Kovarianz der Maxwell-Gleichungen

1. Schritt: inhomogene Wellengleichung

$$\Box \Phi = -\frac{\rho}{\varepsilon_0} \quad , \quad \Box \vec{A} = -\mu_0 \vec{j}$$

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta = \sum_{\mu=0}^{3} \partial_\mu \partial^\mu \qquad \text{(letzte Vorlesung)}$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \Big(\underbrace{\frac{1}{c}\frac{\partial}{\partial t}}_{=\frac{\partial}{\partial x^0}}, \underbrace{\vec{\nabla}}_{\frac{\partial}{\partial x^i}}\Big) \quad , \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \Big(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\Big)$$

$$\partial^\mu = \sum_\nu g^{\mu\nu} \partial_\nu$$

letzte Vorlesung: $\sum_\mu V_\mu V^\mu = \sum_\mu V'_\mu V'^\mu$, für beliebige V_μ

weil $\quad V'^{\mu} = \sum_{\nu} \Lambda^{\mu}{}_{\nu} V^{\nu}$

$\Rightarrow$ $\Box$ ist lorentzinvariant Diff. Op.

d.h. $\quad \Box' = \Box$

<u>Def</u>: $\quad j^{\mu} := (c\rho, \vec{j})$ Vierervektor der Stromdichte

$A^{\mu} := (\frac{1}{c}\Phi, \vec{A})$ Viererpotential

$$\Box A^{\mu} = \begin{cases} \frac{1}{c}\Box\Phi = -\frac{\rho}{c\epsilon_0} = -\frac{c\rho}{\underbrace{c^2\epsilon_0}_{\frac{1}{\mu_0}}} = -\mu_0(c\rho) = -\mu_0 j^0 \\ \Box\vec{A} = -\mu_0\vec{j} = -\mu_0\vec{j} = -\mu_0\vec{j} \end{cases}$$

$\Rightarrow$ $\boxed{\Box A^{\mu} = -\mu_0 j^{\mu}}$ inh. Wellengleichung

A^μ, j^μ transformieren wie 4-er Vektoren, d.h.

$$A^\mu \to A'^\mu = \sum_\nu \Lambda^\mu{}_\nu A^\nu \,, \quad j^\mu \to j'^\mu = \sum_\nu \Lambda^\mu{}_\nu j^\nu$$

$$\boxed{0 = \Box' A'^\mu + \mu_0 j'^\mu} = \Box \sum_\nu \Lambda^\mu{}_\nu A^\nu + \mu_0 \sum_\nu \Lambda^\mu{}_\nu j^\nu$$

$$= \sum_\nu \Lambda^\mu{}_\nu \left(\Box A^\nu + \mu_0 j^\nu \right)$$

$$\Rightarrow \boxed{\Box A^\nu = -\mu_0 j^\nu}$$

<u>Kovarianz</u> der Wellengleichung

$\Updownarrow$

d.h. rechte und linke Gleichung transformieren gleich!

(Check: <u>wichtig</u> mit kovariantem Vektor zu tun)

Prüfe Lorenzeichung: $0 = \vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t}$

$$\boxed{0 = \sum_\mu \partial_\mu A^\mu} \Rightarrow \text{invariant} \checkmark$$

Prüfe Kontinuitätsgleichung:

$$\sum_\mu \partial_\mu j^\mu = \frac{1}{c} \frac{\partial}{\partial t} c\rho + \vec{\nabla} \vec{j} = \frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{j} = 0$$

$$\Rightarrow \boxed{\sum_\mu \partial_\mu j^\mu = 0}$$

Maxwell Gleichung für $\vec{E}$, $\vec{B}$ in kovarianter Form

<u>Def</u>:

Feldstärke Tensor $\quad F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu = -F^{\nu\mu}$

d.h. $F^{\mu\nu}$ hat 6 unabhängige Komponenten

$$F^{\mu\nu} = \begin{pmatrix} 0 & F^{01} & F^{02} & F^{03} \\ -F^{01} & 0 & F^{12} & F^{13} \\ -F^{02} & -F^{12} & 0 & F^{23} \\ -F^{03} & -F^{13} & -F^{23} & 0 \end{pmatrix}, \qquad \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right), \quad A^\mu = \left(\frac{1}{c}\Phi, \vec{A}\right)$$

$$F^{01} = \frac{1}{c}\left(\partial_t A^x + \partial_x \Phi\right) \qquad F^{12} = -\partial_x A^y + \partial_y A^x$$

$$F^{02} = \frac{1}{c}\left(\partial_t A^y + \partial_y \Phi\right) \qquad F^{13} = -\partial_x A^z + \partial_z A^x$$

$$F^{03} = \frac{1}{c}\left(\partial_t A^z + \partial_z \Phi\right) \qquad F^{23} = -\partial_y A^z + \partial_z A^y$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \quad \Rightarrow \quad E_x = -\partial_x \Phi - \frac{\partial A^x}{\partial t} = -c\,F^{01}$$

$$E_y = -\partial_y \Phi - \frac{\partial A^y}{\partial t} = -c\,F^{02}$$

$$E_z = -\partial_z \Phi - \frac{\partial A^z}{\partial t} = -c\,F^{03}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \Rightarrow \quad B_x = \partial_y A^z - \partial_z A^y = -F^{23}$$

$$B_y = \partial_z A^x - \partial_x A^z = F^{13}$$

$$B_z = \partial_x A^y - \partial_y A^x = -F^{12}$$

$$\Rightarrow F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & -B_z & B_y \\ \frac{1}{c}E_y & B_z & 0 & -B_x \\ \frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix},$$

d.h. die 6 unabhängigen Kompnt. sind gerade $\vec{E}, \vec{B}$

$$\sum_\mu \partial_\mu F^{\mu\nu} = \partial_0 F^{0\nu} + \sum_{i=1}^{3} \partial_i F^{i\nu}$$

$$\sum_\mu \partial_\mu F^{\mu 0} = \partial_0 \underbrace{F^{00}}_{=0} + \sum_i \partial_i \underbrace{F^{i0}}_{+\frac{1}{c}E^i} = +\frac{1}{c} \vec{\nabla}\cdot\vec{E}$$

$$\sum_\mu \partial_\mu F^{\mu 1} = \underbrace{\partial_0}_{\frac{1}{c}\partial_t} \underbrace{F^{01}}_{-\frac{1}{c}E^x} + \sum_i \partial_i F^{i1} = -\frac{1}{c^2}\partial_t E_x + \partial_1 \underbrace{F^{11}}_{=0} + \partial_2 \underbrace{F^{21}}_{B_z} + \partial_3 \underbrace{F^{31}}_{-B_y}$$

$$= -\frac{1}{c^2}\partial_t E_x \underbrace{+ \partial_y B_z - \partial_z B_y}_{+(\vec{\nabla}\times\vec{B})_x} = -\frac{1}{c^2}\partial_t E_x + (\vec{\nabla}\times\vec{B})_x$$

$$\sum_\mu \partial_\mu F^{\mu 2} = \dots = -\frac{1}{c^2}\partial_t E_y + (\vec{\nabla}\times\vec{B})_y$$

$$\sum_\mu \partial_\mu F^{\mu 3} = \dots = -\frac{1}{c^2}\partial_t E_z + (\vec{\nabla}\times\vec{B})_z$$

$$\Rightarrow \sum_\mu \partial_\mu F^{\mu\nu} = \begin{cases} \frac{1}{c} \vec{\nabla} \cdot \vec{E} \\ -\frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} + \vec{\nabla} \times \vec{B} \end{cases}$$

inhomogene Maxwell Gleichungen

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \ , \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$$

$$\Rightarrow \boxed{\sum_\mu \partial_\mu F^{\mu\nu} = \mu_0 j^\mu} \ , \quad \left(\sum_\mu \partial_\mu F^{\mu 0} = \frac{1}{c} \vec{\nabla} \cdot \vec{E} = \mu_0 j^0 = \mu_0 c \rho \right.$$

$$\left. \Rightarrow \vec{\nabla} \cdot \vec{E} = \underbrace{\mu_0 c^2}_{\frac{1}{\epsilon_0}} \rho = \rho / \epsilon_0 \ . \right)$$

inhomogene Maxwell-Gleichungen in kovariant Form

Transformation von $F^{\mu\nu} \to F'^{\mu\nu} = \sum_\rho \sum_\sigma \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$

homogene Maxwell Gleichung

$$\boxed{\partial^\nu F^{\mu\kappa} + \partial^\mu F^{\kappa\nu} + \partial^\kappa F^{\nu\mu} = 0}$$

$\nu, \mu, \kappa \neq 0$:

$$\underbrace{\partial^1}_{-\partial_x} F^{23} + \underbrace{\partial^2}_{-\partial_y} F^{31} + \underbrace{\partial^3}_{-\partial_z} F^{12} = -\partial_x(-B_x) - \partial_y(-B_y) - \partial_z(-B_z)$$

$$= \partial_x B_x + \partial_y B_y + \partial_z B_z = \vec{\nabla}\cdot\vec{B} = 0$$

$\nu = 0$

$$\underbrace{\partial^0}_{\frac{1}{c}\frac{\partial}{\partial t}} F^{12} + \underbrace{\partial^1}_{-\partial_x} F^{20} + \underbrace{\partial^2}_{-\partial_y} F^{01} = \frac{1}{c}\frac{\partial}{\partial t}(-B_z) - \partial_x\left(\frac{E_y}{c}\right) - \partial_y\left(-\frac{1}{c}E_x\right)$$

$$= \frac{1}{c}\left(-\frac{\partial B_z}{\partial t} + \underbrace{\partial_y E_x - \partial_x E_y}_{-(\vec{\nabla}\times\vec{E})_z}\right) = 0 \Rightarrow \frac{\partial B_z}{\partial t} = -(\vec{\nabla}\times\vec{E})_z \quad \checkmark$$

$\mu = 0$: ...

$\kappa = 0$: ...

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0}$$

$\Rightarrow$ Maxwell-Gleichungen in kovarianter Form

$$\boxed{\begin{aligned} &\sum_\mu \partial_\mu F^{\mu\nu} = \mu_0 j^\nu && 4 \\ &\partial^\nu F^{\mu\kappa} + \partial^\mu F^{\kappa\nu} + \partial^\kappa F^{\nu\mu} = 0 && 4 \end{aligned}}$$

Eichtransformation

$$\left.\begin{aligned} \vec{A} &\to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \\ \Phi &\to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \end{aligned}\right\} \qquad \boxed{A^\mu \to A'^\mu = A^\mu - \partial^\mu \Lambda}$$

$$A^\mu = \left(\tfrac{1}{c}\Phi, \vec{A}\right)$$

$$\partial^\mu = \left(\tfrac{1}{c}\partial_t, -\vec{\nabla}\right)$$

Prüfe Eichinvarianz der Maxwell-Gleichungen:

$$F^{\mu\nu} \to F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu = \partial^\mu(A^\nu - \partial^\nu\Lambda) - \partial^\nu(A^\mu - \partial^\mu\Lambda)$$

$$= \underbrace{\partial^\mu A^\nu - \partial^\nu A^\mu}_{F^{\mu\nu}} - \underbrace{\partial^\mu\partial^\nu\Lambda + \partial^\nu\partial^\mu\Lambda}_{=0}$$

$$\Rightarrow F'^{\mu\nu} = F^{\mu\nu} \quad \text{unter Eichtrafo.}$$

Lorentztransformation von $F^{\mu\nu}$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F'^{\mu\nu} = \partial'^\mu A'^\nu - \partial'^\nu A'^\mu = \left(\sum_\rho \Lambda^\mu{}_\rho \partial^\rho\right)\left(\sum_\sigma \Lambda^\nu{}_\sigma A^\sigma\right) - \left(\sum_\sigma \Lambda^\nu{}_\sigma \partial^\sigma\right)\left(\sum_\rho \Lambda^\mu{}_\rho A^\rho\right)$$

$$= \sum_\rho \sum_\sigma \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \underbrace{\left(\partial^\rho A^\sigma - \partial^\sigma A^\rho\right)}_{F^{\rho\sigma}}$$

$$\boxed{F'^{\mu\nu} = \sum_\rho \sum_\sigma \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}}$$

Feldstärke transformiert wie Tensor 2. Stufe

$$F' = \Lambda^T F \Lambda$$

explizit:

$$\Lambda^{\mu}{}_{\nu} = \left(\begin{array}{cc|cc} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \qquad \frac{1}{\gamma} = \sqrt{1-\frac{v^2}{c^2}}$$

KS' bewegt sich in x-Richtung mit Geschwindigkeit v gegenüber KS

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ \vdots & 0 & -B_z & B_y \\ \vdots & & 0 & -B_x \\ \vdots & - & & 0 \end{pmatrix}$$

$\Rightarrow$ ...

Ergebnis:

$$\boxed{\begin{aligned} & E'_x = E_x, \quad E'_y = \gamma E_y - \frac{\gamma v}{c} B_z \\ & \qquad\qquad\quad\;\; E'_z = \gamma E_z + \frac{\gamma v}{c} B_y \\ & B'_x = B_x, \quad B'_y = \frac{\gamma v}{c} E_z + \gamma B_y, \quad B'_z = \gamma B_z - \frac{\gamma v}{c} E_y \end{aligned}}$$