Problem 5.1

In QFT I we determined for massless QED ($m_e = 0$)

$$\Sigma_2(p) = \frac{e^2}{(4\pi)^2} \lim_{d\to 4} \int_0^1 dx \left((2-d)(1-x)\not{p}\right) \frac{\Gamma(2-d/2)}{(4\pi)^{d/2-2}} \frac{1}{\Delta^{(2-d/2)}} ,$$
$$\Pi_2(p) = -\frac{8e^2}{(4\pi)^2} \lim_{d\to 4} \int_0^1 dx\, x(1-x) \frac{\Gamma(2-d/2)}{(4\pi)^{d/2-2}} \frac{1}{\Delta^{2-d/2}} ,$$

for $\Delta = -x(1-x)p^2$.

a) Determine the M-dependence of the counterterms δ_2, δ_3 by imposing the renormalization conditions

$$\delta_2 = \left.\frac{d\Sigma_2}{d\not{p}}\right|_{p^2=-M^2} , \qquad \delta_3 = \Pi_2(p^2 = -M^2) .$$

b) Show that the γ-functions in the Callan-Symanzik eqation at lowest order for massless QED are given by

$$\gamma_2 = -\tfrac{1}{2} M\partial_M \delta_2 , \qquad \gamma_3 = -\tfrac{1}{2} M\partial_M \delta_3 ,$$

and compute γ_2, γ_3 explicitly from the results of a).

c) Show that the β-function in the Callan-Symanzik eqation at lowest order is given by

$$\beta = M\partial_M\left(-e\delta_1 + e\delta_2 + \tfrac{1}{2}e\delta_3\right) ,$$

and compute β explicitly from the results of a) and $\delta_1 = \delta_2$.

d) Solve

$$\frac{d\bar{e}(p')}{d\ln(p'/M)} = \frac{\bar{e}^3(p')}{12\pi^2} ,$$

for $\bar{e}(p)$ by separating variables and integrating $p' \in [M, p]$ and $\bar{e} \in [\bar{e}(M), \bar{e}(p)]$.

Problem 5.2

Consider the differential equation

$$\big[\partial_t + v(x)\partial_x - \rho(x)\big]\, D(t,x) = 0 \ .$$

Show that a solution is given by

$$D(t,x) = \hat{D}(\bar{x}(t,x))\, \exp\Big[\int_0^t dt' \rho(\bar{x}(t',x))\Big] \ ,$$

with

$$\partial_{t'}\bar{x}(t',x) = -v(\bar{x}) \ , \qquad \bar{x}(0,x) = x \ , \tag{1}$$

and $\hat{D}$ arbitrary.

Hint: First integrate (1) between x and $\bar{x}(t,x)$ and then differentiate the result with respect to x to show $(\partial_t + v(x)\partial_x)\, \bar{x} = 0$.