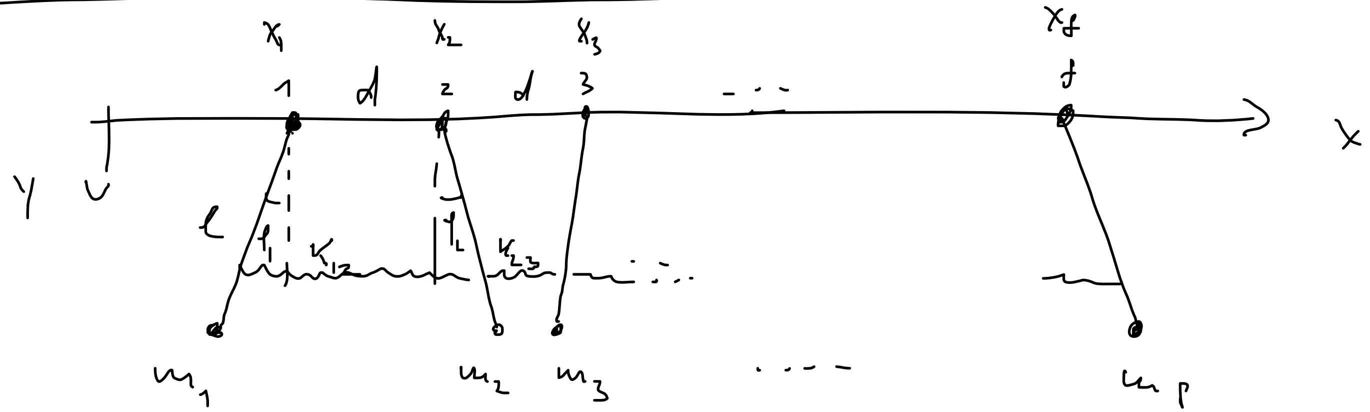
Gekoppelte Schwingungen



$$\vec{F}_a = m_a g \vec{e}_y = -\vec{\nabla}_a U$$

$$U = -\sum_{a=1}^{f} m_a y_a g$$

$$T = \sum_{a=1}^{f} \sum_{b=1}^{f} \frac{1}{2} g_{ab}(q)\, \dot{q}_a \dot{q}_b \overset{\varphi \text{ klein}}{\approx} \sum_a \frac{1}{2} m_a \ell^2 \dot{\varphi}_a^2$$

$$q_a = x_a - (a-1)d = \ell \sin\varphi_a \,, \qquad y_a = \ell \cdot \cos\varphi_a$$

$$U = -\sum_{a=1}^{f} m_a g y_a + \frac{1}{2} \sum_{a=1}^{f-1} k_{a\,a+1} \underbrace{\left((x_{a+1}+ad) - (x_a+(a-1)d)\right)^2}_{(q_{a+1}-q_a)^2} \qquad \left((x_{a+1}+ad) - (x_a+(a-1)d)\right)^2$$

$$= -\sum_a m_a g \ell \cos\varphi_a + \frac{1}{2} \sum_{a=1}^{f-1} k_{a\,a+1} (q_{a+1} - q_a)^2$$

$$\overset{\varphi_a \ll 1}{\simeq} \frac{1}{2} \sum_{a=1}^{f} \sum_{b=1}^{f} M_{ab}(k,m)\, \varphi_a \varphi_b + O(\varphi^3)$$

$$\boxed{L = T - U = \frac{1}{2}\sum_a \sum_b \left(g_{ab}\, \dot{q}_a \dot{q}_b - M_{ab}\, q_a q_b \right)}$$

$$g_{ab} = \ell^2 \begin{pmatrix} m_1 & & & & 0 \\ & m_2 & & & \\ & & m_3 & & \\ & & & \ddots & \\ 0 & & & & m_f \end{pmatrix} = \ell^2 m_a \delta_{ab}$$

$$M_{ab} = \begin{pmatrix} & & \\ & \cdots & \\ & & \end{pmatrix}$$ Hausaufgabe

konstant

Ganz allgemein

$$L = \frac{1}{2} \sum_{a=1}^{f} \sum_{b=1}^{f} \left(g_{ab}\, \dot{q}_a \dot{q}_b - M_{ab}\, q_a q_b \right)$$

mit $g_{ab} = g_{ba}$, $M_{ab} = M_{ba}$

E-L :

$$\frac{\partial L}{\partial q_c} = -\frac{1}{2} \sum_a \sum_b \left(M_{ab} \underbrace{\frac{\partial q_a}{\partial q_c}}_{\delta_{ac}} q_b + M_{ab}\, q_a \underbrace{\frac{\partial q_b}{\partial q_c}}_{\delta_{bc}} \right)$$

$$= -\frac{1}{2} \sum_a \sum_b \left(M_{ab}\, \delta_{ac}\, q_b + M_{ab}\, \delta_{bc}\, q_a \right)$$

$$= -\frac{1}{2} \left(\sum_b M_{cb}\, q_b + \sum_a M_{ac}\, q_a \right)$$

$$= -\frac{1}{2} \left(\sum_{\cancel{b}\,a} M_{c\cancel{b}\,a}\, q_{\cancel{b}\,a} + \sum_a M_{ac}\, q_a \right)$$

$$= -\frac{1}{2} \sum_a \left(M_{ca}\, q_a + \underbrace{M_{ac}}_{M_{ca}}\, q_a \right) = -\sum_a M_{ca}\, q_a$$

$$\frac{\partial L}{\partial \dot{q}_c} = \sum_{a=1}^{f} g_{ca} \dot{q}_a$$

$$\frac{\partial L}{\partial q_c} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_c} = 0$$

$$\Rightarrow \text{E-L}: \quad - \sum_a M_{ca} q_a - \frac{d}{dt} \sum_a g_{ca} \dot{q}_a = 0$$

$$= - \sum_a \left(g_{ca} \ddot{q}_a + M_{ca} q_a \right) = 0 \quad (*)$$

Lösungsansatz: $\quad q_a = q_a^0 \, e^{i\omega t} \quad , \quad \ddot{q}_a = - \omega^2 q_a$

$$\rightsquigarrow (*) \quad \sum_a \left(- g_{ca} \omega^2 + M_{ca} \right) q_a^0 = 0$$

algebraisches Gleichungssystem, hat nur dann nicht-triviale Lösung, wenn

$$\det\left(M - \omega^2 g\right) = 0 \qquad (**)$$

Polynom f-ter Ordnung in ω^2

hat f Wurzeln bzw f Eigenfrequenzen $\omega^2_{(\alpha)}$, $\alpha = 1, \dots, f$

Sind Lösungen von $(**)$

Die zugehörigen Eigenvektoren erfüllen

$$\sum_a \left(M_{ca} - \omega^2_{(\alpha)} g_{ca}\right) q_a^{0(\alpha)} = 0 \qquad \forall\ \alpha = 1, \dots, f$$

und heißen Eigenschwingungen.

$$\boxed{\text{Allg Lösung: } q_a = \sum_{\alpha=1}^{f} \left(A^{(\alpha)} q_a^{0(\alpha)} e^{i\omega_{(\alpha)} t} + A^{*(\alpha)} q_a^{0(\alpha)} e^{-i\omega_{(\alpha)} t} \right)}$$

Satz 14 Die Eigenfrequenzen $\omega_{(\lambda)}^2$ sind reell und $\omega_{(\lambda)}^2 \geq 0$ falls $\sum_a \sum_b q_a^{0(\lambda)} M_{ab} q_b^{0(\lambda)} \geq 0$

Beweis: $\sum_a \left(M_{ca} - \omega_{(\lambda)}^2 g_{ca} \right) q_a^{0(\lambda)} = 0$

$$\Rightarrow \sum_c \sum_a q_c^{0(\lambda)} \left(M_{ca} - \omega_{(\lambda)}^2 g_{ca} \right) q_c^{0(\lambda)} = 0$$

$$\Rightarrow \sum_c \sum_a q_c^{0(\lambda)} M_{ca} q_a^{0(\lambda)} = \omega_{(\lambda)}^2 \underbrace{\sum_c \sum_a q_c^{0(\lambda)} g_{ca} q_c^{0(\lambda)}}_{> 0 \text{ und reell}}$$

$$\Rightarrow \omega_{(\lambda)}^2 = \frac{\sum_c \sum_a q_c^{0(\lambda)} M_{ca} q_a^{0(\lambda)}}{\sum_c \sum_a q_c^{0(\lambda)} g_{ca} q_c^{0(\lambda)}} \geq 0 \text{ reell}$$

da $T = \sum_c \sum_a g_{ca} \dot{q}_c \dot{q}_a$ ≥ 0 und reell ist

q.e.d.

Satz 15: Die Eigenvektoren bilden ein Orthonormalsystem bzgl. der "Metrik" g_{ab}

Beweis:

$$\begin{cases} \sum_a \left(M_{ca} - \omega_{(\alpha)}^2 g_{ca}\right) q_a^{0(\alpha)} = 0 \\ \sum_a \left(M_{ca} - \omega_{(\beta)}^2 g_{ca}\right) q_a^{0(\beta)} = 0 \end{cases}$$

$\Rightarrow$

$$\begin{cases} \sum_c \sum_a q_c^{0(\beta)} \left(M_{ca} - \omega_{(\alpha)}^2 g_{ca}\right) q_a^{0(\alpha)} = 0 \\ \sum_c \sum_a q_c^{0(\alpha)} \left(M_{ca} - \omega_{(\beta)}^2 g_{ca}\right) q_a^{0(\beta)} = 0 \end{cases}$$

Wegen $$\sum_c \sum_a q_c^{0(\beta)} M_{ca}\, q_a^{0(\alpha)} = \sum_c \sum_a q_a^{0(\alpha)} \underbrace{M_{ca}}_{= M_{ac}} q_c^{0(\beta)}$$

$$= \sum_{\cancel{a}\, a} \sum_{\cancel{c}\, c} q_{\cancel{a}\, c}^{0(\alpha)} M_{\cancel{c}\cancel{a}\, ca}\, q_{\cancel{c}\, a}^{(0)(\beta)} = \sum_a \sum_c q_c^{0(\alpha)} M_{ca}\, q_a^{0(\beta)}$$

folgt $$\left(\omega_{(\alpha)}^2 - \omega_{(\beta)}^2\right)\left(\sum_c \sum_a q_c^{0(\beta)} g_{ca}\, q_a^{0(\alpha)}\right) = 0$$

$\alpha = \beta$: ✓

$\alpha \neq \beta$: $$\boxed{\sum_c \sum_a q_c^{0(\beta)} g_{ca}\, q_a^{0(\alpha)} = 0}$$

Sie kennen $\vec{a} \cdot \vec{b} = \sum_a \sum_b a_a b_b\, \delta_{ab} = a_1 b_1 + a_2 b_2 + \dots$

$$\Rightarrow \quad \sum_a \sum_c q_c^{0(\beta)} \, g_{ca} \, q_a^{0(\alpha)} = c \, \delta^{(\alpha)(\beta)}$$

Normierung: $q_c^{0(\beta)} = \hat{q}_c^{0(\beta)} \cdot \sqrt{c}$

$$\rightsquigarrow \quad \sum_a \sum_c \hat{q}_c^{0(\beta)} \, g_{ca} \, \hat{q}_a^{0(\alpha)} = \delta^{(\alpha)(\beta)}$$

$\Rightarrow$ Eigenvektoren sind orthonormal.

L in Normalkoordinaten

$$L = \frac{1}{2} \sum_a \sum_b \left(g_{ab}\, \dot{q}_a \dot{q}_b - M_{ab}\, q_a q_b \right)$$

Normalkoordinaten: $\quad q_a(t) = \sum_{\alpha=1}^{f} Q_\alpha(t)\, q_a^{0(\alpha)}$ ← normiert

$$\Rightarrow L = \frac{1}{2} \sum_a \sum_b \Bigg[g_{ab} \underbrace{\left(\sum_{\alpha=1}^{f} \dot{Q}_\alpha(t)\, q_a^{0(\alpha)} \right)}_{\dot{q}_a} \left(\sum_{\beta=1}^{f} \dot{Q}_\beta(t)\, q_b^{0(\beta)} \right)$$

$$- M_{ab} \left(\sum_{\alpha=1}^{f} Q_\alpha\, q_a^{0(\alpha)} \right) \left(\sum_{\beta=1}^{f} Q_\beta\, q_b^{0(\beta)} \right) \Bigg]$$

$$= \frac{1}{2} \sum_\alpha \sum_\beta \left[\dot{Q}_\alpha \dot{Q}_\beta \sum_a \sum_b g_{ab}\, q_a^{0(\alpha)} q_b^{0(\beta)} - Q_\alpha Q_\beta \sum_a \sum_b M_{ab}\, q_a^{0(\alpha)} q_b^{0(\beta)} \right]$$

$$L = \frac{1}{2}\sum_\alpha \sum_\beta \Big[\dot{Q}_\alpha \dot{Q}_\beta \underbrace{\sum_a \sum_b g_{ab}\, q_a^{0(\alpha)} q_b^{0(\beta)}}_{\delta^{(\alpha)(\beta)}} - Q_\alpha Q_\beta \underbrace{\sum_a \sum_b M_{ab}\, q_a^{0(\alpha)} q_a^{0(\beta)}}_{\omega_{(\alpha)}^2 \sum_{ab} q_a^{0(\alpha)} \underbrace{g_{ab}\, q_b^{0(\beta)}}_{\delta^{(\alpha)(\beta)}}} \Big]$$

$$= \frac{1}{2}\sum_\alpha \sum_\beta \Big[\dot{Q}_\alpha \dot{Q}_\beta\, \delta^{(\alpha)(\beta)} - \omega_{(\alpha)}^2 Q_\alpha Q_\beta\, \delta^{(\alpha)(\beta)} \Big]$$

$$= \frac{1}{2}\sum_\alpha \Big[\dot{Q}_\alpha \dot{Q}_\alpha - \omega_{(\alpha)}^2 Q_\alpha Q_\alpha \Big] = \frac{1}{2}\sum_\alpha \Big[\dot{Q}_\alpha^2 - \omega_{(\alpha)}^2 Q_\alpha^2 \Big]$$

$$= \sum_\alpha L^{(\alpha)} \quad , \quad \boxed{L^{(\alpha)} = \frac{1}{2}\left(\dot{Q}_\alpha^2 - \omega_{(\alpha)}^2 Q_\alpha^2 \right)}$$

andere Schreibweise : Matrixnotation

$$(*) \quad q_a(t) = \sum_\alpha \underbrace{A_{a\alpha}}_{q_a^{0(\alpha)}} Q_\alpha(t) \quad , \quad A = f \times f \text{ Matrix}$$

$$q = A \cdot Q$$

$$L = \frac{1}{2} \underbrace{\sum_a \sum_b \dot{q}_a g_{ab} \dot{q}_b}_{\dot{q}^T \cdot g \cdot \dot{q}} - \frac{1}{2} \underbrace{\sum_a \sum_b q_a M_{ab} q_b}_{q^T \cdot M \cdot q} \qquad q = \begin{pmatrix} q_1 \\ \vdots \\ q_f \end{pmatrix}$$

$$(*) \rightsquigarrow : \quad L = \frac{1}{2} \dot{Q}^T A^T g A Q - \frac{1}{2} Q^T A^T M A Q$$

$A^T g A = g$, d.h. A ist orthogonale Matrix bzgl. g

(Sie kennen $A^T A = \mathbb{1}$)

$$A^T \mathbb{1} A = \underline{\mathbb{1}}$$

$$A^T M A = M_D$$, weil M symmetrisch

<u>Satz der Mathematik</u>

A ist aus Eigenvektoren von M spaltenweise aufgebaut