Lösung der Laplace Gleichung (in Kugelkoordinaten)

(*) $\Delta \Phi = 0$ Laplace Gleichung

aus letzter Vorlesung

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\Lambda}{r^2}, \quad \Lambda = \frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\sin\Theta\frac{\partial}{\partial\Theta} + \frac{1}{\sin^2\Theta}\frac{\partial^2}{\partial\varphi^2}$$

Seperationsansatz: $\Phi(r,\Theta,\varphi) = F(r)\cdot Y(\Theta,\varphi)$

Radialanteil

Kugelflächenfkt.

$$\curvearrowright (*): \left(\frac{\partial^2 F}{\partial r^2}\right)Y + \frac{2}{r}\left(\frac{\partial F}{\partial r}\right)Y + \frac{F}{r^2}(\Lambda Y) = 0 \quad \Big| \frac{r^2}{\Phi}$$

$$\Leftrightarrow \underbrace{r^2\frac{\partial^2 F}{\partial r^2}\frac{1}{F} + 2r\frac{\partial F}{\partial r}\frac{1}{F}}_{=\alpha} + \underbrace{(\Lambda Y)\frac{1}{Y}}_{-\alpha} = 0, \quad \alpha = \text{konstant}$$

$$\Rightarrow \quad \frac{r^2}{F}\frac{\partial^2 F}{\partial r^2} + \frac{2r}{F}\frac{\partial F}{\partial r} = \alpha = \text{konstant} \qquad (1)$$

$$\left.\begin{aligned} &\frac{\Delta Y}{Y} = -\alpha \\ &\frac{1}{Y}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y = -\alpha \end{aligned}\right\} \quad (2)$$

weiterer Separationsansatz: $Y(\theta,\varphi) = P(\theta)\,Q(\varphi)$

$$\Rightarrow (2): \quad \frac{1}{\sin^2\theta}\left(\underbrace{\frac{1}{P}\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}P} + \underbrace{\frac{1}{Q}\frac{\partial^2}{\partial\varphi^2}Q}_{\beta = \text{konstante}}\right) = -\alpha \qquad (2')$$

$$\Rightarrow \quad \frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2} = \beta \iff \frac{\partial^2 Q}{\partial\varphi^2} - \beta Q = 0$$

$$\Rightarrow \quad Q(\varphi) = e^{\pm\sqrt{\beta}\,\varphi}$$

φ ist periodische Variable $\Rightarrow$ $Q(\varphi) = Q(\varphi + 2\pi)$

$$\Rightarrow e^{\pm\sqrt{\lambda}\varphi} = e^{\pm\sqrt{\lambda}(\varphi+2\pi)} \iff 1 = e^{\pm 2\pi\sqrt{\lambda}}$$

$$\Rightarrow \lambda = -m^2 \,, \quad m \in \mathbb{Z} \,, \qquad e^{\pm 2\pi i m}$$

$$\Rightarrow \boxed{Q(\varphi) = e^{\pm i m \varphi} \qquad \forall \; m \in \mathbb{Z}}$$

nächste Schritt: Finde P

$$Q \curvearrowright (2') : \quad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}P + \left(\alpha - \frac{m^2}{\sin^2\theta}\right)P = 0 \qquad (2'')$$

Variable Transformation: $\xi \equiv \cos\theta \,, \quad -1 \leq \xi \leq +1 \,, \quad 0 \leq \theta \leq \pi$

$$\frac{d}{d\theta} = \frac{d\xi}{d\theta}\frac{d}{d\xi} = -\sin\theta\frac{d}{d\xi} \,, \qquad \sin^2\theta = 1 - \cos^2\theta = 1 - \xi^2$$

in (2'') $(-1)^2 \frac{d}{d\xi} \sqrt{1-\xi^2}\sqrt{1-\xi^2} \frac{d}{d\xi} P + \left(\alpha - \frac{m^2}{1-\xi^2}\right) P = 0$

$$\Rightarrow \left(\frac{d}{d\xi} (1-\xi^2) \frac{d}{d\xi} + \alpha - \frac{m^2}{1-\xi^2} \right) P_\alpha^m(\xi) = 0$$

$$\Leftrightarrow \left((1-\xi^2) \frac{d^2}{d\xi^2} - 2\xi \frac{d}{d\xi} + \alpha - \frac{m^2}{1-\xi^2} \right) P_\alpha^m(\xi) = 0$$

Zugeordnete Legendre DGL

$m = 0$

$$\left[(1-\xi^2) \frac{d^2}{d\xi^2} - 2\xi \frac{d}{d\xi} + \alpha \right] P_\alpha(\xi) = 0$$

Legendre DGL

Lösung durch Potenzreihenansatz:

$$P_\alpha = \sum_{j=0}^{\infty} a_j \xi^{j+\rho} \leftarrow \text{wird noch festgelegt}$$

$$\frac{dP_\alpha}{d\xi} = \sum_j (j+\rho) a_j \xi^{j+\rho-1}$$

$$\frac{d^2 P_\alpha}{d\xi^2} = \sum_j (j+\rho)(j+\rho-1) a_j \xi^{j+\rho-2}$$

$\curvearrowright$ LDGL: $$(1-\xi^2) \sum_j (j+\rho)(j+\rho-1) a_j \xi^{j+\rho-2} - 2\xi \sum_j (j+\rho) a_j \xi^{j+\rho-1}$$

$$+ \alpha \sum_j a_j \xi^{j+\rho} = 0$$

$$\Rightarrow \sum_j \Big(\underbrace{\big[-(s+j)(s+j-1) - 2(s+j) + \alpha\big]}_{-(s+j)(s+j+1)} a_j\, \xi^{j+s} + (s+j)(s+j-1)\, a_j\, \xi^{j+s-2} \Big) = 0$$

$j \to j+2$

$$\Rightarrow s(s-1)\, a_0\, \xi^{s-2} + (s+1)s\, a_1\, \xi^{s-1} + \sum_{j=0}^{\infty} \Big\{ \big[\alpha - (s+j)(s+j+1)\big] a_j + (s+j+2)(s+j+1)\, a_{j+2} \Big\}\, \xi^{j+s} = 0$$

$\Rightarrow \quad s(s-1)\, a_0 = 0 \quad \Rightarrow \quad s = 0$ oder 1, $a_0 \neq 0$ oder ($a_0 = 0$, s beliebig)

$\quad\;\; s(s+1)\, a_1 = 0 \quad \Rightarrow \quad s = 0$, a_1 beliebig

$\qquad\qquad\qquad\qquad\qquad\quad s = 1$, $a_1 = 0$ $\leftarrow$ diskutiere diesen Fall zuerst

$$\rightsquigarrow \boxed{a_{j+2} = \frac{(s+j)(s+j+1) - \alpha}{(s+j+2)(s+j+1)}\, a_j \,, \quad j = 0, 1, 2, \ldots}$$

reguläre Lösung sind weiter eingeschränkt

$$\lim_{j\to\infty} \frac{a_{j+2}}{a_j} = \frac{j^2}{j^2} = 1 \Rightarrow \text{Potenzreihe divergiert}$$

$\Rightarrow$ Abbruchbed. $\quad a_{j_0+2} = 0 \Rightarrow \underbrace{(j_0+s)}_{l}\underbrace{(j_0+s+1)}_{l+1} - \alpha = 0$

$$\Rightarrow \alpha = l(l+1), \quad l = 0,1,2,\ldots$$

$\Rightarrow$ reguläre Lösung der L. DGL existieren nur für

$$\alpha = l(l+1), \quad l = 0,1,2,\ldots$$

Konstruktion der P_l über Rekursionsformel möglich

Lösung:

Lösung:

$$P_\ell(\xi) = \frac{1}{2^\ell}\frac{1}{\ell!}\frac{d^\ell}{d\xi^\ell}(\xi^2-1)^\ell \quad , \ \ell = 0,1,2,\dots$$

Legendre Polynome

Beweis: Übungsblatt 10

$$P_0 = 1 \quad , \quad P_1 = \frac{1}{2}\frac{d}{d\xi}(\xi^2-1) = \xi$$

$$P_2 = \frac{1}{4}\frac{1}{2}\frac{d^2}{d\xi^2}(\xi^2-1)^2 = \frac{1}{8}\frac{d^2}{d\xi^2}(\xi^4 - 2\xi^2 + 1) = \frac{3}{2}\xi^2 - \frac{1}{2}$$

$$P_3 = \dots = \frac{1}{2}(5\xi^3 - 3\xi) \quad ,$$

Lösung der zugeordneten L. DGL:

$$P_\ell^m = (-1)^m (1-\xi^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{d\xi^{|m|}} P_\ell(\xi) = \frac{(-1)^m}{2^\ell \ell!} (1-\xi^2)^{\frac{|m|}{2}} \frac{d^{\ell+|m|}}{d\xi^{\ell+|m|}} (\xi^2-1)^\ell$$

$$\ell = 0, 1, 2, \dots \quad , \quad -\ell \leq m \leq \ell$$

Beweis: Hausaufgabe, HR

$$P_0^0 = P_0 = 1, \quad P_1^0 = P_1 = \xi, \quad P_1^{\pm 1} = \mp\sqrt{1-\xi^2}, \dots$$

Def : Kugelflächenfunktion

$$\boxed{Y_{lm}(\theta,\varphi) := \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-|m|)!}{(l+|m|)!}}\; P_l^m(\cos\theta)\, e^{im\varphi}}$$

Konvention

$l = 0, 1, 2, \ldots$

$-l \leq m \leq l$

$$Y_{00} = \frac{1}{\sqrt{4\pi}} P_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_{10} = \frac{1}{\sqrt{4\pi}} \cos\theta$$

$$Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\; e^{\pm i\varphi}$$

$\vdots$

Lösung der Radialgleichung

$$(1): \quad \frac{r^2}{F}\frac{\partial^2 F}{\partial r^2} + \frac{2r}{F}\frac{\partial F}{\partial r} = \ell(\ell+1)$$

$$\Leftrightarrow \quad r^2 \frac{\partial^2 F}{\partial r^2} + 2r\frac{\partial F}{\partial r} - \ell(\ell+1)F = 0 \qquad (1')$$

Ansatz

$$F(r) \equiv \frac{u(r)}{r}, \quad \frac{\partial F}{\partial r} = \frac{u'}{r} - \frac{u}{r^2}, \quad \frac{\partial^2 F}{\partial r^2} = \frac{u''}{r} - 2\frac{u'}{r^2} + 2\frac{u}{r^3}$$

$$\curvearrowright (1'): \quad r u'' - \cancel{2u'} + \cancel{\frac{2u}{r}} + \cancel{2u'} - 2\cancel{\frac{u}{r}} - \ell(\ell+1)F = 0$$

$$\Rightarrow \boxed{u'' - \ell(\ell+1)\frac{u}{r^2} = 0} \qquad (1'')$$

Lösung durch Potenzreihenansatz

$$u(r) = \sum_k b_k r^k \quad , \quad u'' = \sum_k k(k-1) b_k r^{k-2}$$

$$\curvearrowright (1''): \quad \sum_k \left(k(k-1) b_k r^{k-2} - \ell(\ell+1) b_k r^{k-2} \right) = 0$$

$$\Rightarrow \sum_k \left(k(k-1) - \ell(\ell+1) \right) b_k r^{k-2} = 0$$

$$\Rightarrow \quad k(k-1) = \ell(\ell+1)$$

$$\Rightarrow \quad k = \begin{cases} \ell+1 \\ -\ell \end{cases}$$

$$\Rightarrow \boxed{F_\ell = b'_\ell r^\ell + c'_\ell r^{-(\ell+1)}}$$