Energie des elektromag. Feldes

Lorentzkraft $\vec{F} = q\left(\vec{E}(\vec{x}_0(t),t) + \vec{v} \times \vec{B}(\vec{x}_0,t)\right)$

Mechanik: Arbeit W

$$W = \int_{\vec{x}_1}^{\vec{x}_2} \vec{F}\, d\vec{r} = \int_{t_1}^{t_2} \vec{F} \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}} dt = \int_{t_1}^{t_2} \frac{dW}{dt}\, dt$$

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v} = q\left(\vec{E} \cdot \vec{v} + \underbrace{(\vec{v} \times \vec{B}) \cdot \vec{v}}_{=0}\right)$$

$$= \int d^3x' \underbrace{q\, \delta(\vec{x}' - \vec{x}_0)\, \vec{v}}_{\vec{j}(\vec{x}',t)} \cdot \vec{E}(\vec{x}', t)$$

$$\Rightarrow \frac{dW}{dt} = \int d^3x' \, \vec{J}(\vec{x}', t) \cdot \vec{E}(\vec{x}', t) \quad , \quad \vec{J} = q \vec{v} \, \delta(\vec{x} - \vec{x}_0)$$

Maxwell: $\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$

$$\rightsquigarrow \frac{dW}{dt} = \frac{1}{\mu_0} \int d^3x' \left((\vec{\nabla} \times \vec{B}) \cdot \vec{E} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \cdot \vec{E} \right)$$

Espiel: $\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = (\vec{\nabla} \times \vec{E}) \cdot \vec{B} - (\vec{\nabla} \times \vec{B}) \cdot \vec{E}$

$$\rightsquigarrow \frac{dW}{dt} = \frac{1}{\mu_0} \int d^3x' \left[(\vec{\nabla} \times \vec{E}) \cdot \vec{B} - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) - \frac{1}{2c^2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) \right]$$

Maxwell: $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$$\rightsquigarrow \frac{dW}{dt} = \frac{1}{\mu_0} \int d^3x' \left[- \frac{\partial \vec{B}}{\partial t} \cdot \vec{B} - \frac{1}{2} \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} \cdot \vec{E} - \vec{\nabla} (\vec{E} \times \vec{B}) \right]$$

$$\frac{dW}{dt} = \int d^3x' \left[-\frac{1}{2\mu_0} \frac{\partial}{\partial t}(\vec{B}\cdot\vec{B}) - \frac{\epsilon_0}{2} \frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) - \frac{1}{\mu_0} \vec{\nabla}\cdot(\vec{E}\times\vec{B}) \right]$$

Def:

$$\boxed{\begin{aligned} \vec{S} &:= \frac{1}{\mu_0} \vec{E}\times\vec{B} \quad \text{Poynting Vektor} \\ \mathcal{E} &:= \frac{1}{2}\left(\frac{1}{\mu_0} \vec{B}\cdot\vec{B} + \epsilon_0 \vec{E}\cdot\vec{E}\right) \quad \text{Energiedichte des e-m Feldes} \end{aligned}}$$

$$\rightsquigarrow \boxed{\frac{dW}{dt} = -\int_V d^3x' \, \vec{\nabla}\cdot\vec{S} - \frac{d}{dt}\int d^3x' \, \mathcal{E}}$$

$$\boxed{\frac{dW}{dt} = -\int_{\partial V} \vec{S}\cdot d\vec{F} - \frac{d}{dt}\int_V d^3x' \, \mathcal{E}}$$

$$f(x,y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

$$f(x), \quad \frac{df}{dx}$$

$$f(x(y), y), \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial x}, \quad \frac{df}{d\cancel{y}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}$$

$$E = \cos(kx - \omega t), \quad E = \cos(k\cancel{x}x(t) - \omega t)$$

$$\frac{\partial E}{\partial x}, \quad \frac{\partial E}{\partial z}$$

$$\frac{dE}{dt} = -\sin(kx - \omega t)\, k \frac{\partial x}{\partial t} + \omega \sin(kx - \omega t)$$

$$\frac{\partial E}{\partial z} = -\omega \sin(kx - \omega t)$$

Anwendung auf Dipolstrahlung

letzte Vorlesung:

$$(*)\begin{cases} \vec{E}(\vec{x},t) = \mathrm{Re}\left(\vec{E}(\vec{x})\, e^{-i\omega t}\right), & \vec{E}(\vec{x}) \approx c\, \dfrac{\vec{B}\times\vec{x}}{r} \\ \vec{B}(\vec{x},t) = \mathrm{Re}\left(\vec{B}(\vec{x})\, e^{-i\omega t}\right), & \vec{B}(\vec{x}) \approx \dfrac{\mu_0 k\omega}{4\pi}(\vec{x}\times\vec{p})\,\dfrac{e^{ikr}}{r^2} \end{cases}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{4\mu_0}\left(\vec{E}(\vec{x})e^{-i\omega t} + \vec{E}^* e^{i\omega t}\right)\times\left(\vec{B}(\vec{x})\, e^{-i\omega t} + \vec{B}^* e^{i\omega t}\right)$$

$$\overline{\vec{S}} := \frac{1}{T}\int_0^T \vec{S}(\vec{x},t)\, dt, \qquad T = \frac{2\pi}{\omega} \qquad \text{Zeitmittel von } \vec{S}$$

Bemerkung: $\displaystyle \frac{1}{T}\int_0^T dt = 1, \qquad \frac{1}{T}\int_0^T e^{\pm 2i\omega t}\, dt = \frac{1}{\pm 2i\omega}\underbrace{\left(e^{\pm 2i\omega \overbrace{T}^{2\pi}} - 1\right)}_{0}$

$$\overline{\vec{S}} = \frac{1}{4\mu_0}\left(\vec{E}(\vec{x}) \times \vec{B}^{*}(\vec{x}) + \vec{E}^{*}(\vec{x}) \times \vec{B}(\vec{x})\right)$$

$$= \frac{1}{2\mu_0} \mathrm{Re}\left(\vec{E} \times \vec{B}^{*}\right)$$

~~(*)~~

(*) eingesetzt: $\overline{\vec{S}} = \frac{1}{2\mu_0} \mathrm{Re}\left(\left(\frac{\mu_0 k\omega}{4\pi}\right)^2 \frac{c}{r^5} \left[\underbrace{(\vec{x} \times \vec{p}^{*}) \times \vec{x}}_{(\vec{x}\cdot\vec{x})\vec{p} - \vec{x}(\vec{x}\cdot\vec{p})} + (\vec{x} \times \vec{p})\right]\right)$

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$$\boxed{\begin{aligned} \overline{\vec{S}} &= \frac{\mu_0}{32\pi^2} \frac{c^3 k^4}{r^2} \hat{\vec{x}} \left(|\vec{p}|^2 - |\hat{\vec{x}} \cdot \vec{p}|^2\right) \\ \overline{\vec{S}} &= \frac{\mu_0}{32\pi^2} \frac{c^3 k^4}{r^2} \hat{\vec{x}} \, |\hat{\vec{x}} \times \vec{p}|^2 \end{aligned}}$$

Def: Leistung P

$$P := \int_F \overline{\vec{S}} \cdot d\vec{F} = \int d\Omega \, r^2 \, \hat{\vec{x}} \cdot \overline{\vec{S}}$$

$$\int d\Omega = 4\pi \, , \quad d\Omega = -d(\cos\theta)\, d\varphi = \sin\theta \, d\theta \, d\varphi$$

= Zeitmittel der Energie, die in einer Zeiteinheit durch eine Kugeloberfläche mit Radius r transmittiert wird.

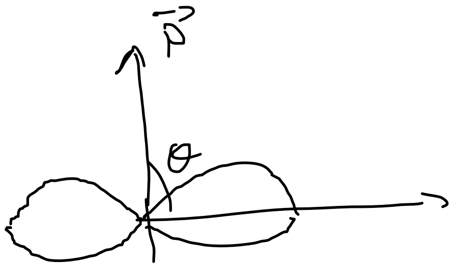
$$\frac{dP}{d\Omega} := r^2 \, \hat{\vec{x}} \cdot \overline{\vec{S}}$$

$$\overset{\text{Dipolstrahlung}}{=} \frac{1}{4\pi\varepsilon_0} \frac{k^4 c}{8\pi} \Big(|\vec{p}|^2 - \underbrace{|\hat{\vec{x}} \cdot \vec{p}|^2}_{|\vec{p}|^2 \cos^2\theta} \Big) \, , \quad \theta = \sphericalangle(\hat{\vec{x}}, \vec{p})$$

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{4\pi\varepsilon_0} \frac{k^4 c}{8\pi} |\vec{p}|^2 \sin^2\theta}$$

$$P = \int d\Omega \, \frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0} \frac{k^4 c}{8\pi} |\vec{p}|^2 \, 2\pi \underbrace{\int_0^\pi \sin^3\theta \, d\theta}_{= 4/3}$$

$$\boxed{P = \frac{1}{4\pi\epsilon_0} \frac{k^4 c}{3} |\vec{p}|^2 = \frac{1}{4\pi\epsilon_0} \frac{\omega^4}{3c^3} |\vec{p}|^2}$$



Winkelverteilung der abgestrahlten Leistung

Bemerkung: $P \sim \omega^4$ Lichtstrahl mit hohem ω (blau) erreicht uns von der Sonne

Selbstenergie des E-M Feldes

Annahme: $\rho(\vec{x}) = \rho_1(\vec{x}) + \rho_2(\vec{x})$, $\quad \vec{j}(\vec{x}) = \vec{j}_1(\vec{x}) + \vec{j}_2(\vec{x})$

$$\Rightarrow \quad \vec{E} = \vec{E}_1 + \vec{E}_2 , \qquad \vec{B} = \vec{B}_1 + \vec{B}_2$$

$$W = \int d^3x' \, \varepsilon = \int d^3x' \left(\frac{1}{2\mu_0} \vec{B}^2 + \frac{\epsilon_0}{2} \vec{E}^2 \right) = W_{el} + W_{mag}$$

$$W_{el} = \frac{\epsilon_0}{2} \int \vec{E}^2 , \qquad W_{mag} = \frac{1}{2\mu_0} \int \vec{B}^2$$

$$W_{el} = \frac{\epsilon_0}{2} \int \left(\vec{E}_1 + \vec{E}_2 \right)^2 = \frac{\epsilon_0}{2} \int \vec{E}_1^2 + \frac{\epsilon_0}{2} \int \vec{E}_2^2 + \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2$$

$$= W_{el}^1 + W_{el}^2 + W_{el}^{12}$$

W^1_{el} = Selbstenergie von ρ_1

W^2_{el} = " " ρ_2

W^{12}_{el} = Wechselwirkungsenergie der beiden Ladungs / Stromverteilungen

analog für W_{mag}

Elektrostatik: $\vec{E}_{1,2} = -\vec{\nabla}\,\Phi_{1,2}$

$$W^1_{el} = \frac{\epsilon_0}{2}\int \vec{E}_1\cdot\vec{E}_1 = -\frac{\epsilon_0}{2}\int (\vec{\nabla}\Phi_1)\cdot\vec{E}_1$$

$$= -\frac{\epsilon_0}{2}\int_V \Big\{ \underbrace{\vec{\nabla}\cdot(\Phi_1\vec{E}_1)}_{\int_{\partial V}\Phi_1\vec{E}_1 = 0} - \Phi_1\underbrace{(\vec{\nabla}\cdot\vec{E}_1)}_{\rho_1/\epsilon_0} \Big\}$$

$$\Rightarrow W^1_{el} = +\frac{1}{2}\int_V \Phi_1\,\rho_1\,d^3x$$

$$W_{el}^{2} = \frac{1}{2}\int \phi_2 \, \rho_2 \, d^3x$$

$$W_{el}^{1,2} = \epsilon_0 \int \phi_1 \, \rho_2 \, d^3x = \underline{q\,\phi_1} \quad , \quad \rho_2 = q\,\delta(\vec{x}\,' - \vec{x}_0)$$