Problem 6.1

The spin-flipped $u^{-s}(p)$ is given by

$$u^{-s}(p) = \frac{1}{N}\begin{pmatrix} (p\cdot\sigma + m)\,(-i\sigma^2\xi^s) \\ (p\cdot\bar{\sigma} + m)\,(-i\sigma^2\xi^s) \end{pmatrix} .$$

a) Show

$$u^{-s}(\tilde{p}) = -\gamma^1\gamma^3\left(u^s(p)\right)^* , \qquad \text{where} \qquad \tilde{p} = (p_0, -\vec{p}) .$$

b) Show

$$T\psi(t,\vec{x})\,T = \gamma^1\gamma^3\psi(-t,\vec{x}) , \qquad \text{for} \qquad Ta^s_{\vec{p}}T = a^{-s}_{-\vec{p}} , \quad Tb^s_{\vec{p}}T = b^{-s}_{-\vec{p}} , \quad T^2 = \mathbf{1} ,$$

and T anti-unitary. (An anti-unitary operator obeys $Tc = c^*T$ for $c \in \mathbb{C}$.)

Hint: Also use $v^{-s}(\tilde{p}) = -\gamma^1\gamma^3(v^s(p))^*$.

c) Compute

$$T\,\bar{\psi}\psi(t,\vec{x})\,T , \qquad T\,\bar{\psi}\gamma^5\psi(t,\vec{x})\,T , \qquad T\,\bar{\psi}\gamma^\mu\psi(t,\vec{x})\,T .$$

Problem 6.2

a) Draw all Feynman-diagrams for the amplitude $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$ to order λ^2 using the interaction $H_I = \frac{\lambda}{4!}\int d^4z\,\phi^4(z)$.

b) Give for each diagram the associated expression in terms of Feynman-propagators (including the symmetry factor).

c) Show that the contribution at order λ is proportional to $I = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}$.

d) Show that in Euclidean $\mathbb{R}^4$ one has $\int_{-\infty}^{\infty} d^4k e^{-k_r^2} = \pi^2$ for $k_r^2 = \sum_{i=1}^4 k_i^2$. By evaluating the same integral in polar coordinates with $d^4k = d\Omega_3 dk_r k_r^3$ infer that $\int d\Omega_3 = 2\pi^2$ holds.

e) Use the result from 2d) and a Wick rotation $k^0 \to ik^0$ to compute the integral I for $k_r \in [0,\Lambda]$. How does I behave in the limit $\Lambda \to \infty$?

Problem 6.3

a) Show that for n odd one has

$$\mathrm{Tr}\big(\gamma^{\mu_1}\dots\gamma^{\mu_n}\big) = 0 \ .$$

Hint: Insert $(\gamma^5)^2$.

b) Show

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\,\eta^{\nu\rho}\,\mathbf{1} \ .$$

c) Show

$$\mathrm{Tr}\big(\gamma^\mu\gamma^\nu\gamma^5\big) = 0 \ .$$

Hint: Insert $(\gamma^\rho)^2$ for $\rho \neq \mu, \nu$.

d) Show

$$\mathrm{Tr}\big(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\big) = -4i\epsilon^{\mu\nu\rho\sigma} \ .$$

Hint: Use 6.2c).