Problem 1.1

a) Show

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2+Jx} = \sqrt{\frac{2\pi}{a}}\; e^{\frac{J^2}{2a}}\ .$$

b) Show

$$Z(J) := \int_{-\infty}^{\infty} dx_1 \dots dx_N\, e^{-\frac{1}{2}x^T\cdot A\cdot x + J^T\cdot x} = \sqrt{\frac{(2\pi)^N}{\det(A)}}\; e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J}\ ,$$

where $x^T \cdot A \cdot x \equiv \sum_{i,j=1}^{N} x_i A_{ij} x_j,\ J^T \cdot x \equiv \sum_{i=1}^{N} J_i x_i$ and A, J are independent of x.

Hint: Diagonalize A and then perform appropriate coordinate transformations.

c) Compute

$$\frac{\partial^4 Z(J)}{\partial J_i \partial J_j \partial J_k \partial J_l}|_{J=0} = \int_{-\infty}^{\infty} dx_1 \dots dx_N\, x_i x_j x_k x_l\, e^{-\frac{1}{2}x^T\cdot A\cdot x}$$

by using the result of b).

Problem 1.2

A one-dimensional harmonic oszillator with external force $J(t)$ is described by the Lagrangian $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 + Jx$.

a) Give the action in the form

$$S = \int dt\, [\tfrac{1}{2}\, xAx + Jx]$$

and determine the differential operator A.

Hint: Assume that surface terms vanish.

b) The Greens function $G(t-t')$ of A satisfies

$$A(t)\, G(t-t') = \delta(t-t')$$

Show

$$S = \tfrac{1}{2}\int dt\, x' A x' - \tfrac{1}{2}\int dt dt'\, J(t) G(t-t') J(t')$$

for $x'(t) = x(t) + \int dt' G(t-t') J(t')$.

c) Give the Fourier representation of $G(t-t')$.

Problem 1.3

The action for a gauge boson A_μ is given by

$$S = -\tfrac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}\ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

a) By using partial integration write the action in the form

$$S = \tfrac{1}{2}\int d^4x A_\mu D^{\mu\nu} A_\nu$$

and determine the differential operator $D^{\mu\nu}$.

b) By using the Ansatz $A_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu e^{-ik\cdot x}$ show that the action in Fourier space takes the form

$$S = \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\tilde{A}_\mu(k)\tilde{D}^{\mu\nu}(k)\tilde{A}_\nu(-k)$$

and compute $\tilde{D}^{\mu\nu}(k)$.

c) The Greens function $G_{\nu\rho}(x-y)$ of $D^{\mu\nu}$ is defined by

$$D^{\mu\nu}G_{\nu\rho}(x-y) = i\delta^\mu_\rho\delta^{(4)}(x-y)\ . \qquad (*)$$

Show that $G_{\nu\rho}(x-y)$ is ill defined by acting with ∂_μ on the defining equation $(*)$.

d) By using the Ansatz $G_{\nu\rho}(x-y) = \int \frac{d^4k}{(2\pi)^4}\tilde{G}_{\nu\rho}(k)e^{-ik\cdot(x-y)}$ determine the analog of $(*)$ for $\tilde{G}_{\nu\rho}$. How does the problem of c) show up in this equation?

e) Add to the action S a term

$$\delta S = -\tfrac{1}{2\xi}\int d^4x(\partial_\mu A^\mu)^2$$

and recompute $D^{\mu\nu}$ and $\tilde{D}^{\mu\nu}$. Show that the problem in c) has gone away.

f) Determine $\tilde{G}_{\nu\rho}$ with the help of the Ansatz

$$\tilde{G}_{\nu\rho} = a(k^2)\,\eta_{\nu\rho} + b(k^2)\,k_\nu k_\rho\ ,$$

and compute a, b.