Problem 9.1

The Γ-function is defined by

$$\Gamma(\alpha) := \int_0^\infty dy\, y^{\alpha-1} e^{-y} \ .$$

a) Show

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \ , \qquad \Gamma(\alpha) = (\alpha-1)! \quad \text{for } \alpha \in \mathbb{N} \ ,$$

and

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 dy y^{\alpha-1}(1-y)^{\beta-1} \ .$$

Hint: The first identity can be shown by appropriate partial integration. The last identity can be proved by writing $\Gamma(\alpha)\,\Gamma(\beta)$ as a double integral (in y and z) and substituting $y = st, z = s(1-t)$.

b) The Γ-function also has a product representation of the form

$$\frac{1}{\Gamma(\alpha)} = \alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} (1 + \frac{\alpha}{n}) e^{-\frac{\alpha}{n}} \ ,$$

where γ is the Euler-Mascheroni constant. Use this representation to show

$$\lim_{\alpha\to 0} \Gamma(\alpha) = \frac{1}{\alpha} - \gamma + \mathcal{O}(\alpha) \ .$$

Problem 9.2

a) Using a Wick rotation show

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^m} = \frac{i(-1)^m}{(4\pi)^{d/2}} \frac{1}{\Delta^{m-d/2}} \frac{\Gamma(m-d/2)}{\Gamma(m)} \ ,$$

and

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^m} = \frac{i(-1)^{m-1}}{(4\pi)^{d/2}} \frac{d}{2} \frac{1}{\Delta^{m-d/2-1}} \frac{\Gamma(m-d/2-1)}{\Gamma(m)} \ .$$

b) Using the results of 9.1 and 9.2a) compute explicitly the divergent and finite pieces of the integral

$$\lim_{d\to 4} \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^3} \ .$$

Problem 9.3

a) Show

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2-\Delta)^m} = \frac{1}{d}\eta^{\mu\nu} \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^m} \ .$$

Analogously argue that

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2-\Delta)^m} = T^{\mu\nu\rho\sigma} \int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^2}{(l^2-\Delta)^m} \ ,$$

and determine $T^{\mu\nu\rho\sigma}$.

b) Show that for $d = 4 - \epsilon$ one has

$$\begin{aligned}
\gamma^\mu\gamma^\nu\gamma_\mu &= (\epsilon-2)\gamma^\nu \ , \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho} - \epsilon\gamma^\nu\gamma^\rho \ , \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + \epsilon\gamma^\nu\gamma^\rho\gamma^\sigma \ ,
\end{aligned}$$

Problem 9.4

In a Pauli-Villars regularization one introduce an additional massive ghost field with a wrong-sign kinetic term. For the electron self-energy computed in class one can show that as a consequence the divergent integral is replaced as follows:

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2-\Delta)^2} \to I = \int \frac{d^4 l}{(2\pi)^4} \left(\frac{1}{(l^2-\Delta)^2} - \frac{1}{(l^2-M)^2} \right) \ .$$

a) Argue that I is convergent in $d = 4$.

b) Compute I by Wick rotating and introducing a UV cut-off. Show

$$I = \frac{i}{(4\pi)^2} \log(M/\Delta) \ .$$

c) In which limit do you now see the UV-divergence?