega_+^{(1,1)} \\
& C_4 = & D_2\cdot\omega_+^{(1,1)} + v_1\cdot\chi_+^{(1,2)} & D_2\cdot\omega_-^{(1,1)} + v_1\cdot\chi_-^{(1,2)}
\end{array}
\tag{6.19}
$$

This results in the spectrum given in Tables 6.1 and 6.2. For type IIA a similar analysis can be found in the literature [6].

gravity multiplet	1	$g_{\mu\nu}$
vector multiplets	$h_+^{(2,1)}$	V
chiral multiplets	$h_-^{(2,1)}$	z
	1	(ϕ, l)
	$h_-^{(1,1)}$	(b, c)
chiral/linear multiplets	$h_+^{(1,1)}$	(v, ρ)

Table 6.1: $N = 1$ spectrum of $O3/O7$-orientifold compactification.

gravity multiplet	1	$g_{\mu\nu}$
vector multiplets	$h_-^{(2,1)}$	V
chiral multiplets	$h_+^{(2,1)}$	z
	$h_+^{(1,1)}$	(v, c)
chiral/linear multiplets	$h_-^{(1,1)}$	(b, ρ)
	1	(ϕ, C_2)

Table 6.2: $N = 1$ spectrum of $O5/O9$-orientifold compactification.

6.3 D-branes on Calabi-Yau manifolds

The worldsheet analysis of modding out by orientation reversal Ω on the worldsheet shows that O-planes carry no physical degree of freedom but do carry tension and RR-charge. In that sense they can be viewed as a topological defect. A consistency condition (tadpole cancellation) implies

$$Q_{O_p} = -2^{p-4} Q_{D_p} \ , \tag{6.20}$$

where Q denotes the RR-charge. Therefore consistent theories can be constructed by adding D-branes and O-planes simultaneously. One commonly requires that the D-branes and O-planes preserve a common $N = 1$ supersymmetry. This BPS-condition translates into geometric conditions on the Calabi-Yau manifold.

In type IIA space-time filling D_6-branes wrap a three-cycle Σ_3. If $N = 1$ supersymmetry is preserved this three-cycle has to be special Lagrangian. This demands

$$J|_{\Sigma_3} = 0 = \mathrm{Im}\,\Omega_3|_{\Sigma_3} \ , \tag{6.21}$$

with the volume of the cycle given by

$$\mathrm{vol}(\Sigma_3) = \int_{\Sigma_3} \mathrm{Re}\,\Omega_3 \ . \tag{6.22}$$

This volume is minimal expressing the supersymmetry condition. Submanifolds where the volume is computed by the integral of a closed, non-degenerate p-form are called calibrated submanifolds.

In type IIB space-time filling D_5/D_7 branes wrap holomorphic two- and four-cycles $\Sigma_{2,4}$. Their volume is

$$\mathrm{vol}(\Sigma_p) = \int_{\Sigma_p} J^{\frac{p}{2}} \ . \tag{6.23}$$

6.4 D-brane model building

Including D-branes it is possible to construct backgrounds which include (generalizations) of the MSSM within type II string theory. This "model building" is vast field which cannot possible be reviewed here. Let us just assemble a few facts/remarks and point the reader to the literature for further detaiils [6].

- IIA

  Many explicit model building is done for toroidal orientifold constructions. Generically one finds a gauge group $G = \prod_a U(N_a)$ with (chiral) matter in the bifundamental $(\mathbf{N_a}, \mathbf{\bar{N}_b})$ whenever the cycles intersect.

- IIB

  To construct realistic (chiral) models a background gauge flux has to be turned on on the D-branes. Therefore we discuss it in some later lecture. As an intermediate step one often discusses/constructs local model where the D-branes configuration is such that the MSSM or some generalization thereof is realized. In a second step this is embedded into a globally consistent Calabi-Yau compactification. This can be achieved by placing the D-branes at collapsed cycles (singularities) and then blow up the singularity.

- In both cases the gauge coupling is given by

$$g_a^{-2} \sim \mathrm{vol}(\Sigma_a) \neq \mathrm{Re}\, S \ . \tag{6.24}$$

  Therefore the problem met in heterotic compactifications of generating the hierarchy $m_{3/2} \ll M_{\mathrm{Pl}}$ and at the same time having a consistent g_{GUT}^{-2} is absent in D-brane models. However, these models have no gauge unification build in and are much less predictive.

7 Flux compactifications

7.1 General discussion

As we already discussed, string theories do have gauge potential C_{p-1} with a field strength $F_p = dC_{p-1}$. It turns out that under certain condition the F_p can have non-trivial background values – called background fluxes. One demands that they obey the Bianchi identity and satisfy the equations of motion, i.e.

$$dF_p = 0 \ , \qquad d^* F_p = 0 \ . \tag{7.1}$$

Here we only consider fluxes that do not break the four-dimensional Lorentz symmetry. Therefore, on $\mathbb{R}_{1,3}$ only F_4 can have a background flux which has been considered as a source for the cosmological constant [13, 14].[13] On the Calabi-Yau manifold Y_3 (7.1) implies that F_p can be expanded in terms of harmonic forms ω_p^I

$$F_p \ = \ e_I \, \omega_p^I \ , \qquad \omega_p \in H^p(Y) \ , \tag{7.2}$$

where the coefficients e_I (often called flux parameters) have to be constant. Integrating (7.2) over a dual p-cycle γ_p^I yields

$$\int_{\gamma_p^J \in Y} F_p = e_I \int_{\gamma_p^J \in Y} \omega_p^I = e^J \ , \tag{7.3}$$

where in the second step we used the duality of the p-cycle.

Before we proceed let us make a few remarks:

- By itself $e^J \neq 0$ is inconsistent on a compact manifold. However, as we will see, if localized sources such as D-branes/O-planes are present it is possible to turn on background fluxes on the Calabi-Yau manifold.
- The e_I have to obey a Dirac-type quantization condition and thus are discrete parameters in string theory. However in the low energy/large volume approximation they appear as continuous parameters which deform the low energy supergravity.
- If one keeps the e_I as small perturbations the light spectrum does not change and they turn the low energy supergravity into a gauged or massive supergravity where the fluxes e_I appear as additional gauge couplings, mass parameters or FI-terms. Furthermore, a potential is generated which potentially lifts the vacuum degeneracy of string theory and can stabilize moduli and spontaneously break supersymmetry.
- The background fluxes e_I introduce many new discrete parameters into string theory. This enlarges the number of consistent vacua or background tremendously and is called the landscape of string vacua.

[13] We return to this mechanism in section ??.

7.2 The no-go theorem

Starting from an Ansatz for a warped space-time

$$ds^2 = e^{2A(y)} ds^2_{\mathbb{R}_{1,3}}(x) + ds^2_{Y_3}(y) \ , \tag{7.4}$$

where $A(y)$ is called the warp-factor, the Einstein equations imply

$$R + \tfrac{1}{2} e^{2A(y)}(-T^{\mu}_{\mu} + +T^m_m + T_{\rm loc}) = 2e^{-2A(y)} \nabla^2_y e^{2A(y)} \ , \tag{7.5}$$

where we also included the possibilty of localized sources (D-branes and O-planes) which contribute to the energy momentum tensor $T_{\rm loc}$. For flux background one can show $T \sim F^2$ and $-T^{\mu}_{\mu} + T^m_m > 0$ while $T_{\rm loc}$ can be negative. Therefore integrating (7.5) yields

$$\int_{Y_3} e^{2A(y)}(R + \tfrac{1}{2} e^{2A(y)}(-T^{\mu}_{\mu} + T^m_m + T_{\rm loc})) = 2 \int_{Y_3} \nabla^2_y e^{2A(y)} = 0 \ , \tag{7.6}$$

where the first term is proportional to the cosmological constant. In the absence of localized sources the second is always positive and thus one can have at best an AdS-background but no Minkowski or de Sitter background is consistent. This is the no-go theorem formulated in refs. [11, 12]. However, once localized sources and in particular O-planes are present Minkowski or de Sitter background can appear [15].

7.3 Supersymmetry in flux background

As we already noted the amount of unbroken supersymmetry can be obtained from inspecting the fermionic transformation laws. For type II they read [10]

$$\begin{aligned}
\delta\Psi_M &= D_M \epsilon + \tfrac{1}{4}\gamma^{NP} H_{MNP} + \frac{1}{16} \sum_n \tfrac{1}{n!} \gamma^{P_1 \dots P_n} F_{P_1 \dots P_n} \gamma_M P_n \epsilon + \dots \ , \\
\delta\lambda &= (\gamma^M \partial_M \phi + \tfrac{1}{2}\gamma^{MNP} H_{MNP})\epsilon + \frac{1}{8} e^{\phi} \sum_n (-1)^n (5-n) \gamma^{P_1 \dots P_n} F_{P_1 \dots P_n} P_n \epsilon + \dots \ ,
\end{aligned} \tag{7.7}$$

where P, P_n are projection operators which can be found in [10].

Before we discuss flux backgrounds let us review the situation for vanishing fluxes, i.e. for $\langle F_p \rangle = \langle H \rangle = \langle \partial\phi \rangle = 0$. In this case the dilatino variation in (7.7) automatically obeys $\langle \delta\lambda \rangle = 0$ while the gravitino variation collapses to $\langle \delta\Psi_M \rangle = \langle D_M \epsilon \rangle$. Unbroken supersymmetry thus implies in this case $\langle D_M \epsilon \rangle = 0$ and as a consequence

$$[D_M, D_N]\epsilon = R_{MNPQ} \gamma^{PQ} \epsilon = 0 \ , \tag{7.8}$$

where we omit the $\langle \cdot \rangle$ henceforth. Approprite contraction and using properties of the γ-matrices one arrives at [2]

$$R_{\mu\nu}\gamma^{\nu}\epsilon = 0 \ , \qquad R_{mn}\gamma^n \epsilon = 0 \ . \tag{7.9}$$

The first equation implies that among the maximally symmetric backgrounds (AdS, dS, Minkowski) with $R_{\mu\nu} \sim \Lambda g_{\mu\nu}$ only a Minkowski background with $\Lambda = 0$ can preserve supersymmetry. In this case ϵ is a constant supersymmetry parameter. The second equation in (7.9) implies that Y_6 has to be Ricci-flat consistent with our discussion in section 3.

In case some of the fluxes are non-vanishing one has two basic options:

1. One imposes $\langle \delta\Psi_M \rangle = \langle \delta\lambda \rangle = 0$ for some supercharges. In this case generically the geometry has to backreact and has to be deformed away from Calabi-Yau manifolds.

2. One allows $\langle \delta\Psi_M \rangle = \langle \delta\lambda \rangle \neq 0$ but insists that a spinor ϵ (or equivalent a supercharges) is globally well defined on Y_6. In this case one obtains backgrounds with spontaneously broken supersymmetry. We will return to this case in the next section.

There is ine exception to option 1 in type IIB. Defining

$$G_3 := \hat{F}_3 - \tau H_3 = F_3 - ie^{-\phi} H_3 \ , \tag{7.10}$$

with $\tau = l + ie^{-\phi}$, $\hat{F}_3 = F_3 + lH_3$ and imposing

$$^*G_3 = iG_3 \ , \qquad G_{(0,3)} = 0 \ , \qquad F_{5\mu\nu\rho\sigma m} = \epsilon_{\mu\nu\rho\sigma}\partial_m A \ , \tag{7.11}$$

one finds $\langle \delta\Psi_M \rangle = \langle \delta\lambda \rangle = 0$. Consistency of the Einstein equation requires D-branes and O-planes to be present and to obey the tadpole cancellation condition

$$N_{D_3} - \tfrac{1}{4} N_{O_3} + \tfrac{1}{(2\pi)^4 \alpha'} \int_{Y_3} H_3 \wedge F_3 = 0 \ . \tag{7.12}$$

7.4 The low energy effective action for type II Calabi-Yau compactification with background fluxes

The KK-reduction in the NS sector uses again (B.16) and (B.22). The NS three-form flux is implemented by

$$H_3 \ = \ m^{\mathrm{NS}\,A} \alpha_A - e_B^{\mathrm{NS}}\, \beta^B \ , \tag{7.13}$$

where (α_A, β^B) is a real, symplectic basis of $H^3(Y)$ satisfying (B.29). In the RR-sector we need to distinguish between type IIA and type IIB.

7.4.1 Type IIA

For the RR gauge potential C_1, C_3 one uses

$$C_1 = A^0_\mu(x) dx^\mu \ , \qquad C_3 = A^\alpha_\mu(x) dx^\mu \omega^\alpha + \xi^A(x)\, \alpha_A - \tilde{\xi}_B(x)\, \beta^B \ , \tag{7.14}$$

where $\alpha = 1, \ldots, h^{(1,1)}, A = 0, \ldots, h^{(2,1)}$. Here $\xi^A, \tilde{\xi}_B$ are four-dimensional scalars while A^0, A^α are vector fields. Without orientifold projection the fields assemble into $N = 2$ multiplets as summarized in table 7.1. The scalar geometry is unchanged and of the form discussed in (D.11).

gravity multiplet	1	$(g_{\mu\nu}, A^0)$
vector multiplets	$h^{(1,1)}$	(A^α, t^α)
hypermultiplets	$h^{(2,1)}$	$(z^a, \xi^a, \tilde{\xi}_a)$
tensor multiplet	1	$(B_2, \phi, \xi^0, \tilde{\xi}_0)$

Table 7.1: Bosonic components of $N = 2$ multiplets for type IIA compactified on a Calabi-Yau threefold

The RR-fluxes are implemented by

$$F_2 = -m^{\mathrm{RR}\,\alpha}\omega_\alpha \ , \qquad F_4 = e^{\mathrm{RR}}_\alpha \tilde{\omega}^\alpha \ , \tag{7.15}$$

where $\tilde{\omega}^\alpha$ are harmonic $(2,2)$-forms which form a basis of $H^{(2,2)}(Y)$ dual to the $(1,1)$-forms ω_α in that

$$\int_{Y_3} \omega_\alpha \wedge \tilde{\omega}^\beta = \delta_\alpha^\beta \ . \tag{7.16}$$

The effect of the fluxes in the effective action can be seen by inspecting

$$|F_2|^2 \sim (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + m^{\mathrm{RR}\,\alpha} m^{\mathrm{RR}\,\bar{\beta}} g_{\alpha\bar{\beta}} \tag{7.17}$$

and

$$\begin{aligned} \hat{F}_4 &= dC_3 - \tfrac{1}{2}B_2 \wedge F_2 - \tfrac{1}{2}H_3 \wedge A_1 \\ &= (\partial_\mu A_\nu - B_{\mu\nu} m^{\mathrm{RR}\,\alpha})\, \omega^\alpha dx^\mu dx^\nu - B_{\mu\nu}\partial_\rho A_\sigma dx^\mu dx^\nu dx^\rho dx^\sigma \\ &\quad + (D_\mu \xi^A \alpha_A - D_\mu \tilde{\xi}_B \beta^B)\, dx^\mu + \mathrm{Re}\, t^\alpha m^{\mathrm{RR}\,\beta} d_{\alpha\beta\gamma} \tilde{\omega}^\gamma \ , \end{aligned} \tag{7.18}$$

where

$$D_\mu \xi^A = \partial_\mu \xi^A - m^{\mathrm{NS}\,A} A^0_\mu \ , \qquad D_\mu \tilde{\xi}_B = \partial_\mu \tilde{\xi}_B - e^{\mathrm{NS}}_B A^0_\mu \ . \tag{7.19}$$

We see that terms contributing to the scalar potential are generated, $\xi^A, \tilde{\xi}_B$ become charged with charges given by the fluxes and $B_{\mu\nu}$ becomes massive via Stueckelberg mechanism. The full effective Lagrangian is discussed in [16]

7.4.2 Type IIB

In type IIB the RR gauge potentials C_2, C_4 are KK expanded as

$$
\begin{aligned}
C_2 &= C^0_{\mu\nu}(x)dx^\mu dx^\nu + \xi^\alpha(x)\,\omega_\alpha \ , \\
C_4 &= C^\alpha_{\mu\nu}(x)dx^\mu dx^\nu \wedge \omega_\alpha + \tilde{\xi}^\alpha(x)\,\tilde{\omega}_\alpha + A^A_\mu(x)dx^\mu \wedge \alpha_A + \tilde{A}_{\mu A}(x)dx^\mu \wedge \beta^A \ .
\end{aligned}
\tag{7.20}
$$

The self-duality condition of F_5 eliminates half of the degrees of freedom in C_4 and one conventionally chooses to eliminate $C^\alpha_{\mu\nu}$ and the magnetic vector $\tilde{A}_{\mu A}$ in favor of $\tilde{\xi}^\alpha$ and A^A_μ. Altogether these fields assemble into $N = 2$ multiplets which are summarized in Table 7.2. The scalar geometry is again unchanged and of the form discussed in (D.11).

gravity multiplet	1	$(g_{\mu\nu}, A^0)$
vector multiplets	$h^{(2,1)}$	(A^a, z^a)
hypermultiplets	$h^{(1,1)}$	$(v^\alpha, b^\alpha, \xi^\alpha, \tilde{\xi}_\alpha)$
double-tensor multiplet	1	(B_2, C^0_2, ϕ, l)

Table 7.2: Bosonic componets of $N = 2$ multiplets for Type IIB compactified on a Calabi-Yau manifold

The RR fluxes are implemented by

$$
F_3 \ = \ m^{\mathrm{RR}\,\mathrm{A}}(\tau)\,\alpha_A - e^{\mathrm{RR}}_A(\tau)\,\beta^A \ , \tag{7.21}
$$

which can be combined with (7.13) to give

$$
G_3 \ = \ m^I(\tau)\,\alpha_I - e_I(\tau)\,\beta^I \ , \tag{7.22}
$$

where

$$
e_I(\tau) \ = \ e^{\mathrm{RR}}_I - \tau\, e^{\mathrm{NS}}_I \ , \qquad m^I(\tau) \ = \ m^{\mathrm{RR}\,I} - \tau\, m^{\mathrm{NS}\,I} \ . \tag{7.23}
$$

and $G_3 \equiv F_3 - \tau H_3, \tau \ = \ l + i\mathrm{e}^{-\phi}$. Inserted into the effective action one finds again a potential, charged scalars and massive two-forms [16].

8 Moduli stabilization and supersymmetry breaking by fluxes

In lecture 6 we discussed type II compactifications on Calabi-Yau manifolds with D-branes and O-planes which lead to $N = 1$ unbroken supersymmetry in $d = 4$ with undetermined moduli. In the previous section (Section 7) we discussed that turning on background fluxes in $N = 2$ Calabi-Yau compactifications does generate a scalar potential. In this lecture we consider fluxes in Calabi-Yau orientifold compactification and discuss the properties of the resulting backgrounds.

8.1 $\mathcal{L}_{\text{eff}}$ for Calabi-Yau orientifold compactification with D-branes

Concretely let us focus on type IIB Calabi-Yau compactification with D_3-branes and O_3-planes. (Other cases are discussed in [6, 10].) The light spectrum contains charged chiral multiplets arising as excitation from the D_3-branes and chiral multiplets arsing from the bulk. The latter are given in Table 6.1. Let us set $h_-^{(1,1)} = 0$ for simplicity or in other words freeze the scalars arising from the KK-expansion of B_2 and C_2. In this case the Kähler potential is found to be

$$K = -\ln(\tau - \bar{\tau}) + K_{\text{ks}}(t, \bar{t}) + K_{\text{cs}}(z, \bar{z}) + K_{\text{m}}(A, \bar{A}, t, \bar{t}, z, \bar{z}) \ , \tag{8.1}$$

where τ is the complexified type II dilaton and $K_{\text{m}}(A, \bar{A}, t, \bar{t}, z, \bar{z})$ the Kähler potential for the chiral multiplets A arising from the D_3-branes. For $K_{\text{ks}}, K_{\text{cs}}$ one finds (cf. (B.27), (B.30))

$$K_{\text{cs}}(z, \bar{z}) = -\ln\Big[-i\int_Y \Omega \wedge \bar{\Omega}\Big] \ , \qquad K_{\text{ks}}(t, \bar{t}) = -\ln Y \ , \tag{8.2}$$

where Y and the chiral coordinates are given by[14]

$$Y = d_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma \ , \qquad t^\alpha = \frac{\partial Y}{\partial v^\alpha} + i\rho^\alpha \ , \tag{8.3}$$

The superpotential reads

$$W = W_{\text{m}}(A, t, z) + W_{\text{GVW}}(z, \tau) \tag{8.4}$$

where

$$W_{\text{m}} = Y_{ijk} A^i A^j A^k + \dots \ , \qquad W_{\text{GVW}} = \int G_3 \wedge \Omega \ , \tag{8.5}$$

and $G_3 = F_3 - \tau H_3$ (cf. (7.22)).

[14] They differ in ordinary Calabi-Yau compactification and Calabi-Yau orientifold compactification. Note that in general one cannot express Y in terms of the t^α or in other words K_{ks} is not explicitly known in terms of the proper chiral coordinates.

For the matter fields $D_{A^i}W = 0$ is solved by $\langle A^i \rangle = 0$ implying $\langle W_{\rm m} \rangle = 0$. Computing the remaining F-terms for $\langle W_{\rm m} \rangle = 0$ yields

$$D_\tau W \sim \int \bar{G}_3 \wedge \Omega \ , \tag{8.6}$$

$$D_{t^\alpha} W \sim v^\alpha W \ , \tag{8.7}$$

$$D_{z^a} W \sim \int G_3 \wedge \omega^a_{2,1} \ , \tag{8.8}$$

where in the last equation we used $\partial_{z^a}\Omega = \partial_{z^a}K\Omega + \omega^a_{2,1}$. A supersymmetric minimum requires that all F-terms vanish. For (8.6) this requires $G_{3(3,0)} = 0$, for (8.7) this requires $W = G_{3(0,3)} = 0$, for (8.8) this requires $G_{3(1,2)} = 0$. Altogether we have a supersymmetric minimum only if $G_3 = G_{3(2,1)} \neq 0$. However, since W does not depend on the Kähler moduli t^α they remain flat directions of such minima. z^a and τ on the other hand are generically fixed.

If flux components in any of the other directions are turned on supersymmetry is spontaneously broken. For $G_3 = G_{3(0,3)} \neq 0$ one has $W \neq 0$ and $D_{t^\alpha} W \neq 0$. However $\langle V \rangle = 0$ still holds due to the no-scale property of $K_{\rm ks}$. Let us pause for a moment and review no-scale supergravity at this point.

8.2 No-scale supergravity

The definition of no-scale supergravity is not unique in the literature and can denote one of the two situations:

(i) $V \equiv 0$ (which one might call "strict/strong no-scale").

(ii) $V \geq 0$ (which one might call "weak no-scale").

Examples of (i) are supergravities with

$$W = \text{const.} \ , \qquad K_i g^{i\bar{j}} K_{\bar{j}} = 3 \ . \tag{8.9}$$

(The second condition is often called the "no-scale" condition.) In this case one has

$$D_i W = K_i W \ , \qquad D_i W g^{i\bar{j}} D_{\bar{j}} \bar{W} = 3|W|^2 \ , \tag{8.10}$$

and thus $V \equiv 0$ for V given by (4.10) is satisfied. For one chiral field the simplest Kähler potential satisfying the no-scale condition is

$$K = -3 \ln(t + \bar{t}) \ , \tag{8.11}$$

which indeed follows from (8.2) for $h^{(1,1)}_+ = 1$.

Examples of (ii) also occur in type IIB flux compactifications. Since K in (8.1) is a sum of independent terms and $K_{\rm ks}$ satisfies the no-scale condition (8.9) the potential in (4.10) reads

$$\begin{aligned} V &= e^{\kappa^2 K}\left(|D_t W|^2 + |D_z W|^2 + |D_\tau W|^2 + |D_A W|^2 - 3\kappa^2 |W|^2\right) \\ &= e^{\kappa^2 K}\left(|D_z W|^2 + |D_\tau W|^2 + |D_A W|^2\right) \geq 0 \ . \end{aligned} \tag{8.12}$$

In this case the minimum is at $\langle V \rangle = 0$ for $D_z W = D_\tau W = D_A W = 0$ but the t^α remain unfixed. This is a generic property of all tree level potential in type IIB.

8.3 Adding quantum corrections

Let us return to type II compactifications. In type IIA the flux superpotential reads [17]

$$W = \int H_3 \wedge \Omega_c + \int F_2 \wedge J_c^2 + \int F_4 \wedge J_c \ , \tag{8.13}$$

where J_c is the complexified Kähler form and $\Omega_c = \mathrm{Re}\,\Omega + iC_3$. Minimization of the potential leads to supersymmetric AdS_4 with $t^\alpha, \mathrm{Re}\, z^a$ fixed but axions from C_3 undetermined.

In both type IIA and type IIB the situation can be improved by

1. deforming the Calabi-Yau manifold,
2. adding quantum corrections.

Let us concentrate on the second point and postpone the discussion of the first point to App. E. In type IIB the superpotential can receive non-perturbative corrections for example from gaugino condensation on (hidden) D7-branes (and other branes instanton effects). Generically one has

$$W_{\rm np} \sim e^{-2\pi n_\alpha t^\alpha} \ . \tag{8.14}$$

The Kähler potential (8.1) is already corrected at one loop with the correction appearing in $K_{\rm ks}$ and given by [18]

$$K_{\rm ks}(t,\bar t) = -2\ln\left(Y + \zeta(3)\chi(Y_3)\left(\frac{-i(\tau-\bar\tau)}{2}\right)^{3/2}\right) , \tag{8.15}$$

where $\chi(Y_3)$ is the Euler number of Y_3 and $\zeta(3)$ the Riemann ζ-function.

In the KKLT analysis only one Kähler modulus is non-trivial, $K_{\rm ks}$ is taken at the tree level while the considered superpotential reads [19]

$$W = W_0 + e^{-2\pi t} \ , \tag{8.16}$$

where $W_0 = W_{\rm GVW}$ is evaluated at $\langle z^a\rangle, \langle\tau\rangle$. In this case one finds the minimum to be supersymmetric $\rm AdS_4$ with $\langle t\rangle \neq 0$. However, one needs $W_0 M_{\rm Pl}^{-3}$ to be exponentially small in order to have $\langle t\rangle$ large which is required for a consistent (supergravity) analysis. A small W_0 can be arranged by a fine-tuning of fluxes and once achieved also leads to a small $m_{3/2}$.

In a second step one has to "uplift" this minimum to $\mathbb{R}_{1,3}$ or $\rm dS_4$. In [19] this is achieved by adding an explicit supersymmetry breaking anti-$\overline{D3}$ brane. This has been critized in that the stability of this non-supersymmetric configuration has been questioned.

Alternative possibilties discussed are D-term uplifts where some non-supersymmetric gauge flux is turned on on some hidden (D7) brane. This generates a D-term which in turn provides a positive contribution to the potential (4.10) [6]. Kähler uplifts use quantum corections to the Kähler potential to provide an extra positive contribution. One of the prominent examples are the large volume scenarios (LVS) which we briefly discuss now.

In LVS one considers $h_+^{(1,1)} = 2$ and couplings

$$\begin{aligned} K &= -2\ln\Big(\tfrac{1}{9\sqrt{2}}\big((\mathrm{Re}\, t_b)^{3/2} - (\mathrm{Re}\, t_s)^{3/2}\big) + \tfrac{1}{2}\zeta g_s^{-3/2}\Big) \\ W &= W_0 + A_s e^{-2\pi n_s t_s} \end{aligned} \tag{8.17}$$

where t_b, t_s are the two Kähler moduli. We abbreviated $\zeta \sim \zeta(3)\chi$ which in LVS has to be positive $\zeta > 0$. In this case minimization leads to non-supersymmetric $\rm AdS_4$ backgrounds with no fine-tuning for W_0 necessary. The competition of exponential terms in t_s with power-law terms of t_b in the potential leads to $\mathrm{Vol}(Y_3) \sim \langle t_b\rangle^{3/2} \gg \langle t_s\rangle^{3/2}$ so that one can trust the supergravity analysis.

In generalizations with $h_+^{(1,1)} > 2$ one can arrange a similar structure such that the overall volume is large but all other cycles are small. Such Calabi-Yau manifolds have been termed "swiss-cheese" Calabi-Yaus.

The next step is to compute the soft supersymmetry breaking terms for the various scenarios. Here we refer to the literature [6, 10].

9 Dualities in string theory

Let us recall the parameters that we encountered so far. First of all there are the string-scale/string-length/tension M_s, l_s, T which are related by (1.6). They are related to the measured value of $M_{\rm Pl}$. Then there are the dimensionless string coupling $g_s \sim e^{\langle\phi\rangle}$ and the background values of the moduli $\langle t^\alpha\rangle, \langle z^a\rangle$ which in Calabi-Yau compactifications are free, countinuous parameters spanning the moduli space $\mathcal{M}$ of a given string background. Finally, there is a (discrete) choice of the background consisting of the choice of the compact manifold Y_d and the background fluxes.

The basic idea of a duality is that there exists map which relates different regions of $\mathcal{M}$. This map might differ in that it relates different regions in $\mathcal{M}$

(i) of the same (string) theory,

(ii) of the different (string) theories.

Furthermore, the map might hold

(A) perturbatively (i.e. at weak string coupling $g_s \ll 1$),

(B) non-perturbatively (i.e. the map involves g_s and includes $g_s = \mathcal{O}(1)$).

Let us discuss examples of these cases in turn [26, 27].

(Ai) Here the standard example is T-duality of the bosonic string in $\mathbb{R}_{1,9-d} \times T^d$. For $d = 1$ the mass spectrum includes states with masses

$$m^2(R, r, s) = r^2 R^{-2} + s^2 M_s^4 R^2 + \text{const.}\ , \qquad r, s \in \mathbb{Z}\ , \tag{9.1}$$

where R is the radius of the circle. The first term corresponds to the familiar masses of KK-states while the second term are masses of the string-specific winding states. This mass spectrum has a symmetry

$$m^2(\tfrac{1}{M_s^2 R}, s, r) = m^2(R, r, s)\ . \tag{9.2}$$

T-duality states that the mass spectrum and all interactions of this theory are invariant under

$$R \leftrightarrow \frac{1}{M_s^2 R}\ , \qquad r \leftrightarrow s\ . \tag{9.3}$$

We see that g_s is not involved in the transformation and therefore it is of type A) and since (9.3) acts within the same theory it is also of type i). $R = M_s^{-1}$ is the fixed point of the transformation and has been discussed as a possible minimal length scale in string theory. For $d > 1$ the T-duality transformation are elements of the group $SO(d, d, \mathbb{Z})$.

(Aii) In this case the examples are:

- IIA in $\mathbb{R}_{1,8} \times S^1(R) \equiv$ IIB in $\mathbb{R}_{1,8} \times S^1(M_s^{-2}R^{-1})$,
- Heterotic $E_8 \times E_8$ in $\mathbb{R}_{1,8} \times S^1(R) \equiv$ Heterotic $SO(32)$ in $\mathbb{R}_{1,8} \times S^1(M_s^{-2}R^{-1})$,
- IIA in $\mathbb{R}_{1,3} \times Y_3 \equiv$ IIB in $\mathbb{R}_{1,3} \times \tilde{Y}_3$ (mirror symmetry).

(Bi) This situation is often called S-duality and occurs in IIB in $\mathbb{R}_{1,9}$. This theory at the tree level has a continuous $SL(2,\mathbb{R})$ symmetry acting on the complex scalar τ and the two forms B_2, C_2 as we discussed in (2.17). It is expected that this symmetry is broken by non-perturbative (space-time instanton) effects and terms of the form $e^{-i\tau}$ appear. It is however conjectured that a discrete $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ with $a, b, c, d \in \mathbb{Z}$ survives. From (2.17) we learn that this includes the two transformations

1. $\tau \to \tau + 1$ for $a = b = d = 1, c = 0$ which is a shift symmetry for the RR-scalar l and which redefines the RR 2-form as $F_3 \to F_3 + H_3$.
2. $\tau \to -1/\tau$ for $a = d = 0, b = -c = 1$. For $l = 0$ it includes a strong-weak duality symmetry $g_s^{-1} \leftrightarrow g_s$ and exchanges $F_3 \leftrightarrow -H_3$. Therefore it implies a relation between the perturbative and the non-perturbative spectrum and interactions. However, this cannot be checked with the present understanding of string theory and thus there is no proof of this conjecture to date.

The evidence for the S-duality conjecture comes from BPS-states. These are states which are anihilated by some of the supercharges $Q|BPS\rangle = 0$. As a consequence the supermultiplets for BPS states are "shorter" then ordinary massive multiplets. For BPS particles one has $M = Z$ where Z is the central charge of the supersymmetry algebra and M the mass of the multiplet. For branes one has $T = Z$ where T is the tension and Z now is the central charge of the extended object. One expects that the BPS condition is respected by non-perturbative physics as it only depends on the existence of Q and its superalgebra. Therefore the duality map should also be realized on BPS-states. In IIB one has the fundamental string F_1 which couples to the NSS two-form B_2 and the odd branes $D_{1,3,5}$ coupling to $C_{2,4,6}$. C_4 is anti self-dual while C_2 is Poincare dual to C_6. Furthermore, the D-branes are non-perturbative BPS states as their tension goes like $T \sim g_s^{-1}$. The conjectured $SL(2,\mathbb{Z})$ relates

$$
\begin{aligned}
B_2 \leftrightarrow C_2 \quad &\Leftrightarrow \quad F_1 \leftrightarrow D_1 \ , \\
C_4 \leftrightarrow C_4^* \quad &\Leftrightarrow \quad D_3 \text{ self-dual} \ , \\
B_6 \leftrightarrow C_6 \quad &\Leftrightarrow \quad F_5 \leftrightarrow D_5 \ ,
\end{aligned}
\tag{9.4}
$$

where F_5 denotes an NS five-brane which indeed can be constructed as a supergravity solution. (It is however, still not well understood.)

Other conjectured examples which display an S-duality are:

- Heterotic in $\mathbb{R}_{1,3} \times T^6$,
- type II in $\mathbb{R}_{1,9-d} \times T^d$ which is conjectured to even have a U-duality that combines the $SL(2,\mathbb{Z})$ S-duality and the $SO(d,d,\mathbb{Z})$ T-duality into a bigger group $E_{d,d}(\mathbb{Z})$ called the U-duality group

$$SL(2,\mathbb{Z}) \times SO(d,d,\mathbb{Z}) \subset E_{d,d}(\mathbb{Z}) \subset E_{d,d}(\mathbb{R}) \ . \tag{9.5}$$

Here $E_{d,d}(\mathbb{R})$ is the continues non-compact symmetry group of supergravities with $q = 32$ supercharges.

(Bii) Examples of this situation are

- Heterotic $SO(32)$ in $\mathbb{R}_{1,9} \equiv$ type I in $\mathbb{R}_{1,9}$, where

$$\begin{aligned} g_{\text{Het}} &\sim g_I^{-1} \\ F_1 &\leftrightarrow D_1 \\ F_5 &\leftrightarrow D_5 \end{aligned} \tag{9.6}$$

  Thus the Heterotic $SO(32)$ theory and the type I theory are only different description of different regimes in the moduli space of one and the same quantum theory.

- IIA in $\mathbb{R}_{1,5}\times$ K3 $\equiv$ Heterotic in $\mathbb{R}_{1,5} \times T^4$, where also the couplings are related by $g_{\text{Het}} \sim g_{IIA}^{-1}$. Both backgrounds have the same moduli space

$$\mathcal{M} = \frac{SO(4,20)}{SO(4) \times SO(20)} \times \mathbb{R}^+ \ , \tag{9.7}$$

  where $\mathbb{R}^+$ is parameterized by the two couplings. The heterotic gauge group G_{het} can be non-Abelian and one has $G_{\text{het}} \subset SO(32)/E_8 \times E_8$. On the type II side the gauge group naively is $G_{\text{II}} = [U(1)]^{16}$. However, K3 has A-D-E-type singular loci in $\mathcal{M}$ where two-cycles shrink. A D2-branes wrapping these two-cycles gives rise to massless gauge bosons in gauge groups of A-D-E-type.

- IIA in $\mathbb{R}_{1,3} \times CY_3 \equiv$ Heterotic in $\mathbb{R}_{1,5} \times$ K3 $\times T^4$. In this case the type IIA coupling g_{IIA} is related to a geometric modulus of K3 while g_{het} is related to a geometric modulus of the Calabi-Yau. This duality is discussed in more detail in appendix F.

10 M-theory

10.1 $d = 11$ Supergravity and its S^1 compactification

In $d = 11$ there is only one supergravity with 32 supercharges and in that sense it is unique. The massless multiplet contains the metric $g_{\hat{M}\hat{N}}$, $\hat{M}, \hat{N} = 0, \ldots, 10$, an antisymmetric 3-index tensor $C_{\hat{M}\hat{N}\hat{P}}$ and a gravitino $\Psi_{\hat{M}}$ which together have $44 + 84$ bosonic and 128 fermionic degrees of freedom. The bosonic action is [7]

$$S = \tfrac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g}\big(R - \tfrac{1}{2}|F_4|^2\big) - \tfrac{1}{6} \int C_3 \wedge F_4 \wedge F_4 \tag{10.1}$$

where $F_4 = dC_3$ is the field strength of the three-form. Apart from diffeomorphism invariance and local supersymmetry there is gauge invariance related to the three-form

$$\delta C_3 = d\Lambda_2 \,, \qquad \delta F_4 = 0 \,, \tag{10.2}$$

where Λ_2 is a 2-form and $dF_4 = 0$.

IIA supergravity can be obtained as an S^1 compactification of $d = 11$ supergravity. In terms of the spectrum one has

$$\begin{aligned} g_{\hat{M}\hat{N}} \quad &\to \quad g_{MN} \ , \quad g_{M10} \sim C_M \ , \quad g_{10,10} \sim \phi \ , \\ C_{\hat{M}\hat{N}\hat{P}} \quad &\to \quad C_{MNP} \ , \quad C_{MN10} \sim B_{MN} \ , \\ \Psi_{\hat{M}} \quad &\to \quad \Psi_M \ , \quad \Psi_{10} \sim \lambda \ . \end{aligned} \tag{10.3}$$

The Lagrangian of $d = 11$ supergravity in the background $\mathbb{R}_{1,9} \times S^1$ agrees with the Lagrangian given in (2.5) with the identification

$$R_{11} = g_s\sqrt{\alpha'} \ , \qquad \kappa_{11}^2 \ = 2\pi R_{11}\kappa_{10}^2 = \tfrac{1}{2}(2\pi)^8 g_s^3 (\alpha')^{9/2} = \tfrac{1}{4\pi}(2\pi l_{11})^9 \ . \tag{10.4}$$

This in turn implies

$$l_{11} = g_s^{1/3}\sqrt{\alpha'} \ , \qquad R_{11} = g_s^{2/3} l_{11} \ , \tag{10.5}$$

which means

$$g_s \to \infty \quad \hat{=} \quad R_{11} \to \infty \ , \quad \text{for } l_{11} \text{ fixed} \ . \tag{10.6}$$

Therefore the strong coupling limit of type IIA is a quantum theory of $d = 11$ supergravity which has been termed M-theory [28].

10.2 The strong coupling limit of type IIA

Type IIA has even D_p-branes with tension $T_p = 2\pi g_s^{-1}(4\pi^2\alpha')^{-\frac{1}{2}(p+1)}$. Thus the lightest D-brane (with the lowest T_p) is a D_0-brane/D-particle with a tension/mass $T_0 = g_s^{-1}(\alpha')^{-\frac{1}{2}}$. Thus a bound state of n D-particles has masses

$$m_n^2 = \frac{n^2}{g_s^2\alpha'} = \frac{n^2}{R_{11}^2} \ . \tag{10.7}$$

This is precisely the KK-tower of the circle compactified $d = 11$ supergravity. Furthermore, one can show from the supersymmetry algebra

$$\{Q, Q\} = \gamma^{\hat{M}} p_{\hat{M}} = \gamma^M p_M + \gamma^{10} p_{10} \ , \tag{10.8}$$

that the term proportional to p_{10} acts like a central charge and thus that the KK-spectrum given in (10.7) is a BPS spectrum. Or in other words, the type IIA D-particles correspond to $d = 11$ KK BPS states. Thus it is legitimate to extrapolate to strong coupling and observe that for $g_s \to \infty$ an infinite tower of BPS states becomes massless and assemble in the massless multiplet of $d = 11$ supergravity. Since there is no string theory with this low energy supergravity the quantum theory behind it must be something other than a string theory [28].

Before we proceed let us note that for the type IIA D-branes one has the following correspondence:

$$\begin{aligned}
D_2 \text{ -brane} \quad &\to \quad d = 11 \text{ membranes } M_2 \\
D_4 \text{ -brane} \quad &\to \quad M_5 \text{ wrapped on } S^1 \\
D_6 \text{ -brane} \quad &\to \quad \text{KK-monopole (magnetic dual of D-particle)}
\end{aligned}$$

$M_{2,5}$ as well as the KK-monopole are known as supergravity solutions.

10.3 Strong coupling limit of the heterotic $E_8 \times E_8$ string

If one compactifies $d = 11$ supergravity on an interval $I = S^1/\mathbb{Z}_2$ with a $\mathbb{Z}_2$ action

$$\mathbb{Z}_2 : \quad X^{10} \to -X^{10} \ , \qquad C_3 \to -C_3 \ , \tag{10.9}$$

the $\mathbb{Z}_2$-invariant states in $R_{1,9}$ are

$$g_{MN} \ , \quad g_{10,10} \sim \phi \ , \quad C_{MN10} \sim B_{MN} \ , \tag{10.10}$$

while g_{M10} and C_{MNP} are projected out. The fields listed in (10.10) correspond to the $N = 1$ gravitational multiplet in $R_{1,9}$.

However, in this situation one also needs to include a so called twisted sector in the Hilbert space where $X(\sigma + 2\pi, \tau) = \theta(X(\sigma, \tau))$ with $\theta \in \mathbb{Z}_2$. Since the quantum theory is unknown one cannot compute this twisted sector. Instead [29] infer from anomaly cancellation that at each endpoint of the interval I there are ten-dimensional fixed planes which each have to support an E_8 gauge theory. Since (10.5) and (10.6) again hold in this compactification one can conclude that in the strong coupling limit of the heterotic $E_8 \times E_8$ string an extra dimensions opens up.

Other strong coupling limits related to $d = 11$ supergravity are:

- Heterotic in $\mathbb{R}_{1,6} \times T^3 \quad \overset{g_s \to \infty}{\longrightarrow} \quad$ M in $\mathbb{R}_{1,6} \times K3$,
- IIB in $\mathbb{R}_{1,5} \times K3 \quad \overset{g_s \to \infty}{\longrightarrow} \quad$ M in $\mathbb{R}_{1,5} \times T^5/\mathbb{Z}_2$,
- Heterotic in $\mathbb{R}_{1,4} \times K3 \times S^1 \quad \overset{g_s \to \infty}{\longrightarrow} \quad$ M in $\mathbb{R}_{1,4} \times Y_3$,

A summary of the various strong coupling limits is depicted in Fig. 10.1.

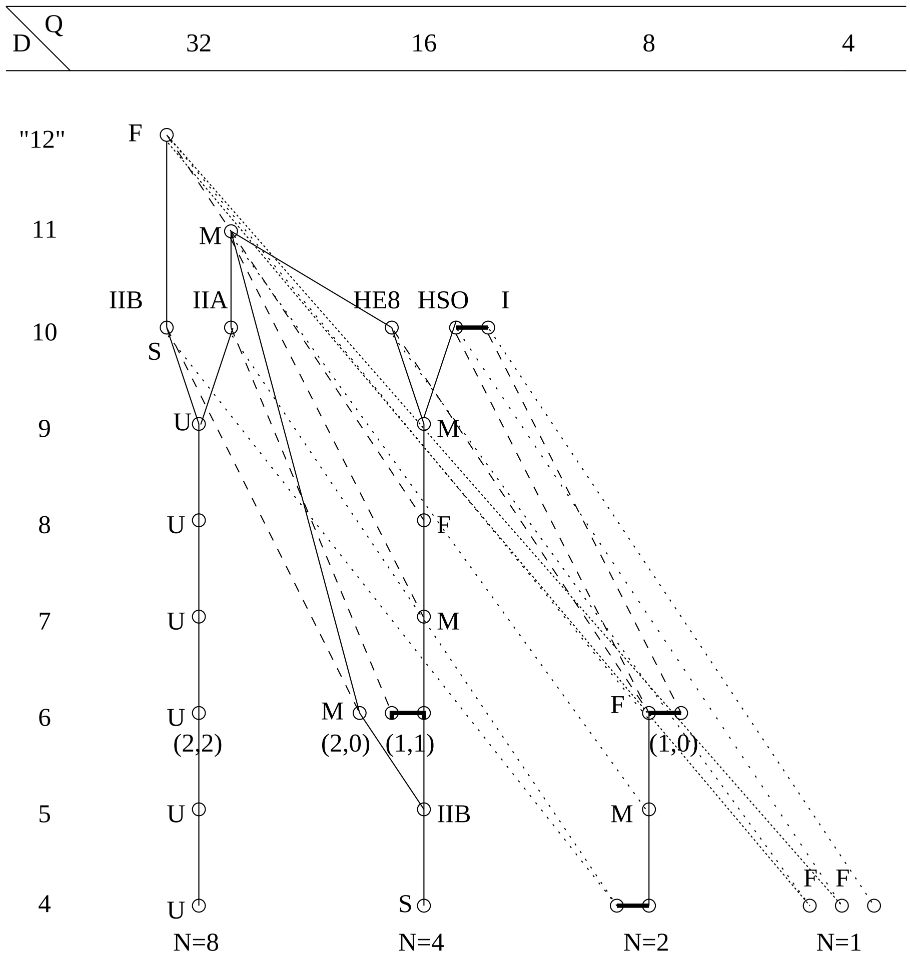


Figure 10.1: String theories and dualities

10.4 What is M-theory

The conjectures of the last two lectures suggest that all string theories are different perturbative limits of one and the same quantum theory called M-theory. Or in other words, M-theory has a moduli space (sketched in Fig. 10.2) where the cusp regions correspond to some parameter becoming small. In that region a string theoretic and/or supergravity description exists. Since there exists a limit where $d = 11$ supergravity appears it is clear that M-theory cannot be a string theory. It is also clear that M-theory does include higher-dimensional objects (D-branes) which become light in certain regions of the moduli space. In [30] it was proposed that M-theory is the quantum theory of D-particles. One of the exciting features of this proposal is that space-time becomes non-commutative.

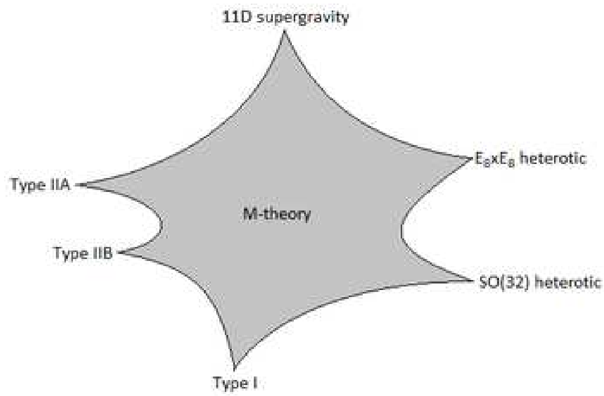


Figure 10.2: Moduli space of M-theory

10.5 Compactification of M-theory on G_2 manifolds

So far we did not find any strong coupling dual of backgrounds in $\mathbb{R}_{1,3}$ with $N = 1$ supersymmetry. Therefore it is interesting to study M-theory in the background $\mathbb{R}_{1,3} \times Y_7$ and demand

$$\delta\psi_{\hat{M}} = D_{\hat{M}}\epsilon + \ldots = 0 \ , \tag{10.11}$$

for one spinor ϵ exactly as we did in Section 7.3. In $d = 11$ supergravity ϵ transform in the $\mathbf{32}$ of $SO(1, 10)$ which has a decomposition under $SO(1, 3)$ analogous to (4.2)

$$\begin{aligned} SO(1, 10)) &\to SO(1, 3) \times SO(7) \\ \mathbf{32} &\to (\mathbf{2}, \mathbf{8}) + (\bar{\mathbf{2}}, \bar{\mathbf{8}}) \ , \end{aligned} \tag{10.12}$$

where $\mathbf{8}$ is a spinor of $SO(7)$. Therefore we need a seven-dimensional manifold Y_7 with a holonomy H such that

$$\mathbf{8} \to \mathbf{7} + \mathbf{1} \ , \tag{10.13}$$

with $\mathbf{7}, \mathbf{1} \in H$. Indeed such a decomposition exists for $H = G_2$ where G_2 is an exceptional group with rk(G_2)=2 and dim(G_2)=14. Seven-dimensional manifolds with G_2-holonomy have been constructed by D. Joyce as orbifolds $T^7/\mathbb{Z}_2^3$ (Joyce-manifold) and are termed G_2-manifolds [31]. These backgrounds break 7/8 of the supercharges and thus leave $N = 1$ (four supercharges) unbroken in $\mathbb{R}_{1,3}$.

Similar to Calabi-Yau manifolds G_2 manifolds are Ricci-flat and have a covariantly constant real three-form ϕ_3 which is closed and co-closed

$$d\phi_3 = d^*\phi = 0 \ . \tag{10.14}$$

As in (E.1) ϕ_3 is constructed as a spinor bi-linear

$$\phi_{mnp} = \epsilon\gamma_{mnp}\epsilon \ . \tag{10.15}$$

Let us close with some remarks:

- G_2 manifolds are difficult to construct explicitly and so far only orbifolds (generalizations of the Joyce manifold) are known.
- Smooth G_2 compactifications have an Abelian gauge group $G = [U(1)]^{b_2}$ where $b_2 = \dim(H_2)$ and a non-chiral spectrum.
- G_2 manifolds can have ADE-singularties leading to non-Abelian gauge groups and a chiral spectrum. These compactifications are related to intersecting D_6-brane models of type IIA [6].

11 F-theory

F-theory was introduced in [32] in order to offer a geometrical understanding of the (conjectured) non-perturbative $SL(2,\mathbb{Z})$ symmetry of type IIB. In addition it serves as a compactification scheme which provides the "missing" strong coupling limits.

Any torus can be characterized as a two-dimensional lattice in a complex plane with coordinate z and the identification

$$z \approx z + n + m\tau \ , \qquad n, m \in \mathbb{Z} \ , \quad \mathrm{Im}\,\tau > 0 \ . \tag{11.1}$$

In this parametrization one of the two periods of the torus has been normalized to 1 and n is the corresponding winding number while the second period is characterized by τ and the corresponding winding number is m. However, two τ's related by an $SL(2,\mathbb{Z})$ transformation

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \ , \quad ad - bc = 1 \ , \quad a, b, c, d \in \mathbb{Z} \ , \tag{11.2}$$

parametrize the same torus. Therefore all inequivalent tori correspond to τ being in the fundamental domain

$$\mathcal{F} = \{-\tfrac{1}{2} \leq \mathrm{Re}\,\tau < \tfrac{1}{2} \ , \ |\tau| \geq 1\} \ . \tag{11.3}$$

Geometrically τ corresponds to the complex structure of the torus and its $SL(2,\mathbb{Z})$ symmetry is also called the modular group.

In F-theory the $SL(2,\mathbb{Z})$ of type IIB is interpreted as the modular group of an (auxiliary) torus. This T^2 is auxiliary in that type IIB cannot be interpreted as a KK-reduction of a theory in $\mathbb{R}_{1,9} \times T^2$. The reason is that there is no representation of supersymmetry in $\mathbb{R}_{1,11}$ with 32 supercharges and the volume of the T^2 is not in the type IIB spectrum in $\mathbb{R}_{1,9}$.

One way to make the definition of F-theory more precise is to use the (conjectured) duality

$$\mathrm{M\ in\ } \mathbb{R}_{1,8} \times T^2 \stackrel{g_s \to 0}{\longrightarrow} \quad \mathrm{IIA\ in\ } \mathbb{R}_{1,8} \times S^1(R) \quad \equiv \quad \mathrm{IIB\ in\ } \mathbb{R}_{1,8} \times S^1(R^{-1}) \ .$$

Sending $R \to 0$ we have

$$\mathrm{M\ in\ } \mathbb{R}_{1,8} \times T^2(\mathrm{vol}(T^2) = 0) \stackrel{g_s \to 0}{\longrightarrow} \quad \mathrm{IIB\ in\ } \mathbb{R}_{1,9} \ .$$

At this point the introduction of F-theory might seem a bit convoluted. However it becomes more interesting in further compactifications and new non-trivial backgrounds can be constructed. Let us consider

$$\mathrm{M\ in\ } \mathbb{R}_{1,6} \times K3 \ ,$$

where the K3 is elliptically fibred. This means the K3 has a base $B = \mathbb{P}_1$ with T^2-fibres. The T^2 varies over the base in that $\tau = \tau(z, \bar{z})$ with z being the complex coordinate of the $\mathbb{P}_1$. Taking the limit $\text{vol}(T^2) \to 0$ we thus have a construction of

$$\text{IIB in } \mathbb{R}_{1,7} \times B \ .$$

It seems that this is a compoactification of type IIB which does not feature a Calabi-Yau manifold. The reason is that the type IIB dilaton τ is not constant but varies over B as $\tau = \tau(z, \bar{z})$. However, as it stands this compactifications is inconsistent. The equation of motion for τ derived from $\mathcal{L} \sim (\text{Im}\,\tau)^{-2} \partial_M \tau \partial^M \bar{\tau}$ reads in the z-direction

$$\partial_z \bar{\partial}_{\bar{z}} \tau - (\text{Im}\,\tau)^{-1} \partial_z \tau \bar{\partial}_{\bar{z}} \bar{\tau} = 0 \ , \tag{11.4}$$

with a solution $\bar{\partial}_{\bar{z}} \bar{\tau} = 0$. This says that the fibration is holomorphic, i.e. $\tau = \tau(z)$. However, there is a complication as τ transforms under (11.2) while z does not! Therefore consider a solution of the form

$$j(\tau) = \left(\frac{z_0}{z}\right)^N \ , \tag{11.5}$$

where $j(\tau)$ is the modular invariant j-function which has a series expansion in $q = e^{2\pi i \tau}$

$$j(\tau) = q^{-1} + 744 + 196884q + \mathcal{O}(q^2) \ . \tag{11.6}$$

Near $z \sim 0$ one thus has

$$\tau = \frac{N}{2\pi i} \ln \frac{z}{z_0} \ . \tag{11.7}$$

Thus $z \to 0$ corresponds to $\text{Im}\,\tau \to \infty$ which is the type IIB weak coupling limit. From (11.5) or (11.7) one sees that τ is multivalued which is physically non-sensible. The way out is to add space-time filling D_7-branes which are points on the $\mathbb{P}_1$-base. They induce a deficit angle into the solution and precisely for 24 D_7-branes a single valued solution can be constructed [2]. $\text{Im}\,\tau =$ constant and large does not exist on the entire $\mathbb{P}_1$ and therefore the solution is inherently non-perturbative.

The nature and the location of the singularity can be seen from the Weierstrass-representation of the torus. One introduces two complex variables x, y with one complex condition

$$y^2 = x^3 + fx + g \ , \qquad f, g \in \mathbb{C} \ . \tag{11.8}$$

f and g are related to τ via

$$j(\tau) = \frac{4(24f)^3}{\Delta} \ , \qquad \text{where} \qquad \Delta = 4f^3 + 27g^2 \ . \tag{11.9}$$

For an elliptic fibration one has

$$y^2 = x^3 + f_8(z)\, x + g_{12}(z) \ , \tag{11.10}$$

where $f_8(z)$ and $g_{12}(z)$ are polynomials of degree 8 and 12 respectively. We thus see that the discriminant Δ has 24 roots where $\Delta = 0$ which corresponds to the location of the D_7-branes.[15] If the 24 branes are at differents points on the base the K3 is smooth. If singularties coincide the K3 is singular and one has a non-Abelian gauge enhancement. To summarize, F-theory can be viewed as non-perturbative IIB compactifications with D7-branes.

Further remarks:

- There is a limit called the Sen-limit where τ =const. almost everywhere on the base B [34]. In this limit the K3 has a description as an orientifold with singular couplings at the point where O_7-planes sit.

- Considering an orientifold of IIB in $\mathbb{R}_{1,7} \times T^2$ there is a T-duality to type I in $\mathbb{R}_{1,7} \times T^2$. Since type I is S-dual to the heterotic SO(32) string which in turn is T-dual to the heterotic $E_8 \times E_8$ one has the following chain of dualities in $\mathbb{R}_{1,7} \times T^2$ [2]:

$$\begin{array}{ccccccc} \text{IIB orientifold} & \overset{T}{\longleftrightarrow} & \text{type I} & \overset{S}{\longleftrightarrow} & \text{het. SO(32)} & \overset{T}{\longleftrightarrow} & \text{het. } E_8 \times E_8 \\ g_s^{\text{IIB}} & & g_s^{\text{I}} \sim \frac{g_s^{\text{IIB}}}{T^{\text{IIB}}} & & g_s^{\text{HSO}} \sim \frac{T^{\text{IIB}}}{g_s^{\text{IIB}}} & & g_s^{\text{HE}} \sim T^{\text{IIB}} \\ T^{\text{IIB}} & & T^{\text{I}} \sim \frac{1}{T^{\text{IIB}}} & & T^{\text{HSO}} \sim \frac{1}{g_s^{\text{IIB}}} & & T^{\text{HE}} \sim g_s^{\text{IIB}} \end{array}$$

  where T is the Kähler modulus of the torus. This implies in particular for the $E_8 \times E_8$ heterotic string

$$\text{Heterotic in } \mathbb{R}_{1,7} \times T^2 \quad \overset{g_s^{\text{HE}} \to \infty}{\longrightarrow} \quad \text{F in } \mathbb{R}_{1,7} \times K3^E,$$

  where g_s^{HE} corresponds to the $\mathbb{P}_1$-base of the elliptic K3.

- Similarly one has

$$\text{Heterotic in } \mathbb{R}_{1,5} \times K3 \overset{g_s \to \infty}{\longrightarrow} \quad \text{F in } \mathbb{R}_{1,5} \times Y_3^E,$$

  and

$$\text{Heterotic in } \mathbb{R}_{1,3} \times Y_3 \overset{g_s \to \infty}{\longrightarrow} \quad \text{F in } \mathbb{R}_{1,3} \times Y_4^E,$$

  where Y_3^E, Y_4^E are elliptic Calabi-Yau threefolds and fourfolds respectively.

  The various F-theory dualities are also summarized in Fig. 10.1.

[15] This is another way to see the necessity of 24 D_7-branes.

For phenomenological applications the last duality is of particular interest. There has been a lot of activity recently in F-theory model building in connection with the construction of Grand Unified Theories (GUTs) [6]. There are mainly two interesting aspects:

1. At the intersection of two D7-branes the **16**-dimensional spinor representation of SO(10) can appear which is not possible within the perturbative heterotic string. Since the matter representation of the Standard Model with an extra right-handed neutrino precisely reside in this representation SO(10) GUTs can be constructed.

2. The up-type Yukawa coupling $\mathbf{10}\cdot\mathbf{5}\cdot\bar{\mathbf{5}}$ of SU(5) GUTs can appear. Again this is not possible within the perturbative heterotic string and thus also SU(5) GUT model building has been pursued recently.

Appendix

A Supersymmetry in arbitrary dimensions

A.1 Spinor representations of $SO(1, D-1)$

The spinor representations of $SO(1, D-1)$ are constructed from Dirac matrices γ^M satisfying the Clifford/Dirac algebra

$$\{\gamma^M, \gamma^N\} = 2\eta^{MN} , \qquad M, N = 0, \ldots, D-1 . \tag{A.1}$$

Then the operators

$$\Sigma^{MN} := \tfrac{1}{4} [\gamma^M, \gamma^N] \tag{A.2}$$

satisfy the $SO(1, D-1)$ algebra and thus are generator of (the spinor representations of) $SO(1, D-1)$.

Concretely let us consider $SO(1, D-1)$ for D even.[16] We choose $D = 2k+2$, $k = 0, 1, 2, \ldots$ and define

$$\begin{aligned} \gamma^{0\pm} &:= \tfrac{1}{2}\left(\pm\gamma^0 + \gamma^1\right) , \\ \gamma^{a\pm} &:= \tfrac{1}{2}\left(\gamma^{2a} \pm i\gamma^{2a+1}\right) , \quad a = 1, \ldots, k , \\ \gamma^{A\pm} &:= \left(\gamma^{0\pm}, \gamma^{a\pm}\right) , \qquad\quad A = 0, \ldots, k . \end{aligned} \tag{A.3}$$

Inserting these definitions into (A.1), one obtains the relations

$$\{\gamma^{A+}, \gamma^{B-}\} = \delta^{AB} , \qquad \{\gamma^{A\pm}, \gamma^{B\pm}\} = 0 . \tag{A.4}$$

This corresponds to the algebra of $k+1$ fermionic creation and annihilation operators (oscillators). One can construct the Dirac representation from the a Clifford vacuum $|\Omega\rangle$ defined by

$$\gamma^{A-}|\Omega\rangle = 0 , \qquad \forall A . \tag{A.5}$$

The states are constructed by acting with γ^{A+} in all possible ways on $|\Omega\rangle$ using $\left(\gamma^{A+}\right)^2 = 0$. The (complex) dimension of the Dirac representation thus is

$$n = \dim_{\mathbb{C}} (\text{Dirac rep.}) = \sum_{i=0}^{k+1} \binom{k+1}{i} = 2^{k+1} . \tag{A.6}$$

For $D = 4$ we have $k = 1$ and thus $n = 2^2 = 4$. For $D = 2$ we have $k = 0$ and thus $n = 2$. Let us exemplary construct the matrix representation for $D = 2$ explicitly. The only non-zero matrices are γ^{0+} and γ^{0-} with

$$\gamma^{0+}|\Omega\rangle = |1\rangle , \qquad \gamma^{0-}|1\rangle = |\Omega\rangle . \tag{A.7}$$

[16] Here we follow Appendix B of Vol II of [7].

Therefore we can read off the matrix representation

$$\gamma^{0+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \gamma^{0-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{A.8}$$

and thus according to (A.3)

$$\gamma^{0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.9}$$

The construction for arbitrary k can be obtained similarly [7].

It is possible to define a 'generalized γ_5' by

$$\gamma_{D+1} := i^k \gamma^0 \gamma^1 \dots \gamma^{D-1}, \tag{A.10}$$

satisfying

$$\{\gamma_{D+1}, \gamma^M\} = 0, \qquad \left[\gamma_{D+1}, \Sigma^{MN}\right] = 0, \qquad (\gamma_{D+1})^2 = 1. \tag{A.11}$$

Then one can define two projection operators, $1 \pm \gamma_{D+1}$, that split the Dirac representation into two Weyl representations with eigenvalues ± 1. The dimension of the Weyl representation thus is

$$\dim_{\mathbb{C}} (\text{Weyl rep.}) = 2^k. \tag{A.12}$$

One can check that $\left(\gamma^M\right)^*$ and $\left(-\gamma^M\right)^*$ both satisfy the Dirac algebra (A.1). Since the previous construction was unique both have to be similar to γ^M itself. Indeed one defines

$$B_1 := \gamma^3 \cdots \gamma^{D-1}, \qquad B_2 := \gamma_{D+1} B_1, \tag{A.13}$$

and shows

$$B_1 \gamma^M B_1^{-1} = (-1)^k \left(\gamma^M\right)^*, \qquad B_2 \gamma^M B_2^{-1} = (-1)^{k+1} \left(\gamma^M\right)^*. \tag{A.14}$$

i.e., for any k a similarity transformation exists. Furthermore

$$B_{1,2} \gamma_{D+1} B_{1,2}^{-1} = (-1)^k \left(\gamma_{D+1}\right)^*, \tag{A.15}$$

so that for k even, i.e., $D = 2, 6, 10, \dots$, the Weyl representation is its own conjugate (s.c.), while for k odd, i.e., $D = 4, 8, \dots$, the Weyl representations are conjugate to each other (c.c.). From

$$B_{1,2} \Sigma^{MN} B_{1,2}^{-1} = -\left(\Sigma^{MN}\right)^* \tag{A.16}$$

it follows that both ψ and $B^{-1}\psi^*$ obey the same Lorentz transformation law, i.e.,

$$\delta\psi = i\omega_{MN} \Sigma^{MN} \psi, \qquad \delta B^{-1}\psi^* = i\omega_{MN} \Sigma^{MN} B^{-1} \psi^*. \tag{A.17}$$

Thus one can impose a Majorana condition and define the Majorana Spinor ψ being a Dirac spinor but with the additional requirement (reality condition)

$$\psi^* = B\psi. \tag{A.18}$$

Thus the dimension of the Majorana representation is

$$\dim_{\mathbb{C}}(\text{Majorana rep.}) = 2^k\,, \qquad \text{or} \qquad \dim_{\mathbb{R}}(\text{Majorana rep.}) = 2^{k+1}\,. \tag{A.19}$$

From (A.18) we find

$$\psi = B^* \psi^* = B^* B \psi\,, \tag{A.20}$$

and thus

$$B B^* = 1\,. \tag{A.21}$$

From the definition (A.13) one computes

$$B_1 B_1^* = (-1)^{\frac{k}{2}(k+1)} \;\Rightarrow\; k = 0, 3, 7, \ldots \; (D = 2, 8, \ldots)\,, \tag{A.22}$$

$$B_2 B_2^* = (-1)^{\frac{k}{2}(k-1)} \;\Rightarrow\; k = 1, 4, 8, \ldots \; (D = 4, 10, \ldots)\,. \tag{A.23}$$

A Majorana-Weyl (MW) representation is only possible if the Weyl representation is self-conjugated, i.e., k is even, and hence, for $k = 0, 4, 8, \ldots$ $(D = 2, 10, \ldots)$. Its dimension is

$$\dim_{\mathbb{R}}(\text{MW}) = 2^k\,. \tag{A.24}$$

For D odd and $D = 2k + 1$ there are no Weyl representation and a Majorana representation is possible only in $D = 1, 3, 9, 11, \ldots$. Its dimension is

$$\dim_{\mathbb{R}}(\text{Majorana rep.}) = 2^k\,. \tag{A.25}$$

In this case the dimension of the Dirac representation is

$$\dim_{\mathbb{R}}(\text{Dirac rep.}) = 2^{k+1}\,. \tag{A.26}$$

All the possible representations are summarized in Table A.1.

A.2 Supersymmetry algebra

The supersymmetry algebra is an extension of the Poincare algebra. In arbitrary space-time dimensions D it depends on the spinor representations of $SO(1, D-1)$. Schematically it reads

$$\begin{aligned} \{Q^I, \overline{Q}^J\} &\sim \gamma^M P_M \delta^{IJ}\,, \qquad \{Q^I, Q^J\} \sim Z^{IJ}\,, \\ \left[L_{MN}, Q^I\right] &\sim \Sigma_{MN} Q^I\,, \qquad \left[P_M, Q^I\right] = 0\,, \end{aligned} \tag{A.27}$$

where $M = 0, \ldots, D-1$. Q^I is a spinor in the smallest spinor representation listed in Table A.1. The Jacobi-identity requires that Z^{IJ} commutes with all generators and this is a central element of the algebra. Positivity requires the BPS-bound

$$M \geq |Z|\,. \tag{A.28}$$

D	k	Majorana	Weyl	M-W	$\dim_R$
2	0	✓	s.c.	✓	1
3	1	✓	-	-	2
4	1	✓	c.c.	-	4
5	2	-	-	-	8
6	2	-	s.c.	-	8
7	3	-	-	-	16
8	3	✓	c.c.	-	16
9	4	✓	-	-	16
10	4	✓	s.c.	✓	16
11	5	✓	-	-	32
12	5	✓	c.c.	-	64

Table A.1: Spinor representations for $2 \leq D \leq 12$.

For arbitrary D it is more convenient to counts real supercharges (which we denote by q) instead of the number of spinor representations. For example, $N = 1$ in $D = 4$ has $q = 4$ real supercharges, or in general $q = 4N$ for arbitrary N in $D = 4$. For this notation the various supersymmetric theories for $4 \leq D \leq 12$ and $4 \leq q \leq 32$ are displayed in Table A.2.[17]

Most of the entries in Table A.2 are self-explanatory. However note that in $D = 6$ the supercharge Q is self-conjugate and two independent Weyl representations of opposite chirality, denoted $\mathbf{8}$ and $\mathbf{8}'$, of $SO(1,5)$ exist. For the theory denoted by $(1,1)$ the two supercharges transform as $Q_1 \in \mathbf{8}$, $Q_2 \in \mathbf{8}'$ and thus the theory is non-chiral while the (2,0) theory has $Q_1 \in \mathbf{8}$, $Q_2 \in \mathbf{8}$ and therefore is chiral.

In $D = 10$ also two Majorana-Weyl representations of opposite chirality $\mathbf{16}, \mathbf{16}'$ exist. Type IIA is non-chiral with $Q_1 \in \mathbf{16}$, $Q_2 \in \mathbf{16}'$ while type IIB is chiral with $Q_1 \in \mathbf{16}$, $Q_2 \in \mathbf{16}$.

In $D = 2$ the Lorentz group is $SO(1,1)$ and the supercharges Q are real one-dimensional Majorana-Weyl spinors. The type (p,q) superalgebra in two dimensions reads

$$\begin{aligned}
\{Q_L^{I_L}, Q_L^{J_L}\} &= \delta^{I_L J_L}\, P^- , \quad I_L, J_L = 1, \ldots, p\,, \\
\{Q_R^{I_R}, Q_R^{J_R}\} &= \delta^{I_R J_R}\, P^+ , \quad I_R, J_R = 1, \ldots, q\,, \\
\{Q_L^{I_L}, Q_R^{I_R}\} &= Z^{I_L I_R} \,.
\end{aligned} \tag{A.29}$$

[17]For $q = 64$ one goes beyond $N = 8$ and thus has higher spin fields in the massless multiplet. For these theories one does not have a consistent interacting quantum field theory in a Minkowski background.

D \ q	4	8	...	16	...	24	...	32	64
4	× (N=1)	○ (N=2)	○	○ (N=4)	○	○	○	○ (N=8)	
5		×		○		○		○	
6		× (1,0)		○ ○ (1,1) (2,0)		○ ○		○ (2,2)	
7				×				○	
8				×				○	
9				×				○	
10				× I				○ ○ IIA IIB	
11								×	
12									×

Table A.2: Table of supersymmetric theories. "×" denotes the theories with the minimal number of supersymmtries.

B Calabi-Yau manifolds and mirror symmetry

B.1 Some basic differential geometry

An n-dimensional complex manifold Y locally looks like $\mathbb{C}^n$. It has an complex structure I which is a map

$$I: \ T(Y) \to T(Y) \ , \qquad v^m \in T(Y) \mapsto I^m_n v^n \ , \quad m, n = 1, \ldots, 2n \ , \tag{B.1}$$

with

$$I^m_n \, I^k_m = -\delta^k_n \ . \tag{B.2}$$

If such an I exists the tanget space $T(Y)$ splits into two eigenspaces with eigenvalues $\pm i$ and locally one can define complex coordinates $z^i, \bar{z}^{\bar{\jmath}}, i, \bar{\jmath} = 1, \ldots, n$.

A one-form $\omega_1 = \omega_m dy^m$ then splits as

$$\omega_1 = \omega_{(1,0)} + \omega_{(0,1)} = \omega_i dz^i + \omega_{\bar{\imath}} d\bar{z}^{\bar{\imath}} \tag{B.3}$$

Similarly the exterior derivative d splits

$$d = \partial + \bar{\partial} = dz^i \partial_i + d\bar{z}^{\bar{\imath}} \partial_{\bar{\imath}} \ . \tag{B.4}$$

One defines (p, q)-forms by

$$\omega_{(p,q)} = \omega_{i_1 \cdots i_p \bar{\jmath}_1 \cdots \bar{\jmath}_q} \, dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{\jmath}_1} \wedge \cdots \wedge d\bar{z}^{\bar{\jmath}_q} \ . \tag{B.5}$$

The properties we discussed so far also hold for almost complex manifold. On a complex manifold I satisfies in addition that its Nijenhuis-tensor N vanishes

$$N^k_{mn}(I) := I^k_l \partial_l I^l_n - I^l_m \partial_l I^K_n - (m \leftrightarrow n) = 0 \ . \tag{B.6}$$

On hermitian manifolds the line element takes the form

$$ds^2 = g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} \tag{B.7}$$

which in real coordinates is equivalent to the property $g_{mn} = I^p_m I^q_n g_{pq}$. On hermitian manifolds one defines a fundamental $(1,1)$-form J by

$$J = i g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \ . \tag{B.8}$$

Kähler manifolds are hermitian manifolds where Kähler form J is closed, i.e.

$$dJ = 0 \ . \tag{B.9}$$

In terms of the metric this is equivalent to the properties

$$\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}} \ , \qquad \partial_{\bar{\imath}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{\imath}} \ , \tag{B.10}$$

which are locally solved by

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(z, \bar{z}) \ . \tag{B.11}$$

K is real and called the Kähler potential. It is not unique as the Kähler transformation $K \to K + f(z) + \bar{f}(\bar{z})$ leave the metric invariant. On Kähler manifolds the Riemann tensor considerably simplifies and only the component with index structure $R_{i\bar{j}k\bar{l}}$ is non-vanishing. The Ricci-tensor in turn obeys

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \ln \det g \ . \tag{B.12}$$

Calabi-Yau manifolds are Ricci-flat Kähler manifolds defined in Section 2.

B.2 The moduli space of Calabi-Yau threefolds

It is of interest to study the deformation of a Calabi-Yau metric which preserves the Ricci-flatness and which are not coordinate transformations. Or in other words one looks for the solutions of

$$R_{mn}(g^0 + \delta g) = 0 \tag{B.13}$$

subject to the gauge fixing condition $\nabla^m \delta g_{mn} - \frac{1}{2} \nabla_n \delta g^m_m = 0$. Expanding R_{mn} to first order in δg one obtains the Lichnerowicz equation

$$\nabla^l \nabla_l \, \delta g_{mn} + 2 R_{mknl} \, \delta g^{kl} = 0 \tag{B.14}$$

One can check that on Kähler manifolds $\delta g_{i\bar{j}}$ and δg_{ij} independently satisfy (B.14). For $\delta g_{i\bar{j}}$ one finds that (B.14) coincides

$$\Delta \delta g_{i\bar{j}} = 0 \ , \tag{B.15}$$

where $\Delta = dd^* + d^* d$ is the Laplace operator acting on differential forms.[18] The solution of (B.15) are harmonic $(1,1)$-forms which are in turn elements of the Dolbeault cohomology group $H^{(1,1)}(Y)$ defined in (3.12). Therefore $\delta g_{i\bar{j}}$ can be expanded in a basis (denoted by $\omega^\alpha_{i\bar{j}}$) of $H^{(1,1)}(Y)$ according to

$$\delta g_{i\bar{j}} = i \sum_{\alpha=1}^{h^{(1,1)}} v^\alpha \, \omega^\alpha_{i\bar{j}} \ , \qquad \alpha = 1, \ldots, h^{(1,1)} \ , \tag{B.16}$$

where the v^α denote $h^{(1,1)}$ Calabi-Yau moduli. Exactly as in Kaluza-Klein compactification these moduli appear as scalar fields in the effective action in that we can assign an arbitrary dependence on the space-time coordinates x by replacing $v^\alpha \to v^\alpha(x)$.

The deformations δg_{ij} change the complex structure of the original metric and (B.14) leads to

$$\Delta \delta g^i = 0 \ , \tag{B.17}$$

where $\delta g^i \equiv g^{i\bar{k}} \delta g_{\bar{k}\bar{j}} d\bar{z}^{\bar{j}}$ is a $(0,1)$ form with values in the holomorphic tangent bundle $T^{1,0}(Y)$. By using the $(3,0)$-form $\Omega = \Omega_{ijk} dz^i \wedge dz^j \wedge dz^k$ one can show that

$$\Omega_{ijk} \delta g^k_{\bar{l}} dz^i \wedge dz^j \wedge d\bar{z}^{\bar{l}} \in H^{(2,1)}(Y) \tag{B.18}$$

Therefore one has an expansion

$$\delta g_{ij} = \frac{i}{||\Omega||^2} \sum_{a=1}^{h^{(1,2)}} \bar{z}^a(x) \, \bar{\omega}^a_{i\bar{i}\bar{j}} \, \Omega^{\bar{i}\bar{j}}{}_j \ , \quad a = 1, \ldots, h^{(1,2)} \ , \tag{B.19}$$

where $\omega^a_{i\bar{i}\bar{j}}$ is a basis of $H^{(1,2)}$ and z^a are $h^{(1,2)}$ complex moduli. (We abbreviate $||\Omega||^2 \equiv \frac{1}{3!} \Omega_{ijk} \bar{\Omega}^{ijk}$.) So altogether there are $h^{(1,1)} + 2h^{(1,2)}$ real moduli of the Calabi-Yau metric.

For the other p-form gauge fields which occur in string theory and which we discussed in Section 2 the equations of motion in the gauge $d^* C_p = 0$ also read

$$\Delta C_p = 0 \ , \tag{B.20}$$

and thus the solutions are $C_p \in H^{(p)}$. In particular for the NS two-form B one has $B \in H^2 = H^{(1,1)}$ and thus one can expand

$$\delta B_{i\bar{j}} = \sum_\alpha b^\alpha(x) \, \omega^\alpha_{i\bar{j}} \ . \tag{B.21}$$

[18] d^* is the adjoint of d and maps p-forms to $(p-1)$-forms.

It turns out to be convenient to complexify the Kähler-form $J \to J_c = B + iJ$ so that in components

$$\delta J_c = \delta B_{i\bar{\jmath}} + i\delta g_{i\bar{\jmath}} = \sum_\alpha t^\alpha(x)\,\omega^\alpha_{i\bar{\jmath}}\ , \qquad t^\alpha = b^\alpha + iv^\alpha\ . \tag{B.22}$$

The moduli itself span a space – the moduli space – with a metric which is the metric on the space of metrics (and B-fields). The fact the deformations $\delta g_{i\bar{\jmath}}$ and δg_{ij} are independent says that the moduli space is the procuct

$$\mathcal{M} = \mathcal{M}^{h^{(1,2)}}_{\text{cs}}(z) \times \mathcal{M}^{h^{(1,1)}}_{\text{ks}}(t)\ . \tag{B.23}$$

$\mathcal{M}^{h^{(1,2)}}_{\text{cs}}$ is the complex $h^{(1,2)}$-dimensional component spanned by the complex structure deformations z^a while $\mathcal{M}^{h^{(1,1)}}_{\text{k}}$ is the complex $h^{(1,1)}$-dimensional component spanned by the complexified Kähler deformations t^α. Both components turn out to be special Kähler manifolds.

A special Kähler manifold is Kähler manifold where the Kähler potential is of the specific form [**?**, **?**, **?**, **?**, **?**].

$$K = -\ln i\left[\bar{Z}^A F_A(Z) - Z^A \bar{F}_A(\bar{Z})\right]\ , \qquad A = 0,\ldots,h \tag{B.24}$$

with

$$F_A := \frac{\partial F}{\partial Z^A} \qquad \text{and} \qquad Z^A F_A = 2F\ , \tag{B.25}$$

i.e. F is homogeneous of degree 2. One defines special coordinates as $z^a = \frac{Z^a}{Z^0}$, so that $F = (Z^0)^2\mathcal{F}(z^a)$ and K then can be also expressed as:

$$K = -\ln\left[2i\left(\mathcal{F} + \bar{\mathcal{F}}\right) - \left(\mathcal{F}_a + \bar{\mathcal{F}}_a\right)(z^a - \bar{z}^a)\right]\ , \tag{B.26}$$

where $\mathcal{F}(z^a)$ is an arbitrary holomorphic function with no homogeneity property.

The metric on $\mathcal{M}^{h^{(1,2)}}_{\text{cs}}$ turns out to be a special Kähler metric with a Kähler potential given by [?]

$$g_{a\bar{b}} = \partial_{z^a}\partial_{\bar{z}^{\bar{b}}}\ K_{\text{cs}}\ , \qquad K_{\text{cs}} = -\ln\Big[-i\int_Y \Omega\wedge\bar{\Omega}\Big] = -\ln i\Big[\bar{Z}^A F_A - Z^B\bar{F}_B\Big]\ . \tag{B.27}$$

The second form of K_{cs} is obtained from the expansion of Ω

$$\Omega(z) = Z^A(z)\,\alpha_A - F_B(z)\,\beta^B\ , \tag{B.28}$$

where (α_A, β^B) is a real, symplectic basis of $H^3(Y)$ satisfying

$$\int_Y \alpha_A\wedge\beta^B = \delta^B_A\ , \qquad \int_Y \alpha_A\wedge\alpha_B = 0 = \int_Y \beta^A\wedge\beta^B\ . \tag{B.29}$$

The moduli space $\mathcal{M}^{h^{(1,1)}}_{\text{ks}}$ spanned by the coordinates t^α also is a special Kähler manifold with a Kähler potential and prepotential $\mathcal{F}(t)$ given by

$$K_{\text{ks}} = -\ln d_{\alpha\beta\gamma}v^\alpha v^\beta v^\gamma\ , \qquad \mathcal{F}^0(t) = d_{\alpha\beta\gamma}t^\alpha t^\beta t^\gamma\ , \tag{B.30}$$

where $d_{\alpha\beta\gamma} = \int_Y \omega_\alpha \wedge \omega_\beta \wedge \omega_\gamma$ are topological intersection numbers. $\mathcal{F}^0$ represents the leading contribution in a large volume limit. There are, however, perturbative and non-perturbative α' corrections. The perturbative corrections can be understood as arising from loop corrections of the 2d SCFT which also give rise to higher derivative interactions in $\mathcal{L}^{10}_{\text{eff}}$. The non-perturbative corrections correspond to topological non-trivial embeddings of the worldsheet into space-time and they are termed worldsheet instanton corrections. For the case at hand the worldsheet can wrap a two-cycle in Y_3 which give amplitudes suppressed by $e^{-n_\alpha v^\alpha}$ where v^α parametrizes the volume of the two-cycle in question. Including both types of corrections results in

$$\mathcal{F}(t) = d_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma + \tfrac{2}{(2\pi)^3}\chi(Y_3) + \mathcal{F}^{\text{np}} \ , \tag{B.31}$$

where $\chi(Y_3) = 2(h^{(1,2)} - h^{(1,1)})$ is the Euler number of Y_3 and $\mathcal{F}^{\text{np}}$ denotes the non-perturbative effects. They are more easily expressed by the third derivative

$$\partial_\alpha \partial_\beta \partial_\gamma \mathcal{F}^{\text{np}} = \sum_{n_1 \cdots n_{h^{(1,1)}}=1}^{\infty} N_{n_1 \cdots n_{h^{(1,1)}}}\, n_\alpha n_\beta n_\gamma \frac{\prod_\delta q^{n_\delta}}{1 - \prod_\delta q^{n_\delta}} \ , \qquad q_\delta := e^{2\pi i t^\delta} \ , \tag{B.32}$$

N is the instanton number which counts how often a worldsheet wraps around a 2-cycle, $n_\alpha n_\beta n_\gamma$ is a combinatorical factor and the last factor aroses from the instanton action [?].

B.3 Mirror Symmetry

Mirror symmetry is not yet a symmetry but rather the conjecture about a not yet rigorously defined space of Calabi-Yau threefolds [?]. It has been established on a subspace of Calabi-Yau manifolds [?] and is a very useful concept in order to compute certain couplings in the effective action. It states that for 'every' Calabi-Yau Y there exists (at least) one mirror manifold $\tilde{Y}$ with reversed Hodge numbers, i.e.

$$h^{1,1}(Y) = h^{1,2}(\tilde{Y}) \ , \qquad h^{1,2}(Y) = h^{1,1}(\tilde{Y}) \ . \tag{B.33}$$

In terms of the Hodge diamond (3.13) this corresponds to a reflection along the diagonal or in other words the third cohomology $H^{(3)} = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$ is interchanged with the even cohomologies $H^{(\text{even})} = H^{(0,0)} \oplus H^{(1,1)} \oplus H^{(1,2)} \oplus H^{(3,3)}$.

Furthermore, the respective (complexified) moduli spaces of (B.23) are identified under mirror symmetry

$$\mathcal{M}^{h^{(1,2)}}_{\text{cs}}(Y) \equiv \mathcal{M}^{h^{(1,1)}}_{\text{ks}}(\tilde{Y}) \ , \qquad \mathcal{M}^{h^{(1,1)}}_{\text{ks}}(Y) \equiv \mathcal{M}^{h^{(1,2)}}_{\text{cs}}(\tilde{Y}) \ . \tag{B.34}$$

This in turn means that the underlying prepotentials are identical

$$\mathcal{F}(Y) \equiv \mathcal{F}(\tilde{Y}) \ , \qquad \mathcal{F}(Y) \equiv \mathcal{F}(\tilde{Y}) \ . \tag{B.35}$$

This fact has been used to compute instanton corrections to the prepotential $\mathcal{F}$ of the Kähler moduli (B.30) which only in the large volume approximation is a cubic polynomial.

In type II string theory mirror symmetry manifests itself by the equivalence of the two different type II string theories, called type IIA and type IIB, in mirror symmetric background or in other words the following equivalence holds

$$\text{IIA in background } \mathcal{M}_4 \times Y \quad \equiv \quad \text{IIB in background } \mathcal{M}_4 \times \tilde{Y} \; . \tag{B.36}$$

Therefore one can focus the attention on one of the two string theories and infer couplings of the other one by mirror symmetry. However, depending on the precise question it might be easier to ask it either in IIA or IIB.

C The holomorphic anomaly and soft supersymmetry breaking

C.1 The holomorphic anomaly

As we saw in section 5 it is of interest to compute $f^{(1)}(\Phi)$ in string theory. This is possible essentially in two ways:

1. directly via the computation of a string loop diagram,
2. indirectly via the holomorphic anomaly.

The problem with method 1 is that the entire massive string spectrum contributes in the loop and therefore $f^{(1)}(\Phi)$ is difficult to compute. Similarly, the result depends on the chosen background and thus relatively few generic properties can be identified.

The direct computation has been done for orbifold compactification of the heterotic string with the result [**?**]

$$\Delta \sim \ln \left[|\eta(it)|^4 (t+\bar{t}) \right] , \tag{C.1}$$

where t are the moduli in the untwisted sector of the orbifold and η is the Dedekind η-function. This result has two distinct features:

- Δ is invariant under an $SL(2,\mathbb{Z})$ transformations of the form

  $$t \;\to\; \frac{at-ib}{ict+d} , \tag{C.2}$$

  where $a,b,c,d \in \mathbb{Z}$ and $ad-bc=1$.

- Δ is non-harmonic in that

  $$\partial_t \bar{\partial}_{\bar{t}} \Delta \sim \partial_t \bar{\partial}_{\bar{t}} \log(t+\bar{t}) \neq 0 , \tag{C.3}$$

  and therefore

  $$\Delta \neq \mathrm{Re}\, f^{(1)}(t) \tag{C.4}$$

  as one would naively expect for a consistent supersymmetric effective theory. The failure of eq. (C.4) is known as the holomorphic anomaly but as we will see shortly this anomaly has nothing to do with string theory but rather occurs in any supersymmetric field theory with massless fermions in the spectrum.

It can be shown that supersymmetric theories the threshold correction have two contributions

$$\Delta = \Delta_m + \Delta_0 , \tag{C.5}$$

where massive particles contribute

$$\Delta_m = \mathrm{Re}\, f^{(1)} \ . \tag{C.6}$$

On the other hand massless particles with non-trivial non-renormalizable couplings contribute

$$\Delta_0 = -\ \frac{c}{16\pi^2}\,\hat{K}(t,\bar{t}) - \sum_{\mathbf{r}} \frac{T(\mathbf{r})}{8\pi^2}\ \log\det Z_{(\mathbf{r})}\ , \tag{C.7}$$

where $\mathbf{r}$ runs over the representations of the gauge group, $c = T(ad) - \sum_{\mathbf{r}} T(\mathbf{r})$ and we expand the Kähler potential as

$$K(t,\bar{t},A,\bar{A}) = \hat{K}(t,\bar{t}) + Z_{A\bar{B}}(t,\bar{t})A^A\bar{A}^{\bar{B}} + \dots\ , \tag{C.8}$$

and $Z_{(r)}$ is the block of the matrix Z_{AB} referring to the "flavor" indices of the matter multiplets A in the representation $\mathbf{r}$. For orbifolds (C.1) splits as

$$f^{(1)} \sim \ln \eta(it)^2\ , \qquad \Delta_0 \sim \ln(t+\bar{t})\ . \tag{C.9}$$

Altogether we thus have

$$g^{-2}(\mu) = \mathrm{Re}\,(f^{(0)} + f^{(1)}) - \frac{b}{8\pi^2}\ \ln\frac{M_{\rm Pl}}{\mu} - \frac{c}{16\pi^2}\,\hat{K}(t,\bar{t}) - \sum_{\mathbf{r}} \frac{T(\mathbf{r})}{8\pi^2}\ \log\det Z_{(\mathbf{r})}\ . \tag{C.10}$$

The second, indirect method to determine $f^{(1)}$ uses (C.10) and an exact quantum symmetries such as the $SL(2,\mathbb{Z})$ of orbifolds. One computes Δ_0 from (C.7) and solely tree-level couplings and then determines the harmonic piece Δ_m by requiring that the physical gauge couplings $g^{-2}(\mu)$ is invariant. For the Standard Embedding this method is used in refs. [**?**, 25].

Before we come to soft supersymmetry breaking let us briefly update gaugino condensation discussed in section 5. For a single pure gauge group in orbifold compactifications one finds

$$f^{(1)} = \frac{T(G)}{4\pi^2}\log\eta(iT)\ , \tag{C.11}$$

which leads via eq. (5.13) to

$$W(S,T) \sim M_{\rm Pl}^3\, e^{-\frac{8\pi^2}{T(G)}S}\,\eta(it)^{-2}\ . \tag{C.12}$$

For large t one has $W \to e^{\frac{\pi}{12}(t+\bar{t})}$ and thus a minimum at finite t. The explicit minimization of V reveals that $\langle t\rangle = \mathcal{O}(1)$ and supersymmetry is broken since $\langle D_t W\rangle \neq 0$. Unfortunately, this minimum has a large negative cosmological constant. The analysis of refs. [**?**, **?**, **?**, **?**] showed that the moduli dependence of $f^{(1)}$ can lead to a stabilization of the moduli vacuum expectation values and the breaking of supersymmetry. However, the dilaton problem and the problem of the cosmological constant remain unsolved in this class of models.

The next step is to include a moduli dependent $f^{(1)}$ into the racetrack scenarios. Indeed one finds [?] that one can simultaneously achieve the stabilization of the dilaton and moduli and break supersymmetry in the moduli directions. However, within this approach there always is a cosmological constant induced.

C.2 Soft Supersymmetry Breaking

Let us consider a generic $N = 1$ effective theory with an observable and a hidden sector specified by a superpotential

$$W = W_{\rm obs}(t, A) + W_{\rm hid}(t) \ . \tag{C.13}$$

Here t generically denotes the moduli fields while A denotes the observable (charged) matter fields.[19] $W_{\rm obs}(t, A)$ should be thought of as generated at the string tree level while $W_{\rm hid}(t)$ arises non-perturbatively. For $W_{\rm obs}(t, A)$ we make the general Ansatz

$$W_{\rm obs}(t, A) = \tfrac{1}{2}\mu_{AB}(t) A^A A^B + \tfrac{1}{3} Y_{ABC}(t) A^A A^B A^C + \mathcal{O}(A^4) \ . \tag{C.14}$$

For simplicity we are interested in the situation where $\langle A \rangle = 0$ and the gauge group is unbroken. Therefore we expand the tree level Kähler potential as in (C.8) around $\langle A \rangle = 0$ to obtain

$$K(t, \bar t, A, \bar A) = \hat K(t, \bar t) + Z_{A\bar B}(t, \bar t) A^A \bar A^{\bar B} + H_{AB}(t, \bar t) A^A A^B + h.c. + \mathcal{O}(A^3) \ , \tag{C.15}$$

We further assume:

1. $\langle F_A \rangle = 0$, ie. no supersymmetry breaking in the observable sector.
2. $\langle F_\alpha \rangle \neq 0$, ie. supersymmetry breaking in the hidden sector.
3. $\langle V \rangle = 0$, ie. a vanishing cosmological constant.
4. $m_{3/2} \ll M_{\rm Pl}$, ie. hierarchical supersymmetry breaking.

In the $N = 1$ potential (4.10) we now take the limit $M_{\rm Pl} \to \infty$ with $m_{3/2}$ fixed or in other words we keep the leading order contributions of the supersymmetry breaking. One finds that the (canonically normalized) gaugino masses turn out to be

$$\tilde m = \ F^\alpha \partial_\alpha \ln g^{-2} + \frac{b g^2}{16\pi^2}\, m_{3/2} \ , \tag{C.16}$$

whereas the (un-normalized) masses of the observable matter fermions and their (un-normalized) Yukawa couplings are given by

$$\begin{aligned} \tilde\mu_{AB} &\equiv e^{\hat K/2} \mu_{AB} \ + m_{3/2} H_{AB} \ - \ \bar F^{\bar\alpha} \bar\partial_{\bar\alpha} H_{AB} \ , \\ \tilde Y_{ABC} &\equiv e^{\hat K/2}\, Y_{ABC} \ . \end{aligned} \tag{C.17}$$

[19]Of course there is the possibility of hidden matter which we ignore for this discussion.

It is convenient to combine both terms into an effective superpotential

$$W^{\rm eff}(A) \ \equiv \ \tfrac{1}{2}\tilde{\mu}_{AB}A^AA^B \ + \ \tfrac{1}{3}\tilde{Y}_{ABC}A^AA^BA^C, \tag{C.18}$$

but one should remember that this is a superpotential of the observable sector and not of the full theory. In the latter context, (C.18) would not make sense as a superpotential because $\tilde{\mu}_{AB}$ and $\tilde{Y}_{ABC}$ are non-holomorphic functions of the moduli. The potential V splits into a potential of global $N=1$ supersymmetry (denoted as $V^{N=1}_{\rm global}$) and the soft supersymmetry breaking terms $V_{\rm soft}$

$$V = V^{N=1}_{\rm global} + V_{\rm soft}\ , \tag{C.19}$$

where

$$\begin{aligned} V^{N=1}_{\rm global} &= \tfrac{1}{2}D^2 + \ \partial_A W^{\rm eff} Z^{A\bar{B}}\partial_{\bar{B}}\bar{W}^{\rm eff}\ , \\ V_{\rm soft} &= m^2_{A\bar{B}}A^A\bar{A}^{\bar{B}} + \ \left(\tfrac{1}{3}A_{ABC}A^AA^BA^C \ + \ \tfrac{1}{2}B_{AB}A^AA^B \ + \ \text{h.c.}\right)\ . \end{aligned} \tag{C.20}$$

The first line here gives the scalar potential of an effective theory with unbroken rigid supersymmetry while the second line is comprised of the soft supersymmetry-breaking terms. The coefficients of these soft terms are as follows

$$\begin{aligned} m^2_{A\bar{B}} &= m^2_{3/2}Z_{A\bar{B}} \ - \ F^\alpha\bar{F}^{\bar{\beta}}R_{\alpha\bar{\beta}A\bar{B}}\ , \\ A_{ABC} &= F^\alpha D_\alpha\tilde{Y}_{ABC}\ , \\ B_{AB} &= F^\alpha D_\alpha\tilde{\mu}_{AB} \ - \ m_{3/2}\tilde{\mu}_{AB}\ , \end{aligned} \tag{C.21}$$

where

$$\begin{aligned} R_{\alpha\bar{\beta}A\bar{B}} &\equiv \partial_\alpha\bar{\partial}_{\bar{\beta}}Z_{A\bar{B}} \ - \ \Gamma^D_{\alpha A}Z_{D\bar{C}}\overline{\Gamma}^{\bar{C}}_{\bar{\beta}\bar{B}}\ , \qquad \Gamma^D_{\alpha A} \ = \ Z^{D\bar{B}}\partial_\alpha Z_{\bar{B}A}\ , \\ D_\alpha\tilde{Y}_{ABC} &\equiv \partial_\alpha\tilde{Y}_{ABC} \ + \ \tfrac{1}{2}\hat{K}_\alpha\tilde{Y}_{ABC} \ - \ \Gamma^D_{\alpha(A}\tilde{Y}_{BC)D}\ , \\ D_\alpha\tilde{\mu}_{AB} &\equiv \partial_\alpha\tilde{\mu}_{AB} \ + \ \tfrac{1}{2}\hat{K}_\alpha\ \tilde{\mu}_{AB} \ - \ \Gamma^D_{\alpha(A}\ \tilde{\mu}_{B)D}\ . \end{aligned} \tag{C.22}$$

(When evaluating $\partial_\alpha\tilde{\mu}_{AB}$ or $\partial_\alpha\tilde{Y}_{ABC}$, one should apply ∂_{t^α} to all quantities on the right-hand side of eqs. (C.17), including $m_{3/2}$ and $\bar{F}^{\bar{\beta}}$.) Notice that all quantities appearing in eqs. (C.16), (C.17) and (C.21) are covariant with respect to the supersymmetric reparametrization of matter and moduli fields as well as covariant under Kähler transformations.

According to eq. (C.21), $m^2_{A\bar{B}} \sim m^2_{3/2}$, $A_{ABC} \sim m_{3/2}\tilde{Y}_{ABC}$, and $B_{AB} \sim m_{3/2}\tilde{\mu}_{AB}$; nevertheless, the soft terms are generally not universal, ie. $A_{ABC} \neq \text{const}\cdot m_{3/2}\tilde{Y}_{ABC}$ and $m^2_{A\bar{B}} \neq \text{const}\cdot m^2_{3/2}Z_{A\bar{B}}$, even at the tree level. In the context of the minimal supersymmetric standard model, this non-universality means that the absence of flavor-changing neutral currents is not an automatic feature of supergravity but a non-trivial constraint that has to be satisfied by a fully realistic theory.

To summarize, the displayed formulae express all the couplings of the observable sector in terms of a few perturbative parameters of the effective supergravity, namely $\hat{K}(t,\bar{t})$, $Z_{A\bar{B}}(t,\bar{t})$, $H_{AB}(t,\bar{t})$, $Y_{ABC}(t)$ and $f(t)$, and even fewer non-perturbative parameters induced by the hidden sector, namely $m_{3/2}$ and F^α.

Nothing so far relied in any way on the stringy nature of the fundamental theory behind the effective supergravity and are equally valid for any other unified theory that gives rise to an effective supergravity below the Planck scale. However, in the context of string theory, one can make again use of the special properties of the dilaton S. Let us recall from eqs. (5.7) that at the tree level $K^{(0)} = -\ln(S+\bar{S}) + \hat{K}^{(0)}(t,\bar{t}), f^{(0)} = S$ while $Z_{A\bar{B}}(t,\bar{t})$, $H_{AB}(t,\bar{t})$, and $Y_{ABC}(t)$ are independent of S. (Their t dependence cannot be further constrained unless one chooses to focus on a particular class of string vacua).[20]

Generically, the dynamics of the hidden sector can give rise to both $\langle F^S\rangle$ and $\langle F^t\rangle$, but one type of F-term often dominates over the other. Therefore, it is instructive to concentrate on the two limiting cases $\langle F^S\rangle \gg \langle F^t\rangle$ and $\langle F^S\rangle \ll \langle F^t\rangle$ and discuss the phenomenological implications of the two scenarios. The main feature of the $\langle F^S\rangle \gg \langle F^t\rangle$ scenario is the great simplicity of the resulting soft terms before string loops and renormalization are taken into account. Specifically, one finds

$$\tilde{m} = \sqrt{3}m_{3/2}\,, \qquad m^2_{A\bar{B}} = m^2_{3/2}Z_{A\bar{B}}\,, \qquad A_{ABC} = -\sqrt{3}m_{3/2}\tilde{Y}_{ABC}\,, \tag{C.23}$$

whereas $\tilde{\mu}_{AB}$ and B_{AB} are independent parameters. Thus, in the context of the minimal supersymmetric standard model the masses of all super-particles as well as the Higgs VEVs are determined in terms of the three independent parameters $m_{3/2}$, $\tilde{\mu}$ and B, and if we further assume that $\mu = 0$, then only $m_{3/2}$ and $\tilde{\mu}$ are independent while $B = 2m_{3/2}\tilde{\mu}$. Numerical study of the electroweak phenomenology produced by these soft terms shows that for $\mu = 0$ the Higgs particle is too light for all allowed values of the other parameters; the general case ($\mu \neq 0$) is slightly more involved and nor ruled out by current data.

When the dominant non-perturbative effect in the hidden sector is the formation of gaugino (and possibly) other condensates, the resulting effective $W^{(\mathrm{np})}(S,t)$ is more likely to give rise to $\langle F^t\rangle$ than to $\langle F^S\rangle$ as we saw in section 5. However, the analysis of this scenario is much more model-dependent since the t-dependence of various couplings is quite different for different string vacua; nevertheless, even without choosing a particular vacuum it is possible to make some generic statements about the soft terms. First of all, the usual assumption of the universality of the soft terms in the minimal supersymmetric

[20] At the string loop level, $\hat{K}, Z_{A\bar{B}}$ and H_{AB} receive an S-dependent but generically small threshold correction, which we neglect in the following discussion. f is corrected by the one-loop t-dependent (but S-independent) term $f^{(1)}(t)$ which we discussed above.

standard model does not automatically hold in this case: $m^2_{A\bar{B}}$ is not flavor-blind or even generation-blind; instead, we have a non-universality parameterized by the field-space curvature $R_{\alpha\bar{\beta}A\bar{B}}$ (see eqs. (C.21)), which generically does not vanish. The absence of flavor-changing neutral currents imposes strong phenomenological constraints on this curvature term and thus on string model building. Equations (C.21) also reveals that the trilinear couplings A_{ABC} are not strictly proportional to the Yukawa couplings $\tilde{Y}_{ABC}$, nor is B_{AB} proportional to $\tilde{\mu}_{AB}$.

Despite the lack of universality in the $\langle F^t \rangle$-driven scenario, we can still make an order-of-magnitude estimate of the supersymmetry-breaking masses and couplings. The scalar masses are typically $\mathcal{O}(m_{3/2})$. Similarly, the trilinear couplings $A_{ABC} = \mathcal{O}(m_{3/2}\tilde{Y}_{ABC})$. On the other hand, because the gauge couplings depend on the dilaton S more strongly than on the other moduli t^i, the gaugino masses come out rather light, $\mathcal{O}(\frac{\alpha}{4\pi} m_{3/2})$ (see eq. (C.16)). Furthermore, eq. (C.7) allows us to estimate the magnitude of the gaugino masses after the renormalization, ie. just above $m_{3/2}$. The result is

$$\tilde{m}(\mu) \; = \; C \, \frac{\alpha(mu)}{4\pi} \, m_{3/2} \ll m_{3/2} \, , \tag{C.24}$$

where the coefficients C is model-dependent but generally $O(1)$. Therefore, in this scenario we expect the gaugino masses to be close to their experimental lower bounds, while the squarks and the sleptons heavy.

The two scenarios we just analyzed lead to distinct signals at the weak scale. It is important to stress that such signals do not depend on the detailed mechanism for supersymmetry breaking nor do they depend on the chosen string vacuum. Rather, they are a mere consequence of which F-term is the dominant seed of the breaking.

D Supergravity actions for $4 \leq d \leq 9$

In this section we discuss the couplings of (bosonic) supergravity actions in dimensions $4 \leq d \leq 9$. The effective actions derived from string theory have to satisfy the constraints and properties of these actions.

A generic bosonic Lagrangian reads

$$L = -\frac{1}{2\kappa^2} R - \frac{1}{4} g_{ab}^{-2} F_{\mu\nu}^a F^{\mu\nu b} - \frac{1}{2} G_{IJ}(\Phi) D_\mu \Phi^I D^\mu \Phi^J - V(\Phi) + \dots \ , \tag{D.1}$$

where R is the Einstein-Hilbert-term and $G_{IJ}(\Phi)$ is the metric on the scalar manifold M. The ... stand for additional topological terms and/or kinetic terms and couplings of higher p-form gauge potentials. These terms differ in various dimensions.

$\mathcal{L}$ is gauge invariant under the gauge transformations

$$\delta_\alpha \Phi^I = \alpha^a(x)\, k_a^I(\Phi) \ , \quad a = 1, \dots, n_v \ , \tag{D.2}$$

where n_v is the number of vector multiplets or equivalently the dimension of the Lie algebra assocciated to the Lie group G, $\alpha^a(x)$ is the (local) parameter of the gauge transformation and k_a^I are Killing vector fields. Correspondingly, the covariant derivatives are given by

$$D_\mu \Phi^I = \partial_\mu \Phi^I - A_\mu^a\, k_a^I(\Phi) \ . \tag{D.3}$$

Gauge invariance requires that the metric is invariant $\delta_\alpha G_{i\bar{j}} = 0$ which implies the Killing equations

$$\nabla_I k_J^a + \nabla_J k_I^a = 0 \ . \tag{D.4}$$

D.1 $N = 1$ supergravity in $d = 4$

The $N = 1$ multiplets are summarized in Table D.1 where $[s]$ denotes a field of spin (helicity) s.[21]

$N = 1$	$d = 4$
Gravitational multiplet	$\left([2], [\frac{3}{2}]\right)$
Vector multiplet	$\left([1], [\frac{1}{2}]\right)$
Chiral/Linear multiplet	$\left([\frac{1}{2}], 2[0]\right)$

Table D.1: $N = 1, d = 4$ multiplets

[21] The linear multiplet contains an antisymmetric tensor $B_{\mu\nu}$ and a real scalar ϕ. $B_{\mu\nu}$ can be dualized to a second scalar a so that the entire multiplet becomes dual to a chiral multiplet.

In $N=1$ the scalars in the chiral multiplets are complex $\Phi^i, \bar{\Phi}^{\bar{\jmath}}, i, \bar{\jmath} = 1, \ldots, n_c$ and M is a Kähler manifold, i.e. the metric obeys

$$G_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K \ . \tag{D.5}$$

In a addition, a topological term $\frac{\theta_{ab}}{32\pi^2}\,\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^b_{\rho\sigma}$ is present and the (inverse) gauge couplings and the θ-angle combine into the holomorphic gauge kinetic function

$$f_{ab} = g_{ab}^{-2} + \tfrac{i}{8\pi^2}\,\theta_{ab} \ . \tag{D.6}$$

The potential is given by

$$V = e^{\kappa^2 K}\left[(D_i W)\, G^{-1 i\bar{\jmath}} \left(D_{\bar{\jmath}} \bar{W} \right) - 3\kappa^2 |W|^2 \right] + \tfrac{1}{2} g_{ab}^{-1} D^a D^b \ , \tag{D.7}$$

where W is the holomorphic superpotential and

$$D_i W := \frac{\partial W}{\partial \Phi^i} + \kappa^2 \left(\frac{\partial K}{\partial \Phi^i} \right) W \ . \tag{D.8}$$

The D-terms D^a are the Killing prepotentials. On a Kähler manifold the solution of (D.4) is

$$\partial_{\bar{\jmath}} k^{ai} = 0 \ , \qquad k^a_{\bar{\imath}} = i \partial_{\bar{\imath}} D^a \ . \tag{D.9}$$

The first equation states that k^{ai} is holomorphic while the second determines $k^a_{\bar{\imath}}$ in terms of the Killing prepotentials (or moment maps) D^a. Using $k^a_{\bar{\imath}} = G_{\bar{\imath}j} k^{aj}$ one finds

$$D^a = -i (\partial_j K) k^{aj} + \xi \delta^{aU(1)} \ . \tag{D.10}$$

ξ is a Fayet-Illiopoulos parameter which arises for any $U(1)$-factor in the gauge group G as an (undetermined) integration constant of the Killing prepotentials. The Lagrangian (D.1) is thus characterized by four functions $K(\Phi, \bar{\Phi}), f(\Phi), W(\Phi)$ and D^a.

D.2 $N=2$ supergravity in $d=4,5,6$

The $N=2$ multiplets in dimensions $d=4,5,6$ are summarized in Table D.2. The scalar field space is locally the product

$$\mathcal{M} = \mathcal{M}^{4n_h}_{h,QK} \times \begin{cases} \mathcal{M}^{2n_v}_{v,SK} & d=4 \\ \mathcal{M}^{n_v}_{v,RSK} & d=5 \\ \frac{O(1,n_t)}{O(n_t)} & d=6 \end{cases} \ , \tag{D.11}$$

where $\mathcal{M}^{4n_h}_{h,QK}$ is a $4n_h$-dimensional quaternionic-Kähler manifold, $\mathcal{M}^{2n_v}_{v,SK}$ is a $2n_v$-dimensional special Kähler manifold, $\mathcal{M}^{n_v}_{v,RSK}$ is n_v-dimensional real special Kähler manifold and n_t counts the number of tensor multiplets. Let us discuss these geometries in turn [4].

$N=2$(eight supercharges)	$d=4$	$d=5$	$d=6$
Gravitational multiplet	$\big([2],2[\frac{3}{2}],[1]\big)$	$\big([2],[\frac{3}{2}],[1]\big)$	$\big([2],[\frac{3}{2}],[1],B^-_{\mu\nu}\big)$
Vector multiplet	$\big([1],2[\frac{1}{2}],2[0]\big)$	$\big([1],[\frac{1}{2}],[0]\big)$	$\big([1],[\frac{1}{2}]\big)$
Hypermultiplet	$\big(2[\frac{1}{2}],4[0]\big)$	$\big([\frac{1}{2}],4[0]\big)$	$\big([\frac{1}{2}],4[0]\big)$
Tensor multiplet	dual to hyper	dual to vector	$\big(B^+_{\mu\nu},([\frac{1}{2}],[0]\big)$

Table D.2: $N=2, d=4,5,6$ multiplets

D.2.1 Quaternionic-Kähler geometry

$\mathcal{M}^{4n_h}_{h,QK}$ is not a Kähler manifold bur rather quaternionic-Kähler manifold. This means that it admits three almost complex structures $(J^x)^v_u$, $x=1,2,3$, $u,v=1,\ldots,4n_h$, which satisfy

$$J^xJ^y = -\delta^{xy}\mathbf{1} + i\epsilon^{xyz}J^z \ , \tag{D.12}$$

and the metric G_{uv} is Hermitian with respect to all three of them. They are also covariantly closed with respect to an $SU(2)$ connection ω

$$DJ^x = 0 \ . \tag{D.13}$$

The associated Kähler two-forms $K^x_{uv} = G_{uw}(J^x)^w_v$ obey

$$DK^x = dK^x + \epsilon^{xyz}w^y \wedge K^z = 0 \ . \tag{D.14}$$

On this geometry the Killing vectors can be expressed in terms of Killing prepotential P^x_A by

$$k^u_A K^x_{uv} = -D_v P^x_A = -\partial_\nu P^x_A - \epsilon^{xyz} w^y_v P^z_A \ , \tag{D.15}$$

where the index A takes the values $A=(0,a)$ and the 0-direction denotes the graviphoton.

Explicit quaternionic-Kähler manifolds are sparsely known. A prominent example appearing at the tree-level of type II compactifications are the quaternionic-Kähler manifolds in the image of the c-map. The metric depends on the coordinates $(z^a,\xi^A,\tilde{\xi}_A,\phi,a)$ with index ranges $a=1,\ldots,n_h-1, A=(0,a)$. It reads

$$\begin{aligned} ds^2 = {} & G_{a\bar{b}}(z,\bar{z})\,\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} + (\partial_\mu\phi)^2 + \tfrac{1}{4}\big(\partial_\mu a - (\tilde{\xi}_A\partial_\mu\xi^A - \xi^A\partial_\mu\tilde{\xi}_A)\big)^2 \\ & -\tfrac{1}{2}e^{2\phi}(\mathrm{Im}\,\mathcal{N}(z,\bar{z}))^{-1AB}(\partial_\mu\tilde{\xi}_A - \mathcal{N}_{AC}\partial_\mu\xi^C)(\partial_\nu\tilde{\xi}_B - \mathcal{N}_{BD}\partial_\mu\xi^D) \ , \end{aligned} \tag{D.16}$$

where $G_{a\bar{b}}(z,\bar{z})$ is the metric on a special Kähler manifold $\mathcal{M}_{\mathrm{SK}}$ while $\mathcal{N}$ is the gauge kinetic function on $\mathcal{M}_{\mathrm{SK}}$. (Both are discussed in the next section.) Thus the c-map associates to every special Kähler manifold a quaternionic Kähler manifold

$$\mathrm{c}: \quad \mathcal{M}^{2(n_h-1)}_{\mathrm{SK}} \times \frac{SU(1,1)}{U(1)} \quad \to \quad \mathcal{M}^{4n_h}_{\mathrm{QK}} \ , \tag{D.17}$$

where (ϕ, a) are the coordinates on the $SU(1,1)/U(1)$ component. In string theory one finds $\mathcal{M}_{\rm SK} = \mathcal{M}_\Omega(\mathcal{M}_J)$ for IIA (IIB) and $(\phi, a$ are dilaton and axion while $(\xi^A, \tilde{\xi}_A)$ are the RR-scalars.

D.2.2 Special Kähler geometry

In Appendix B.2 we already discussed special Kähler geometry in the context of the Calabi-Yau moduli spaces as they are examples of special Kähler manifolds.

For special Kähler manifolds the Kähler potential is given by

$$K = -\ln i\left[\bar{Z}^A F_A(Z) - Z^A \bar{F}_A(\bar{Z})\right] , \qquad A = 0, \dots, n_v \tag{D.18}$$

with

$$F_A := \frac{\partial F}{\partial Z^A} \qquad \text{and} \qquad Z^A F_A = 2F , \tag{D.19}$$

i.e. F is homogeneous of degree 2. One defines special coordinates as $z^a = \frac{Z^a}{Z^0}$, so that $F = (Z^0)^2 \mathcal{F}(z^a)$ and K then can be also expressed as

$$K = -\ln\left[2i\left(\mathcal{F} + \bar{\mathcal{F}}\right) - \left(\mathcal{F}_a + \bar{\mathcal{F}}_a\right)(z^a - \bar{z}^a)\right] , \tag{D.20}$$

where $\mathcal{F}(z^a)$ is an arbitrary holomorphic function with no homogeneity property.

The gauge kinetic matrix f is given by:

$$f_{AB} = F_{AB} - \frac{(\mathrm{Im}F)_{AC}\, \bar{Z}^C\, (\mathrm{Im}F)_{BD} \bar{Z}^D}{\bar{Z}^C\, (\mathrm{Im})_{CD}\, \bar{Z}^D} , \tag{D.21}$$

where the second term is not holomorphic and arises due to the mixing with the graviphoton.

The Killing vectors can again be expressed in terms of Killing prepotential P_0^B by

$$k_a^B = i\partial_a P_0^B . \tag{D.22}$$

Together with the Killing vectors $k_A^u(q)$ and Killing prepotentials P_A^x on $\mathcal{M}_{h,QK}$ discussed in the previous section the covariant derivatives are

$$D_\mu q^u = \partial_\mu q^u - A_\mu^A k_A^u(q) , \qquad D_\mu z^a = \partial_\mu z^a - A_\mu^B\, k^{Ba}(z) , \tag{D.23}$$

while the potential is given by

$$V = e^K\Big(G_{a\bar{b}} k_A^a \bar{k}_B^{\bar{b}} Z^A \bar{Z}^B + 4h_{uv} k_A^u k_B^v Z^A \bar{Z}^B + G^{a\bar{b}}(\partial_a Z^A)(\bar{\partial}_{\bar{b}} \bar{Z}^B) P_A^x P_B^x - 3Z^A \bar{Z}^B P_A^x P_B^x\Big) . \tag{D.24}$$

Before we continue let us mention one caveat. The situation discussed here only features multiplets which are charged with respect to electric gauge bosons but not their magnetic duals. In string theory it is sometimes convenient to go to a different symplectic basis and includes magnetic charges. This can be done via the embedding tensor formalism [**?**].

D.2.3 Real special Kähler geometry

In $d = 5$ $\mathcal{M}_{h,QK}$ is unchanged while $\mathcal{M}_v$ becomes a real special Kähler manifold. The vector multiplets contain a real instead of a complex scalar and the geometry is constrained by

$$d_{ABC}\Phi^A\Phi^B\Phi^C = 1 \ . \tag{D.25}$$

The physical scalars φ^a are the solutions of this constraint with a metric

$$G_{ab} = -3\,(\frac{\partial\Phi^A}{\partial\varphi^a})(\frac{\partial\Phi^B}{\partial\varphi^b})\, d_{ABC}\Phi^C \ . \tag{D.26}$$

In $d = 6$ $\mathcal{M}_h$ is again is unchanged, the vector multiplets have no scalar but the tensor multiplets have a real scalar spanning the geometry

$$\mathcal{M}_t = \frac{O(1, n_t)}{O(n_t)} \ . \tag{D.27}$$

D.3 Supergravities with 16 supercharges

In theories with 16 supercharges there is the gravitational multiplet and the vector multiplet. Their bosonic components are

$$\begin{aligned} \text{gravitational multiplet :} \qquad & \Big([2], (10-D)[1], [0], B_{MN}\Big) \ , \\ \text{vector multiplet :} \qquad & \Big([1], (10-D)[0]\Big) \ , \end{aligned} \tag{D.28}$$

plus an appropriate number of gravitinos and $s = 1/2$-fermions. (For more details see [?].)

The scalar field space is

$$M = \frac{SO\,(10-D, n_v)}{SO\,(10-D) \times SO\,(n_v)} \times \begin{cases} R^+ & \text{for} \quad D = 5, \ldots, 10 \ , \\ \frac{SU(1,1)}{U(1)} & \text{for} \quad D = 4 \end{cases} \tag{D.29}$$

where n_v is the number of vector multiplets. The first component of the product is spanned by the scalars of the vector multiplet and the second by the scalar(s) of the gravity multiplet.

A special case is the $(2, 0)$ theory in $d = 6$ where the scalar manifold is given by

$$M = \frac{SO\,(5, 21)}{SO\,(5) \times SO(21)} \ . \tag{D.30}$$

D.4 Supergravities with 32 supercharges

For 32 supercharges there only is the gravitational multiplet which in $D = 4$ has the field content

$$\Big([2], 8[3/2], 28[1], 56[1/2], 70[0]\Big) . \tag{D.31}$$

In $D > 4$ the field content can be found, for example, in [?].

We summarize all geometries in the following Table D.3 [**?**, ?]. We abbreviate

$$SO_{m,n} \equiv \frac{SO(m,n)}{SO(m) \times SO(n)} \times \begin{cases} R^+ & \text{for} \quad D = 5, \ldots, 10 \ , \\ \frac{SU(1,1)}{U(1)} & \text{for} \quad D = 4 \end{cases} .$$

D/q	4	8	16	32
4	$\mathcal{M}_K$	$\mathcal{M}_{SK} \times \mathcal{M}_{QK}$	$SO_{6,n}$	$\frac{E_{7(7)}}{SU(8)}$
5		$\mathcal{M}_{RSK} \times \mathcal{M}_{QK}$	$SO_{5,n}$	$\frac{E_{6(6)}}{U_{sp}(8)}$
6		$\frac{O(1,n_t)}{O(n_t)} \times \mathcal{M}_{QK}$	$SO_{4,n}/SO_{5,21}$	$\frac{E_{5(5)}}{U_{sp}(4)\times U_{sp}(4)}$
7			$SO_{3,n}$	$\frac{E_{4,4}}{U_{sp}(4)}$
8			$SO_{2,n}$	$\frac{E_{3,3}}{U(2)}$
9			$SO_{1,n}$	$\frac{GL(2)}{SO(2)}$
10			$\mathbb{R}^+$	$\mathbb{R}^+$, $\frac{SU(1,1)}{U(1)}$
11			-	-
		"$\mathcal{M}_{SK}$" $\times \mathcal{M}_{QK}$	$SO_{10-D,n}$	$\frac{E_{11-D}}{H_R}$

Table D.3: Scalar geometries in supergravity

E Compactifications on generalized geometries

E.1 Manifolds with G-structure

Recall the discussion of supersymmetry in compactification with backgrounds $\mathbb{R}_{1,3} \times Y_6$ in section 3.2. The decomposition of the Lorentz group is given in eq. (3.2) while the decomposition of the spinor representation is given in (4.2). The existence of a supercharge in $\mathbb{R}_{1,3}$ requires the existence of a nowhere vanishing and globally defined spinor ϵ on Y_6. This requires that ϵ is a singlet of $SO(6)$ but it does not require it to be covariantly constant, i.e. $D_m\epsilon \neq 0$ is possible. Manifolds which admit a globally defined tensor or spinor have been studied in the mathematical literature and are called manifolds with G-structure. Here G denotes the subgroup of the structure group $SO(6)$ which leaves the tensor or spinor invariant. Generically G does not coincide with the holonomy group precisely because the spinor does not have to be covariantly constant with respect to the Levi-Civita connection. However, one can show that a different connection – a connection with torsion $D^{(T)}$ – always exists which satisfies $D^{(T)}_m \epsilon = 0$.

In this section we focus on the example of one globally defined spinor. Using $SO(6) \sim SU(4)$ we see that a globally defined spinor is left invariant by an $SU(3) \subset SU(4)$ and thus we have $G = SU(3)$ or in other words we need to study manifolds with $SU(3)$-structure.

On manifolds with $SU(3)$-structure it is possible to build a two-form J and a three-form Ω from ϵ

$$J_{mn} = -\tfrac{i}{2}\,\bar\epsilon\gamma_{[mn]}\epsilon\ , \qquad \Omega_{mnp} = -\tfrac{i}{2}\,\epsilon\gamma_{[mnp]}\epsilon\ . \tag{E.1}$$

Due to Fierz identities they obey the relation

$$J\wedge J\wedge J = \tfrac{3i}{4}\Omega\wedge\bar\Omega\ . \tag{E.2}$$

Raising one index on J_{mn} with the metric one can show that J_n^m is an almost complex structure in that it satisfies $J^2 = -1$. Using the definition (E.1) and $D^{(T)}_m\epsilon = 0$ implies

$$\begin{aligned} dJ &= \tfrac{3i}{4}\big(W_1\bar\Omega - \bar W_1\Omega\big) + W_4\wedge J + W_3\ ,\\ d\Omega &= W_1 J^2 + W_2\wedge J + \bar W_5\wedge\Omega\ , \end{aligned} \tag{E.3}$$

and

$$W_3\wedge J \;=\; W_3\wedge\Omega \;=\; W_2\wedge J^2 \;=\; 0\ , \tag{E.4}$$

where the W's are five different torsion classes which can be characterized by their $SU(3)$ representation or equivalent their form-degree. W_1 is a zero-form, W_4, W_5 are one-forms, W_2 is a two-form and W_3 is a three-form. Generically manifolds with $SU(3)$ structure are neither complex, nor Kähler, nor Ricci-flat. Only for a particular choice of the torsion such that some of the W_α vanish one has manifolds with additional properties. For example Calabi-Yau manifolds are manifolds of $SU(3)$ structure where all five torsion classes vanish $W_{1,\dots,5} = 0$. Complex manifolds have $W_{1,2} = 0$ while Kähler manifolds have $W_{1,\dots,4} = 0$. Half-flat manifolds play a special role later on and their are characterized by $\mathrm{Im}\, W_1 = \mathrm{Im}\, W_2 = W_4 = W_5 = 0$ or in other words

$$dJ \sim \mathrm{Im}\,\Omega\ , \qquad d\Omega \sim J^2\ . \tag{E.5}$$

E.2 $\mathcal{L}_{\text{eff}}$ on manifolds with $SU(3)$-structure

The KK reduction on manifolds with $SU(3)$-structure leads to an effective action with $N = 2$ supersymmetry which can be spontaneously broken. The scalar geometry is unchanged compared to the Calabi-Yau case and the Kähler potentials for the geometric moduli are given by

$$K_{\text{ks}}(t,\bar{t}) = -\ln \int_{Y_6} J \wedge J \wedge J\ , \qquad K_{\text{cs}}(z,\bar{z}) = -\ln \Big[-i \int_{Y_6} \Omega \wedge \bar{\Omega} \Big]\ . \tag{E.6}$$

The Killing vectors and the potential on the other hand do depend on dJ and $d\Omega$.

There is no globally defined one-form which can be build from ϵ so that we continue to have the vanishing of the first Betti-number $b_1 = b_5 = 0$. The existence of J and Ω implies that $b_{2,3,4} \neq 0$. Let us define a finite basis of light modes by a set of two-forms $\omega^\alpha, \alpha = 1, \dots, b_2$, a symplectic set of three-forms $(\alpha_A, \beta^B), A, B = 1, \dots, \frac{1}{2}b_3$ and a set of four-forms $\tilde{\omega}_\alpha$ dual to the two-forms. To ensure the vanishing of the five-forms they are required to obey

$$\omega^\alpha \wedge \alpha_A = 0 = \omega^\alpha \wedge \beta^B\ . \tag{E.7}$$

Now one can parametrize the torsion by the parameters $(e^\alpha_A, m^{\alpha A})$ which appear as

$$\begin{aligned} d\omega^\alpha &= m^{\alpha A}\alpha_A - e^\alpha_B \beta^B\ , \\ d\alpha_A &= e^\alpha_B \tilde{\omega}_\alpha\ , \\ d\beta^B &= m^{\alpha B}\tilde{\omega}_\alpha\ , \\ d\tilde{\omega}_\alpha &= 0\ . \end{aligned} \tag{E.8}$$

Here the consistency condition

$$\omega^\alpha \wedge d\alpha_A = -d\omega^\alpha \wedge \alpha_A\ , \qquad \omega^\alpha \wedge d\beta^B = -d\omega^\alpha \wedge \beta^B\ , \tag{E.9}$$

has been already implemented. In addition $d^2 = 0$ implies

$$m^{\alpha A} e^\beta_A - e^\alpha_A m^{\beta A} = 0\ . \tag{E.10}$$

By using this basis one can compute the Killing vectors and the potential which turn out to be consistent with the constraints of $N = 2$ supergravity. However before we display the result let us pause and discuss mirror symmetry in compactification with fluxes.

E.3 Mirror symmetry in flux compactifications

Recall that in Calabi-Yau compactifications of type IIA we turned on RR-fluxes for F_2 and F_4 in (7.15) and in IIB for F_3 in (7.21). In type IIA one can add flux for F_0 and F_6 where F_0 denotes the flux in the space-time part of F_4 and F_6 can be identified as an additional parameter of ten-dimensional type IIA supergravity. Thus altogether there are $2(h^{(1,1)}+1)$ fluxes in IIA and $2(h^{(1,2)}+1)$ fluxes in IIB. Mirror symmetry exchanges $h^{(1,1)} \leftrightarrow h^{(1,2)}$ and we can see that that the number of fluxes is such that it could be extended to Calabi-Yau compactifications with RR-flux. This can indeed be verified in the effective Lagrangian.

However, the NS-flux H_3 is identical in IIA and IIB with no obvious mirror dual. In manifolds with $SU(3)$-structure the torsion can play the role of mirror fluxes for H_3 in that one can have

$$H_3 + i dJ \leftrightarrow d\Omega \ . \tag{E.11}$$

A detailed analysis shows that on half-flat manifolds discussed in (E.5) one obtains mirror symmetric compactifications for electric three-form flux [10]. However, including also magnetic fluxes we immediately see that the left hand side corresponds to $2b_3$ fluxes while the right hand side only has b_4 fluxes. These missing fluxes are provided on manifolds with $SU(3) \times SU(3)$-structure.

E.4 Manifolds with $SU(3) \times SU(3)$-structure

The notion of generalized geometry was introduced by Hitchin [20–23]. He suggested to combine the sum of the tangent bundle and the cotangent bundle into one generalized tangent bundle. In addition he demanded an action of $SO(d,d)$ on this $2d$-dimensional generalized tangent bundle. However, this is not the structure group of a manifold as it includes T-duality type transformations.[22] Manifolds of $G \times G$-structure are defined to have a pair of globally defined spinors/tensors where each one is left invariant by a (different) $G \subset SO(6) \subset SO(6,6)$. Here the case of interest is a pair of spinors each left invariant by $SU(3) \subset SO(6) \subset SO(6,6)$. If the two $SU(3)$'s coincide one has a manifold with $SU(3)$-structure. For this generalized tangent bundle one can develop notions of generalized differential geometry and define generalized complex structures or generalized Kähler structures.

It turns to be convenient to express the couplings of the effective Lagrangian in terms of spinors Φ of $SO(6,6)$. As in ordinary differential geometry one has a one-to-one correspondence between bi-spinors and differential forms. The correspondence $\omega_p \sim \epsilon\gamma^{[i_1} \cdots \gamma^{i_p]}\epsilon$ is generalized as

$$\Phi\Gamma \cdots \Gamma\Phi \sim \sum_p \omega_p \ , \tag{E.12}$$

[22] It also is tailored for the split into left- and right-movers on the string worldsheet.

where Γ are generalized Γ-matrices and the left hand side now is a poly-form. A Majorana condition on Φ implies that the poly-form is real while a Weyl-condition splits the poly-form into even and odd parts

$$\Phi^{\pm} \sim \sum_{p \text{ even/odd}} \omega_p \ . \tag{E.13}$$

For manifolds with $SU(3)$-structure one finds

$$\Phi^{+} \sim e^{J_c} \ , \qquad \Phi^{-} \sim \Omega \ . \tag{E.14}$$

The metric on the deformation space is again special Kähler with Kähler potentials

$$K^{\pm} = -\ln i \langle \Phi^{\pm}, \bar{\Phi}^{\pm} \rangle \ , \tag{E.15}$$

where $\langle \cdot, \cdot \rangle$ is the Mukai-pairing defined by

$$\begin{aligned} \langle \Phi^{+}, \bar{\Phi}^{+} \rangle &= \omega_0 \wedge \bar{\omega}_6 - \omega_2 \wedge \bar{\omega}_4 + \omega_4 \wedge \bar{\omega}_2 - \omega_6 \wedge \bar{\omega}_0 \ , \\ \langle \Phi^{-}, \bar{\Phi}^{-} \rangle &= \omega_1 \wedge \bar{\omega}_5 - 2\omega_3 \wedge \bar{\omega}_3 - \omega_5 \wedge \bar{\omega}_1 \ . \end{aligned} \tag{E.16}$$

The Killing vectors and the potential can also be expressed in terms of $\Phi^{\pm}$ and expressions like $\langle \Phi^{-}, d\Phi^{+} \rangle, \langle \Phi^{+}, d\Phi^{-} \rangle$ appear. The quantities $\langle d\Phi^{+} \rangle, \langle d\Phi^{-} \rangle$ can be viewed as the generalized fluxes. Expanding in a symplectic basis (α_A, β^B) for the odd-forms and $(\omega^{\alpha}, \tilde{\omega}_{\alpha})$ for the even forms. They generalize (E.8) and obey

$$\begin{aligned} d\omega^{\alpha} &= m^{\alpha A} \alpha_A - e^{\alpha}_B \beta^B \ , \\ d\tilde{\omega}_{\alpha} &= -q^A_{\alpha} \alpha_A + p_{\alpha B} \beta^B \ , \\ d\alpha_A &= p_{\alpha A} \omega^{\alpha} + e^{\beta}_A \tilde{\omega}_{\beta} \ , \\ d\beta^A &= q^A_{\alpha} \omega^{\alpha} + m^{A\beta} \tilde{\omega}_{\beta} \ . \end{aligned} \tag{E.17}$$

$d^2 = 0$ again imposes additional relations among the fluxes. However, d is no longer an exterior derivative but a nilpotent operator ($d^2 = 0$) which maps even-forms $\leftrightarrow$ odd-forms. With this generalization mirror symmetry can be established which simply amounts to

$$\Phi^{+} \leftrightarrow \Phi^{-} \ . \tag{E.18}$$

Finally orientifolding such manifolds leads to superpotentials of the form

$$W_{IIB/O3} = -\int \langle \Phi^{-}, d\Pi^{+} \rangle \ , \qquad W_{IIA/O6} = -\int \langle \Phi^{+}, d\Pi^{-} \rangle \ , \tag{E.19}$$

where

$$\Pi^{+} = C_0 + C_2 + C_4 + C_6 + i\mathrm{Re}\,\Phi^{+} \ , \qquad \Pi^{-} = C_1 + C_3 + C_5 + i\mathrm{Re}\,\Phi^{-} \ . \tag{E.20}$$

This leads to additional terms in the potential and helps moduli stabilization and supersymmetry breaking.

F Heterotic–type IIA duality in $\mathbb{R}_{1,3}$

In this appendix we discuss the conjecture

$$\text{Heterotic in } \mathbb{R}_{1,3} \times K3 \times T^2 \quad \equiv \quad \text{IIA in } \mathbb{R}_{1,3} \times Y_3 \quad (\equiv \text{ IIB in } \mathbb{R}_{1,3} \times \tilde{Y}_3) \ . \tag{F.1}$$

The second equality is the already familiar (perturbative) mirror symmetry with $\tilde{Y}_3$ being the mirror Calabi-Yau of Y_3. The first equality is non-perturbative and the topic of this lecture.

Let us first recall the massless bosonic spectrum on both sides. The IIA spectrum is summarized in Table 7.1, the heterotic spectrum for Calabi-Yau compactification we discussed in Table 4.1 but we now need to redo the analysis for compactifications in $\mathbb{R}_{1,3} \times K3 \times T^2$.

The Hodge diamond for K3 reads

$$\begin{array}{ccccc} & & h^{(0,0)} & & \\ & h^{(1,0)} & & h^{(0,1)} & \\ h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} \\ & h^{(2,1)} & & h^{(1,2)} & \\ & & h^{(2,2)} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \quad , \tag{F.2}$$

i.e. all Hodge numbers are fixed and therefore also the Euler number $\chi = \sum_r (-1)^r b_r = 1 + b_2 + 1 = 24$ is fixed. The metric on the moduli space of K3 surfaces has been studied in mathematics extensively and is know to be the metric on the 58-dimensional coset space

$$\mathcal{M}_{\text{K3}} = \frac{SO(3,17)}{SO(3) \times SO(17)} \times \mathbb{R}^+ \ . \tag{F.3}$$

The NS two-form is expanded as

$$B_2 = B_{\mu\nu} dx^\mu dx^\nu + b^\alpha \omega_2^\alpha \ , \quad \alpha = 1, \ldots, 22 \ , \tag{F.4}$$

where ω_2 is a basis of $H^2(K3)$. The K3 metric and the B-field together have the moduli space

$$\mathcal{M}_{\text{K3}+B} = \frac{SO(4,20)}{SO(4) \times SO(20)} \ , \tag{F.5}$$

which is a quaternionic-Kähler manifold.

As for Calabi-Yau compactifications we need to implement the heterotic constraint (2.22). On K3 it implies

$$\int_{\text{K3}} d\hat{H}_3 = -\tfrac{1}{4}\alpha' \int_{\text{K3}} (\text{Tr} F \wedge F - \text{Tr} R \wedge R) = 0 \ . \tag{F.6}$$

Using

$$\int_{\rm K3} {\rm Tr} R\wedge R = \chi(K3) = 24\ , \qquad \int_{\rm K3} {\rm Tr} F\wedge F = n_{\rm inst}\ , \tag{F.7}$$

one infers that on the K3 there has to be gauge bundle with instanton number 24.

In the standard embedding one breaks $E_8 \to E_7 \times SU(2)$ and embeds the instanton background in the $SU(2)$ so that E_7 appears as the unbroken gauge group in $\mathbb{R}_{1,3}$. More generally one breaks $E_8 \to G\times H$, embeds the instanton background in H and is left with G as the unbroken gauge group in $\mathbb{R}_{1,3}$. The instanton solutions on K3 have a moduli space $\mathcal{M}_{\rm HK}$ which is hyper-Kähler but otherwise unknown. In the heterotic compactification discussed here is is fibred over $\mathcal{M}_{{\rm K3}+B}$ given in (F.5) and this total moduli space is known to be quaternionic-Kähler but otherwise is unknown.

Before we proceed let us discuss the bosonic spectrum of the heterotic string in $\mathbb{R}_{1,5}\times K3$. It features the gravity multiplet containing $(g_{\hat\mu\hat\nu}, B^-_{\hat\mu\hat\nu}), \hat\mu,\hat\nu = 0,\dots 5$, one tensor multiplet containing $(B^+_{\hat\mu\hat\nu},\phi)$, $n_v = \dim(G)$ vector multiplets containing $A^a_{\hat\mu}$ and $n_h = 20 + n_h^{\rm inst} = 20 + \dim(\mathcal{M}_{\rm HK})$ hypermultiplets each containing four scalars.

Further compactification on T^2 gives one Kähler modulus T, one complex structure modulus U and dilaton ϕ and axion a (dual of $B_{\mu\nu}$) combine again to $S = e^{-\phi} + ia$. There are also four KK gauge fields arising from $G_{\mu i}, B_{\mu i}, \mu,\nu = 0,\dots 3, i = 1,2$. The 6d gauge fields $A^a_{\hat\mu}$ split into (A^a_μ, A^a_i). The scalars A^{CSA}_i in the Cartan subalgebra of G are flat direction of the potential and thus parametrize part of the moduli space. For generic $\langle A^{CSA}_i\rangle$ the gauge group is broken $G\to [U(1)]^{{\rm rank}(G)}$. At such points the spectrum contains the gravity multiplet $(g_{\mu\nu}, A^0_\mu)$, $n_v = {\rm rank}(G)$ vector multiplets (A^{CSA}_μ, A^{CSA}_i) plus three vector multiplets out of the four $G_{\mu i}, B_{\mu i}$. (The fourth is the graviphoton A^0_μ.) Finally the n_h hypermultiplets are exactly as in $d = 6$.

The scalar field space was discussed in section D.2. The scalar geometry is the product space given in (D.11). Recall that the $M^{2n_v}_{v,SK}$ component is a special Kähler manifold specified by a holomorphic prepotential $\mathcal{F}$ as given in (D.20) and (B.31). Since the dilaton in type IIA is part of a hypermultiplet, $\mathcal{F}$ receives no quantum correction and thus is exact at the string tree-level.

As we just saw on the heterotic side the dilaton is in a vector multiplet and thus $\mathcal{F}$ is corrected at one-loop and non-perturbatively as follows

$$\mathcal{F} = \mathcal{F}^0(S,t) + \mathcal{F}^{1-{\rm loop}}(t) + \mathcal{F}^{\rm np}(e^{-S},t)\ , \tag{F.8}$$

where $t = (T, U, A^{CSA})$. For $\mathcal{F}^0(S,t)$ one finds

$$\mathcal{F}^0(S,t) = S\eta_{ij}t^it^j\ , \tag{F.9}$$

where η is the flat metric of $SO(2, n_v - 1)$. The Kähler potential derived from this $\mathcal{F}^0$ (with the help of (D.20)) reads

$$K^0 = -\ln(S+\bar S) - \ln \eta_{ij}(t+\bar t)^i(t+\bar t)^j\ , \tag{F.10}$$

which is the Kähler potential on the space

$$\mathcal{M} = \frac{SU(1,1)}{U(1)} \times \frac{SO(2,n_v-1)}{SO(2)\times SO(n_v-1)} \ . \tag{F.11}$$

Recall that the second derivative $\mathcal{F}_{ij}$ is related to the gauge couplings via (D.21). Therefore $\mathcal{F}^{1-\text{loop}}(t)$ is related to the threshold corrections of the gauge couplings which we discussed for $N=1$ in Section C.1.

As in $N=1$ it is difficult to compute $\mathcal{F}^{1-\text{loop}}(t)$ in general. As in $N=1$ one has two option: a direct computation via an explicit string loop diagram or indirectly via the holomorphic anomaly which in $N=2$ reads

$$\Delta_0 = -\,\tfrac{1}{16\pi^2}\, b\,\hat{K}(t,\bar{t}) \ , \tag{F.12}$$

where $b = 2(T(ad) - \sum_{\mathbf{r}} T(\mathbf{r}))$ is the one-loop coefficient of the $N=2$ β-function.

For the toroidal moduli T, U one finds [**?**]

$$\begin{aligned}
\partial_T^3 \mathcal{F}^{1-\text{loop}} &= \frac{1}{2\pi}\frac{E_4(iT)E_4(iU)E_6(iU)}{(j(iT)-j(iU))\eta(iU)} \ , \\
\partial_U^3 \mathcal{F}^{1-\text{loop}} &= -\frac{1}{2\pi}\frac{E_4(iU)E_4(iT)E_6(iT)}{(j(iT)-j(iU))\eta(iT)} \ .
\end{aligned} \tag{F.13}$$

Here E_r are modular forms which means they are holomorphic and transform under $SL(2,\mathbb{Z})$ as

$$E_r(iT) \to (icT+d)^r E_r(iT) \ . \tag{F.14}$$

j is the unique holomorphic, $SL(2,\mathbb{Z})$ invariant but singular j-function. The Dedekind η-function we already intoduced in Section C.1. The singularities in (F.13) correspond to the gauge enhancement $[U(1)]^2 \to SU(2)\times U(1) \to SU(3)$ on a torus. Before we proceed let us note that the expressions given in (F.13) can be integrated to give $\mathcal{F}^{1-\text{loop}}$ [**?**].

Now we are prepared to discuss the duality (F.1). For a dual pair the massless spectrum has to agree, i.e. one has to have $n_v^{\text{het}} = n_v^{\text{IIA}}$, $n_h^{\text{het}} = n_h^{\text{IIA}}$ and there has to be a "mirror map" $t^\alpha \leftrightarrow (S, t^i)$ such that

$$\mathcal{F}_{\text{het}}(S,t^i) \equiv \mathcal{F}_{\text{IIA}}(t^\alpha) \ . \tag{F.15}$$

From (F.8) and (F.9) we see that the dilaton plays a special role and there has to be one Kähler modulus t^s which is dual to the heterotic dilaton. Comparing (F.9) and (B.31) we see that this requires

$$d_{t^s t^s t^s} = 0 = d_{t^s t^s t^i} \ . \tag{F.16}$$

This condition is known in the mathematics literature and states that the Calabi-Yau Y_3 is K3-fibred. This means that it has a $\mathbb{P}_1$ as a base and K3 manifolds as fibers. One requirement is that there are only a finite number of points on the $\mathbb{P}_1$ where the K3 is allowed to degenerate. For these classes of manifolds the Calabi-Yau intersection numbers $d_{\alpha\beta\gamma}$ obey (F.16) with t^s being the volume of the $\mathbb{P}_1$. Via mirror symmetry one can compute $\mathcal{F}_{\text{IIA}}$ exactly in specific cases, evaluate it in the large t^s limit and compare with $\mathcal{F}_{\text{het}}$ computed via (F.13). In all known examples (F.15) holds for an infinite number of terms. Conversely, if one accepts the duality (F.1) one can use (F.15) to compute $\mathcal{F}_{\text{het}}$ exactly including all non-perturbative terms.

The scalars in the hypermultiplets live on a quaternionic-Kähler geometry $\mathcal{M}^{4n_h}_{h,QK}$ as discussed in section D.2. This geometry is more constrained but at the same time more difficult to describe. (For example, there is no (easy) holomorphic function which characetrizes it.) As a consequence the checks performed so far are much weaker. A similar analysis as we just described for the vector multiplets has been partially performed for hypermultiplets in [?]. One of the resulting conjectures is that the duality (F.15) also requires that the mirror Calabi-Yau $\tilde{Y}_3$ has to be a K3-fibration [**?**].

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