**This problem set is a former written exam. If you hand it in by 12.00h on 22.11.10, it counts towards the bonus. (Note that handing it in jointly does not count.)**

**Problem 5.1** (20 points)

Consider a complex scalar field $\phi$ with a Lagrangian

$$\mathcal{L} = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi \phi^* - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$D_\mu \phi := \partial_\mu \phi + iqA_\mu \phi \ , \qquad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \ , \qquad q \in \mathbb{R} \ .$$

a) Compute the Euler-Lagrange equations for $\phi, \phi^*$ and $A_\mu$.

b) Show that

$$\phi \to \phi' = e^{iq\alpha} \phi \ , \qquad \alpha \in \mathbb{R} \ ,$$

is a symmetry of $\mathcal{L}$, compute the associated Noether current and check its conservation.

**Problem 5.2** (20 points)

a) Show in the chiral representation

$$\gamma^2 (\gamma^\mu)^* \gamma^2 = a \gamma^\mu \ , \qquad \gamma^0 (\gamma^\mu)^* \gamma^0 = b\, (\gamma^\mu)^T \ ,$$

and compute $a$ and $b$.

*Hint*: You might find $\gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger$ useful.

b) Determine the transformation law of $\psi_c := \gamma^2 \psi^*$ to be

$$\delta \psi_c = -\tfrac{i}{2}\, d\, \omega_{\mu\nu} S^{\mu\nu} \psi_c \ ,$$

and compute $d$.

*Hint*: Use the result of a) and $\delta\psi = -\frac{i}{2}\, \omega_{\mu\nu} S^{\mu\nu} \psi$.

c) Which covariant first-order differential equation does $\psi_c$ satisfy?

**Problem 5.3** (20 points)

For a four-component Dirac spinor $\psi$ one defines

$$\psi = \psi_L + \psi_R \ , \qquad \psi_L := \tfrac{1}{2}(1-\gamma^5)\,\psi \ , \qquad \psi_R := \tfrac{1}{2}(1+\gamma^5)\,\psi \ .$$

a) Show

$$\bar\psi_L = \tfrac{1}{2}\,\bar\psi\,(1+\gamma^5) \ , \qquad \bar\psi_R = \tfrac{1}{2}\,\bar\psi\,(1-\gamma^5) \ ,$$

and

$$\bar\psi_R\psi_R = 0 = \bar\psi_L\psi_L \ , \qquad \bar\psi_R\gamma^\mu\psi_L = 0 = \bar\psi_L\gamma^\mu\psi_R \ .$$

*Hint*: $(\gamma^5)^2 = 1, \{\gamma^5, \gamma^\mu\} = 0, (\gamma^5)^\dagger = \gamma^5$.

b) Show

$$\gamma^5\psi_L = -\psi_L \ , \qquad \gamma^5\psi_R = \psi_R \ ,$$

and express $\bar\psi\psi$ and $\bar\psi\gamma^5\psi$ in terms of $\psi_L, \psi_R, \bar\psi_L, \bar\psi_R$.

*Hint*: Use the results of a).

c) Give the Dirac Lagrangian expressed in terms of $\psi_L, \psi_R, \bar\psi_L, \bar\psi_R$.

*Hint*: Use the results of a).

**Problem 5.4** (20 points)

Consider the operator

$$\vec P := -i\int d^3x\,\psi^\dagger\vec\nabla\psi \ .$$

Use

$$\psi(x, t=0) = \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec p}}}\sum_s\left(a^s_{\vec p}\,u^s(p)\,e^{i\vec p\cdot\vec x} + b^{\dagger s}_{\vec p}\,v^s(p)\,e^{-i\vec p\cdot\vec x}\right) \ ,$$

to show

$$\vec P = \int\frac{d^3p}{(2\pi)^3}\,\vec f\sum_s\left(a^{\dagger s}_{\vec p}\,a^s_{\vec p} + g\,b^{\dagger s}_{\vec p}b^s_{\vec p}\right) + c \ ,$$

and compute $\vec f, g$ and $c$.

*Hint*: You might find the following identities useful:

$$u^{s\dagger}(p)\,u^{s'}(p) = 2E_{\vec p}\,\delta^{ss'} = v^{s\dagger}(p)\,v^{s'}(p) \ , \qquad u^{s\dagger}(\vec p)\,v^{s'}(-\vec p) = 0 = v^{s\dagger}(\vec p)\,u^{s'}(-\vec p) \ .$$

Furthermore $\int d^3x e^{i(\vec p-\vec p')\cdot\vec x} = (2\pi)^3\delta^3(\vec p-\vec p')$.