Problem 3.1

Consider the Lagragian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 + j\,\phi\ ,$$

for a classical external source $j(x)$.

a) Compute the Euler-Lagrange equation and determine the solution in terms of the retarded Greens function G_{ret}.

b) Show that for a j which is non-vanishing only in the past (i.e. for $y^0 < x^0$) one has

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(\tilde{a}_{\vec{p}}e^{-ip\cdot x} + \tilde{a}^\dagger_{\vec{p}}e^{ip\cdot x}\right)_{p^0=E_{\vec{p}}}\ ,$$

where $\tilde{a}_{\vec{p}} \equiv \left(a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}}\tilde{\jmath}(p)\right)$ and $\tilde{\jmath}(p) \equiv \int d^4y e^{ip\cdot y} j(y)$.

c) Compute $\langle 0|H|0\rangle$ after j is turned off.

Hint: Compute H from $\mathcal{L}$ with $j=0$ but use the ϕ computed in b).

Problem 3.2

Consider the Lagragian $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$.

a) Check that $\mathcal{L}$ is invariant under the symmetry $\psi \to \psi' = e^{-i\alpha}\psi$ for $\alpha \in \mathbb{R}$.

b) Compute the corresponding Noether current and Noether charge and check the current conservation.

c) Under what condition is $\psi \to \psi' = e^{-i\alpha\gamma^5}\psi$ a symmetry? Compute for that case the Noether current and check its conservation.

Problem 3.3

The γ-matrices in the chiral representation are defined as

$$\gamma_c^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} , \qquad \text{where} \qquad \sigma^\mu := (\mathbf{1}, \sigma^i), \quad \overline{\sigma}^\mu := (\mathbf{1}, -\sigma^i).$$

a) Show $\{\gamma_c^\mu, \gamma_c^\nu\} = 2\eta^{\mu\nu}$.

b) Compute

$$\gamma_D^\mu = M\gamma_c^\mu M^{-1} \qquad \text{for} \qquad M = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{pmatrix} .$$

c) For $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ show

$$(\gamma^5)^2 = \mathbf{1} \ , \qquad \{\gamma^\mu, \gamma^5\} = 0 \ ,$$

without using any explicit representation of the γ-matrices.

d) Show

$$\gamma_c^5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} , \qquad \gamma_D^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} .$$

e) Define $P_{L,R} := \frac{1}{2}(1 \pm \gamma^5)$ and show without using an explicit representation

$$P_L^2 = P_L \ , \qquad P_R^2 = P_R \ , \qquad P_R P_L = 0 \ .$$

Problem 2.4

Consider the differential operator

$$J^{\mu\nu} = i\big(x^\mu\partial^\nu - x^\nu\partial^\mu\big) \ .$$

a) Show that for infinitesimal rotations parameterized by the three angles $\theta_i, i = 1, 2, 3$ and for infinitesimal Lorentz transformations parameterized by the three velocities v_i one has

$$\delta x^i = -\tfrac{i}{2}\theta_k\epsilon^{kmn}J^{mn}x^i \ , \qquad \delta x^\mu = -iv_k J^{0k}x^\mu \ .$$

b) Show

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\mu\sigma}J^{\nu\rho}\right)$$