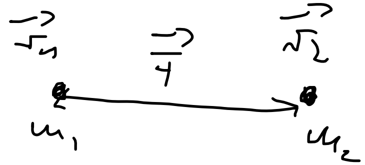
# Zusammenfassung der letzten Vorlesung

2 Körper-Problem




Zentralkraft

$$\vec F = f(|\vec r|)\cdot\frac{\vec r}{|\vec r|}$$

$$= -\vec\nabla U$$

$$= -\frac{k\,\vec r}{|\vec r|^3}$$

Kepler

$$\underline{\vec r = \vec r_1 - \vec r_2}$$

$$\left.\begin{aligned} m_1\ddot{\vec r}_1 &= \vec F \\ m_2\ddot{\vec r}_2 &= -\vec F \end{aligned}\right\}\ \text{6 DGL}$$

Lösung durch Seperation von Schwerpunkt + Relativbew.

$$\vec R := \frac{m_1\vec r_1 + m_2\vec r_2}{M} \qquad \vec r := \vec r_1 - \vec r_2$$

↯ $\Rightarrow$ $\ddot{\vec R} = 0$ SP. bewegt sich geradlinig gleichförmig

Benutze Energieerhaltung + Drehimpulserhaltung

$\searrow$ $\vec{L} = \ell \, \vec{e}_z$

Relativbewegung in Ebene $\perp \vec{L}$

$\swarrow$

Polarkoordinaten $(r, \varphi)$

Energieerhaltung $\Rightarrow$ Reduktion auf 1-dim Problem

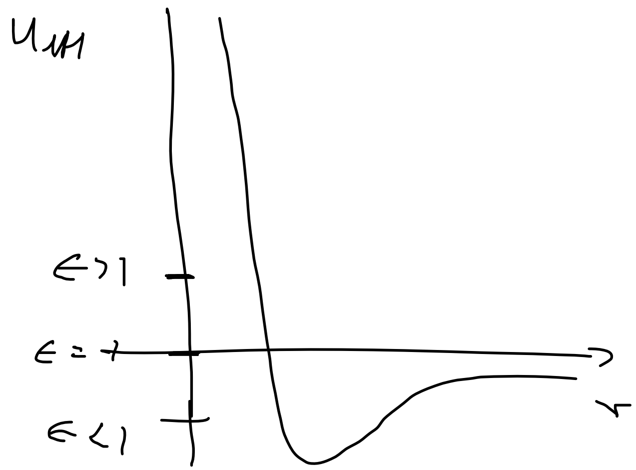
$$\varphi - \varphi_0 = \pm \int_{r_0}^{r} \frac{dr'}{r'^2 \sqrt{E - U_{eff}(r')}} \, , \qquad U_{eff} = \frac{\ell^2}{2\mu r'^2} + \underbrace{U(r')}_{= -\frac{k}{r'} \text{ (Kepler)}}$$

für Kepler: $\underline{r(\varphi) = \frac{p}{1 + \epsilon \cos\varphi}}$ , $p \equiv \frac{\ell^2}{\mu k}$ , $\epsilon = \sqrt{1 + \frac{2\ell^2 E}{\mu k^2}}$

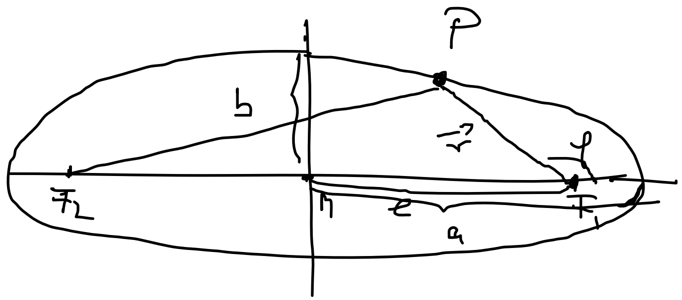
Wir wollen zeigen:

$[H, R]$

$$\left\{\begin{array}{lll} \epsilon < 1 & \Rightarrow \text{Ellipsenbahn} & (E < 0) \\ \epsilon = 1 & \Rightarrow \text{Parabelbahn} & (E = 0) \\ \epsilon > 1 & \Rightarrow \text{Hyperbelbahn} & (E > 0) \end{array}\right.$$




Ellipse




Ursprung im M

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

<u>Kepler</u>: Ursprung im Brennpunkt, z.B. $F_1$

<u>Def</u>: $\left|\overrightarrow{F_2 P}\right| + \left|\overrightarrow{F_1 P}\right| = 2a$ (*)

Ursprung in $F_1$: $\vec{r} = \overrightarrow{F_1 P}$, $\vec{e} := \overrightarrow{MF_1}$

$\Rightarrow \vec{r} - \overrightarrow{F_2 P} = -2\vec{e} \Rightarrow \overrightarrow{F_2 P} = 2\vec{e} + \vec{r}$

In (*): $|2\vec{e} + \vec{r}| + |\vec{r}| = 2a$

$\Rightarrow (2\vec{e} + \vec{r})^2 = (2a - |\vec{r}|)^2$, $r \equiv |\vec{r}|$

$\Rightarrow \cancel{4}e^2 + \cancel{4}\underbrace{\vec{e}\,\vec{r}}_{er\cos\varphi} + \cancel{r^2} = \cancel{4}a^2 - \cancel{4}ar + \cancel{r^2}$

$\Rightarrow r(a + e\cos\varphi) = a^2 - e^2 \Rightarrow r = \frac{a^2 - e^2}{a}\frac{1}{(1 + \frac{e}{a}\cos\varphi)}$

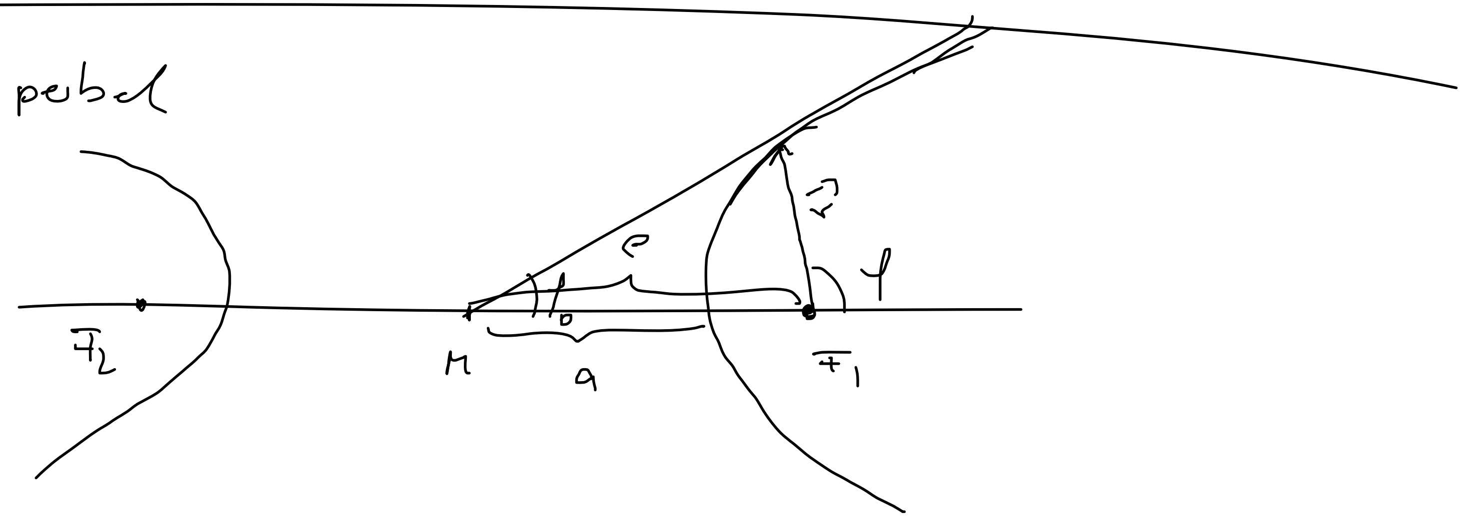
$\Rightarrow r = \frac{p}{1 + \epsilon\cos\varphi}$, $p = \frac{a^2 - e^2}{a}$, $\epsilon = \frac{e}{a}$

Für $|\overrightarrow{F_1P}| = |\overrightarrow{F_2P}|$ gilt $|\overrightarrow{F_1P}| = a$ , $b^2 + e^2 = |\overrightarrow{F_1P}|^2 = a^2$

$\Rightarrow$ $\underline{a^2 - e^2 = b^2}$ $\curvearrowright$ $p = \frac{b^2}{a} = \frac{a^2 - e^2}{a}$

1. Kepler Gesetz bewiesen !

Hyperbel




$\underline{\text{Ursprung}}$: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , $b^2 = e^2 - a^2$

~~Ursprung in $F_2$~~

<u>Def</u> $\quad |\overrightarrow{F_2P}| - |\overrightarrow{F_1P}| = \pm 2a$

Recht Ast ($+$)

Link Ast ($-$)

Ursprung in $F_1$ : $\overrightarrow{F_1P} = \vec{r}$ , $\overrightarrow{F_2P} = 2\vec{e} + \vec{r}$

$$|2\vec{e} + \vec{r}| - |\vec{r}| = \pm 2a$$

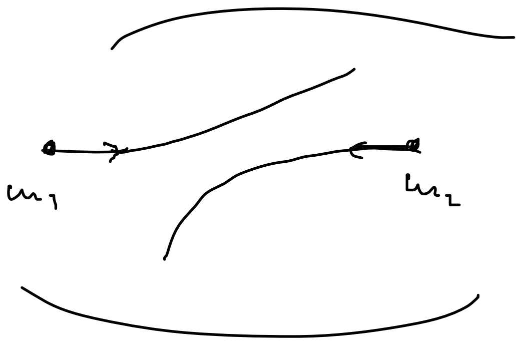
$$\Rightarrow \dots \Rightarrow \quad r = \frac{\pm p}{1 \mp \epsilon \cos\varphi} \qquad p = \frac{e^2 - a^2}{a} > 0$$

$$\epsilon = \frac{e}{a} > 1$$

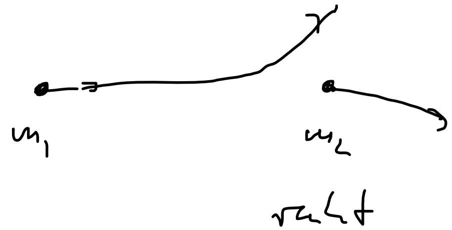
oben berechnet $\quad p = \dfrac{\ell^2}{\mu k} = \begin{cases} < 0 & \text{für } k < 0 \\ > 0 & \text{—}\ \ k > 0 \end{cases} \qquad k = \begin{cases} \gamma m_1 m_2 \\ q_1 q_2 \end{cases}$

$$\epsilon = \sqrt{1 + \frac{2\ell^2 E}{\mu k^2}} = \begin{cases} > 1 & \text{für } E > 0 \\ < 1 & \text{für } E < 0 \end{cases}$$

# Streuung von Teilchen




Streuung




Stoß

Annahme: elastische Streuung: Teilchen verändern sich nicht!

$\vec{v}_1, \vec{v}_2$ = Geschwindigkeiten lange vor der Wechselwirkung

$\vec{v}_1{}', \vec{v}_2{}'$ = " " nach "

Energieerhaltung: $\frac{1}{2} m_1 \vec{v}_1^{\,2} + \frac{1}{2} m_2 \vec{v}_2^{\,2} = \frac{1}{2} m_1 \vec{v}_1{}'^2 + \frac{1}{2} m_2 \vec{v}_2{}'^2$

Impulserhaltung : $m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}_1' + m_2 \vec{v}_2'$

Zusammen 4 Gleichungen, $\vec{v}_1, \vec{v}_2$ bekannt + suche

$$\underbrace{\vec{v}_1', \vec{v}_2'}_{6 \text{ Größen}}$$

$\Rightarrow$ $6 - 4 = 2$ Unbekannte

$\Rightarrow$ Drehimpulserhaltung $\Rightarrow$ Bewegung verläuft in einer Ebene

$\Rightarrow$ 1 Unbekannte $\equiv$ Streuwinkel

Übergang zu Relativ + Schwerpunktkoordinaten

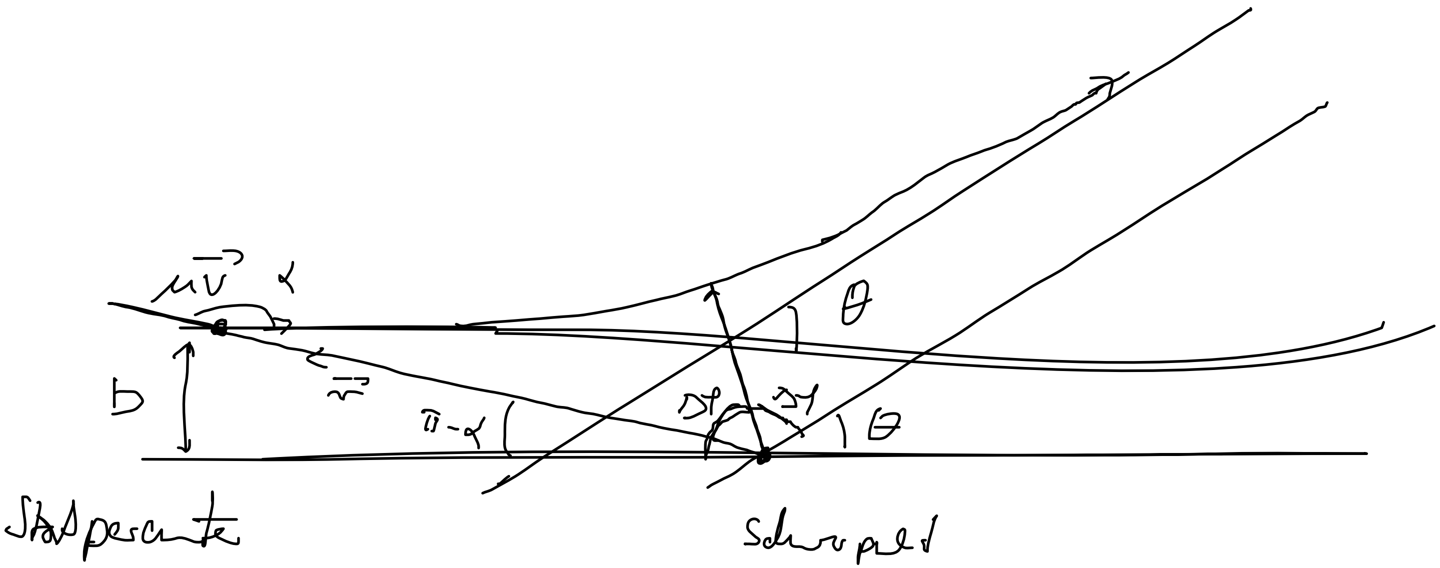
$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad , \quad \dot{\vec{r}} = \dot{\vec{r}}_1 - \dot{\vec{r}}_2 \xrightarrow[t \to -\infty]{} \vec{v} = \vec{v}_1 - \vec{v}_2$$

$$\xrightarrow[t \to +\infty]{} \vec{v}' = \vec{v}_1' - \vec{v}_2'$$

$$E_{rel} = \frac{1}{2}\mu \dot{\vec{r}}^{\,2} + U(r) \;, \quad \dot{E}_{rel} = 0 \;, \quad \lim_{t\to\pm\infty} U(r(\pm\infty)) = 0$$

$$\lim_{t\to\pm\infty} E_{rel} = \lim_{t\to\pm\infty} \frac{1}{2}\mu \dot{\vec{r}}^{\,2} = \begin{cases} \frac{1}{2}\mu \vec{V}'^{2} & t\to+\infty \\ \frac{1}{2}\mu \vec{V}^{2} & t\to-\infty \end{cases}$$

$$\Rightarrow \underline{\underline{|\vec{V}| = |\vec{V}'|}}$$ (folgt aus Energieerhaltung)



$\theta$ = Streuwinkel

$\vec{L}_{rel} = \ell \vec{e}_z \quad , \ell = \mu \left| \vec{r} \times \dot{\vec{r}} \right| = \mu \underbrace{|\vec{r}|}_{r} \underbrace{|\dot{\vec{r}}|}_{|v|} \sin\alpha$

Es gilt $\quad b = r \sin(\pi - \alpha) = r \sin\alpha$

$\Rightarrow \ell = \mu\, r |v| \cdot \frac{b}{r} = \mu\, b |v| \quad \Rightarrow \quad \boxed{\ell = \mu\, b\, |v|}$

$$\Delta\varphi = \frac{\ell}{2\mu} \int_{r_{min}}^{r_{max}} \frac{dr'}{r'^2 \sqrt{E - U_{eff}(r')}}$$

$\nearrow$ oben berechnet

Es gilt: $\quad \theta + 2\Delta\varphi = \pi \quad \checkmark$

$$\theta = \pi - \frac{2\ell}{2\mu} \int_{r_{min}}^{\infty} \frac{dr'}{r'^2 \sqrt{E - U_{eff}}}$$

Rutherford : Helium kerne auf Goldfolie
(positiv geladen) (positiv geladene Atomkerne)

$$U = -\frac{k}{r}, \quad k = -q_1 q_2 < 0$$

Integral oben berechnet : $r = \frac{-p}{1+\epsilon\cos\varphi}$, $p = \frac{\ell^2}{\mu k} < 0$

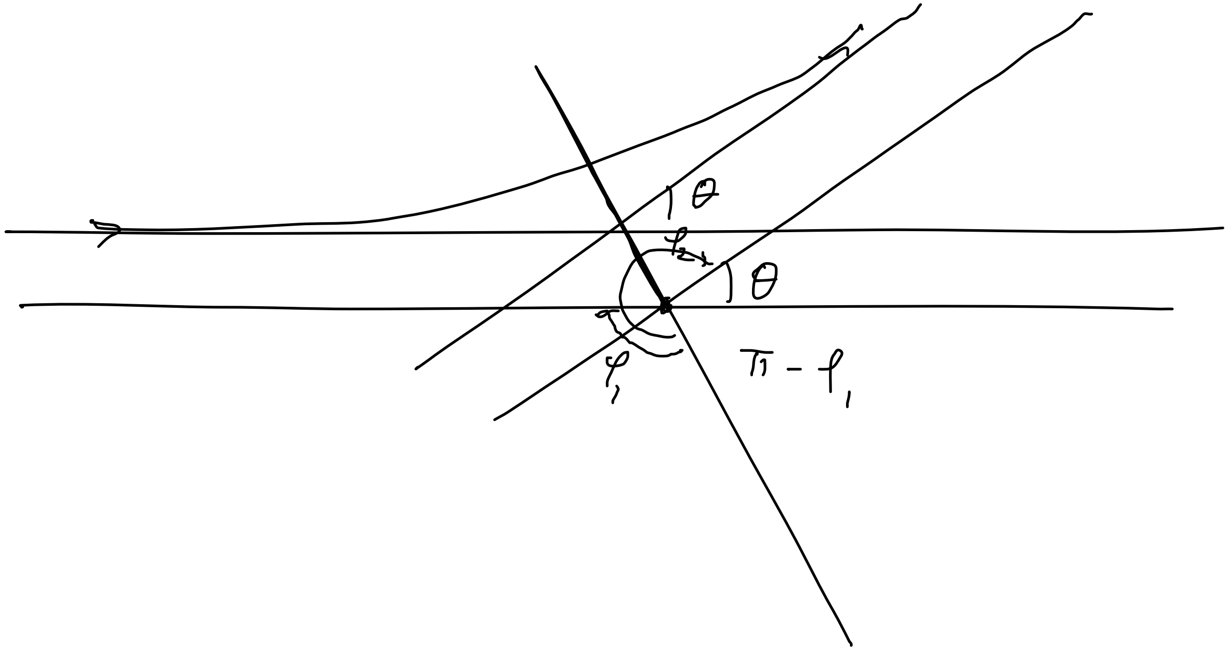
Hyperbelbahn $\epsilon = \sqrt{1 + \frac{2\ell^2 E}{\mu k^2}} > 1$

$r > 0 \Rightarrow 1 + \epsilon\cos\varphi > 0 \Rightarrow \cos\varphi > 0$

$\Rightarrow \varphi_1 \leq \varphi \leq \varphi_2$ mit $1 + \epsilon\cos\varphi_1 = 1 + \epsilon\cos\varphi_2 = 0$

$\Rightarrow \cos\varphi_{1,2} = -\frac{1}{\epsilon}$, $r(\varphi_1) = r(\varphi_2) = \infty$

asymptotische Winkel!




Exempel: i) $\pi - \varphi_1 = 2\pi - (\varphi_2 + \theta) \Rightarrow \theta = \pi + \varphi_1 - \varphi_2 - \theta$

ii) $\varphi_1 + 2(\pi - \varphi_1) = \varphi_2 \Rightarrow 2\pi - \varphi_1 = \varphi_2$

$$\Rightarrow \theta = \pi + \varphi_1 - \varphi_2 = \pi + \varphi_1 - (2\pi - \varphi_1) = -\pi + 2\varphi_1$$

$$\Rightarrow \varphi_1 = \frac{\theta + \pi}{2} \Rightarrow -\frac{1}{2} = \cos\varphi_1 = \cos\frac{\theta+\pi}{2} = -\sin\frac{\theta}{2}$$

$$\Rightarrow \boxed{\sin\frac{\theta}{2} = \frac{1}{2}}$$

$$1 = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = \sin^2\frac{\theta}{2}\left(1 + \cot^2\frac{\theta}{2}\right)$$

$$\Rightarrow \sin\frac{\theta}{2} = \frac{1}{\sqrt{1+\cot^2\frac{\theta}{2}}} = \frac{1}{\varepsilon} = \frac{1}{\sqrt{1+\frac{2\ell^2 E}{\mu k^2}}}$$

$$\Rightarrow \cot^2\frac{\theta}{2} = \frac{2\ell^2 E}{\mu k^2}$$

von oben:

$$\left.\begin{aligned} \ell &= \mu b |v| \\ E &= \tfrac{1}{2}\mu |v|^2 \end{aligned}\right\} \Rightarrow \cot^2\frac{\theta}{2} = \frac{\mu^2 b^2 v^4}{k^2}$$

$$\boxed{\tan\frac{\theta}{2} = \frac{|k|}{\mu b v^2}}$$