SUPERSYMMETRY AND DUALITIES IN VARIOUS DIMENSIONS

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1. Introduction

Since the seventies string theory has been discussed as a possible candidate for a theory which unifies all known particle interactions including gravity. Until recently, however, string theory has only been known in its perturbative regime. That is, the (particle) excitations of a string theory are computed in the free theory ($g_{\rm s} = 0$), while their scattering processes are evaluated in a perturbative series for $g_{\rm s} \ll 1$. The string coupling constant $g_{\rm s}$ is a free parameter of string theory but for $g_{\rm s} = \mathcal{O}(1)$ no method of computing the spectrum or the interactions had been known. This situation dramatically changed during the past three years. For the first time it became possible to go beyond the purely perturbative regime and to reliably compute some of the nonperturbative properties of string theory. The central point of these developments rests on the idea that the strong-coupling limit of a given string theory can be described in terms of another, weakly coupled, 'dual theory'. This dual theory can take the form of either a different string theory, or the same string theory with a different set of perturbative excitations, or a new theory termed M-theory.

The precise nature of the strong-coupling limit sensitively depends on the number of (Minkowskian) spacetime dimensions and the amount of supersymmetry. Supersymmetry has played a major role in the recent developments in two respects. First of all, it is difficult (and it has not been satisfactorily accomplished) to rigourously prove a string duality, since it necessitates a full nonperturbative formulation, which is not yet available. Nevertheless it has been possible to perform nontrivial checks of the conjectured dualities for quantities or couplings whose quantum corrections are under (some) control. It is a generic property of supersymmetry that it protects a subset of the couplings and implies a set of nonrenormalization theorems. The recent developments heavily rely on the fact that the mass (or tension) of BPS-multiplets is protected and that holomorphic couplings obey a nonrenormalization theorem. Thus, they can be computed in the perturbative regime of string theory and, under the assumption of unbroken supersymmetry, reliably extrapolated into the nonperturbative region. It is precisely for these BPS-states and holomorphic couplings that the conjectured dualities have been successfully verified.

Second of all, for a given spacetime dimension D and a given representation of supersymmetry there can exist perturbatively different string theories. For example, the heterotic SO(32) string in $D = 10$ and the type-I string in $D = 10$ share the same supersymmetry, but their interactions are different in perturbation theory. However, once nonperturbative corrections are taken into account, it is believed that the two theories are identical and merely different perturbative limits of the same underlying quantum theory. A similar phenomenon is encountered with other string theories in different dimensions and the moduli space of string theory is much smaller than was previously known. In the course of these lectures we will see that many nontrivial relations exist among perturbatively distinct string vacua and furthermore, that, what were thought to be disconnected components of the moduli space, can in fact be different perturbative regions of one and the same component. Thus, in determining the properties of the underlying quantum theory, supersymmetry seems to play a much more prominent role than had previously been appreciated.

The purpose of these lectures is to review some of these recent developments with particular emphasis on the role played by supersymmetry.[1] In section 2 we collect the representations of supersymmetry in spacetime dimensions $3 \leq D \leq 11$ from a common point of view. Many features are only displayed in appropriate tables but we present slightly more detail in the dimensions $D = 11, 10$ and 6 as representative cases. We explain a number of features of the dimensional reduction of supergravity, such as the emergence of hidden symmetries, the low-energy action in different frames and other aspects relevant in the string context. In section 3 we first recall the different perturbative string theories, their (Calabi-Yau) compactifications and the dualities which already exist at the perturbative level. Then we discuss the various types of possible strong-coupling limits (S-duality,

[1] This set of lectures notes is an expanded version of [1].

U-duality, M-theory, F-theory) and the corresponding string vacua. This leads to a 'web' of interrelations which we attempt to visualize at the end of section 3. Finally, in appendix A we review some basic properties of the free-field representation for states of different spin, while in appendix B we present a more detailed discussion of the relation between the parameters of string theory and those of the corresponding low-energy effective field theory.

2. Supersymmetry in various dimensions

2.1. THE SUPERSYMMETRY ALGEBRA

An enormous amount of information about supersymmetric theories is contained in the structure of the underlying representations of the supersymmetry algebra [2]. Here we distinguish the supermultiplet of the fields, which transforms irreducibly under the supersymmetry transformations, and the supermultiplet of states described by the theory. The latter will depend on the dynamics encoded in a supersymmetric action or Hamiltonian. The generators of the super-Poincaré algebra comprise the supercharges, transforming as spinors under the Lorentz group, the energy and momentum operators, the generators of the Lorentz group, and possibly additional generators that commute with the supercharges. For the moment we ignore these additional charges, often called *central* charges[2]. There are other relevant superalgebras, such as the supersymmetric extension of the anti-de Sitter algebra. These will not be considered here, but they are unavoidable when considering supersymmetry in theories with a cosmological term.

Ignoring the central extensions for the moment, the most important anti-commutation relation is the one between the supercharges,

$$\{Q_\alpha, \bar{Q}_\beta\} = -2iP^\mu\,(\Gamma_\mu)_{\alpha\beta}\,. \tag{1}$$

Here Γ_μ are the gamma matrices that generate the Clifford algebra with Minkowskian signature $(-,+,+,\cdots)$.

The size of a supermultiplet depends sensitively on the number of independent supercharge components Q. The first step is therefore to determine Q for any given number of spacetime dimensions D. The result is summarized in Table 1. As shown, there exist five different sequences of spinors, corresponding to spacetimes of particular dimensions. When this dimension is odd, it is possible in certain cases to have Majorana spinors. These cases constitute the first sequence. The second one corresponds to those odd dimensions where Majorana spinors do not exist. The spinors are then Dirac

[2]The terminology adopted in the literature is not always very precise. Usually, all charges that commute with the supercharges, but not necessarily with all the generators of the Poincaré algebra, are called 'central charges'. We will adhere to this nomenclature.

D	$Q_{\rm irr}$	$H_{\rm R}$	type
1, 3, 9, 11, mod 8	$2^{(D-1)/2}$	SO(N)	M
5, 7, mod 8	$2^{(D+1)/2}$	USp($2N$)	D
4, 8, mod 8	$2^{D/2}$	U(N)	M
6, mod 8	$2^{D/2}$	USp($2N_+$)×USp($2N_-$)	W
2, 10, mod 8	$2^{D/2-1}$	SO(N_+)×SO(N_-)	MW

TABLE 1. The supercharges in various spacetime dimensions D. In the second column, $Q_{\rm irr}$ specifies the real dimension of an irreducible spinor in a D-dimensional Minkowski spacetime. The third column specifies the group $H_{\rm R}$ for N-extended supersymmetry, defined in the text, acting on N-fold reducible spinor charges. The fourth column denotes the type of spinors: Majorana (M), Dirac (D), Weyl (W) and Majorana-Weyl (MW).

spinors. In even dimension one may distinguish three sequences. In the first one, where the number of dimensions is a multiple of 4, charge conjugation relates positive- with negative-chirality spinors. All spinors in this sequence can be restricted to Majorana spinors. For the remaining two sequences, charge conjugation preserves the chirality of the spinor. Now there are again two possibilities, depending on whether Majorana spinors can exist or not. The cases where we cannot have Majorana spinors, whenever $D = 6$ mod 8, comprise the fourth sequence. For the last sequence Majorana spinors exist and we restrict the charges to Majorana-Weyl spinors.

One can consider *extended supersymmetry*, where the spinor charges transform reducibly under the Lorentz group and comprise N irreducible spinors. For Weyl charges, one can consider combinations of N_+ positive- and N_- negative-chirality spinors. In all these cases there exists a group $H_{\rm R}$ that rotates the irreducible components such that the supersymmetry algebra is left invariant. This group acts exclusively on the spinor charges and commutes with the Lorentz transformations. The group $H_{\rm R}$ is thus the part of the automorphism group of the supersymmetry algebra that commutes with the Lorentz group. This group is often realized as a manifest invariance group of a supersymmetric field theory.

Another way to present the various cases is shown in Table 2. Here we list the real dimension of an irreducible spinor charge and its corresponding spacetime dimension. In addition we have included the number of states of the shortest[3] supermultiplet of massless states, written as a sum of bosonic and fermionic states.

[3] By the *shortest* multiplet, we mean the multiplet with the helicities of the states as low as possible. This is usually (one of) the smallest possible supermultiplet(s).

$Q_{\rm irr}$	D	shortest supermultiplet
32	$D = 11$	$128 + 128$
16	$D = 10, 9, 8, 7$	$8 + 8$
8	$D = 6, 5$	$4 + 4$
4	$D = 4$	$2 + 2$
2	$D = 3$	$1 + 1$

TABLE 2. Simple supersymmetry in various dimensions. We present the dimension of the irreducible spinor charge with $2 \leq Q_{\rm irr} \leq 32$ and the corresponding spacetime dimensions D. The third column represents the number of bosonic + fermionic massless states for the shortest supermultiplet.

2.2. MASSLESS REPRESENTATIONS

Because the momentum operators P^μ commute with the supercharges, we may consider the states at arbitrary but fixed momentum P^μ, which, for massless representations, satisfies $P^2 = 0$. The matrix $P^\mu \Gamma_\mu$ on the right-hand side of (1) has therefore zero eigenvalues. In a positive-definite Hilbert space some (linear combinations) of the supercharges must therefore vanish. To exhibit this more explicitly, let us rewrite (1) as

$$\{Q_\alpha, Q^\dagger_\beta\} = 2\,(\not{P}\Gamma^0)_{\alpha\beta}\,. \tag{2}$$

For light-like $P^\mu = (P^0, \vec{P})$ the right-hand side is proportional to a projection operator $(\mathbf{1} + \Gamma_{||}\Gamma^0)/2$. Here $\Gamma_{||}$ is the gamma matrix along the spatial momentum $\vec{P}$ of the states. The supersymmetry anti-commutator can then be written as

$$\{Q_\alpha, Q^\dagger_\beta\} = 2\,P^0 \Big(\mathbf{1} + \tilde\Gamma_D \tilde\Gamma_\perp\Big)_{\alpha\beta}\,. \tag{3}$$

Here $\tilde\Gamma_D$ consists of the product of all D independent gamma matrices, and $\tilde\Gamma_\perp$ of the product of all $D-2$ gamma matrices in the transverse directions (i.e., perpendicular to $\vec{P}$), with phase factors such that

$$(\tilde\Gamma_D)^2 = (\tilde\Gamma_\perp)^2 = \mathbf{1}\,, \qquad [\tilde\Gamma_D, \tilde\Gamma_\perp] = 0\,. \tag{4}$$

This shows that the right-hand side of (3) is proportional to a projection operator, which projects out half of the spinor space. Consequently, half the spinors must vanish on physical states, whereas the other ones generate a Clifford algebra. Denoting the real dimension of the supercharges by Q, the representation space of the charges decomposes into the two chiral spinor representations of SO($Q/2$). When confronting these results with the last column in Table 2, it turns out that the dimension of the shortest supermultiplet is not just equal to $2^{Q_{\rm irr}/4}$, as one might naively expect. For $D = 6$, this is so because the representation is complex. For $D = 3,4$ the representation is twice as big because it must also accommodate fermion number (or, alternatively, because it must satisfy the correct CPT properties). The derivation for $D = 4$ is presented in many places. For $D = 3$ we refer to [3].

The two chiral spinor spaces correspond to the bosonic and fermionic states, respectively. For the massless multiplets, the dimensions are shown in Table 2. Bigger supermultiplets can be obtained by combining various irreducible multiplets in a nontrivial way. We will demonstrate this below in three relevant cases, corresponding to $D = 11, 10$ and 6 spacetime dimensions. For the convenience of the reader we present Fig. 1, which lists the pure supergravity theories in dimensions $4 \leq D \leq 11$ with $Q = 32, 16, 8, 4$.[4] Some of these theories will be discussed later in more detail (in particular supergravity in $D = 11$ and 10 dimensions).

2.2.1. $D = 11$

In 11 dimensions we are dealing with 32 independent real supercharges. In odd-dimensional spacetimes irreducible spinors are subject to the eigenvalue condition $\tilde\Gamma_D = \pm\mathbf{1}$. Therefore (3) simplifies and shows that the 16 nonvanishing spinor charges transform according to the chiral spinor representation of the helicity group SO(9).

On the other hand, when regarding the 16 spinor charges as gamma matrices, it follows that the representation space constitutes the spinor representation of SO(16), which decomposes into two chiral subspaces, one corresponding to the bosons and the other one to the fermions. To determine the helicity content of the bosonic and fermionic states, one considers the embedding of the SO(9) spinor representation in the SO(16) vector transformation. It then turns out that one of the **128** representations branches into helicity representations according to $\mathbf{128} \to \mathbf{44} + \mathbf{84}$, while the second one transforms irreducibly according to the **128** representation of the helicity group.

The above states comprise precisely the massless states corresponding to $D = 11$ supergravity [5]. The graviton states transform in the **44**, the antisymmetric tensor states in the **84** and the gravitini states in the **128**

[4]In $D = 4$ there exist additional theories with $Q = 12, 20$ and 24; in $D = 5$ there exists a theory with $Q = 24$ [4] and most likely there also exits a $Q = 24$ supergravity in $D = 6$. So far these supergravities have played no role in string theory and hence we omit them from our discussion here.

D \ Q	32	16	8	4
11	○			
10	○ IIB, ○ IIA	○		
9	○	○		
8	○	○		
7	○	○		
6	○ (2,2)	○ (2,0), ○ (1,1)	○ (1,0)	
5	○	○	○	
4	○ N=8	○ N=4	○ N=2	○ N=1

Figure 1. Pure supergravity theories in dimensions $4 \leq D \leq 11$ with the number of independent supercharges equal to $Q = 32$, 16, 8 and 4. In 3 spacetime dimensions, pure supergravity does not describe propagating degrees of freedom and is a topological theory.

representations of SO(9). Rather than showing all this in detail, we continue with other cases, where the representations are smaller and the group theory is more transparent. The helicity representations of the graviton, gravitino and tensor gauge fields are discussed in the appendix. Bigger supermultiplets consist of multiples of 256 states. For instance, without central charges, the smallest massive supermultiplet comprises $32768 + 32768$ states. These multiplets will not be considered here.

2.2.2. $D = 10$

In 10 dimensions the supercharges are both Majorana and Weyl spinors. The latter means that they are eigenspinors of $\tilde{\Gamma}_D$. According to (3), when we have simple (i.e., nonextended) supersymmetry with 16 charges, the nonvanishing charges transform in a chiral spinor representation of the SO(8) helicity group. With 8 nonvanishing supercharges we are dealing with an 8-dimensional Clifford algebra, whose irreducible representation space corresponds to the bosonic and fermionic states, each transforming according to a chiral spinor representation. Hence we are dealing with three 8-dimensional representations of SO(8), which are inequivalent. One is the representation to which we assign the supercharges, which we will denote by $\mathbf{8}_s$; to the other two, denoted as the $\mathbf{8}_v$ and $\mathbf{8}_c$ representations, we assign the bosonic and fermionic states, respectively. The fact that SO(8) representations appear in a three-fold variety is known as *triality*, which is a characteristic property of the group SO(8). With the exception of certain representations, such as the adjoint and the singlet representation, the three types of representation are inequivalent. They are traditionally distinguished by labels s, v and c (see, for instance, [6]).

The smallest massless supermultiplet has now been constructed with 8 bosonic and 8 fermionic states and corresponds to the vector multiplet of supersymmetric Yang-Mills theory in 10 dimensions [7]. Before constructing the supermultiplets that are relevant for $D = 10$ supergravity, let us first discuss some other properties of SO(8) representations. One way to distinguish the inequivalent representations, is to investigate how they decompose into representations of an SO(7) subgroup. Each of the 8-dimensional representations leaves a different SO(7) subgroup of SO(8) invariant. Therefore there is an SO(7) subgroup under which the $\mathbf{8}_v$ representation branches into

$$\mathbf{8}_v \longrightarrow \mathbf{7} + \mathbf{1}.$$

Under this SO(7) the other two 8-dimensional representations branch into

$$\mathbf{8}_s \longrightarrow \mathbf{8}\,, \qquad \mathbf{8}_c \longrightarrow \mathbf{8}\,,$$

where $\mathbf{8}$ is the spinor representation of SO(7). Corresponding branching rules for the 28-, 35- and 56-dimensional representations are

$$\begin{array}{rclrcl} \mathbf{28} & \longrightarrow & \mathbf{7}+\mathbf{21}\,, & & & \\ \mathbf{35}_v & \longrightarrow & \mathbf{1}+\mathbf{7}+\mathbf{27}\,, & \mathbf{56}_v & \longrightarrow & \mathbf{21}+\mathbf{35}\,, \\ \mathbf{35}_{c,s} & \longrightarrow & \mathbf{35}\,, & \mathbf{56}_{c,s} & \longrightarrow & \mathbf{8}+\mathbf{48}\,. \end{array} \tag{5}$$

In order to obtain the supersymmetry representations relevant for supergravity we consider tensor products of the smallest supermultiplet consisting of $\mathbf{8}_v + \mathbf{8}_c$, with one of the 8-dimensional representations. There are thus three different possibilities, each leading to a 128-dimensional super-

supermultiplet	bosons	fermions
vector multiplet	$\mathbf{8}_v$	$\mathbf{8}_c$
graviton multiplet	$\mathbf{1}+\mathbf{28}+\mathbf{35}_v$	$\mathbf{8}_s+\mathbf{56}_s$
gravitino multiplet	$\mathbf{1}+\mathbf{28}+\mathbf{35}_c$	$\mathbf{8}_s+\mathbf{56}_s$
gravitino multiplet	$\mathbf{8}_v+\mathbf{56}_v$	$\mathbf{8}_c+\mathbf{56}_c$

TABLE 3. Massless $N=1$ supermultiplets in $D=10$ spacetime dimensions containing $8+8$ or $64+64$ bosonic and fermionic degrees of freedom.

multiplet. Using the multiplication rules for SO(8) representations,

$$\begin{aligned}
\mathbf{8}_v\times\mathbf{8}_v &= \mathbf{1}+\mathbf{28}+\mathbf{35}_v\,, & \mathbf{8}_v\times\mathbf{8}_s &= \mathbf{8}_c+\mathbf{56}_c\,,\\
\mathbf{8}_s\times\mathbf{8}_s &= \mathbf{1}+\mathbf{28}+\mathbf{35}_s\,, & \mathbf{8}_s\times\mathbf{8}_c &= \mathbf{8}_v+\mathbf{56}_v\,,\\
\mathbf{8}_c\times\mathbf{8}_c &= \mathbf{1}+\mathbf{28}+\mathbf{35}_c\,, & \mathbf{8}_c\times\mathbf{8}_v &= \mathbf{8}_s+\mathbf{56}_s\,,
\end{aligned}\tag{6}$$

it is straightforward to obtain these new multiplets. Multiplying $\mathbf{8}_v$ with $\mathbf{8}_v+\mathbf{8}_c$ yields $\mathbf{8}_v\times\mathbf{8}_v$ bosonic and $\mathbf{8}_v\times\mathbf{8}_c$ fermionic states, and leads to the second supermultiplet shown in Table 3. This supermultiplet contains the representation $\mathbf{35}_v$, which can be associated with the states of the graviton in $D=10$ dimensions (the field-theoretic identification of the various states will be clarified in the appendix). Therefore this supermultiplet will be called the *graviton multiplet*. Multiplication with $\mathbf{8}_c$ or $\mathbf{8}_s$ goes in the same fashion, except that we will associate the $\mathbf{8}_c$ and $\mathbf{8}_s$ representations with fermionic quantities (note that these are the representations to which the fermion states of the Yang-Mills multiplet and the supersymmetry charges are assigned). Consequently, we interchange the boson and fermion assignments in these products. Multiplication with $\mathbf{8}_c$ then leads to $\mathbf{8}_c\times\mathbf{8}_c$ bosonic and $\mathbf{8}_c\times\mathbf{8}_v$ fermionic states, whereas multiplication with $\mathbf{8}_s$ gives $\mathbf{8}_s\times\mathbf{8}_c$ bosonic and $\mathbf{8}_s\times\mathbf{8}_v$ fermionic states. These supermultiplets contain fermions transforming according to the $\mathbf{56}_s$ and $\mathbf{56}_c$ representations, respectively, which can be associated with gravitino states (see the appendix for the helicity assignment of gravitino states), but no graviton states as those transform in the $\mathbf{35}_v$ representation. Therefore these two supermultiplets are called *gravitino multiplets*. We have thus established the existence of two inequivalent gravitino multiplets. The explicit SO(8) decompositions of the vector, graviton and gravitino supermultiplets are shown in Table 3.

By combining a graviton and a gravitino multiplet it is possible to construct an $N=2$ supermultiplet of $128+128$ bosonic and fermionic states. However, since there are two inequivalent gravitino multiplets, there will also be two inequivalent $N=2$ supermultiplets containing the states corresponding to a graviton and two gravitini. According to the construction presented above, one $N=2$ supermultiplet may be be viewed as the tensor product of two identical supermultiplets (namely $\mathbf{8}_v+\mathbf{8}_c$). Such a multiplet follows if one starts from a supersymmetry algebra based on *two* Majorana-Weyl spinor charges Q with the *same* chirality. The states of this multiplet decompose as follows:

$$\textit{Chiral } N=2 \textit{ supermultiplet } \text{(IIB)} \qquad (\mathbf{8}_v+\mathbf{8}_c)\times(\mathbf{8}_v+\mathbf{8}_c) \implies \begin{cases} \textbf{bosons}: \\ \mathbf{1}+\mathbf{1}+\mathbf{28}+\mathbf{28}+\mathbf{35}_v+\mathbf{35}_c \\ \\ \textbf{fermions}: \\ \mathbf{8}_s+\mathbf{8}_s+\mathbf{56}_s+\mathbf{56}_s \end{cases} \tag{7}$$

This is the multiplet corresponding to IIB supergravity [8]. Because the supercharges have the same chirality, one can perform rotations between these spinor charges which leave the supersymmetry algebra unaffected. Hence the automorphism group $H_{\rm R}$ is equal to SO(2). This feature reflects itself in the multiplet decomposition, where the $\mathbf{1}$, $\mathbf{8}_s$, $\mathbf{28}$ and $\mathbf{56}_s$ representations are degenerate and constitute doublets under this SO(2) group.

A second supermultiplet may be viewed as the tensor product of a $(\mathbf{8}_v+\mathbf{8}_s)$ supermultiplet with a second supermultiplet $(\mathbf{8}_v+\mathbf{8}_c)$. In this case the supercharges constitute two Majorana-Weyl spinors of opposite chirality. Now the supermultiplet decomposes as follows:

$$\textit{Nonchiral } N=2 \textit{ supermultiplet } \text{(IIA)} \qquad (\mathbf{8}_v+\mathbf{8}_s)\times(\mathbf{8}_v+\mathbf{8}_c) \implies \begin{cases} \textbf{bosons}: \\ \mathbf{1}+\mathbf{8}_v+\mathbf{28}+\mathbf{35}_v+\mathbf{56}_v \\ \\ \textbf{fermions}: \\ \mathbf{8}_s+\mathbf{8}_c+\mathbf{56}_s+\mathbf{56}_c \end{cases} \tag{8}$$

This is the multiplet corresponding to IIA supergravity [9]. It can be obtained by a straightforward reduction of $D=11$ supergravity. The latter follows from the fact that two $D=10$ Majorana-Weyl spinors with opposite chirality can be combined into a single $D=11$ Majorana spinor. The formula below summarizes the massless states of IIA supergravity from an 11-dimensional perspective. The massless states of 11-dimensional supergravity transform according to the $\mathbf{44}$, $\mathbf{84}$ and $\mathbf{128}$ representation of the helicity group SO(9). They correspond to the degrees of freedom described by the metric, a 3-rank antisymmetric gauge field and the gravitino field, respectively. We also show how the 10-dimensional states can subsequently be branched into 9-dimensional states, characterized in terms of representations of the helicity group SO(7):

$$
\begin{aligned}
\mathbf{44} &\Longrightarrow \begin{cases} \mathbf{1} & \longrightarrow \mathbf{1} \\ \mathbf{8}_v & \longrightarrow \mathbf{1}+\mathbf{7} \\ \mathbf{35}_v & \longrightarrow \mathbf{1}+\mathbf{7}+\mathbf{27} \end{cases} \\
\mathbf{84} &\Longrightarrow \begin{cases} \mathbf{28} & \longrightarrow \mathbf{7}+\mathbf{21} \\ \mathbf{56}_v & \longrightarrow \mathbf{21}+\mathbf{35} \end{cases} \\
\mathbf{128} &\Longrightarrow \begin{cases} \mathbf{8}_s & \longrightarrow \mathbf{8} \\ \mathbf{8}_c & \longrightarrow \mathbf{8} \\ \mathbf{56}_s & \longrightarrow \mathbf{8}+\mathbf{48} \\ \mathbf{56}_c & \longrightarrow \mathbf{8}+\mathbf{48} \end{cases}
\end{aligned}
\qquad (9)
$$

Clearly, in $D = 9$ we have a higher degeneracy of states, related to the automorphism group SO(2). We note the presence of graviton and gravitino states, transforming in the **27** and **48** representations.

One could also take the states of the IIB supergravity and decompose them into $D = 9$ massless states. This leads to precisely the same supermultiplet as the reduction of the states of IIA supergravity. Indeed, the reductions of IIA and IIB supergravity to 9 dimensions, yield the same theory [10, 11, 12]. To see this at the level of the Lagrangian requires certain duality transformations, which we discuss in section 3. Hence $Q = 32$ supergravity is unique in all spacetime dimensions, except for $D = 10$. Maximal supergravity will be discussed in subsection 2.4. The field-content of the maximal $Q = 32$ supergravity theories for dimensions $3 \leq D \leq 11$ will be presented in two tables (cf. Table 6 and 7).

2.2.3. $D = 6$

In 6 dimensions we have chiral spinors, which are not Majorana. The charge conjugated spinor has the same chirality, so that the chiral rotations of the spinors can be extended to the group $\mathrm{USp}(2N_+)$, for N_+ chiral spinors. Likewise N_- negative-chirality spinors transform under $\mathrm{USp}(2N_-)$. This is already incorporated in Table 1. In principle we have N_+ positive- and N_- negative-chirality charges, but almost all information follows from first considering the purely chiral case. In Table 4 we present the decomposition of the various helicity representations of the smallest supermultiplets based on $N_+ = 1, 2, 3$ or 4 supercharges. In $D = 6$ dimensions the helicity group SO(4) decomposes into the product of two SU(2) groups: $\mathrm{SO}(4) \cong (\mathrm{SU}_+(2) \times \mathrm{SU}_-(2))/Z_2$. When we have supercharges of only one chirality, the smallest supermultiplet will only transform under one SU(2) factor of the helicity group, as is shown in Table 4.[5]

[5]The content of this table also specifies the smallest *massive* supermultiplets in four dimensions. The SU(2) group is then associated with spin in three space dimensions.

$SU_+(2)$	$N_+ = 1$	$N_+ = 2$	$N_+ = 3$	$N_+ = 4$
5				1
4			1	8
3		1	6	27
2	1	4	14	48
1	2	5	14	42
	$(2+2)_{\mathbf{C}}$	$(8+8)_{\mathbf{R}}$	$(32+32)_{\mathbf{C}}$	$(128+128)_{\mathbf{R}}$

TABLE 4. Shortest massless supermultiplets of $D = 6$ N_+-extended chiral supersymmetry. The states transform both in the $SU_+(2)$ helicity group and under a $USp(2N_+)$ group. For odd values of N_+ the representations are complex, for even N_+ they can be chosen real. Of course, an identical table can be given for negative-chirality spinors.

Let us now consider specific supermultiplets. All these multiplets are summarized in Table 5. The helicity assignments of the states described by gravitons, gravitini, vector and tensor gauge fields, and spinor fields are presented in the appendix. The simplest case is $(N_+, N_-) = (1, 0)$, where the smallest supermultiplet is the (1,0) *hypermultiplet*, consisting of a complex doublet of spinless states and a chiral spinor. Taking the tensor product of the smallest supermultiplet with the $(2, 1)$ helicity representation gives the (1,0) *tensor multiplet*, with a selfdual tensor, a spinless state and a doublet of chiral spinors. The tensor product with the $(1, 2)$ helicity representation yields the (1,0) *vector multiplet*, with a vector state, a doublet of chiral spinors and a scalar. Multiplying the latter with the (2,3) helicity representation, one obtains the states of (1,0) *supergravity*. Observe that the selfdual tensor fields in the tensor and supergravity supermultiplet are of opposite selfduality phase.

Next consider $(N_+, N_-) = (2, 0)$ supersymmetry. The smallest multiplet, shown in Table 4, then corresponds to the (2,0) *tensor multiplet*, with the bosonic states decomposing into a selfdual tensor, and a five-plet of spinless states, and a four-plet of chiral fermions. Multiplication with the (1,3) helicity representation yields the (2,0) supergravity multiplet, consisting of the graviton, four chiral gravitini and five selfdual tensors [13]. Again, the selfdual tensors of the tensor and of the supergravity supermultiplet are of opposite selfduality phase.

Of course, there exists also a nonchiral version with 16 supercharges, namely the one corresponding to $(N_+, N_-) = (1, 1)$. The smallest multiplet is now given by the tensor product of the supermultiplets with (1,0) and (0,1) supersymmetry. This yields the vector multiplet, with the vector state

multiplet	#	bosons	fermions
(1,0) hyper	$4+4$	$(1,1;2,1)+\text{h.c.}$	$(2,1;1,1)$
(1,0) tensor	$4+4$	$(3,1;1,1)+(1,1;1,1)$	$(2,1;2,1)$
(1,0) vector	$4+4$	$(2,2;1,1)$	$(1,2;2,1)$
(1,0) supergravity	$12+12$	$(3,3;1,1)+(1,3;1,1)$	$(2,3;2,1)$
(2,0) tensor	$8+8$	$(3,1;1,1)+(1,1;5,1)$	$(2,1;4,1)$
(2,0) supergravity	$24+24$	$(3,3;1,1)+(1,3;5,1)$	$(2,3;4,1)$
(1,1) vector	$8+8$	$(2,2;1,1)+(1,1;2,2)$	$(2,1;1,2)+(1,2;2,1)$
(1,1) supergravity	$32+32$	$(3,3;1,1)$ $+(1,3;1,1)+(3,1;1,1)$ $+(1,1;1,1)+(2,2;2,2)$	$(3,2;1,2)+(2,3;2,1)$ $+(1,2;1,2)+(2,1;2,1)$
(2,2) supergravity	$128+128$	$(3,3;1,1)$ $+(3,1;1,5)+(1,3;5,1)$ $+(2,2;4,4)+(1,1;5,5)$	$(3,2;4,1)+(2,3;1,4)$ $+(2,1;4,5)+(1,2;5,4)$

TABLE 5. Some relevant $D = 6$ supermultiplets with (N_+, N_-) supersymmetry. The states $(n,m;\tilde{n},\tilde{m})$ are assigned to (n,m) representations of the helicity group $\text{SU}_+(2)\times\text{SU}_-(2)$ and $(\tilde{n},\tilde{m})$ representations of $\text{USp}(2N_+)\times\text{USp}(2N_-)$. The second column lists the number of bosonic + fermionic states for each multiplet.

and four scalars, the latter transforming with respect to the (2,2) representation of USp(2) $\times$ USp(2). There are two doublets of chiral fermions with opposite chirality, each transforming as a doublet under the corresponding USp(2) group. Taking the tensor product of the vector multiplet with the (2,2) representation of the helicity group yields the states of the (1, 1) *supergravity* multiplet. It consists of 32 bosonic states, corresponding to a graviton, a tensor, a scalar and four vector states, where the latter transform under the (2,2) representation of USp(2) $\times$ USp(2). The 32 fermionic states comprise two doublets of chiral gravitini and two chiral spinor doublets, transforming as doublets under the appropriate USp(2) group.

Finally, we turn to the case of $(N_+, N_-) = (2,2)$. The smallest supermultiplet is given by the tensor product of the smallest (2,0) and (0,2) supermultiplets. This yields the 128 + 128 states of the (2,2) *supergravity* multiplet. These states transform according to representations of USp(4) $\times$ USp(4).

In principle, one can continue and classify representations for other values of (N_+, N_-). As is obvious from the construction that we have presented, this will inevitably lead to states transforming in higher-helicity representations. As we will discuss in subsection 2.4, the higher-spin gauge fields associated with these representations can not be coupled to gravity. Although the representations exist and can be described by appropriate free-field theories, they have no future as nontrivial quantum field theories.

2.3. MAXIMAL SUPERSYMMETRY: $Q \leq 32$

In the above we have restricted ourselves to (massless) supermultiplets based on $Q \leq 32$ supercharge components. From the general analysis it is clear that increasing the number of supercharges leads to higher and higher helicity representations. Obviously some of these representations will also occur in lower-Q supermultiplets, by multiplying shorter multiplets by suitable helicity representations. It is not so easy to indicate in arbitrary dimension what we mean by a higher helicity representation, but we have in mind those representations that are described by gauge fields that are *symmetric* Lorentz tensors. Symmetric tensor gauge fields for arbitrary helicity states can be constructed (in four dimensions, see, for instance, [14]). However, it turns out that symmetric gauge fields cannot consistently couple to themselves or to other fields. An exception is the graviton field, which can interact with itself as well as to low-spin matter, but not to other higher-spin gauge fields. By consistent, we mean that their respective gauge invariances of the higher-spin fields (or appropriate deformations thereof) cannot be preserved at the interacting level.

There have been many efforts to circumvent this apparent no-go theorem. What seems clear, is that one needs a combination of the following ingredients in order to do this (for a recent review, see [15]): (i) an infinite tower of higher-spin gauge fields; (ii) interactions that are inversely proportional to the cosmological constant; (iii) extensions of the super-Poincaré or the super-de Sitter algebra with additional fermionic and bosonic charges.

Conventional supergravity theories are not of this kind. This is the reason why we have avoided (i.e. in Table 5) to list supermultiplets with states transforming in higher-helicity representations. The bound $Q \leq 32$ originates from the necessity of avoiding the higher-spin fields. It implies that supergravity does not exist for spacetime dimensions $D \geq 11$ (at least, if one assumes a single time coordinate), because Lorentz spinors have more than 32 components beyond $D = 11$ [16].

Most of the search for interacting higher-spin fields was performed in four spacetime dimensions [17]. When one increases the number of supercharges beyond $Q = 32$, then a supermultiplet will contain several massless states of spin-2 and at least spin-5/2 fermions. In higher spacetime dimensions, more than 32 supercharges are excluded (in the absence of higher-spin gauge fields), because, upon dimensional reduction, these theories would give rise to theories that are inconsistent in $D = 4$. There is also direct

D	$H_{\rm R}$	graviton	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$
11	1	1	0	0	0	1	0
10A	1	1	1	1	1	1	0
10B	SO(2)	1	2	0	2	0	1^*
9	SO(2)	1	$2+1$	$2+1$	2	1	
8	U(2)	1	$5+1+\bar{1}$	$3+\bar{3}$	3	[1]	
7	USp(4)	1	14	10	5		
6	USp(4)×USp(4)	1	(5,5)	(4,4)	$(5,1)+(1,5)$		
5	USp(8)	1	42	27			
4	U(8)	1	$35+\overline{35}$	[28]			
3	SO(16)	1	128				

TABLE 6. Bosonic field content for maximal supergravities. The $p=3$ gauge field in $D=10$B has a self-dual field strength. The representations [1] and [28] (in $D=8,4$, respectively) are extended to U(1) and SU(8) representations through duality transformations on the field strengths. These transformations can not be represented on the vector potentials. In $D=3$ dimensions, the graviton does not describe propagating degrees of freedom.

evidence in $D=3$, where graviton and gravitini fields do not describe dynamic degrees of freedom. Hence, one can write down supergravity theories based on a graviton field and an arbitrary number of gravitino fields, which are topological. However, when coupling matter to this theory, described by scalars and spinors, the theory supports not more than 32 supercharges. Beyond $Q=16$ there are four unique theories with $Q=18,20,24$ and 32 [3].

2.4. MAXIMAL SUPERGRAVITIES

In this section we review the maximal supergravities in various dimensions. These theories have precisely $Q=32$ supercharge components. We restrict our discussion to $3\leq D\leq 11$.

The bosonic fields always comprise the metric tensor for the graviton field and a number of antisymmetric gauge fields. For the antisymmetric gauge fields, it is a priori unclear whether to choose a $(p{+}1)$-rank gauge field or its dual $(D-3-p)$-rank partner, but it turns out that the interactions often prefer the rank of the gauge field to be as small as possible. Therefore, in Table 6, we restrict ourselves to $p\leq 3$, as in $D=11$ dimensions, $p=3$ and $p=4$ are each other's dual conjugates. This table presents all the field configurations for maximal supergravity in various dimensions. Obviously, the problematic higher-spin fields are avoided, because the only symmetric

D	$H_{\rm R}$	gravitini	spinors
11	1	1	0
10A	1	1+1	1+1
10B	SO(2)	2	2
9	SO(2)	2	$2+2$
8	U(2)	$2+\bar{2}$	$2+\bar{2}+4+\bar{4}$
7	USp(4)	4	16
6	USp(4)×USp(4)	$(4,1)+(1,4)$	$(4,5)+(5,4)$
5	USp(8)	8	42
4	U(8)	$8+\bar{8}$	$56+\overline{56}$
3	SO(16)	16	128

TABLE 7. Fermionic field content for maximal supergravities. For $D=5,6,7$ the fermion fields are counted as symplectic Majorana spinors. For $D=4,8$ we include both chiral and antichiral spinor components, which transform in conjugate representations of $H_{\rm R}$. In $D=3$ dimensions the gravitino does not correspond to propagating degrees of freedom.

gauge field is the one describing the graviton. In Table 7 we also present the fermionic fields, always consisting of gravitini and simple spinors. All these fields are classified as representations of the automorphism group $H_{\rm R}$. In order to compare these tables to similar tables in the literature, one may need to use the (local) equivalences: USp(4)∼SO(5), USp(2)∼SU(2) and SU(4)∼SO(6).

The supersymmetry algebra of the maximal supergravities comprises general coordinate transformations, local supersymmetry transformations and the gauge transformations associated with the antisymmetric gauge fields[6]. These gauge transformations usually appear in the anticommutator of two supercharges, and may be regarded as central charges. In perturbation theory, the theory does not contain charged fields, so these central charges simply vanish on physical states. However, at the nonperturbative level, there may be solitonic or other states that carry charges. An example are magnetic monopoles, dyons, or extremal black holes. On such states, some of the central charges may take finite values. Without further knowledge about the kind of states that may emerge at the nonperturbative level,

[6]There may be additional gauge transformations that are of interest to us. As we discuss in subsection 2.5.1, it is possible to have (part of the) the automorphism group $H_{\rm R}$ realized as a local invariance. However, the corresponding gauge fields are then composite and do not give rise to physical states (at least, not in perturbation theory).

we can generally classify the possible central charges, by considering a decomposition of the anticommutator. This anticommutator carries at least two spinor indices and two indices associated with the group $H_{\rm R}$. Hence we may write

$$\{Q_\alpha, Q_\beta\} \propto \sum_r (\Gamma^{\mu_1\cdots\mu_r}C)_{\alpha\beta}\, Z_{\mu_1\cdots\mu_r}\,, \tag{10}$$

where $\Gamma^{\mu_1\cdots\mu_r}$ is the antisymmetrized product of r gamma matrices, C is the charge-conjugation matrix and $Z_{\mu_1\cdots\mu_r}$ is the central charge, which transforms as an antisymmetric r-rank Lorentz tensor and depends on possible additional $H_{\rm R}$ indices attached to the supercharges. The central charge must be symmetric or antisymmetric in these indices, depending on whether the product of the gamma matrices with C is symmetric or antisymmetric, so that the product is symmetric in the combined indices of the supercharges. For given spacetime dimension all possible central charges can be classified.[7] For the maximal supergravities in spacetime dimensions $3 \leq D \leq 11$ this classification is given in Table 8. Because we have 32 supercharge components, the sum of the independent momentum operators and the central charges must be equal to $(32 \times 33)/2 = 528$.

2.4.1. $D = 11$

Supergravity in 11 spacetime dimensions is based on an "elfbein" field E_M^A, a Majorana gravitino field Ψ_M and a 3-rank antisymmetric gauge field C_{MNP}. With chiral (2,0) supergravity in 6 dimensions, it is the only $Q \geq 16$ supergravity theory without a scalar field. Its Lagrangian can be written as follows [5],

$$\begin{aligned}\mathcal{L}_{11} \;=\; \frac{1}{\kappa_{11}^2}\Big[&-\tfrac{1}{2}E\,R(E,\Omega) - \tfrac{1}{2}E\bar\Psi_M\Gamma^{MNP}D_N(\Omega)\Psi_P - \tfrac{1}{48}E(F_{MNPQ})^2 \\ &-\tfrac{1}{3456}\sqrt{2}\,\varepsilon^{MNPQRSTUVWX}\,F_{MNPQ}\,F_{RSTU}\,C_{VWX} \\ &-\tfrac{1}{192}\sqrt{2}E\Big(\bar\Psi_R\Gamma^{MNPQRS}\Psi_S + 12\,\bar\Psi^M\Gamma^{NP}\Psi^Q\Big)F_{MNPQ} + \cdots\Big]\,,\end{aligned} \tag{11}$$

where the ellipses denote terms of order Ψ^4, $E = \det E_M^A$ and $\Omega_M{}^{AB}$ denotes the spin connection. The supersymmetry transformations are equal to

$$\begin{aligned}\delta E_M^A &= \tfrac{1}{2}\,\bar\epsilon\,\Gamma^A\Psi_M\,, \\ \delta C_{MNP} &= -\tfrac{1}{8}\sqrt{2}\,\bar\epsilon\,\Gamma_{[MN}\Psi_{P]}\,, \\ \delta\Psi_M &= D_M(\hat\Omega)\,\epsilon + \tfrac{1}{288}\sqrt{2}\,\Big(\Gamma_M{}^{NPQR} - 8\,\delta_M^N\,\Gamma^{PQR}\Big)\,\epsilon\,\hat F_{NPQR}\,.\end{aligned} \tag{12}$$

[7] For a related discussion see for example [18, 19] and references therein.

D	$H_{\rm R}$	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
11	1			1 [55]			1 [462]
10A	1	1 [1]	1 [10]	1 [45]		1 [210]	$1+1$ [126]
10B	SO(2)		2 [10]		1 [120]		$1+2$ [126]
9	SO(2)	$1+2$ [1]	2 [9]	1 [36]	1 [84]	$1+2$ [126]	
8	U(2)	$3+\bar{3}$ [1]	3 [8]	$1+\bar{1}$ [28]	$1+3$ [56]	$3+\bar{3}$ [35]	
7	USp(4)	10 [1]	5 [7]	$1+5$ [21]	10 [35]		
6	USp(4)×USp(4)	$(4,4)$ [1]	$(1,1)+(5,1)$ $+(1,5)$ [6]	(4,4) [15]	$(10,1)$ $+(1,10)$ [10]		
5	USp(8)	$1+27$ [1]	27 [5]	36 [10]			
4	U(8)	$28+\overline{28}$ [1]	63 [4]	$36+\overline{36}$ [3]			
3	SO(16)	120 [1]	135 [3]				

TABLE 8. Decomposition of the central extension in the supersymmetry algebra with $Q=32$ supercharge components in terms of r-rank Lorentz tensors. The second row specifies the number of independent components for each r-rank tensor charge. The total number of central charges is equal to $528-D$, because we have not listed the D independent momentum operators

Here the covariant derivative is covariant with respect to local Lorentz transformation

$$D_M(\Omega)\,\epsilon = \Big(\partial_M - \tfrac{1}{4}\Omega_M{}^{AB}\Gamma_{AB}\Big)\epsilon\,, \tag{13}$$

and $\hat{F}_{MNPQ}$ is the supercovariant field strength

$$\hat{F}_{MNPQ} = 24\,\partial_{[M}C_{NPQ]} + \tfrac{3}{2}\sqrt{2}\,\bar{\Psi}_{[M}\Gamma_{NP}\Psi_{Q]}\,. \tag{14}$$

Note the presence in the Lagrangian of a Chern-Simons-like term $F\wedge F\wedge C$, so that the action is only invariant up to surface terms. We also wish to point out that the quartic-Ψ terms can be included into the Lagrangian (11) by replacing the spin-connection field Ω by $(\Omega+\hat{\Omega})/2$ in the covariant derivative of the gravitino kinetic term and by replacing F_{MNPQ} in the last line by $(\hat{F}_{MNPQ} + F_{MNPQ})/2$. These substitutions ensure that the field equations corresponding to (11) are supercovariant. The Lagrangian is derived in the context of the so-called "1.5-order" formalism, in which the spin connection is defined as a dependent field determined by its (algebraic) equation of motion, whereas its supersymmetry variation in the action is treated as if it were an independent field [20]. The supercovariant spin connection is the solution of the following equation,

$$D_{[M}(\hat{\Omega})\, E_{N]}^{A} - \tfrac{1}{4}\bar{\Psi}_M \Gamma^A \Psi_N = 0\,. \tag{15}$$

The left-hand side is the supercovariant torsion tensor.

We have the following bosonic field equations and Bianchi identities,

$$\begin{aligned} R_{MN} &= \tfrac{1}{72} g_{MN}\, F_{PQRS} F^{PQRS} - \tfrac{1}{6} F_{MPQR}\, F_N{}^{PQR}\,, \\ \partial_M\Big(E\, F^{MNPQ}\Big) &= \tfrac{1}{1152}\sqrt{2}\, \varepsilon^{NPQRSTUVWXY} F_{RSTU}\, F_{VWXY}\,, \\ \partial_{[M} F_{NPQR]} &= 0\,, \end{aligned} \tag{16}$$

which no longer depend on the antisymmetric gauge field. An alternative form of the second equation is [21]

$$\partial_{[M} H_{NPQRSTU]} = 0\,, \tag{17}$$

where $H_{MNPQRST}$ is the dual field strength,

$$H_{MNPQRST} = \frac{1}{7!} E\, \varepsilon_{MNPQRSTUVWX} F^{UVWX} - \tfrac{1}{2}\sqrt{2}\, F_{[MNPQ}\, C_{RST]}\,. \tag{18}$$

One could imagine that the third equation of (16) and (17) receive contributions from charges that would give rise to source terms on the right-hand side of the equations. These charges are associated with the 'flux'-integral of $H_{MNPQRST}$ and F_{MNPQ} over the boundary of an 8- and a 5-dimensional spatial volume, respectively. This volume is transverse to a $p = 2$ and $p = 5$ brane configuration, and the corresponding charges are 2- and 5-rank Lorentz tensors. These are just the charges that can appear as central charges in the supersymmetry algebra, as one can verify in Table 8. Solutions of 11-dimensional supergravity that contribute to these charges were considered in [22, 23, 24, 18].

It is straightforward to evaluate the supersymmetry algebra on these fields. The commutator of two supersymmetry transformations yields a general-coordinate transformation, a local Lorentz transformation, a supersymmetry transformation and a gauge transformation associated with the tensor gauge field,

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_{\rm gct}(\xi^M) + \delta_Q(\epsilon_3) + \delta_L(\lambda^{AB}) + \delta_A(\xi_{MN})\,. \tag{19}$$

The parameters of the transformations on the right-hand side are given by

$$
\begin{aligned}
\xi^M &= \tfrac{1}{2}\,\bar\epsilon_2\Gamma^M\epsilon_1\,,\\
\epsilon_3 &= -\xi^M\Psi_M\,,\\
\lambda^{AB} &= -\xi^M\hat\Omega_M^{AB} + \tfrac{1}{288}\sqrt{2}\,\bar\epsilon_2\Big[\Gamma^{ABCDEF}\hat F_{CDEF} + 24\,\Gamma_{CD}\hat F^{ABCD}\Big]\epsilon_1\,,\\
\xi_{MN} &= -\tfrac{1}{8}\sqrt{2}\,\bar\epsilon_2\Gamma_{MN}\epsilon_1\,. \qquad (20)
\end{aligned}
$$

Note that the normalizations differ from the ones used in the supersymmetry algebra in previous subsections. The tensor gauge field transforms under gauge transformations as $\delta C_{MNP} = \partial_{[M}\xi_{NP]}$.

Finally, the constant $1/\kappa_{11}^2$ in front of the Lagrangian (11), which has the dimension $[\mathrm{length}]^{-9} \sim [\mathrm{mass}]^9$, is undetermined and depends on fixing some length scale. To see this consider a continuous rescaling of the fields,

$$E_M^A \to \mathrm{e}^{-\alpha}E_M^A\,,\qquad \Psi_M \to \mathrm{e}^{-\alpha/2}\Psi_M\,,\qquad C_{MNP} \to \mathrm{e}^{-3\alpha}C_{MNP}\,. \qquad (21)$$

Under this rescaling the Lagrangian changes according to

$$\mathcal{L}_{11} \to \mathrm{e}^{-9\alpha}\mathcal{L}_{11}\,. \qquad (22)$$

This change can then be absorbed into a redefinition of κ_{11},[8]

$$\kappa_{11}^2 \to \mathrm{e}^{-9\alpha}\kappa_{11}^2\,. \qquad (23)$$

The indetermination of κ is not a special property of $D = 11$ but occurs in any spacetime dimension where the Einstein-Hilbert action displays a similar scaling property,

$$g_{\mu\nu}^D \to \mathrm{e}^{-2\alpha}g_{\mu\nu}^D\,,\qquad \mathcal{L}_D \to \mathrm{e}^{(2-D)\alpha}\mathcal{L}_D\,,\qquad \kappa_D^2 \to \mathrm{e}^{(2-D)\alpha}\kappa_D^2\,. \qquad (24)$$

Newton's constant, $(\kappa_D^2)^{\mathrm{physical}}$, does not necessarily coincide with the parameter κ_D^2 but also depends on the precise value adopted for the (flat) metric in the ground state of the theory. Up to certain convention-dependent normalization factors one defines

$$(\kappa_D^2)^{\mathrm{physical}} := \kappa_D^2\,\lambda^{(2-D)/2}\,, \qquad (25)$$

where $g_{\mu\nu}^D$ is expanded about $\lambda\,\eta_{\mu\nu}^D$, with $\eta_{\mu\nu}^D$ equal to the Lorentz-invariant flat metric with diagonal elements equal to ± 1. Note that $(\kappa_D^2)^{\mathrm{physical}}$ is invariant under the scale transformations (24) and thus a physically meaningful scale.

[8]Note that the rescalings also leave the supersymmetry transformation rules unchanged, provided the supersymmetry parameter ϵ is changed accordingly.

When the Lagrangian contains additional terms, for instance, of higher order in the Riemann tensor, then the corresponding coupling constant will scale differently under (24). Its physical value will therefore depend in a different way on the parameter λ that parametrizes the (flat) metric in the ground state. An even simpler example is a scalar massive field, added to the Einstein-Hilbert Lagrangian. Its physical mass is equal to λ times the mass parameter in the Lagrangian. However, we should stress that the physics never depends *explicitly* on λ, provided one expresses all physical quantities in terms of physical parameters, all determined for the same value of λ. We return to the issue of frames and scales in section 2.5.1 and in appendix B.

2.5. DIMENSIONAL REDUCTION AND HIDDEN SYMMETRIES

The maximal supergravities in various dimensions are related by dimensional reduction. Here some of the spatial dimensions are compactified on a hyper-torus whose size is shrunk to zero. In this situation some of the gauge symmetries that are related to the compactified dimensions survive and take the form of internal symmetries. The aim of our discussion here is to elucidate a number of features related to these symmetries, mainly in the context of the reduction of $D = 11$ supergravity to $D = 10$ dimensions.

We denote the compactified coordinate by x^{11} which now parameterizes a circle of length L.[9] The fields are thus decomposed as periodic functions in x^{11} on the interval $0 \leq x^{11} \leq L$. This results in a spectrum of massless modes and an infinite tower of massive modes. The massless modes form the basis of the lower-dimensional supergravity theory. Because a toroidal background does not break supersymmetry, the resulting supergravity has the same number of supersymmetries as the original one. For compactifications on less trivial spaces than the hyper-torus (which we will discuss in section 3) this is not necessarily the case and the number of independent supersymmetries can be reduced. Actually, fully supersymmetric compactifications are rare. For instance, in 11-dimensional supergravity 7 coordinates can be compactified in precisely two ways such that all supersymmetries remain unaffected [25]. One is the compactification on a torus T^7, the other one the compactification of a sphere S^7. However, in the latter case the resulting 4-dimensional supergravity theory acquires a cosmological term. In the context of these lectures, such compactifications are less relevant and will not be discussed.

[9] Throughout these lectures we enumerate spacetime coordinates by $0, 1, \ldots, D-1$. Nevertheless, we denote the compactified coordinate by x^{11}, to indicate that it is the eleventh spacetime coordinate.

In the formulation of the compactified theory, it is important to decompose the higher-dimensional fields in such a way that they transform covariantly under the lower-dimensional gauge symmetries, and in particular under diffeomorphisms of the lower-dimensional spacetime. This ensures that various complicated mixtures of massless modes with the tower of massive modes will be avoided. It is a key element in ensuring that solutions of the lower-dimensional theory remain solutions of the original higher-dimensional one. Another point of interest concerns the nature of the massive supermultiplets. Because these originate from supermultiplets that are massless in higher dimensions, these multiplets must be shortened by the presence of central charges. The central charge here originates from the momentum operator in the compactified dimension. We return to this issue shortly.

The emergence of new internal symmetries in theories that originate from a higher-dimensional setting, is a standard feature of Kaluza-Klein theories [26]. Following the discussion in [27] we distinguish between symmetries that have a direct explanation in terms of the symmetries in higher dimensions, and symmetries whose origin is obscure from a higher-dimensional viewpoint. Let us start with the symmetries associated with the metric tensor. The 11-dimensional metric can be decomposed according to

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{4\phi/3}(\mathrm{d}x^{11} + V_\mu \mathrm{d}x^\mu)(\mathrm{d}x^{11} + V_\nu \mathrm{d}x^\nu)\,, \qquad (26)$$

where the indices μ, ν label the 10-dimensional coordinates and the factor multiplying ϕ is for convenience later. The massless modes correspond to the x^{11}-independent parts of the 10-dimensional metric $g_{\mu\nu}$, the vector field V_μ and the scalar ϕ. Here the x^{11}-independent component of V_μ acts as a gauge field associated with reparametrizations of the circle coordinate x^{11} with an arbitrary function $\xi(x)$ of the 10 remaining spacetime coordinates x^μ. Specifically, we have $x^{11} \to x^{11} - \xi(x)$ and $x^\mu \to x^\mu$, corresponding to

$$V_\mu(x) \to V_\mu(x) + \partial_\mu \xi(x)\,. \qquad (27)$$

The massive modes, which correspond to the Fourier modes in terms of x^{11}, couple to this gauge field with a charge that is a multiple of

$$e_{\mathrm{KK}} = \frac{2\pi}{L}\,. \qquad (28)$$

Another symmetry of the lower-dimensional theory is more subtle to identify.[10] In the previous subsection we identified certain scale transfor-

[10] There are various discussions of this symmetry in the literature. Its existence in 10-dimensional supergravity was noted long ago (see, e.g. [4, 28]) and an extensive discussion can be found in [12]. Our derivation here was alluded to in [27], which deals with isometries in $N = 2$ supersymmetric Maxwell-Einstein theories in $D = 5, 4$ and 3 dimensions.

mations of the $D = 11$ fields, which did not leave the theory invariant but could be used to adjust the coupling constant κ_{11}. In the compactified situation we can also involve the compactification length into the dimensional scaling. The integration over x^{11} introduces an overall factor L in the action (we do not incorporate any L-dependent normalizations in the Fourier sums, so that the 10-dimensional and the 11-dimensional fields are directly proportional). Therefore, the coupling constant that emerges in the 10-dimensional theory equals

$$\frac{1}{\kappa_{10}^2} = \frac{L}{\kappa_{11}^2} \,, \tag{29}$$

and has the dimension $[\mathrm{mass}]^8$. However, because of the invariance under diffeomorphisms, L itself has no intrinsic meaning. It simply expresses the length of the periodicity interval of x^{11}, which itself is a coordinate without an intrinsic meaning. Stated differently, we can reparameterize x^{11} by some diffeomorphism, as long as we change L accordingly. In particular, we may rescale L according to

$$L \to \mathrm{e}^{-9\alpha} L \,, \tag{30}$$

corresponding to a reparametrization of the 11-th coordinate,

$$x^{11} \to \mathrm{e}^{-9\alpha} x^{11} \,, \tag{31}$$

so that κ_{10} remains invariant. Consequently we are then dealing with a symmetry of the Lagrangian.

In the effective 10-dimensional theory, the scale transformations (21) are thus suitably combined with the diffeomorphism (31) to yield an invariance of the Lagrangian. For the fields corresponding to the 11-dimensional metric, these combined transformations are given by[11]

$$e_\mu^a \to \mathrm{e}^{-\alpha} e_\mu^a \,, \qquad \phi \to \phi + 12\alpha \,, \qquad V_\mu \to \mathrm{e}^{-9\alpha} V_\mu \,. \tag{32}$$

The tensor gauge field C_{MNP} decomposes into a 3- and a 2-rank tensor in 10 dimensions, which transform according to

$$C_{\mu\nu\rho} \to \mathrm{e}^{-3\alpha} C_{\mu\nu\rho} \,, \qquad C_{11\mu\nu} \to \mathrm{e}^{6\alpha} C_{11\mu\nu} \,. \tag{33}$$

The presence of the above scale symmetry is confirmed by the resulting 10-dimensional Lagrangian for the massless (i.e., x^{11}-independent) modes.

[11]Note that this applies to all Fourier modes, as they depend on x^{11}/L, which is insensitive to the scale transformation.

Its purely bosonic terms read

$$\mathcal{L}_{10} = \frac{1}{\kappa_{10}^2}\Big[-\tfrac{1}{2} e\, \mathrm{e}^{2\phi/3} R(e,\omega) - \tfrac{1}{8} e\, \mathrm{e}^{2\phi} (\partial_\mu V_\nu - \partial_\nu V_\mu)^2 \\ -\tfrac{1}{48} e\, \mathrm{e}^{2\phi/3} (F_{\mu\nu\rho\sigma})^2 - \tfrac{3}{4} e\, \mathrm{e}^{-2\phi/3} (H_{\mu\nu\rho})^2 \\ +\tfrac{1}{1152}\sqrt{2}\, \varepsilon^{\mu_1 - \mu_{10}}\, C_{11\mu_1\mu_2}\, F_{\mu_3\mu_4\mu_5\mu_6}\, F_{\mu_7\mu_8\mu_9\mu_{10}} \Big] , \tag{34}$$

where $H_{\mu\nu\rho} = 6\partial_{[\mu} C_{\nu\rho]11}$ is the field strength tensor belonging to the 2-rank tensor gauge field.

The above reduction allows us to discuss a number of characteristic features. First of all, the metric tensor produces an extra vector and a scalar, when dimensionally reducing the dimension by one unit. The scalar is invariant under certain shift symmetries, as shown above, which act multiplicatively on the other fields. Secondly, tensor fields generate tensor fields of a rank that is one unit lower. When this lower-rank field is a scalar field (or equivalent to it by a duality transformation), it will be subject to shifts by a constant parameter, which is simply associated with a gauge transformation that is linearly proportional to the extra higher-dimensional coordinate. Because of this, these shifts must leave the Lagrangian invariant. This last feature is still missing in the above discussion, as the rank-3 tensor decomposes into a rank-3 and a rank-2 tensor. But when descending to lower dimensions than 10, additional scalars will emerge and this phenomenon will be present.

When consecutively reducing the dimension, this pattern repeats itself.[12] In this way, for each scalar field that is generated by the dimensional reduction, there is also an extra symmetry. The dimension of the isometry group is thus (at least) equal to the dimension of the manifold. Furthermore it is easy to see that the symmetries indicated above act *transitively* on the manifold, so that this manifold is homogeneous. The corresponding algebra of these isometries is solvable and the rank of the algebra is equal to $\mathbf{r} = 11 - D$, where D is the spacetime dimension to which we reduce. This is because its Cartan subalgebra is precisely associated with the scale symmetries connected with the scalars that originate from the metric.

[12] For instance, in (34) one considers the scale transformations (for the bosonic fields),

$$\begin{array}{llll} e_\mu{}^a \to \mathrm{e}^{-\beta}\, e_\mu{}^a\,, & \qquad C_{\mu\nu 11} \to \mathrm{e}^{-2\beta}\, C_{\mu\nu 11}\,, \\ V_\mu \to \mathrm{e}^{-\beta}\, V_\mu\,, & \qquad C_{\mu\nu\rho} \to \mathrm{e}^{-3\beta}\, C_{\mu\nu\rho}\,, \end{array} \tag{35}$$

while ϕ remains invariant. These transformations change the Lagrangian by an overall factor $\exp[-8\beta]$, which can be absorbed into $1/\kappa_{10}^2$. When compactifying one more dimension to a circle, these scale transformations yield another isometry in 9 spacetime dimensions, that commutes with the scale transformations (32,33).

D	G	H	dim [G] − dim [H]
11	1	1	$0-0=0$
10A	$\mathrm{SO}(1,1)/Z_2$	1	$1-0=1$
10B	SL(2)	SO(2)	$3-1=2$
9	GL(2)	SO(2)	$4-1=3$
8	$\mathrm{E}_{3(+3)} \sim \mathrm{SL}(3)\times\mathrm{SL}(2)$	U(2)	$11-4=7$
7	$\mathrm{E}_{4(+4)} \sim \mathrm{SL}(5)$	USp(4)	$24-10=14$
6	$\mathrm{E}_{5(+5)} \sim \mathrm{SO}(5,5)$	USp(4)×USp(4)	$45-20=25$
5	$\mathrm{E}_{6(+6)}$	USp(8)	$78-36=42$
4	$\mathrm{E}_{7(+7)}$	SU(8)	$133-63=70$
3	$\mathrm{E}_{8(+8)}$	SO(16)	$248-120=128$

TABLE 9. Homogeneous scalar manifolds G/H for maximal supergravities in various dimensions. The type-IIB theory cannot be obtained from reduction of 11-dimensional supergravity and is included for completeness. The difference of the dimensions of G and H equals the number of scalar fields, listed in Table 6.

The above isometries not only leave the scalar manifold invariant but the whole supergravity Lagrangian. In $D=4$, or 8, these symmetries do not leave the Lagrangian, but only the field equations invariant. The reason for this is that the isometries act by means of a duality transformation on the field strengths associated with the vector or 3-rank gauge field, respectively. They cannot be implemented directly on the gauge fields themselves. The presence of these duality invariances is a well-known feature of supergravity theories, which was first observed many years ago [29, 30, 31, 32, 27]. However, it is easy to see that the scalar manifold (as well as the rest of the theory) must possess additional symmetries, simply because the isometries corresponding to the solvable algebra do not yet contain the automorphism group H_{R} of the underlying supermultiplet. We expect that H_{R} is realized as a symmetry, because the maximal supergravity theories have no additional parameters, so there is nothing that can break this symmetry. So we expect an homogeneous space with an isometry group whose algebra is the sum of the solvable algebra and the one corresponding to (part of) H_{R}. A counting argument (of the type first used in [30]) then usually reveals what the structure of the homogeneous space is. In Table 9 we list the isometry and isotropy groups of these scalar manifolds for maximal supergravity in dimensions $3 \leq D \leq 11$. Earlier version of such tables can, for instance, be found in [4, 28]. A more recent discussion of these isometry groups can be found in, for example, [34, 33]. We return to this discussion and related issues in section 3.

We should add that it is generally possible to realize the group $H_{\rm R}$ as a *local* symmetry of the Lagrangian. The corresponding connections are then composite connections, governed by the Cartan-Maurer equations. In such a formulation most fields (in particular, the fermions) do not transform under the duality group, but only under the local $H_{\rm R}$ group. The scalars transform linearly under both the rigid duality group as well as under the local $H_{\rm R}$ group. After fixing a gauge, the isometries become nonlinearly realized. The fields which initially transform only under the local $H_{\rm R}$ group, will now transform under the duality group through field-dependent $H_{\rm R}$ transformations. This phenomenon is also realized for the central charges, which transform under the group $H_{\rm R}$ as we have shown in Table 8.

2.5.1. *Frames and scales*

The Lagrangian (34) does not contain the standard Einstein-Hilbert term for gravity, while a standard kinetic term for the scalar field ϕ is lacking. This does not pose a serious problem. In this form the gravitational field and the scalar field are entangled and one has to deal with the scalar-graviton system as a whole. To separate the scalar and gravitational degrees of freedom, one may apply a so-called Weyl rescaling of the metric $g_{\mu\nu}$ by an appropriate function of ϕ. In the case that we include the massive modes, this rescaling may depend on the extra coordinate x^{11}. In the context of Kaluza-Klein theory this factor is therefore known as the 'warp factor'. For these lectures two different Weyl rescalings are particularly relevant, which lead to the so-called Einstein and the string frame, respectively. They are defined by

$$e_\mu^a = {\rm e}^{-\phi/12}\,[e_\mu^a]^{\rm Einstein}\,, \qquad e_\mu^a = {\rm e}^{-\phi/3}\,[e_\mu^a]^{\rm string}\,. \tag{36}$$

After applying the first rescaling (36) to the Lagrangian (34) one obtains the Lagrangian in the Einstein frame. This frame is characterized by a standard Einstein-Hilbert term and by a graviton field that is invariant under the scale transformations (32,33). The corresponding Lagrangian reads[13]

$$\begin{aligned}\mathcal{L}_{10}^{\rm Einstein} \;=\; \frac{1}{\kappa_{10}^2}\Big[& e\Big[-\tfrac{1}{2}R(e,\omega)-\tfrac{1}{4}(\partial_\mu\phi)^2\Big]-\tfrac{1}{8}e\,{\rm e}^{3\phi/2}(\partial_\mu V_\nu-\partial_\nu V_\mu)^2 \\ & -\tfrac{3}{4}e\,{\rm e}^{-\phi}(H_{\mu\nu\rho})^2-\tfrac{1}{48}e\,{\rm e}^{\phi/2}(F_{\mu\nu\rho\sigma})^2 \\ & +\tfrac{1}{1152}\sqrt{2}\,\varepsilon^{\mu_1-\mu_{10}}\,C_{11\mu_1\mu_2}\,F_{\mu_3\mu_4\mu_5\mu_6}\,F_{\mu_7\mu_8\mu_9\mu_{10}}\Big]\,. \end{aligned} \tag{37}$$

[13] Note that under a local scale transformation $e_\mu^a \to {\rm e}^{\Lambda} e_\mu^a$, the Ricci scalar in D dimensions changes according to

$$R \to {\rm e}^{-2\Lambda}\Big[R + 2(D-1)D^\mu\partial_\mu\Lambda + (D-1)(D-2)g^{\mu\nu}\,\partial_\mu\Lambda\,\partial_\nu\Lambda\Big]\,.$$

Supergravity theories are usually formulated in this frame, where the isometries of the scalar fields do not act on the graviton.

The second rescaling (36) leads to the Lagrangian in the string frame,

$$\begin{aligned}\mathcal{L}_{10}^{\text{string}} \;=\; \frac{1}{\kappa_{10}^2}\Big[& e\,\mathrm{e}^{-2\phi}\Big[-\tfrac{1}{2}R(e,\omega)+2(\partial_\mu\phi)^2-\tfrac{3}{4}(H_{\mu\nu\rho})^2\Big] \\ & -\tfrac{1}{8}e\,(\partial_\mu V_\nu-\partial_\nu V_\mu)^2-\tfrac{1}{48}e\,(F_{\mu\nu\rho\sigma})^2 \\ & +\tfrac{1}{1152}\sqrt{2}\,\varepsilon^{\mu_1-\mu_{10}}\,C_{11\mu_1\mu_2}\,F_{\mu_3\mu_4\mu_5\mu_6}\,F_{\mu_7\mu_8\mu_9\mu_{10}}\Big]\,. \qquad (38)\end{aligned}$$

This frame is characterized by the fact that R and $(H_{\mu\nu\rho})^2$ have the same coupling to the scalar ϕ, or, equivalently, that $g_{\mu\nu}$ and $C_{11\mu\nu}$ transform with equal weights under the scale transformations (32,33). In string theory ϕ coincides with the dilaton field that couples to the topology of the worldsheet and whose vacuum-expectation value defines the string coupling constant according to $g_\mathrm{s}=\exp(\langle\phi\rangle)$. We shall return to this in section 3, but here we already indicate the significance of the dilaton factors in the Lagrangian above. The metric $g_{\mu\nu}$, the antisymmetric tensor $C_{\mu\nu 11}$ and the dilaton ϕ always arise in the Neveu-Schwarz sector and couple universally to $\mathrm{e}^{-2\phi}$. On the other hand the vector V_μ and the 3-form $C_{\mu\nu\rho}$ describe Ramond-Ramond (R-R) states and the specific form of their vertex operators forbids any tree-level coupling to the dilaton [33, 12]. In particular the Kaluza-Klein gauge field V_μ corresponds in the string context to the R-R gauge field of type-II string theory. The infinite tower of massive Kaluza-Klein states carry a charge quantized in units of e_{KK}, defined in (28). In the context of 10-dimensional supergravity, states with a R-R charge are solitonic. In string theory, the R-R charges are carried by the D-brane states.

As we already discussed in the previous section, Newton's constant is only defined after a choice of the metric in the ground state is made. Expanding the metric in the Einstein frame around $\lambda\,\eta_{\mu\nu}$ one obtains from (24,37)

$$(\kappa_{10}^2)^{\text{physical}}=\kappa_{10}^2\,\lambda^{-4}\,, \qquad (39)$$

while expanding the metric in the string frame around $\lambda\,\eta_{\mu\nu}$ leads to

$$(\kappa_{10}^2)^{\text{physical}}=\kappa_{10}^2\,\lambda^{-4}\,\mathrm{e}^{2\langle\phi\rangle}\,. \qquad (40)$$

Note that one cannot expand both metrics simultaneously around $\lambda\,\eta_{\mu\nu}$.

For later purposes let us note that the above discussion can be generalized to arbitrary spacetime dimensions. The Einstein frame in any dimension is defined by a gravitational action that is just the Einstein-Hilbert action, whereas in the string frame the Ricci scalar is multiplied by a dilaton term $\exp(-2\phi)$, as in (37) and (38), respectively. The Weyl rescaling which connects the two frames is given by,

$$[e_\mu^a]^{\rm string} = {\rm e}^{2\phi/(D-2)}\,[e_\mu^a]^{\rm Einstein}\,. \tag{41}$$

In arbitrary dimensions (39) and (40) read

$$\begin{aligned} \text{Einstein frame}: &\quad (\kappa_D^2)^{\rm physical} = \kappa_D^2\,\lambda^{(2-D)/2}\,,\\ \text{string frame}: &\quad (\kappa_D^2)^{\rm physical} = \kappa_D^2\,\lambda^{(2-D)/2}\,{\rm e}^{2\langle\phi\rangle}\,. \end{aligned} \tag{42}$$

This frame dependence does not only apply to κ_D^2 but to any dimensionful quantity. For example, a mass, when measured in the same flat metric but specified in the two frames, is related by

$$M^{\rm string} = {\rm e}^{-2\langle\phi\rangle/(D-2)}\,M^{\rm Einstein}\,. \tag{43}$$

Of course this is consistent with the relation (42). The physical masses in the above relation depend again on the value of λ. In the remainder of this subsection we choose $\lambda = 1$ for convenience.

Let us now return to 11-dimensional supergravity with the 11-th coordinate compactified to a circle so that $0 \leq x^{11} \leq L$. As we stressed already, L itself has no intrinsic meaning and it is better to consider the geodesic radius of the 11-th dimension, which reads

$$R_{11} = \frac{L}{2\pi}\,{\rm e}^{2\langle\phi\rangle/3}\,. \tag{44}$$

This result applies to the frame specified by the 11-dimensional theory[14]. In the string frame, the above result reads

$$(R_{11})^{\rm string} = \frac{L}{2\pi}\,{\rm e}^{\langle\phi\rangle}\,. \tag{45}$$

It shows that a small 11-th dimension corresponds to small values of $\exp\langle\phi\rangle$ which in turn corresponds to a weakly coupled string theory. We come back to this crucial observation in section 3. Observe that L is fixed in terms of κ_{10} and κ_{11} (cf. (29)).

From the 11-dimensional expressions,

$$E_a^M\partial_M = e_a^{\,\mu}(\partial_\mu - V_\mu\,\partial_{11})\,, \qquad E_{11}^M\partial_M = {\rm e}^{-2\phi/3}\,\partial_{11}\,, \tag{46}$$

where a and μ refer to the 10-dimensional Lorentz and world indices, we infer that, in the frame specified by the 11-dimensional theory, the Kaluza-Klein masses are multiples of

$$M^{\rm KK} = \frac{1}{R_{11}}\,. \tag{47}$$

[14]This is the frame specified by the metric given in (26), which leads to the Lagrangian (34).

Hence Kaluza-Klein states have a mass and Kaluza-Klein charge (cf. (28)) related by

$$M^{\rm KK} = |e_{\rm KK}|\,{\rm e}^{-2\langle\phi\rangle/3}\,. \tag{48}$$

In the string frame, this result becomes

$$(M^{\rm KK})^{\rm string} = |e_{\rm KK}|\,{\rm e}^{-\langle\phi\rangle}\,. \tag{49}$$

Massive Kaluza-Klein states are always BPS states, meaning that they are contained in supermultiplets that are 'shorter' than the generic massive supermultiplets because of nontrivial central charges. The central charge here is just the 11-th component of the momentum, which is proportional to the Kaluza-Klein charge.

The surprising insight that emerged in recent years, is that the Kaluza-Klein features of 11-dimensional supergravity have a precise counterpart in string theory [34, 35, 33]. There one has nonperturbative (in the string coupling constant) states which carry R-R charges. We return to this phenomenon in section 3.

2.6. NONMAXIMAL SUPERSYMMETRY

In previous subsections we discussed a large number of supermultiplets. Furthermore, in Fig. 1 we presented an overview of all supergravity theories in spacetime dimensions $4 \leq D \leq 11$ with $Q = 32, 16, 8$ or 4 supercharges. In this section we summarize a number of results on nonmaximal supersymmetric theories with $Q = 16$ supercharges, which are now restricted to dimensions $D \leq 10$.

For $Q = 16$ the automorphism group $H_{\rm R}$ is smaller. Table 10 lists this group in various dimensions and shows the field representations for the vector multiplet in dimension $3 \leq D \leq 10$. This multiplet comprises $8 + 8$ physical degrees of freedom.

We also consider the $Q = 16$ supergravity theories. The Lagrangian can be obtained by truncation of (34). However, unlike in the case of maximal supergravity, we now have the option of introducing additional matter fields. For $Q = 16$ the matter will be in the form of vector supermultiplets, possibly associated with some nonabelian gauge group. Table 11 summarizes $Q = 16$ supergravity for dimensions $3 \leq D \leq 10$. In $D = 10$ dimensions the bosonic terms of the supergravity Lagrangian take the form [36],

$$\begin{aligned} \mathcal{L}_{10} \;=\; \frac{1}{\kappa_{10}^2}\Big[&-\tfrac{1}{2}e\,{\rm e}^{2\phi/3}R(e,\omega) \\ &-\tfrac{3}{4}e\,{\rm e}^{-2\phi/3}(H_{\mu\nu\rho})^2 - \tfrac{1}{4}e(\partial_\mu A_\nu - \partial_\nu A_\mu)^2\Big]\,, \end{aligned} \tag{50}$$

D	$H_{\rm R}$	A_μ	ϕ	χ
10	1	1	0	1
9	1	1	1	1
8	U(1)	1	$1+\bar 1$	$1+\bar 1$
7	USp(2)	1	3	2
6	USp(2)×USp(2)	1	(2, 2)	(2, 1) + (1, 2)
5	USp(4)	1	5	4
4	U(4)	1	6^*	$4+\bar 4$
3	SO(8)		8	8

TABLE 10. Field content for maximal super-Maxwell theories in various dimensions. All supermultiplets contain a gauge field A_μ, scalars ϕ and spinors χ. In $D=3$ dimensions the vector field is dual to a scalar. The 6^* representation of SU(4) is a selfdual 2-rank tensor.

where, for convenience, we have included a single vector gauge field, representing an abelian vector supermultiplet. A feature that deserves to be mentioned, is that the field strength $H_{\mu\nu\rho}$ associated with the 2-rank gauge field acquires a Chern-Simons term $A_{[\mu}\partial_\nu A_{\rho]}$. Chern-Simons terms play an important role in the anomaly cancellations of this theory. Note also that the kinetic term for the Kaluza-Klein vector field in (34), depends on ϕ, unlike the kinetic term for the matter vector field in the Lagrangian above. This reflects itself in the extension of the symmetry transformations noted in (32, 33),

$$e_\mu^a \to {\rm e}^{-\alpha} e_\mu^a\,, \quad \phi \to \phi + 12\alpha\,, \quad C_{11\mu\nu} \to {\rm e}^{6\alpha} C_{11\mu\nu}\,, \quad A_\mu \to {\rm e}^{3\alpha} A_\mu\,. \tag{51}$$

where A_μ transforms differently from the Kaluza-Klein vector field V_μ.

In this case there are three different Weyl rescalings that are relevant, namely

$$\begin{aligned} e_\mu^a &= {\rm e}^{-\phi/12}\,[e_\mu^a]^{\rm Einstein}\,, \qquad e_\mu^a = {\rm e}^{-\phi/3}\,[e_\mu^a]^{\rm string}\,, \\ e_\mu^a &= {\rm e}^{\phi/6}\,[e_\mu^a]^{\rm string'}\,. \end{aligned} \tag{52}$$

It is straightforward to obtain the corresponding Lagrangians. In the Einstein frame, the graviton is again invariant under the isometries of the scalar field. The bosonic terms read

$$\begin{aligned} \mathcal{L}_{10}^{\rm Einstein} &= \frac{1}{\kappa_{10}^2}\Big[-\tfrac12 e\, R(e,\omega) - \tfrac14 e(\partial_\mu\phi)^2 \\ &\qquad -\tfrac34 e\, {\rm e}^{-\phi}(H_{\mu\nu\rho})^2 - \tfrac14 e\, {\rm e}^{-\phi/2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2\Big]\,. \end{aligned} \tag{53}$$

D	$H_{\rm R}$	graviton	$p=-1$	$p=0$	$p=1$
10	1	1	1		1
9	1	1	1	1	1
8	U(1)	1	1	$1+\bar{1}$	1
7	USp(2)	1	1	3	1
6A	USp(2)×USp(2)	1	1	(2,2)	(1,1)
6B	USp(4)	1			5^*
5	USp(4)	1	1	5	
4	U(4)	1	$1+\bar{1}$	[6]	
3	SO(8)	1	$8k$		

TABLE 11. Bosonic fields of nonmaximal supergravity with $Q=16$. In 6 dimensions type-A and type-B correspond to (1,1) and (2,0) supergravity. Note that, with the exception of the 6B and the 4-dimensional theory, all these theories contain precisely one scalar field. In $D=4$ dimensions, the SU(4) transformations cannot be implemented on the vector potentials, but act on the (abelian) field strengths by duality transformations. In $D=3$ dimensions supergravity is a topological theory and can be coupled to scalars and spinors. The scalars parameterize the coset space $\mathrm{SO}(8,k)/\mathrm{SO}(8)\times\mathrm{SO}(k)$, where k is an arbitrary integer.

The second Weyl rescaling leads to the following Lagrangian,

$$\begin{aligned}\mathcal{L}_{10}^{\rm string} &= \frac{1}{\kappa_{10}^2}\mathrm{e}^{-2\phi}\Big[-\tfrac{1}{2}e\,R(e,\omega)+2e(\partial_\mu\phi)^2\\ &\qquad -\tfrac{3}{4}e\,(H_{\mu\nu\rho})^2-\tfrac{1}{4}e(\partial_\mu A_\nu-\partial_\nu A_\mu)^2\Big]\,, \qquad (54)\end{aligned}$$

which shows a uniform coupling with the dilaton. This is the low-energy effective Lagrangian relevant to the heterotic string. Eventually the matter gauge field has to be part of an nonabelian gauge theory based on the group SO(32) or $\mathrm{E}_8\times\mathrm{E}_8$, in order to be anomaly-free.

Finally, the third Weyl rescaling yields

$$\begin{aligned}\mathcal{L}_{10}^{\rm string\prime} &= \frac{1}{\kappa_{10}^2}\Big[e\,\mathrm{e}^{2\phi}\Big[-\tfrac{1}{2}R(e,\omega)+2(\partial_\mu\phi)^2\Big]\\ &\qquad -\tfrac{3}{4}e\,(H_{\mu\nu\rho})^2-\tfrac{1}{4}e\,\mathrm{e}^{\phi}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2\Big]\,. \qquad (55)\end{aligned}$$

Here the dilaton seems to appear with the wrong sign. As it turns out, this is the low-energy effective action of the type-I string, where the type-I dilaton must be associated with $-\phi$. This will be further elucidated in section 3.

3. String theories in various dimensions

3.1. PERTURBATIVE STRING THEORIES IN $D = 10$

The perturbative expansion. In string theory the fundamental objects are one-dimensional strings which, as they move in time, sweep out a 2-dimensional worldsheet Σ [37]. Strings can be open or closed and their worldsheet is embedded in some higher-dimensional target space which is identified with a Minkowskian spacetime. States in the target space appear as eigenmodes of the string and their scattering amplitudes are generalized by appropriate scattering amplitudes of strings. These scattering amplitudes are built from a fundamental vertex, which for closed strings is depicted in Fig. 2. It represents the splitting of a string or the joining of two strings and

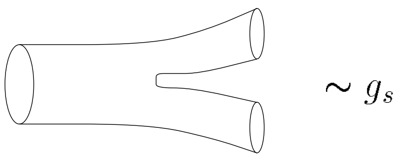

Figure 2. The fundamental closed string vertex.

the strength of this interaction is governed by a dimensionless string coupling constant $g_{\rm s}$. Out of the fundamental vertex one composes all possible closed string scattering amplitudes $\mathcal{A}$, for example the four-point amplitude shown in Fig. 3. The expansion in the topology of the Riemann surface

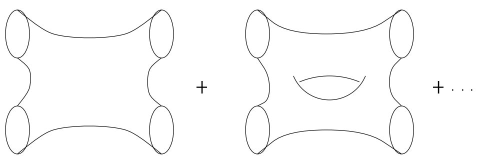

Figure 3. The perturbative expansion of string scattering amplitudes. The order of g_s is governed by the number of holes in the world sheet.

(i.e. the number of holes in the surface) coincides with a power series expansion in the string coupling constant formally written as

$$\mathcal{A} = \sum_{n=0}^{\infty} g_{\rm s}^{-\chi} \mathcal{A}^{(n)} , \tag{56}$$

where $\mathcal{A}^{(n)}$ is the scattering amplitude on a Riemann surface of genus n and $\chi(\Sigma)$ is the Euler characteristic of the Riemann surface

$$\chi(\Sigma) = \frac{1}{4\pi} \int_\Sigma R^{(2)} = 2 - 2n - b \ . \tag{57}$$

$R^{(2)}$ is the curvature on Σ and b the number of boundaries of the Riemann surface (for the four-point amplitude of Fig. 3 one has $b = 4$).[15]

In all string theories there is a massless scalar field ϕ called the dilaton which couples to $\sqrt{h}R^{(2)}$ and therefore its vacuum-expectation value determines the size of the string coupling; one finds [38, 37]

$$g_\mathrm{s} \;=\; e^{\langle\phi\rangle} \ . \tag{58}$$

g_s is a free parameter since ϕ is a flat direction (a modulus) of the effective potential. Thus, string perturbation theory is defined in that region of the parameter space (which is also called the moduli space) where $g_\mathrm{s} < 1$ and the tree-level amplitude (genus-0) is the dominant contribution with higher-loop amplitudes suppressed by higher powers of g_s. Until three years ago this was the only regime accessible in string theory.

The spacetime spectrum of the string. The propagation of a free string ($g_\mathrm{s} = 0$) is governed by the 2-dimensional action

$$S_\mathrm{free} = -\frac{1}{4\pi\alpha'} \int_\Sigma \partial_i X^\mu(\sigma,\tau)\, \partial^i X^\nu(\sigma,\tau)\ \eta_{\mu\nu} \ , \tag{59}$$

where ∂_i denotes $\partial/\partial\sigma$ and $\partial/\partial\tau$. Here σ parameterizes the spatial direction on Σ while τ denotes the 2-dimensional time coordinate. The coordinates of the D-dimensional target spacetime in which the string moves, are represented by X^μ, with $\mu = 0, \ldots, D-1$; in terms of the 2-dimensional field theory they appear as D scalar fields. For S to be dimensionless α' has dimension $[\mathrm{length}]^2 \sim [\mathrm{mass}]^{-2}$; it is the fundamental mass scale of string theory which is also denoted by M_s with the identification $\alpha' = M_s^{-2}$. The mass of all perturbative string states is a multiple of M_s. Just as the coupling constant κ in the supergravity Lagrangians in section 2, this scale has no intrinsic meaning and must be fixed by some independent criterion. Demanding that string theory contains Einstein gravity as its low-energy limit relates the characteristic scales of the two theories. By comparing for example physical graviton-graviton scattering amplitudes in both theories

[15] For open strings different diagrams contribute at the same order of the string loop expansion. See [37] for further details.

one finds in the following expression for Newton's constant in D dimensions [39],[16]

$$(\kappa_D^2)^{\rm physical} = \alpha'^{(D-2)/2}\, {\rm e}^{2\langle\phi\rangle}\,, \tag{60}$$

where we dropped convention-dependent numerical proportionality factors.

The equations of motion of the action (59) are given by

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0\ , \tag{61}$$

with the solutions

$$X^\mu = X_L^\mu(\sigma+\tau) + X_R^\mu(\sigma-\tau)\ . \tag{62}$$

A closed string satisfies the boundary condition $X^\mu(\sigma) = X^\mu(\sigma+2\pi)$, which does not mix X_L^μ and X_R^μ and leaves them as independent solutions. This splitting into left (L) and right (R) moving fields has the consequence that upon quantizing the 2-dimensional field theory, also the Hilbert space splits into a direct product $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$ where $\mathcal{H}_L(\mathcal{H}_R)$ contains states built from oscillator modes of $X_L(X_R)$. These states also carry a representation of the D-dimensional target space Lorentz group and thus can be identified as perturbative states in spacetime of a given spin and mass.[17]

In open string theory one has a choice to impose at the end of the open string either Neumann (N) boundary conditions, $\partial_\sigma X^\mu = 0$, or Dirichlet (D) boundary conditions, $X^\mu =$ constant. The boundary conditions mix left- and right-movers and the product structure of the closed string is not maintained. As a consequence a perturbative spectrum of states is built from a single Hilbert space. Neumann boundary conditions leave the D-dimensional Lorentz invariance unaffected.

Dirichlet boundary conditions, on the other hand, lead to very different types of objects and a very different set of states (D-branes) in spacetime [40]. In this case the end of an open string is constrained to only move in a fixed spatial hyper-plane. This plane must be regarded as a dynamical object with degrees of freedom induced by the attached open string. A careful analysis shows that the corresponding states in spacetime are not part of the perturbative spectrum but rather correspond to nonperturbative solitonic type excitations[18]. It is precisely these states which dramatically

[16] The relation (60) holds in arbitrary dimensions with ${\rm e}^{\langle\phi\rangle}$ being the string coupling constant for a string moving in D spacetime dimensions. Later in these lectures we consider compactifications of string theory and then there is a volume-dependent relation between the string couplings defined in different dimensions. This relation is discussed in appendix B.

[17] These are perturbative states since the quantization procedure is a perturbation theory around the free string theory with $g_s = 0$.

[18] They are nonperturbative in that their mass (or rather their tension for higher-dimensional D-branes) goes to infinity in the weak coupling limit $g_s \to 0$.

affect the properties of string theory in its nonperturbative regime. These aspects will be subject of section 3.3.

So far we discussed the free string governed by the action (59); its interactions are incorporated by promoting $S_{\rm free}$ to a nonlinear 2-dimensional σ-model. The amplitude $\mathcal{A}$ can be interpreted as a unitary scattering amplitude in the target space whenever this 2-dimensional field theory is conformally invariant. The action is found to be

$$\begin{aligned} S \;=\; & -\frac{1}{4\pi\alpha'}\int_\Sigma \partial_i X^\mu(\sigma,\tau)\,\partial^i X^\nu(\sigma,\tau)\;g_{\mu\nu}(X(\sigma,\tau)) \\ & -\frac{1}{4\pi\alpha'}\int_\Sigma \varepsilon^{ij}\,\partial_i X^\mu(\sigma,\tau)\,\partial_j X^\nu(\sigma,\tau)\;b_{\mu\nu}(X(\sigma,\tau)) \\ & +\frac{1}{4\pi}\int_\Sigma R^{(2)}\,\phi(X(\sigma,\tau)) + \cdots\;, \end{aligned} \tag{63}$$

where $g_{\mu\nu}(X)$ is the metric[19] of the target space, $b_{\mu\nu}(X)$ is the antisymmetric target-space tensor and $R^{(2)}$ is the curvature scalar of the 2-dimensional worldsheet Σ. The target-space field $\phi(X)$ represents a scalar coupling and corresponds to the dilaton, since the coefficient of its constant vacuum-expectation value $\langle\phi\rangle$ is the Euler number $\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma R^{(2)}$. The ellipses denote further terms depending on the type of string theory and the number of spacetime dimensions.

The spacetime properties of a string theory significantly change once one introduces supersymmetry on the worldsheet. In two dimensions the irreducible supercharges are Majorana-Weyl spinors (see Table 1). In addition there are independent left- and right-moving supercharges Q_L, Q_R, so that in general one can have p supercharges Q_L and q supercharges Q_R; this is also termed (p,q) supersymmetry. A supersymmetric version of the action (63) requires the presence of Majorana-Weyl worldsheet fermions χ^μ with appropriate couplings; for example a scalar supermultiplet of $(1,0)$ supersymmetry contains the fields $(X_L(\sigma+\tau), \chi_L(\sigma+\tau))$. Depending on the amount of worldsheet supersymmetry one defines the different *closed* string theories: the bosonic string, the superstring and the heterotic string (see Table 12).

For *open* string theories the left- and right-moving worldsheet supercharges are not independent. One can either have a bosonic open string (with no worldsheet supersymmetry) or an open superstring with one supercharge which is a linear combination of Q_L and Q_R. The latter string theory is called type-I. It contains (unoriented) open and closed strings with SO(32) Chan-Paton factors coupling to the ends of the open string.

[19] As we mentioned already in subsection 2.5, the metric $g_{\mu\nu}(X)$ in (63) is the metric in the string frame.

closed string theories	worldsheet supersymmetry	$D_{\max}$
bosonic string	$(0,0)$	26
superstring	$(1,1)$	10
heterotic string	$(0,1)$	10

TABLE 12. The closed-string theories, their worldsheet supersymmetry and the maximal possible spacetime dimension.

The bosonic string (open or closed) is tachyonic and cannot accommodate spacetime fermions; for these reasons we omit it from our subsequent discussion. The superstring, the heterotic string and the type-I string can all be tachyon-free and do have spacetime fermions in the massless spectrum. In addition, in most cases they are also spacetime supersymmetric and contain (at least) a massless gravitino. There are also tachyon-free non-supersymmetric string theories [41] but they have a dilaton tadpole at one-loop and thus do not seem to correspond to stable vacuum configurations.[20] For this reason we solely focus on supersymmetric string theories henceforth.

The worldsheet fermions χ^μ can have two distinct type of boundary conditions when transported around the closed string,

$$\chi^\mu(\sigma) = \begin{cases} +\chi^\mu(\sigma+2\pi) & \text{Ramond (R)}\,, \\ -\chi^\mu(\sigma+2\pi) & \text{Neveu-Schwarz (NS)}\,. \end{cases} \tag{64}$$

Consequently the states of the closed string Hilbert space can arise in four different sectors of fermion boundary conditions:

$$\left.\begin{array}{lcl} \mathrm{NS}_L & \otimes & \mathrm{NS}_R \\ \mathrm{R}_L & \otimes & \mathrm{R}_R \end{array}\right\} \quad \text{spacetime bosons}$$

$$\left.\begin{array}{lcl} \mathrm{NS}_L & \otimes & \mathrm{R}_R \\ \mathrm{R}_L & \otimes & \mathrm{NS}_R \end{array}\right\} \quad \text{spacetime fermions}\,.$$

The first two sectors contain the spacetime bosons, while the last two sectors generate spacetime fermions. The bosons from the R-R sector are built from bi-spinors and thus the representation theory of the Lorentz group constrains these bosons to always be antisymmetric Lorentz tensors of varying rank. Furthermore, in the effective action and in all scattering processes

[20] For a recent discussion of non-supersymmetric string theories, see [42].

these tensors can only appear via their field strength and thus there are no states in perturbative string theory which carry any charge under the antisymmetric tensors in the R-R sector. However, it turns out that this is an artifact of perturbation theory and states carrying R-R charge do appear in the nonperturbative spectrum; they are precisely the states generated by appropriate D-brane configurations [40].

Conformal invariance on the worldsheet (or equivalently unitarity in spacetime) imposes a restriction on the maximal number of spacetime dimensions and the spacetime spectrum. All supersymmetric string theories necessarily have $D \leq 10$ and they are particularly simple in their maximal possible dimension $D = 10$.[21]

In $D = 10$ there are only five consistent spacetime supersymmetric string theories: type-IIA, type-IIB, heterotic $E_8 \times E_8$ (HE8), heterotic SO(32) (HSO) and the type-I SO(32) string. The first two have $Q = 32$ supercharges and thus there is a unique massless multiplet in each case with a field content given in Table 6. As we already indicated perturbative string theory distinguishes between states arising in the NS-NS sector from states of the R-R sector in that the coupling to the dilaton is different. In the type-IIA theory one finds the graviton $g_{\mu\nu}$, an antisymmetric tensor $b_{\mu\nu}$ and the dilaton ϕ in the NS-NS sector while an abelian vector V_μ and a 3-form $C_{\mu\nu\rho}$ appear in the R-R sector. The corresponding low-energy effective Lagrangian was already given in section 2.5.1. In type-IIB one has exactly the same states in the NS-NS sector, but in the R-R sector one has a 2-form $b'_{\mu\nu}$, an additional scalar ϕ' and a 4-form $c^*_{\mu\nu\rho\sigma}$ whose field strength is selfdual. Its field equations can be found in [8].

The other three string theories all have $Q = 16$ supercharges. In this case, the supersymmetric representation theory alone does not completely determine the spectrum. The gravitational multiplet is unique (see Table 11), but the gauge group representation of the vector multiplets (see Table 10) is only fixed if also anomaly cancellation is imposed. The low-energy effective Lagrangian for the two heterotic theories is displayed in (53) and (54), with the abelian vector appropriately promoted to vector fields of $E_8 \times E_8$ or SO(32), respectively. The type-I string has the same supersymmetry but $b_{\mu\nu}$ arises in the R-R sector and thus has different (perturbative) couplings to the dilaton. The corresponding low-energy effective Lagrangian is given by (55) with ϕ replaced by $-\phi$. In Table 13 we summarize the bosonic massless spectra for the five string theories, which is in

[21] For closed strings an additional constraint arises from the requirement of modular invariance of one-loop amplitudes which results in an anomaly-free spectrum of the corresponding low-energy effective theory [43]. For open strings anomaly cancellation is a consequence of the the absence of tadpole diagrams [37].

type	Q	bosonic spectrum	
IIA	32	NS-NS	$g_{\mu\nu}$, $b_{\mu\nu}$, ϕ
		R-R	V_μ, $C_{\mu\nu\rho}$
IIB	32	NS-NS	$g_{\mu\nu}$, $b_{\mu\nu}$, ϕ
		R-R	$c^*_{\mu\nu\rho\sigma}$, $b'_{\mu\nu}$, ϕ'
HE8	16	$g_{\mu\nu}$, $b_{\mu\nu}$, ϕ A_μ in adjoint of $E_8 \times E_8$	
HSO	16	$g_{\mu\nu}$, $b_{\mu\nu}$, ϕ A_μ in adjoint of SO(32)	
I	16	NS-NS	$g_{\mu\nu}$, ϕ
		R-R	$b_{\mu\nu}$
		open string	A_μ in adjoint of SO(32)

TABLE 13. Supersymmetric string theories in $D = 10$ and their fields describing the bosonic massless spectrum.

direct correspondence with some of the material collected in the Tables 6, 10 and 11, presented in section 2.

3.2. CALABI–YAU COMPACTIFICATIONS AND PERTURBATIVE DUALITIES

So far we discussed the various string theories in 10 spacetime dimensions. Lower-dimensional theories can be obtained by compactifying the $D = 10$ theories on an internal, 'curled up' compact manifold Y.[22] Unitarity in spacetime requires Y to be a Calabi-Yau manifold [37].[23] Calabi-Yau manifolds are Ricci-flat Kähler manifolds of vanishing first Chern class ($c_1(Y) = 0$) with holonomy group SU(M) where M is the complex dimension of Y. A one dimensional (complex) Calabi-Yau manifold is topologically always a torus T^2 and toroidal compactifications leave all supercharges intact. For $M = 2$ all Calabi-Yau manifolds are topologically equivalent to the 4-dimensional $K3$ surface [44] and as a consequence of the nontrivial SU(2) holonomy half of the supercharges are broken. For $M = 3$ there exist

[22] There are also string vacua which cannot be viewed as a compactification of the 10-dimensional string theories. Their duality properties have been much less investigated and for a lack of space we neglect them in our discussion here and solely focus on string vacua with a geometrical interpretation.

[23] By a slight abuse of terminology we include in our discussion here also the circle T^1, which strictly speaking is not a Calabi-Yau manifold but does give unitary S-matrices.

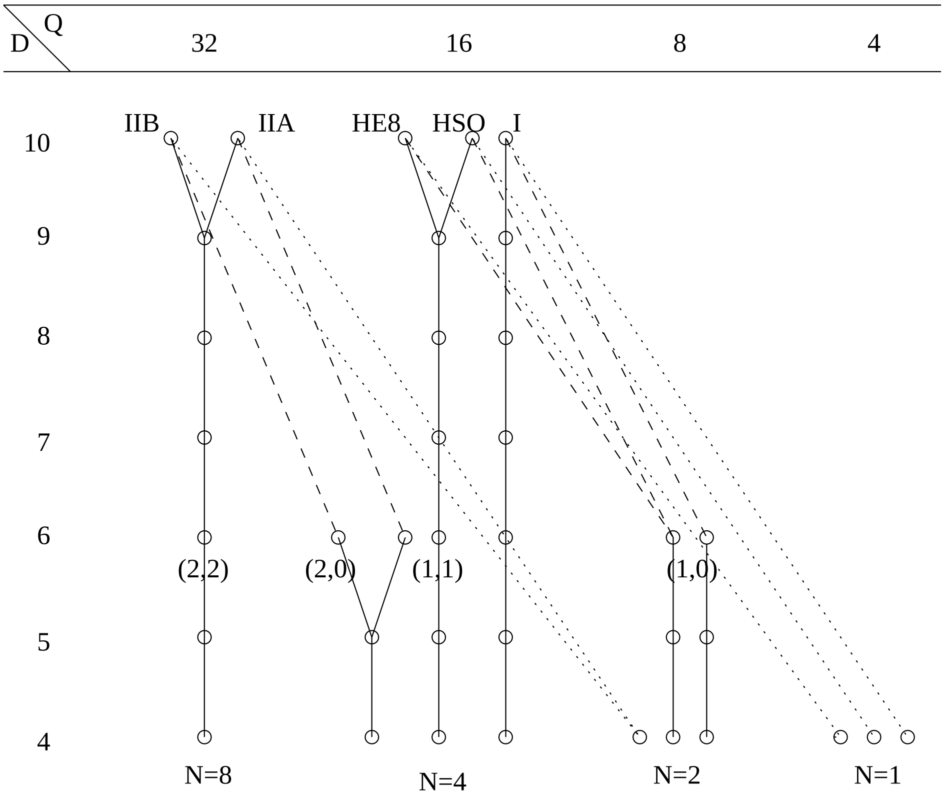

Figure 4. Calabi-Yau compactifications of the 10-dimensional string theories. The solid line (−) denotes toroidal compactification, the dashed line (−−) denotes $K3$ compactifications and the dotted line ($\cdots$) denotes Y_3 compactifications. Whenever two compactifications (two lines) terminate in the same point, the two string theories are related by a perturbative duality. (A line crossing a circle is purely accidental and has no physical significance.)

many topologically distinct Calabi–Yau threefolds Y_3 and they all break $\frac{3}{4}$ of the supercharges [45]. We summarize this situation in the following table:

$M = 1:$	T^2	breaks no supercharges
$M = 2:$	$K3$	breaks 1/2 of the supercharges
$M = 3:$	Y_3	breaks 3/4 of the supercharges

The different theories obtained by compactifying on such Calabi–Yau manifolds are depicted in Fig. 4. Note that different compactifications can have the exact same representation of supersymmetry.

3.2.1. *Toroidal compactifications and T-duality.*
Compactifying the 10-dimensional string theories on an n-dimensional torus T^n leads to string theories with $D = 10 - n$. Supersymmetry remains unbroken and thus one moves down vertically in the D-Q plane of Fig. 4. The massless spectrum can be obtained by dimensional reduction of the appropriate 10-dimensional theories. For simplicity we start by considering closed string theories with one compact dimension which we take to be a circle T^1. In this case there are nine spacetime coordinates X^μ satisfying the boundary conditions[24]

$$X^\mu(\sigma = 2\pi, \tau) = X^\mu(\sigma = 0, \tau) , \tag{65}$$

and one internal coordinate X^{10}, which can wrap m times around the T^1 of radius R,

$$X^{10}(\sigma = 2\pi, \tau) = X^{10}(\sigma = 0, \tau) + 2\pi m R . \tag{66}$$

The massless spectrum of the 9-dimensional theory includes the two abelian Kaluza-Klein gauge bosons $g_{\mu\, 10}$ and $b_{\mu\, 10}$ as well as a massless scalar field $g_{10\, 10}$ which is a flat direction of the effective potential and whose vacuum-expectation value parameterizes the size of the internal T^1 (cf. section 2.5) The appearance of flat directions is a generic feature of string compactifications and such scalar fields are called moduli. For the case at hand the moduli space is one-dimensional and hence there is a one-parameter family of inequivalent string vacua.[25] The boundary condition (66) leads to a quantization of the internal momentum component $p^{10} = k/R$ and a whole tower of massive Kaluza–Klein states labelled by the integer k. In addition there are also massive winding modes labelled by m and altogether one finds a mass spectrum

$$M^2 = \tfrac{1}{2}(P_L^2 + P_R^2) + \cdots , \qquad P_{L,R} = \frac{k}{R} \pm \frac{mR}{\alpha'} . \tag{67}$$

The ellipses stand for R-independent contributions of the oscillator modes. The mass spectrum is invariant under the exchange of $R \leftrightarrow \alpha'/R$ if simultaneously one also exchanges $m \leftrightarrow k$. This Z_2 invariance of the spectrum is the first example of a *T–duality* [46].

These considerations can be generalized to higher-dimensional toroidal compactifications on a torus T^n [47]. The boundary conditions on the n compact coordinates are

$$X^i(\sigma = 2\pi, \tau) = X^i(\sigma = 0, \tau) + 2\pi R^i , \tag{68}$$

[24] Throughout this section the indices μ, ν always denote the uncompactified spacetime directions.

[25] The different solutions of a given string theory are often referred to as the vacuum states of that string theory or simply as the string vacua.

where $R^i = \sum_I m_I e^i_I$ are vectors on an n-dimensional lattice with basis e^i_I.[26] The momenta p^i live on the dual lattice and one finds

$$M^2 = \tfrac{1}{2}(P_L^2 + P_R^2) + \cdots , \qquad P^i_{L,R} = p^i \pm \frac{R^i}{\alpha'} . \tag{69}$$

Modular invariance constrains the lattice to be an even, selfdual Lorentzian lattice. The inequivalent lattices of this type can be labelled by points in the coset space [47]

$$\mathcal{M} = \frac{\mathrm{SO}(n,n)}{\mathrm{SO(n)} \times \mathrm{SO}(n)} \Big/ \Gamma_T \ , \tag{70}$$

where

$$\Gamma_T = \mathrm{SO}\,(n,n,Z) \tag{71}$$

is the T–duality group identifying equivalent lattices or in other words equivalent toroidal compactifications. The mass formula (69) (as well as the entire partition function) shares the invariance under Γ_T.

Type-II compactified on T^n. Toroidal compactifications of type-II string theory all have the maximal number of supercharges $Q = 32$. The associated supergravities have been discussed in section 2.4 and for each of these cases there is a unique gravitational multiplet containing all massless fields.[27] However in perturbative string theory there is a clear distinction between states arising from the NS-NS versus the R-R sector. In the NS-NS sector one finds the Kaluza–Klein gauge bosons $g_{\mu i}$ and $b_{\mu i}$ of a gauge group $G = U(1)^{2n}$, the dilaton ϕ and the moduli g_{ij}, b_{ij}; the latter are precisely the n^2 coordinates of the toroidal moduli space $\mathcal{M}$ given in (70). The dilaton and the scalars of the R-R sector are not part of this $\mathcal{M}$. However, all scalar fields – NS-NS and R-R – reside in the same gravitational multiplet and so the supergravity considerations discussed in section 2.5 suggest that they combine into a larger moduli space with nontrivial mixings. For a long time this state of affairs seemed incompatible with perturbative string theory since the specific form of the vertex operator of the R-R scalars implies

[26] The index i runs over the internal dimensions, i.e. $i = 1, \dots, n = 10 - D$.

[27] We briefly mentioned in section 2.2.2 that below $D = 10$ the type-IIA and type-IIB supergravities are equivalent. A careful analysis in $D = 9$ reveals a flip of the chiralities of the space-time fermions in the limits $R \to 0$ versus $R \to \infty$ and thus the non-chiral type-IIA and the chiral type-IIB theory can be viewed as two distinct limits of one and the same type-II theory in $D = 9$ [10, 11]. One also often refers to this fact by stating that in $D = 9$ type-IIA and type-IIB are T-dual to each other in that type-IIA at a large compactification radius is equivalent to type-IIB at a small compactification radius and vice versa.

that they can have no nontrivial couplings to the NS-NS scalars [40]. However, nonperturbative corrections alter this conclusion and by now it is believed that taking perturbative and nonperturbative contributions together exactly reproduces the geometrical structures suggested by supergravity. We will return to this point in more detail in section 3.3.

Heterotic string compactified on T^n. The heterotic string compactified on T^n has 16 supercharges and there are n additional scalars A_i^a transforming in the adjoint representation of $E_8 \times E_8$ or $SO(32)$. However, only the $16n$ scalars in the Cartan subalgebra are flat directions and their (generic) vacuum expectation values break the non-abelian gauge symmetry to $U(1)^{16}$. Together with the $2n$ Kaluza–Klein gauge bosons $g_{\mu i}$ and $b_{\mu i}$ they form[28] the total gauge group $G = U(1)^{2n+16}$. On special subspaces of the moduli space there can be nonabelian enhancement of the $U(1)^{16}$, at most up to the original $E_8 \times E_8$ or SO(32) (at least perturbatively).

The $16n$ scalars in the Cartan subalgebra parametrize together with the toroidal moduli g_{ij}, b_{ij} and the dilaton ϕ the $n(n+16)+1$ dimensional moduli space [47]

$$\mathcal{M} \;=\; R^+ \;\times\; \frac{\mathrm{SO}(n, n+16)}{\mathrm{SO}(n) \times \mathrm{SO}(n+16)} \Big/ \Gamma_T, \tag{72}$$

with the T–duality group

$$\Gamma_T = \mathrm{SO}\,(n, n+16, Z)\ . \tag{73}$$

The toroidal moduli all reside in $n+16$ (abelian) vector multiplets and the heterotic dilaton is the unique scalar in the gravitational multiplet (see Table 11). Supergravity implies that there can be no mixing between the dilaton and the other $n(n+16)$ scalars and thus the dilaton spans the R^+ component of $\mathcal{M}$.[29] Locally, the moduli space is uniquely determined by supersymmetry [32] and so already from this point of view the moduli space (72) is likely to be exact.

It has also been shown that below ten dimensions the heterotic $E_8 \times E_8$ theory and the heterotic SO(32) theory are continuously connected in the moduli space. That is, the two theories sit at different points of the same

[28] $n+16$ of these vectors reside in vector multiplets while the remaining n vectors are part of the gravitational multiplet (see Tables 10 and 11).

[29] In $D=4$ the antisymmetric tensor in the gravitational multiplet is dual to a pseudoscalar and thus can be combined with the dilaton into one complex scalar field. Since this scalar still resides in the gravitational multiplet it does not mix with the other toroidal moduli and again spans a separate component of the moduli space which is found to be locally equivalent to the $SU(1,1)/U(1)$ coset space, which replaces R^+ in (72).

moduli space of one and the same heterotic string theory [48]. The continuous path which connects the two theories in $D = 9$ involves a transformation $R \to \alpha'/R$ and hence they are also called T–duals of each other.

Type-I compactified on T^n. Toroidal compactifications of the type-I theory are slightly more involved. Locally the moduli space is dictated by supersymmetry to be the coset $R^+ \times \mathrm{SO}(n, n+16)/\mathrm{SO}(n) \times \mathrm{SO}(n+16)$ but perturbatively there is no T–duality symmetry and thus the global moduli space does not coincide with (72). However, once D-branes are included as possible open string configurations, type-I theories also have T–duality and the moduli space is given by (72) and (73) [11, 49, 40]. In fact, establishing T–duality was a guiding motive in the original discovery of D-branes [11]. From the open string point of view T–duality is not a perturbative symmetry since it necessarily involves the presence of solitonic-like excitations.

3.2.2. *$K3$ compactifications*

So far we discussed toroidal compactifications which leave all supercharges intact. Compactifications on a 4-dimensional $K3$ surface break half of the supercharges and hence one moves one column to the right and four rows down in the D-Q plane of Fig. 4; the resulting string theories therefore have $D = 6$ and either 16 or 8 supercharges. The massless modes of $K3$ compactifications arise from nontrivial deformations of the metric and from nontrivial harmonic forms on the $K3$ surface [44]. The moduli space of nontrivial metric deformations is known to be 58-dimensional and given by the coset space

$$\mathcal{M} \;=\; R^+ \;\times\; \frac{\mathrm{SO}(3,19)}{\mathrm{SO}(3)\times\mathrm{SO}(19)}\;, \tag{74}$$

where the second factor is the Teichmüller space for Einstein metrics of unit volume and the first factor is associated with the size of the $K3$.

On any (complex) Kähler manifold, the differential forms can be decomposed into (p, q)-forms with p holomorphic and q antiholomorphic differentials. The harmonic (p, q)-forms form the cohomology groups $H^{p,q}$ of dimension $h^{p,q}$ and for $K3$ one has a Hodge diamond

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} \;=\; \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}\;. \tag{75}$$

Thus there are 22 harmonic $p+q = 2$–forms which represent the nontrivial deformations of an antisymmetric tensor b_{ij}. On a 4-dimensional manifold an antisymmetric tensor can be constrained to a selfdual or an anti-selfdual tensor and on $K3$ one finds that the 22 2-forms decompose into 3 selfdual and 19 anti-selfdual 2-forms [44]. For later reference we also record that the Euler number of $K3$ is found to be

$$\chi(K3) = \sum_{p,q} (-)^{p+q}\, h^{p,q} = 24. \tag{76}$$

Type-IIA compactified on $K3$ Compactifying the type-IIA string on $K3$ breaks half of the 32 (non-chiral) supercharges in $D = 10$ and thus results in a $(1,1)$ supergravity in $D = 6$ coupled to vector multiplets which we already discussed in section 2.2.3. The massless bosonic modes of such compactifications are given by $g_{\mu\nu}$, $b_{\mu\nu}$, ϕ, g_{ij}, b_{ij} all of which arise from the NS-NS sector. g_{ij} denotes the 58 zero modes of the metric on $K3$ and b_{ij} are the 22 harmonic 2-forms.[30] In the R-R sector one finds $V_\mu, C_{\mu\nu\rho}$ and 22 vectors $C_{\mu ij}$. In $D = 6$ a 3-form is dual to a vector field so that altogether there are 24 vector fields in the R-R sector and 81 scalars in the NS-NS sector. The multiplets of $(1,1)$ supergravity are discussed in section 2.2.3. and one infers that the bosonic states of the $K3$ compactification fill out one gravity multiplet and 20 vector multiplets.

All 81 scalars arise in the NS-NS sector. The deformations of the metric span the moduli space given in (74). Together with the 22 harmonic 2-forms they combine into the 81-dimensional moduli space [50, 51]

$$\mathcal{M} \;=\; R^+ \;\times\; \frac{\mathrm{SO}(4,20)}{\mathrm{SO}(4)\times\mathrm{SO}(20)}\Big/ \Gamma_T \;, \tag{77}$$

where the factor R^+ is again parameterized by the single scalar in the gravitational multiplet which can be identified with the 6-dimensional dilaton. The second coset factor is spanned by the scalar fields of the vector multiplets. Similar to toroidal compactifications one finds perturbative identifications of the parameter space which are directly related to the properties of the underlying 2-dimensional conformal field theory. Such equivalences are also termed T-duality and for the case at hand the T–duality group is found to be [50, 51]

$$\Gamma_T = \mathrm{SO}(4,20,Z) \;. \tag{78}$$

Type-IIB compactified on $K3$. Compactifying the type-IIB string on $K3$ breaks half of the 32 chiral supercharges in $D = 10$ and thus results in a $(2,0)$ supergravity in $D = 6$ coupled to tensor multiplets which we

[30] Contrary to toroidal compactifications there are no massless vectors $g_{\mu i}$ or $b_{\mu i}$ since there are no one-forms on $K3$ (cf. (75)).

already discussed in section 2.2.3. The massless bosonic modes of such compactifications are given by $g_{\mu\nu}, b_{\mu\nu}, \phi, g_{ij}, b_{ij}$ in the NS-NS sector and $b'_{\mu\nu}, \phi', b'_{ij}, C_{\mu\nu\rho\sigma}, C_{\mu\nu ij}$ in the R-R sector; both b_{ij} and b'_{ij} are 22 harmonic 2-forms on $K3$. In $D=6$ a 4-form tensor $C_{\mu\nu\rho\sigma}$ describes only one physical degree of freedom and is dual to a real scalar. Furthermore, there are 22 spacetime tensors $C_{\mu\nu ij}$, proportional to the 22 harmonic forms on $K3$. Since C is chosen selfdual in $D=10$, the tensor fields $C_{\mu\nu ij}$ are selfdual or anti-selfdual and their selfduality phase is correlated with the (anti)selfduality of the corresponding $K3$ harmonic forms. Hence the $C_{\mu\nu ij}$ decompose into 3 selfdual and 19 anti-selfdual $D=6$ antisymmetric tensors. Altogether there are thus 81 NS-NS and 24 R-R scalars and 5 selfdual and 21 anti-selfdual R-R antisymmetric tensors. The corresponding supermultiplets were already discussed in section 2.2.3.; we immediately infer that the massless modes arising from the $K3$ compactification combine into one gravitational and 21 tensor multiplets of $(2,0)$ supersymmetry. This theory being chiral is potentially anomalous; however, it was shown in [13] that precisely this combination of multiplets is anomaly free.

The 81 scalars of NS-NS sector span the same moduli space as in (77) and similar to toroidal compactifications the scalars from the R-R cannot have any nontrivial mixing with the NS-NS scalars at the perturbative level. However, the $(2,0)$ gravitational multiplet contains no scalar at all but rather all scalars appear in the 21 tensor multiplets. On the basis of supersymmetry it was conjectured [52] that all 105 scalars locally parameterize the moduli space

$$\mathcal{M} = \frac{\mathrm{SO}(5,21)}{\mathrm{SO}(5)\times \mathrm{SO}(21)}\ . \tag{79}$$

Indeed, once nonperturbative corrections of string theory are taken into account this moduli space is generated; we return to this point in section 3.3.4.

The heterotic string compactified on $K3$ The heterotic string compactified on $K3$ has 8 unbroken supercharges or $(1,0)$ supergravity in $D=6$ coupled to vector-, tensor- and hypermultiplets. Contrary to the previously discussed type-II compactifications this heterotic string theory does not uniquely fix the massless spectrum but instead one finds distinct families of string vacua with different contents of massless states.[31] However, $(1,0)$ supersymmetry is chiral and thus gauge and gravitational anomaly cancellation do impose some constraints on the allowed spectra of supermultiplets. One finds the condition [53]

$$n_H - n_V + 29\, n_T - 273 = 0, \tag{80}$$

[31] Of course, it is a general property of supersymmetry and string theory that fewer unbroken supercharges lead to a much richer variety of low-energy spectra.

where n_H, n_V and n_T are the numbers of hyper, vector and tensor multiplets, respectively. (The specific field content of these multiplets is displayed in Table 5.) In the perturbative spectrum of the heterotic string there is only one tensor multiplet which contains the selfdual part of $b_{\mu\nu}$ (the anti-selfdual piece resides in the gravitational multiplet) and the dilaton and hence anomaly cancellation in any perturbative heterotic string vacua demands $n_H - n_V = 244$. In addition the Green–Schwarz mechanism requires a modified field strength H for the antisymmetric tensor $H = db + \omega^L - \sum_a v_a\, \omega_a^{YM}$ where ω^L is a Lorentz–Chern–Simons term and ω_a^{YM} are the Yang–Mills Chern–Simons terms [53]. The index a labels the factors G_a of the gauge group $G = \otimes_a G_a$ and v_a are some computable constants which depend on the specific massless spectrum. In order to ensure a globally well-defined H on the compact $K3$ the integral $\int_{K3} \mathrm{d}H$ has to vanish. This implies

$$\sum_a n_a \equiv \sum_a \int_{K3} \left(\mathrm{tr}\, F^2\right)_a = \int_{K3} \mathrm{tr}\, R^2 = 24, \tag{81}$$

where the last equation used the fact that 24 is the Euler number of $K3$. From (81) we learn that in any compactification of the heterotic string on $K3$ the original 10-dimensional gauge symmetry ($\mathrm{E}_8 \times \mathrm{E}_8$ or SO(32)) is necessarily broken since consistency requires a non-vanishing instanton number n_a.

As before we can also ask for perturbative equivalences on the space of heterotic $K3$ compactifications. It has been shown that the $K3$ compactifications of the 10-dimensional heterotic string with gauge group SO(32) lie in the same moduli space as (particular) $K3$ compactifications of the 10-dimensional heterotic string with gauge group $\mathrm{E}_8 \times \mathrm{E}_8$ [54, 55]. More precisely, the gauge group is really Spin(32)$/Z_2$ and one has to distinguish two different types of instantons which are characterized by the second Stieffel–Whitney class [55]. The corresponding distinct compactifications of the Spin(32)$/Z_2$ heterotic string are also called compactifications with and without vector structure. It has been shown that compactifications with vector structure have a common moduli space with $\mathrm{E}_8 \times \mathrm{E}_8$ compactifications of instanton numbers $n_1 = 8, n_2 = 16$ [54] while compactifications without vector structure have a common moduli space with $\mathrm{E}_8 \times \mathrm{E}_8$ compactifications of instanton numbers $n_1 = n_2 = 12$ [55]. The continuous path which connects the two pairs of theories involves a transformation $R \to \alpha'/R$ and hence they are also called T–dual. Furthermore, the $\mathrm{E}_8 \times \mathrm{E}_8$ compactifications with instanton numbers $n_1 = n_2 = 12$ are part of the same moduli space as the compactifications with instanton numbers $n_1 = 10, n_2 = 14$ [56, 54].

Type-I compactified on $K3$. Also this compactification leads to $(1,0)$ supersymmetry and thus anomaly cancellation imposes the same constraint (80) on the massless spectrum. However, in type-I compactification there can be more than one tensor multiplet and as a consequence also a generalized Green-Schwarz mechanism can be employed [57]. The resulting spectra are much less investigated and we refer the reader to the literature for further details [57, 58].

3.2.3. *Calabi-Yau threefolds compactifications*

Compactifications on a 6-dimensional Calabi-Yau threefold Y_3 break 3/4 of the supercharges present in $D = 10$ and hence one moves two columns to the right and six rows down in the D-Q plane of Fig. 4; the resulting string theories therefore have $D = 4$ and either 8 or 4 supercharges. The massless modes of such compactification arise from the nontrivial harmonic forms on Y_3 which again are most conveniently summarized by the Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \quad , \tag{82}$$

where $h^{1,1}$ and $h^{1,2}$ are arbitrary integers [37, 45]. The corresponding $(1,1)$ and $(1,2)$ forms are the deformations of the Calabi-Yau metric and the complex structure, respectively. Using the definition (76) one finds the Euler number of a threefold to be $\chi(Y_3) = 2(h^{1,1} - h^{1,2})$. It is believed that most (if not all) Calabi–Yau threefolds Y_3 have an associated mirror manifold $\tilde{Y}_3$ with the property that its Hodge numbers are exactly reversed, i.e. $h^{1,1}(\tilde{Y}_3) = h^{1,2}(Y_3)$ and $h^{1,2}(\tilde{Y}_3) = h^{1,1}(Y_3)$, so that $\chi(Y_3) = -\chi(\tilde{Y}_3)$ [59].

The moduli space of Calabi–Yau threefolds is locally a direct product space

$$\mathcal{M}_{Y_3} = \mathcal{M}_{(1,1)} \otimes \mathcal{M}_{(1,2)} \ , \tag{83}$$

where $\mathcal{M}_{(1,1)}$ ($\mathcal{M}_{(1,2)}$) is the moduli space parameterized by the $(1,1)$-forms ($(1,2)$-forms). Both factors are constrained to be special Kähler manifolds [31, 60, 61, 62]. The metric $G_{i\bar{j}}$ of a Kähler manifold is determined by a real scalar function, the Kähler potential K

$$G_{i\bar{j}} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^{\bar{j}}} K\left(z, \bar{z}\right), \tag{84}$$

	IIA	IIB
n_H	$h^{1,2}+1$	$h^{1,1}+1$
n_V	$h^{1,1}$	$h^{1,2}$
G	$\mathrm{U}(1)^{h^{1,1}+1}$	$\mathrm{U}(1)^{h^{1,2}+1}$

TABLE 14. Massless spectra of type-II vacua.

where z^i are the (complex) coordinates on the moduli space. For a special Kähler manifold the Kähler potential satisfies the additional constraint

$$K = -\log\left[2(\mathcal{F}(z)+\bar{\mathcal{F}}(\bar{z})) - (z^i - \bar{z}^{\bar{i}})(\mathcal{F}_i(z) - \bar{\mathcal{F}}_{\bar{i}}(\bar{z}))\right] , \quad \mathcal{F}_i \equiv \frac{\partial \mathcal{F}}{\partial z^i} . \tag{85}$$

That is, K is determined by a single holomorphic function $\mathcal{F}(z)$.

Type-II compactified on Y_3. Such compactifications have 8 unbroken supercharges which is also called $N=2$ supersymmetry in $D=4$. The multiplets are the gravitational multiplet, the vector multiplet and the hypermultiplet.

Compactifications of the type-IIA string results in the massless modes $g_{\mu\nu}, b_{\mu\nu}, \phi, g_{ij}, b_{ij}$ from the NS-NS sector and $A_\mu, C_{\mu ij}, C_{ijk}$ from the R-R sector. From g_{ij}, b_{ij} one obtains $h^{1,1}+h^{1,2}$ complex massless scalar fields in the NS-NS sector. $C_{\mu ij}$ leads to $h^{1,1}$ abelian vectors while $h^{1,2}$ complex scalars arise from C_{ijk} (all in the R-R sector) [50, 63]. These states (together with their fermionic partners) combine into $h^{1,1}$ vector multiplets and $h^{1,2}$ hypermultiplets. Furthermore, in $D=4$ an antisymmetric tensor is dual to a scalar and hence $\phi, b_{\mu\nu}$ and two R-R scalars from C_{ijk} form an additional hypermultiplet. Thus the total number of vector multiplets is $n_V = h^{1,1}$ while the number of hypermultiplets is given by $n_H = h^{2,1}+1$.

For type-IIB vacua one also has $h^{1,1}+h^{1,2}$ complex massless scalar fields in the NS-NS sector but now $h^{1,2}$ abelian vectors together with $h^{1,1}$ complex scalars in the R-R sector (the universal hypermultiplet containing the dilaton is again present) [50, 63]. Hence, $n_V = h^{2,1}$ and $n_H = h^{1,1}+1$ holds for the type-IIB theory.

The gauge group is always abelian and given by $(h^{1,1}+1)$ U(1) factors in type-IIA and $(h^{1,2}+1)$ $U(1)$ factors in type-IIB (the extra U(1) is the graviphoton of the gravitational multiplet). We summarize the spectrum of type-II vacua in Table 14.

As we see the role of $h^{1,1}$ and $h^{1,2}$ is exactly interchanged between type-IIA and type-IIB. Therefore, compactification of type-IIA on a Calabi–Yau threefold Y_3 is equivalent to compactification of type-IIB on the mirror Calabi–Yau $\tilde{Y}_3$. This is another example of a perturbative equivalence of two entire classes of string vacua.

Heterotic string compactified on Y_3. Such compactification have 4 unbroken supercharges which corresponds to $N = 1$ supersymmetry in $D = 4$. Now the $(1,1)$ and $(1,2)$ forms both correspond to massless chiral multiplets. Generically, there are many distinct families of string vacua with varying gauge groups and matter content and relatively little can be said in general about their properties.

As in $K3$ compactifications of the heterotic string the field strength H of the antisymmetric tensor has to be modified by appropriate Lorentz– and Yang–Mills Chern–Simons terms. However, the requirement for a globally defined H on Y_3 is slightly more involved since the compact manifold is now 6-dimensional and the integral over $\mathrm{d}H$ is no longer a topological invariant.

A special class of consistent compactifications is obtained by embedding the spin connection in the gauge connection of $\mathrm{E}_8 \times \mathrm{E}_8$ or SO(32) [64]. In the first case one obtains a gauge group $\mathrm{E}_8 \times \mathrm{E}_6$ with $h^{1,1}$ chiral multiplets in the **27** and $h^{1,2}$ chiral multiplets in the $\overline{\mathbf{27}}$ representation of E_6. In addition there are also $h^{1,1} + h^{1,2}$ gauge neutral moduli multiplets.

Similarly, compactifications of the SO(32) heterotic string lead to a gauge group $\mathrm{SO}(26) \times \mathrm{U}(1)$ with $h^{1,1}$ chiral multiplets in the $\mathbf{26}_1 \oplus \mathbf{1}_{-2}$ and $h^{1,2}$ chiral multiplets in the $\mathbf{26}_{-1} \oplus \mathbf{1}_2$ representation of $\mathrm{SO}(26) \times \mathrm{U}(1)$. In addition, there are $h^{1,1} + h^{1,2}$ gauge neutral moduli multiplets.

Again we encounter a perturbative equivalence: A heterotic string compactified on a given Y_3 leads to the exact same string vacuum as a compactification on the mirror manifold $\tilde{Y}_3$. This can be seen from the above assignment of the massless spectrum (it is only convention what is called **27** versus $\overline{\mathbf{27}}$) but also has been shown more generally for the full string theory [59].

There are many compactifications with different embeddings than the one discussed above and they can lead to very different spectra. The space of heterotic Y_3 compactifications displays a much bigger variety of spectra than any of the compactifications discussed so far and many of the properties can only be discussed on a case-by-case basis [65].

Type-I string compactified on Y_3. This class of string compactification also has 4 supercharges ($N = 1$) but has been much less investigated as the heterotic string. There are also many distinct families of string vacua with varying spectra and couplings [66].

3.3. DUALITY IN STRING THEORY

The concept of duality is very common in physics. Generically it means that there are two (or more) different descriptions of the same physical system. Frequently the different descriptions are only valid in specific domains of the parameter space and only together they can be used to cover the entire parameter space of the physical system. The past few years have shown [67, 18, 40] that also the various string theories of Fig. 4 are interrelated by a complicated 'web' of duality relations; they are not at all independent but instead are different regions of a common parameter space. In fact, it seems that a given representation of the supersymmetry algebra (with a given number of supersymmetries and spacetime dimensions) lead only to one distinct quantum theory with a parameter space that can encompass various perturbatively distinct string theories. In Fig. 4 we plotted the different perturbative compactifications, some of which share the same representation of supersymmetry. As we will see in this section they all turn out to be different regions in a common parameter space.

One distinguishes perturbative and nonperturbative dualities. Perturbative dualities already hold at weak string coupling and the map which identifies the perturbative theories does not involve the dilaton. On the other hand nonperturbative dualities identify regions of the parameter space which are not simultaneously at weak coupling and the duality map involves the dilaton in a nontrivial way. Such nonperturbative dualities are of utmost importance since they map the strong-coupling region of a given (string) theory to the weak-coupling region of a dual theory where perturbative methods are applicable and hence the strong-coupling limit gets (at least partially) under quantitative control.

The perturbative dualities are well established and we have already seen them in the previous section. The nonperturbative dualities are more difficult to deal with and they cannot be proven at present. Rather their validity has only been checked for quantities or couplings which do not receive quantum corrections. Such quantities or couplings exist in supersymmetric (string) theories; they are the BPS states of the theory as well as the holomorphic couplings (such as the prepotential $\mathcal{F}(z)$ of $N = 2$ supergravity in $D = 4$) of the effective action. It is precisely for this reason that supersymmetry has played such an important (technical) role in establishing nonperturbative dualities.

Let us first briefly discuss the perturbative dualities from a common point of view. Then we focus on the nonperturbative dualities and discuss the various relations in turn.

3.3.1. *T–dualities*

All perturbative dualities are now called T–dualities but one can divide them into two classes. In toroidal compactifications (which we discussed in section 3.2.1.) different points in the parameter space of the compactification correspond to equivalent theories with the exact same spectrum and interactions. As a consequence there is a discrete symmetry Γ_T acting on the space of toroidal compactifications. The same situation is encountered in $K3$ compactifications of type-IIA string theories where also a discrete symmetry Γ_T has been identified (cf. (78)).

A different situation occurs in T^1 or $K3$ compactifications of the heterotic string. What was thought are two distinct perturbative string theories – the $E_8 \times E_8$ heterotic string compactified on T^1 or $K3$ and the SO(32) string compactified on T^1 or $K3$ – turn out to be merely different regions of a common parameter space. In other words there is a continuous path which connects the two theories and thus also their respective parameter spaces are continuously connected. Finally, the equivalence of type-IIA compactified on Y_3 with type-IIB compactified on the mirror $\tilde{Y}_3$ identifies compactifications on geometrically distinct manifolds as identical and hence maps the parameter space of type-IIA compactifications onto the parameter space of type-IIB compactifications. The common feature of all of these examples is a perturbative equivalence between string compactifications. Let us now turn to nonperturbative equivalences.

3.3.2. *S–dualities*

Let A and B be two perturbatively distinct string theories each with its own string coupling g_A and g_B, respectively. However, it is possible that once all quantum corrections (including the nonperturbative corrections) are taken into account A and B are equivalent as quantum theories and one has

$$A \;\equiv\; B\ . \qquad (86)$$

This situation can occur in two different ways:

(a) The strong-coupling limit of A is mapped to the weak coupling limit of B

$$\lim_{g_A\to\infty} A \;\to\; \lim_{g_B\to 0} B\ , \qquad (87)$$

or in other words $g_A \sim g_B^{-1}$. Using (58) one finds in terms of the corresponding dilatons the identification

$$\phi_A = -\phi_B\ . \qquad (88)$$

Along with this strong-weak coupling relation goes a map of the elementary excitations of theory A to the nonperturbative, solitonic excitations of theory B and vice versa. The theories A and B are called S–dual and one also refers to this situation as a 'string–string duality'. Examples of S–dual string theories are:

- The heterotic SO(32) string and the type-I string are S–dual in $D = 10$. The evidence for this duality is the agreement of the low-energy effective actions[32] once one identifies $\phi_{\rm HSO} = -\phi_{\rm I}$ [33, 68, 69] and the fact that the perturbative heterotic SO(32) string has been identified as the D–string of the type-I theory [70]. In the limit of strong coupling in the type-I theory ($g_{\rm I} \to \infty$) the heterotic SO(32) string becomes the 'lightest' and thus perturbative object.
- The type-IIA string compactified on $K3$ and the heterotic string compactified on T^4 are S–dual [71, 34, 33, 72, 73]. Both theories have $(1,1)$ supersymmetry in $D = 6$ with exactly the same massless spectrum. Furthermore, from (72, 77) one learns that also the moduli spaces of the two string compactifications coincide. The effective actions of the two perturbative theories agree if one identifies [33]

$$\begin{aligned} \phi_{\rm H} &= -\phi_{\rm IIA} \ , \\ H_{\rm H} &= e^{-2\phi_{\rm IIA}}\, {}^*H_{\rm IIA} \ , \\ (g_{\mu\nu})_{\rm H} &= e^{-2\phi_{\rm IIA}}\, (g_{\mu\nu})_{\rm IIA} \ , \end{aligned} \tag{89}$$

where $H = {\rm d}b$ is the field strength of the antisymmetric tensor and ${}^*H$ is its Poincare dual. The first equation in (89) again implies a strong-weak coupling relation while the second is the equivalent of an electric-magnetic duality. Further evidence for this S-duality arises from the observation that the zero modes in a solitonic string background of the type-IIA theory compactified on $K3$ have the same structure as the Kaluza–Klein modes of the heterotic string compactified on T^4 [72, 73].

(b) There is a variant of the above situation where the dilaton of theory A is not mapped to the dilaton of theory B as in (88), but rather to any of the other perturbative moduli R_B of theory B. In this case one has the identifications

$$\phi_A \sim R_B \ , \qquad \phi_B \sim R_A \ , \tag{90}$$

or in other words the strong-coupling limit of A is independent of g_B

$$\lim_{g_A \to \infty} A \ = \ \text{independent of } g_B \ . \tag{91}$$

[32] In (53)–(55) we displayed the heterotic Lagrangian in different frames. The last frame (55) shows the equivalence with type-I.

As in case (a) also here the strong-coupling limit of A is controlled by the perturbative regime of theory B and thus accessible in perturbation theory (at least in principle).
This situation is found in the following examples:

- The type-II string compactified on Y_3 and the heterotic string compactified on $K3 \times T^2$ are S–dual in the sense just defined [74, 75]. The heterotic dilaton is a member of a vector multiplet and mapped to one of the geometrical moduli of the Calabi-Yau threefold Y_3.[33] Conversely, the type-II dilaton is part of a hyper-multiplet and mapped to one of the geometrical moduli of the $K3$. The validity of this duality has been checked in a variety of ways for quite a number of dual string vacua [76, 77, 78, 79]. In particular it has been shown that the prepotential $\mathcal{F}(z)$ appearing in (85) agrees for dual pairs of string vacua.
- In the same sense the heterotic string compactified on $K3$ and the type-I string compactified on $K3$ are S–dual [55, 80].

Let us summarize the known S–dualities in the following table

$$
\begin{array}{llcll}
D=10 & : & \mathrm{HSO} & \stackrel{S}{\longleftrightarrow} & \mathrm{I} \\
D=6 & : & \mathrm{IIA}/K3 & \stackrel{S}{\longleftrightarrow} & \mathrm{H}/T^4 \\
 & & \mathrm{H}/K3 & \stackrel{S}{\longleftrightarrow} & \mathrm{I}/K3 \\
D=4 & : & \mathrm{II}/Y_3 & \stackrel{S}{\longleftrightarrow} & \mathrm{H}/K3 \times T^2
\end{array}
$$

In Fig. 5 these S-dualities are denoted by a horizontal bar (**—**).

3.3.3. *Self-duality and U-duality*

Another situation is encountered when the strong-coupling limit of a theory A is controlled not by a distinct theory B, but rather by a different perturbative region of the same theory A. That is, the strong-coupling regime of A has an alternative weakly-coupled description within the same theory A but in terms of a different set of elementary degrees of freedom. The new perturbation theory is often called the magnetic theory and its perturbative degrees of freedom are called magnetic degrees of freedom. This stems from the fact that the first duality (in $D = 4$) put forward by Montonen and Olive [81] suggested that an electric U(1) gauge theory is dual to a magnetic U(1) gauge theory with a magnetic photon and magnetic monopoles as perturbative degrees of freedom. This situation is more general and can appear also for extended objects. However, for such a self-duality to hold

[33] More precisely, Y_3 has to be a $K3$-fibration and the heterotic dilaton is mapped to the modulus parameterizing the size of the $\mathbf{P}^1$ base of the fibration [76, 77, 78].

the theory A has to have a nontrivial (discrete) symmetry which maps the strong-coupling region to a region of weak coupling and simultaneously the different elementary excitations onto each other. One has to make a further subdivision of this case:

(a) The symmetry group is $\Gamma_S = \mathrm{SL}(2, Z)$ which acts on a single complex scalar field containing the dilaton as its real (or imaginary) part. Unfortunately this situation is also called S–duality and the associated symmetry group Γ_S is called the S–duality group. (We prefer to call it a special case of a U–duality.)
Examples of this case are:

- The type-IIB string in $D = 10$ is conjectured to have $\Gamma_S = \mathrm{SL}(2, Z)$ [34, 33, 69]. The corresponding supergravity theory has a $\mathrm{SL}(2, R)$ as a symmetry group [8] (see also Table 9) but quantum corrections break this continuous symmetry to its discrete version $\mathrm{SL}(2, Z)$. This exact symmetry predicts an infinite number of equivalent weakly coupled type-IIB strings which carry R-R charge; such strings have indeed been identified as appropriate D-strings [82, 83].
- A second example is the heterotic string compactified on T^6 which has $D = 4$ and also $\Gamma_S = \mathrm{SL}(2, Z)$. In toroidal compactifications of the heterotic string the dilaton is the unique scalar in the gravitational multiplet and parameterizes the R^+ component of the heterotic moduli space (72). However, in $D = 4$ an antisymmetric tensor of rank 2 is on-shell equivalent to a pseudoscalar and thus can be combined together with the dilaton into one complex scalar of the gravitational multiplet spanning the SU(1,1)/U(1) component of the moduli space. It is this complex scalar on which Γ_S acts, leaving all other toroidal moduli invariant.[34] The vacuum-expectation value of the pseudoscalar plays the role of the θ-angle and so this duality is nothing but the string theoretical version of the original electric–magnetic Montonen–Olive duality which in many respects started the subject of string dualities [81, 84, 85, 86].

In Fig. 5 we mark the theories with $\Gamma_S = \mathrm{SL}(2, Z)$ by an 'S' next to it. There also is a nontrivial generalization of this case:

(b) The product of Γ_S and the T-duality group Γ_T is contained as a maximal subgroup in a bigger group Γ_U, called the U–duality group [34].

[34] Note that this is rather different than the previous case where Γ_S acted on the two scalars of the IIB string.

D	Γ_T	Γ_U
10A	1	1
10B	1	SL(2,Z)
9	Z_2	SL$(2, Z) \times Z_2$
8	SO$(2, 2, Z)$	SL$(3, Z) \times$SL$(2, Z)$
7	SO$(3, 3, Z)$	SL$(5, Z)$
6	SO$(4, 4, Z)$	SO$(5, 5, Z)$
5	SO$(5, 5, Z)$	E$_{6(+6)}(Z)$
4	SO$(6, 6, Z)$	E$_{7(+7)}(Z)$

TABLE 15. U–duality groups for type-II strings.

This situation is encountered in toroidal T^n compactifications of the type-II string. Γ_S is 'inherited' from the type-IIB string in $D = 10$ and Γ_T already exists at the perturbative level (cf. (71)). As we already discussed extensively in section 2.5, the corresponding supergravities do have a much larger continuous symmetry group which transforms all scalar fields into each other irrespective of their NS-NS or R-R origin (cf. Table 9). This is a consequence of the fact that the supergravities have a unique gravitational multiplet, which contains all scalar fields on an equal footing. Furthermore, they are constructed as toroidal compactifications of the 11-dimensional supergravity while the string vacua arise as compactifications of 10-dimensional string theories. Within the perturbative regime there can never be a symmetry which mixes NS-NS scalars with their R-R 'colleagues' due to their rather different dilaton couplings. However, nonperturbatively, when also D-brane configurations are taken into account, such a symmetry is no longer forbidden and evidence in its favour has been accumulated [34, 33]. The necessary states carrying R-R charge have been identified and the nonperturbative BPS-spectrum assembles in representations of Γ_U. The U–duality groups in arbitrary dimensions are summarized in Table 15 [34]; they are just the discrete version of the group G in Table 9. In Fig. 5 these theories are marked with a 'U'.

3.3.4. *M-theory*

The various dualities discussed so far relate different perturbative string theories. In these cases the strong-coupling limit of a given string theory is controlled by another (or the same) perturbative string theory. However, not all strong-coupling limits are of this type. Instead it is possible that the strong-coupling limit of a given theory is something entirely new, not any of the other string theories [33]. Only limited amount of information is so far known about this new theory which is called M-theory. Examples of this situation are:

- The strong-coupling limit of the type-IIA theory in $D = 10$. The low-energy effective action was discussed in section 2.5 where we also indicated how it can be constructed as a T^1 compactification of 11-dimensional supergravity. This implied a relation between the radius R_{11} of the 11-th dimension and the string coupling constant $g_s = e^{\langle\phi\rangle}$ [33] (cf. (44))

$$R_{11} = \frac{L}{2\pi}\, g_s^{\frac{2}{3}} \ , \tag{92}$$

  where L is the length of the 11-th dimension introduced in section 2.5. Moreover, the Kaluza-Klein spectrum of this theory obeys (in the string frame)

$$M^{\rm KK} = \frac{2\pi|n|}{g_s L} \ , \tag{93}$$

  where n is an arbitrary integer (cf. (49)). These KK-states are not part of the perturbative type-IIA spectrum since they become heavy in the weak-coupling limit $g_s \to 0$. However, in the strong-coupling limit $g_s \to \infty$ they become light and can no longer be neglected in the effective theory. This infinite number of light states (which can be identified with D-particles of type-IIA string theory, or extremal black holes of IIA supergravity) signals that the theory effectively decompactifies, which can also be seen from (92). Supersymmetry is unbroken in this limit and hence the KK-states assemble in supermultiplets of the 11-dimensional supergravity. Since there is no string theory which has 11-dimensional supergravity as the low-energy limit, the strong-coupling limit of type-IIA string theory has to be a new theory, called M-theory, which cannot be a theory of (only) strings. M-theory is supposed to capture all degrees of freedom of all known string theories, both at the perturbative and the nonperturbative level [35, 33, 87, 18]. There exists a conjecture according to which the degrees of freedom of M-theory are captured in U(N) supersymmetric matrix models in the $N \to \infty$ limit [88]. These matrix models have been known for some time [89] and were also known to describe supermembranes [90] in the lightcone gauge [91]. The same quantum-mechanical models describe the short-distance dynamics of N D-particles, caused by the exchange of open strings [83, 40]. A review of these developments is beyond the scope of these lectures and we refer the reader to the literature [92].

- A second and maybe even more surprising result shows that also the strong-coupling limit of the heterotic $E_8 \times E_8$ string is captured by M-theory. In this case, 11-dimensional supergravity is not compactified on a circle but rather on a Z_2 orbifold of the circle [87]. The space coordinate x^{11} is odd under the action of Z_2 and hence the three-form $C_{\mu\nu\rho}$ as well as $g_{\mu 11}$ are also odd. The Z_2-invariant spectrum in $D = 10$ consists of the metric $g_{\mu\nu}$, the antisymmetric tensor $C_{\mu\nu 11}$ and the scalar $g_{11\,11}$. Up to the gauge degrees of freedom this is precisely the massless spectrum of the 10-dimensional heterotic string. The $E_8 \times E_8$ Yang-Mills fields have to arise in the twisted sector of the orbifold. One way to see this is to note that the Z_2 truncation of 11-dimensional supergravity is inconsistent in that it gives rise to gravitational anomalies [93]. In order to cancel such anomalies non-abelian gauge fields have to be present with appropriate couplings to the antisymmetric tensor such that a Green-Schwarz mechanism can be employed [94]. Such additional states can only appear in the twisted sectors of the orbifold theory which are located at the orbifold fixed points $x^{11} = 0$ and $x^{11} = L/2$. However, due to the Z_2 symmetry, these two 10-dimensional hyperplanes have to contribute equally to the anomaly. This can only be achieved for a gauge group which is a product of two factors and thus $E_8 \times E_8$ with one E_8 factor on each hyperplane is the only consistent candidate for such a theory [87]. Just as in the type-IIA case one has $R_{11} = g_{\mathrm{H}}^{3/2} L/2\pi$ and thus weak coupling corresponds to small R_{11} and the two 10-dimensional hyperplanes sit close to each other; in the strong-coupling limit the two 10-dimensional hyperplanes move far apart (to the end of the world). Using the previous terminology the heterotic $E_8 \times E_8$ string theory can be viewed as M-theory compactified on T^1/Z_2.
- There is an immediate corollary of the dualities discussed so far. The strong-coupling limit of the type-IIA string compactified on $K3$ is simultaneously governed by M-theory compactified on $K3 \times T^1$ and the heterotic string compactified on T^4. Since there is a T^1 in both theories one concludes that the strong-coupling limit of the heterotic string compactified on T^3 is governed by M-theory compactified on $K3$ [33]. From (72) and (74) we learn that indeed the moduli spaces of both theories agree if the heterotic dilaton is related to the overall size of the $K3$. A detailed comparison of the respective effective actions reveals that the strong-coupling limit on the heterotic side corresponds to the large-radius limit of the $K3$ on the M-theory side [33].
- The exact same argument can be repeated in $D = 5$. The strong-coupling limit of the type-IIA string compactified on (a $K3$-fibred) Y_3 is simultaneously governed by M-theory compactified on $Y_3 \times T^1$ and

the heterotic string compactified on $K3 \times T^2$. By the same argument used above one concludes that the strong-coupling limit of the heterotic string compactified on $K3 \times T^1$ is governed by M-theory compactified on (a $K3$-fibred) Y_3 [95]. In this case the heterotic dilaton is directly related to the size of the $\mathbf{P^1}$ base of the $K3$-fibration.

- A slightly more involved analysis is necessary to conclude that the strong-coupling limit of the IIB string compactified on $K3$ is governed by M-theory compactified on T^5/Z_2 [96]. Compactifying 11-dimensional supergravity on the orbifold T^5/Z_2 one obtains the chiral $(2,0)$ supergravity with one gravity multiplet and five tensor multiplets from the untwisted sector. The twisted sector is again inferred by anomaly cancellation and provides 16 further tensor multiplets. The weakly-coupled type-IIB theory on $K3$ corresponds to a 'smashed' T^5/Z_2 where the 32 fixed points degenerate into 16 pairs and the 16 tensor multiplets are equally distributed among those pairs. The 81 scalars from the NS-NS sector combine with the 24 scalars from the R-R sector to form the moduli space [52, 97, 98]

$$\mathcal{M} = \frac{\mathrm{SO}(5,21)}{\mathrm{SO}(5)\times\mathrm{SO}(21)}\Big/\Gamma_U, \tag{94}$$

  with a U–duality group $\Gamma_U = \mathrm{SL}(5,21,Z)$. The local structure of this moduli space is already fixed by supergravity (cf. (79)) while the global structure follows from M-theory.

Let us summarize the nontrivial compactifications of M-theory:

$$\begin{array}{lcl}
\mathrm{M}/T^1 & \rightarrow & \mathrm{IIA} \\
\mathrm{M}/T^1/Z_2 & \rightarrow & \mathrm{HE8} \\
\mathrm{M}/K3 & \rightarrow & \mathrm{H}/T^3 \\
\mathrm{M}/T^5/Z_2 & \rightarrow & \mathrm{IIB}/K3 \\
\mathrm{M}/Y_3 & \rightarrow & \mathrm{H}/K3\times T^1
\end{array}$$

Theories whose strong-coupling limit is governed by M-theory are marked with an 'M' in Fig. 5.

3.3.5. *F-theory*

As we discussed previously the type-IIB theory in 10 spacetime dimensions is believed to have an exact $\mathrm{SL}(2,Z)$ quantum symmetry which acts on the complex scalar $\tau = e^{-2\phi} + i\phi'$, where ϕ and ϕ' are the two scalar fields of type-IIB theory (c.f. Table 13). This fact led Vafa to propose that the type-IIB string could be viewed as the toroidal compactification of a twelve-dimensional theory, called F-theory, where τ is the complex structure modulus of a two-torus T^2 and the Kähler-class modulus is frozen [99].

Apart from having a geometrical interpretation of the SL$(2, Z)$ symmetry this proposal led to the construction of new, nonperturbative string vacua in lower space-time dimensions. In order to preserve the SL$(2, Z)$ quantum symmetry the compactification manifold cannot be arbitrary but has to be what is called an elliptic fibration. That is, the manifold is locally a fibre bundle with a two-torus T^2 over some base B but there are a finite number of singular points where the torus degenerates. As a consequence nontrivial closed loops on B can induce a SL$(2, Z)$ transformation of the fibre. This implies that the dilaton is not constant on the compactification manifold, but can have SL$(2, Z)$ monodromy [100]. It is precisely this fact which results in nontrivial (nonperturbative) string vacua inaccessible in string perturbation theory.

F-theory can be compactified on elliptic Calabi–Yau manifolds and each of such compactifications is conjectured to capture the nonperturbative physics of an appropriate string vacua. One finds:

- The IIB string in $D = 10$ can be viewed as F-theory compactified on T^2 with a frozen Kähler modulus.
- F-theory compactified on an elliptic $K3$ yields an 8-dimensional vacuum with 16 supercharges which is quantum equivalent to the heterotic string compactified on T^2 [99, 101].
- F-theory compactified on an elliptic Calabi–Yau threefold has 8 unbroken supercharges and is quantum equivalent to the heterotic string compactified on $K3$ [54]. In fact there is a beautiful correspondence between the heterotic vacua labelled by the instanton numbers (n_1, n_2) and elliptically fibred Calabi-Yau manifolds with the base being the Hirzebruch surfaces $\mathbb{F}_{n_2-12}$ (we have chosen $n_2 \geq n_1$) [54].
- Finally, the heterotic string compactified on a Calabi–Yau threefold Y_3 is quantum equivalent to F-theory compactified on an elliptic Calabi–Yau fourfold [102]. Calabi-Yau fourfolds are Calabi-Yau manifolds of complex dimension four and holonomy group SU(4).

Let us summarize the nontrivial compactifications of F-theory:

$$\begin{array}{lcl} \mathrm{F}/T^2 & \to & \mathrm{IIB} \\ \mathrm{F}/K3 & \to & \mathrm{H}/T^2 \\ \mathrm{F}/Y_3 & \to & \mathrm{H}/K3 \\ \mathrm{F}/Y_4 & \to & \mathrm{H}/Y_3 \end{array}$$

The theories governed by F-theory are marked with an 'F' in Fig. 5.

3.3.6. *Summary of all strong-coupling limits.*

So far we tried to systematically discuss the different possible strong-coupling limits of string theories along with the relevant examples. In this final section let us one more time summarize all strong-coupling relations but

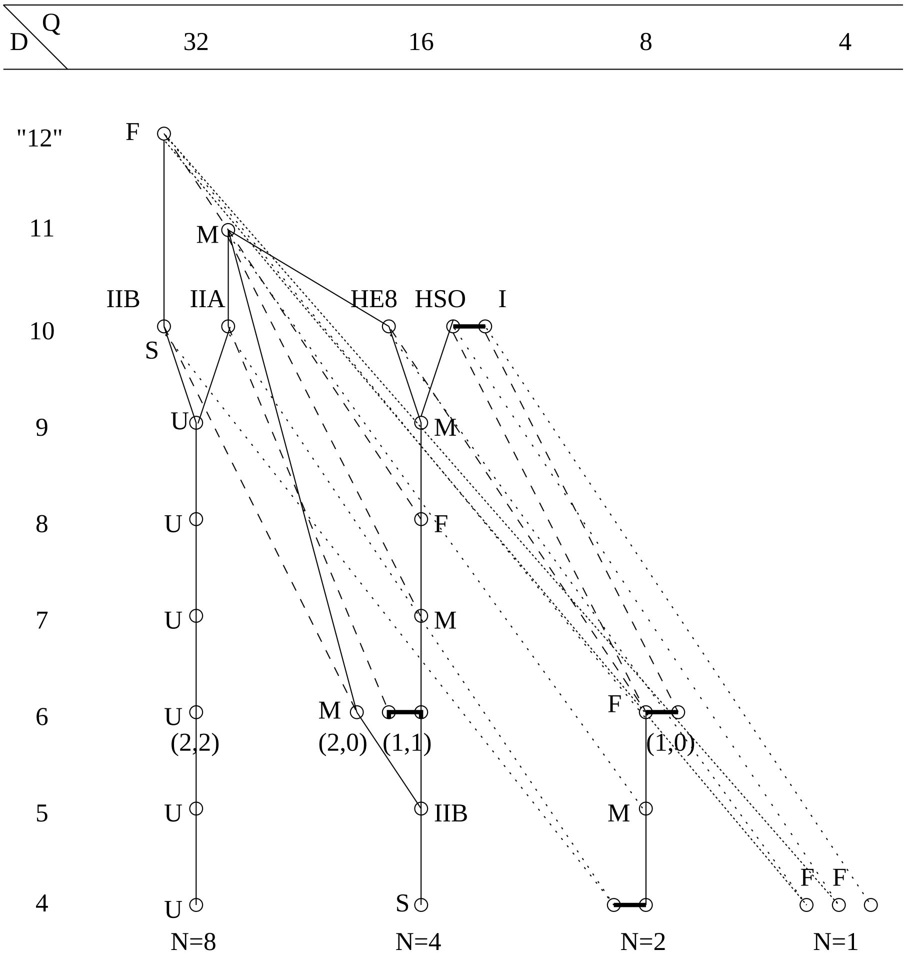

Figure 5. The distinct string theory and their strong-coupling limit. As in Fig. 4 the solid line (−) denotes toroidal compactification, the dashed line (−−) denotes $K3$ compactifications and the dotted line $(\cdots)$ denotes Y_3 compactifications. The fine-dotted line (···) denotes Y_4 compactifications while the horizontal bar (**—**) indicates a string-string duality. The theories marked with a 'U' ('S') have a U–duality (S–duality); the strong-coupling limit of the theories marked by 'M' ('F') are controlled by M-theory (F-theory). With a slight abuse of convention, we also denote the two orbifold compactification $\mathrm{M}/T^1/Z_2$ and $\mathrm{M}/T^5/Z_2$ by a solid line.

now organized by the number of supercharges. The following discussion is visualized in Fig. 5.

$Q = 32$. Theories with $Q = 32$ have a unique massless multiplet which contains all scalar fields on an equal footing and does not single out a string coupling constant. As a consequence there is a discrete symmetry group Γ_U (listed in Table 15) which leads to global identifications in the moduli space and in any given region a different scalar plays the role of the perturbative expansion parameter. A special situation occurs in $D = 10$ where the type-IIB string has $\Gamma_U = \Gamma_S = \mathrm{SL}(2, Z)$ while the strong-coupling limit of the type-IIA string cannot be a string theory but is something new – M-theory – whose low-energy limit is 11-dimensional supergravity.

$Q = 16$. Theories with $Q = 16$ contain, besides the gravitational supermultiplet, also a set of vector supermultiplets. The gravitational multiplet always contains one scalar (cf. Table 11), which can be uniquely identified to play the role of the coupling constant. In $D = 10$ the heterotic SO(32) string and the type-I string are S-dual while the strong-coupling limit of the $\mathrm{E}_8 \times \mathrm{E}_8$ string is governed by M-theory compactified on an orbifold T^1/Z_2. In $D = 9$ the two heterotic theories are perturbatively equivalent and their strong coupling limit is governed both by M-theory and type-I. In $D = 8$ the strong coupling limit is governed both by F-theory and type-I; in $D = 7$ the strong-coupling limit is governed by M-theory compactified on $K3$. In $D = 6$ the heterotic string compactified on T^4 and the type-IIA string compactified on $K3$ are S-dual and the strong-coupling limit of type-IIB compactified on $K3$ is captured by M-theory compactified on T^5/Z_2. In $D = 5$ the strong-coupling limit of the heterotic string compactified on T^5 is governed by type-IIB compactified on $K3 \times T^1$ [33]. In $D = 4$ the antisymmetric tensor and the dilaton can be combined into one complex scalar with an appropriate action of $\Gamma_U = \Gamma_S = \mathrm{SL}(2, Z)$.

$Q = 8$. Theories with $Q = 8$ supercharges exist in $D = 6$ and below. In $D = 6$ the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ and the heterotic SO(32) string are perturbatively equivalent and their nonperturbative regime is governed by F-theory compactified on elliptic Calabi–Yau threefolds. Furthermore, there also exists an S-duality with type-I compactified on $K3$. In $D = 5$ the strong-coupling limit is controlled by M-theory while in $D = 4$ the heterotic string compactified on $K3 \times T^2$ is S-dual to the type-II string compactified on Y_3.

$Q = 4$. Finally, theories with $Q = 4$ only exist in $D = 4$. Both heterotic string theories are non-perturbatively equivalent to F-theory compactified on an elliptic Calabi–Yau fourfold Y_4, while the strong-coupling limit of the type-I theory is not yet completely understood. Some of the type-I models seem to be S-dual to the heterotic vacua [103]. It might well be that all three theories are part of a larger moduli space.

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Appendix

A. Field representations

In section 2 we have outlined the derivation of various supermultiplets of states. At the noninteracting level, these states can easily be described in terms of local fields. The purpose of this appendix is to present suitable field representations for the relevant states. With the help of these field representations one can then write down free massless supersymmetric field theories. Interactions can be introduced separately, for instance, by iteration or by some more systematic procedure. We should stress that there are sometimes ambiguities, because different field representation can describe the same massless free states. At the interacting level, these ambiguities will usually disappear. So the proper choice of the field representation may be subtle. Our strategy is to discuss a number of standard field representations, in D spacetime dimensions, with their corresponding free wave equations and exhibit the behaviour of the corresponding states under helicity rotations. The supermultiplets discussed previously can then be converted into supersymmetric actions, quadratic in the fields. For selfdual tensor fields, the action must augmented by a duality constraint on the corresponding field strength.

A.1. GRAVITON FIELDS

The linearized Einstein equation for $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ implies that (for $D \geq 3$)

$$R_{\mu\nu} \propto \partial^2 h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\rho h_{\rho\nu} - \partial_\nu \partial^\rho h_{\rho\mu} = 0\,, \tag{95}$$

where $h \equiv h_{\mu\mu}$ and $R_{\mu\nu}$ is the Ricci tensor. To analyze the number of states implied by this equation, one may count the number of plane-wave solutions with given momentum q^μ. It then turns out that there are D arbitrary solutions, corresponding to the linearized gauge invariance, $h_{\mu\nu} \rightarrow$

$h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, which can be discarded. Many other components vanish and the only nonvanishing ones require the momentum to be lightlike. These reside in the fields h_{ij}, where the components i, j are in the transverse $(D-2)$-dimensional subspace. In addition, the trace of h_{ij} must be zero. Hence, the relevant plane-wave solutions are massless and have polarizations (helicities) characterized by a symmetric traceless 2-rank tensor. This tensor comprises $\frac{1}{2}D(D-3)$, which transform irreducibly under the $\mathrm{SO}(D-2)$ helicity group of transverse rotations. For the special case of $D=6$ spacetime dimensions, the helicity group is SO(4), which factorizes into two SU(2) groups. The symmetric traceless representation then transforms as a doublet under each of the SU(2) factors and it is thus denoted by (2,2).

As is well known, for $D=3$ there are obviously no dynamic degrees of freedom associated with the gravitational field. When $D=2$ there are again no dynamic degrees of freedom, but here (95) should be replaced by $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$.

A.2. ANTISYMMETRIC TENSOR GAUGE FIELDS

Antisymmetric tensor gauge fields have field-strength tensors which are antisymmetric and of rank $p+2$. They satisfy field equations and Bianchi identities, generalizations of the Maxwell equations, which read

$$\partial_{[\mu_1} F_{\mu_2 \cdots \mu_{p+3}]} = 0\,, \qquad \partial_\mu F^{\mu\nu_1 \cdots \mu_{p+1}} = 0\,. \tag{96}$$

A trivial example is the case $p=-1$, which describes an ordinary scalar field. For $p=-1$ the solution of the first equation of (96) yields $F_\mu = \partial_\mu \phi$, so that the second equation yields the Klein-Gordon equation for ϕ. Another example is the case of a vector gauge field, which corresponds to $p=0$, where (96) are just the Maxwell equations.

There are two ways of dealing with (96). One is to solve the first equation in terms of an antisymmetric tensor gauge field $A_{\mu_2 \ldots \mu_{p+1}}$ of rank $p+1$,

$$F_{\mu_1 \cdots \mu_{p+2}} = (p+2)\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+2}]}\,, \tag{97}$$

and then impose the second equation. The alternative is to first solve the second equation in terms of an antisymmetric gauge field $B_{\nu_1 \cdots \nu_{D-p+3}}$ of rank $D-p+3$,

$$F_{\mu_1 \cdots \mu_{p+2}} = \frac{1}{(D-p-1)!}\, \varepsilon_{\mu_1 \cdots \mu_{p+2}\rho\nu_1 \cdots \nu_{D-p-3}}\, \partial^\rho B^{\nu_1 \cdots \nu_{D-p-3}}\,, \tag{98}$$

after which one imposes the first equation. The second procedure coincides with the first one, but it is based on the dual field strength defined by

$$F_{\mu_1 \cdots \mu_{p+2}} = \frac{1}{(D-p-2)!}\, \varepsilon_{\mu_1 \cdots \mu_{p+2}\nu_1 \cdots \nu_{D-p-2}}\, \tilde{F}^{\nu_1 \cdots \nu_{D-p-2}}\,, \tag{99}$$

which can be written as

$$\tilde{F}_{\nu_1\cdots\nu_{D-p-2}} = (D-p-2)\,\partial_{[\nu_1}B_{\nu_2\cdots\nu_{D-p-3}]}\,. \tag{100}$$

For $\tilde{F}$ the two equations (96) are interchanged and the solution in terms of $B_{\nu_1\ldots\nu_{p'+1}}$, with $p'+p=D-4$ is the dual formulation of the one in terms of $A_{\mu_1\ldots\mu_{p+1}}$. As is well known, in a so-called "first-order" formulation it is possible to have a Lagrangian that encompasses both descriptions.

Let us now examine the plane-wave solutions for the equations (96). We will be somewhat more explicit here and start from a decomposition of $F_{\mu_1\cdots\mu_{p+2}}$ in the momentum representation, with a fixed momentum vector q^μ. Introducing $D-2$ transverse polarization vectors ϵ^i_μ, with $i=1,2,\ldots,D-2$, and an additional vector $\bar{q}^\mu=(-q^0,\vec{q})$, we decompose the field strength according to

$$\begin{aligned} F_{\mu_1\ldots\mu_{p+2}}(q) \;\;\propto\;\; & a_{i_1\cdots i_{p+2}}(q)\,\epsilon^{i_1}_{[\mu_1}\cdots\epsilon^{i_{p+2}}_{\mu_{p+2}]} \\ & + \Big(b_{i_1\cdots i_{p+1}}(q)\,\bar{q}_{[\mu_1} + c_{i_1\cdots i_{p+1}}(q)\,q_{[\mu_1}\Big)\,\epsilon^{i_1}_{\mu_2}\cdots\epsilon^{i_{p+1}}_{\mu_{p+2}]} \\ & + d_{i_1\ldots i_p}(q)\,\epsilon^{i_1}_{[\mu_1}\cdots\epsilon^{i_p}_{\mu_p}\,\bar{q}_{\mu_{p+1}}\,q_{\mu_{p+2}]}\,. \end{aligned} \tag{101}$$

Imposing (96) yields

$$a_{i_1\cdots i_{p+2}}(q) = b_{i_1\cdots i_{p+1}}(q) = d_{i_1\cdots i_p}(q) = 0\,, \qquad q^2\,c_{i_1\cdots i_{p+1}}(q) = 0\,, \tag{102}$$

so that the dynamic degrees of freedom are massless and reside in the antisymmetric $(p+1)$-th rank tensors $c_{i_1\cdots i_{p+1}}(q)$ living in the transverse $(D-2)$-dimensional space. Hence the number of degrees of freedom is equal to $(D-2)!/[(p+1)!(D-p-3)!]$, which is, as expected, invariant under $p\to p'=D-4-p$.

If $D=2$ mod 4 and $p+1=\frac{1}{2}(D-2)$, it is possible to restrict the tensor $F_{\mu_1\cdots\mu_{p+2}}$ to be selfdual or antiselfdual, viz.

$$F_{\mu_1\ldots\mu_{p+2}} = \pm\frac{1}{(p+2)!}\,\varepsilon_{\mu_1\cdots\mu_{2p+4}}\;F^{\mu_{p+3}\cdots\mu_{2p+4}}\,. \tag{103}$$

For such tensors the two equations (96) are no longer independent. The above duality condition on the field strength induces a corresponding $(D-2)$-dimensional duality condition (but now in the space of transverse momenta, which is Euclidean) on the coefficients $c_{i_1\cdots i_{p+1}}(q)$,

$$c_{i_1\cdots i_{p+1}}(q) = \mp\frac{1}{(p+1)!}\,\varepsilon_{i_1\cdots i_{2p+2}}\;c^{i_{p+1}\cdots i_{2p+2}}(q)\,. \tag{104}$$

Consequently the number of independent solutions associated with the antisymmetric tensor is reduced by a factor 2. For $D=6$ where the helicity group factorizes, the representation of the (anti-)selfdual tensor gauge fields, correspond to (3,1) and (1,3). In $D=10$ the (anti-)selfdual tensors correspond to the $\mathbf{35}_s$ and $\mathbf{35}_c$ representations.

A.3. SPINOR FIELDS

Consider a spinor $u(q)$ in D space-time dimensions, satisfying the massless Dirac equation (in momentum space),

$$\not{q}\, u(q) = 0, \tag{105}$$

The Dirac equation implies that $q^2 = 0$. Using the same manipulations as those leading to (3), we rewrite the Dirac equation as

$$q^0 \Big(\mathbf{1} - \tilde\Gamma_D \tilde\Gamma_\perp\Big) u(q) = 0\,, \tag{106}$$

where $\tilde\Gamma_D$ and $\tilde\Gamma_\perp$ were defined in section 2.2.

In odd dimensions $\tilde\Gamma_D$ is proportional to the unit matrix, so that the above condition determines that the spinors are reduced to a subspace where $\tilde\Gamma_\perp = \pm\mathbf{1}$. For even dimensions the states constitute a spinor representation of the helicity group whose chirality is related to the D-dimensional chirality of the spinor field. For instance, for $D=6$ dimensions a chiral spinor will transform under only one of the SU(2) groups of the helicity group, so we have either (2,1) of (1,2). For Majorana-Weyl spinors in $D=10$, the states transform as $\mathbf{8}_c$ or $\mathbf{8}_s$, depending on the chirality of the spinor field.

A.4. GRAVITINO FIELDS

The gravitino field is a vector-spinor ψ_μ and acts as the gauge field of local supersymmetry transformations. Free gravitini satisfy the Rarita-Schwinger equation

$$\Gamma^\mu(\partial_\mu\psi_\nu - \partial_\nu\psi_\mu) = 0\,. \tag{107}$$

To examine the nature of plane-wave solutions, we again consider the momentum representation and decompose $\psi(q)$ as

$$\psi_\mu(q) = u_i(q)\,\epsilon^i_\mu + v(q)\,\bar q_\mu + w(q)\,q_\mu\,, \tag{108}$$

where the coefficient functions $u_i(q)$, $v(q)$ and $w(q)$ are spinors. The field equation (107) takes the form

$$\not{q}\, u_i(q)\,\epsilon^i_\nu - [\not\epsilon^i u_i(q) - \not{\bar q}\, v(q)]\, q_\nu - \not{q}\, v(q)\,\bar q_\nu = 0\,, \tag{109}$$

where $\not\epsilon^i = \epsilon^i_\mu \Gamma^\mu$. The spinor $w(q)$, which is subject to gauge transformations $\delta\psi_\mu = \partial_\mu \epsilon$, is not determined by the gauge invariant field equation (107) and can be discarded. The remaining spinors $u_i(q)$ and $v(q)$ satisfy

$$\not q u_i(q) = \not q v(q) = 0\,, \qquad \not\bar q v(q) = \not\epsilon^i u_i(q)\,. \tag{110}$$

Multiplying the last equation with $\not q$ and using the first two equations and $\epsilon^i \cdot q = 0$, one derives $q \cdot \bar q v(q) = 0$. Hence $v(q) = 0$, so that we are left with two equations for $u_i(q)$,

$$\not q u_i(q) = \not\epsilon^i u_i(q) = 0\,. \tag{111}$$

Hence the gravitino states transform under transverse rotations according to the highest helicity representation contained in the product of a vector and a spinor representation.

For instance, in $D = 6$ dimensions $u_i(q)$ transforms as a chiral vector-spinor, which is a product of (2,2) with (2,1) or (1,2). This product decomposes into $(3,2) + (1,2)$, or $(2,3) + (2,1)$, respectively. The second representation is again suppressed by virtue of the second condition in (111), so that we are left with (3,2) or (2,3).

In $D = 10$ spacetime dimensions a chiral gravitino field $u_i(q)$ transforms as a chiral vector-spinor, which constitutes a tensor product $\mathbf{8}_v \times \mathbf{8}_c$ (or $\mathbf{8}_v \times \mathbf{8}_s$, depending on the chirality). According to the multiplication rules (6), this product decomposes into $\mathbf{8}_s + \mathbf{56}_s$ (or, $\mathbf{8}_c + \mathbf{56}_c$). However, the second equation in (111), which is SO(8) covariant, imposes eight conditions thus suppressing the $\mathbf{8}_s$ or $\mathbf{8}_c$ representation. Consequently chiral gravitini transform according to the $\mathbf{56}_s$ or $\mathbf{56}_c$ representations of SO(8).

B. Coupling constants of low-energy effective field theories

In section 2 we discussed various field theories that play a role as effective low-energy field theories for superstrings. The effective field theories can be rigorously derived from the underlying string theory and in this process the free parameters of the field theories are expressed in terms of the parameters of the string theory itself. The purpose of this appendix is to briefly recall the various possibilities of deriving the low-energy effective action.

One method to obtain the low-energy effective action is known as the 'S-matrix approach', which was pioneered in [39]. Here one computes physical scattering amplitudes in both string theory and the low-energy field theory and demands their equality in the limit $p \ll M_s$ where p is the characteristic momentum of the scattering process. This method is carried out most conveniently in the Einstein frame. Alternatively one can use the 'σ-model approach' which was pioneered in [104]. One imposes conformal invariance on the 2-dimensional σ-model specified by the action (63). This requirement leads to field equations in spacetime which coincide with the field equations obtained from an action in the string frame. (The σ-model approach is not applicable to all string compactifications.)

Let us first outline how the relation (60) emerges in the S-matrix approach. From (57) and (63) we know that the dilaton couples to the topology of the world sheet, so that in leading order (genus-0), the N-particle S-matrix elements are proportional to g_s^{N-2} multiplied by an appropriate power of α', in accordance with dimensional counting. On the other hand, the corresponding S-matrix elements when calculated from the effective field theory, are expressed in terms of Newton's constant. Comparing the S-matrix elements one obtains (suppressing numerical factors) the relation (60),

$$(\kappa_D^2)^{\text{physical}} = \kappa_D^2\, \lambda^{(2-D)/2} = \alpha'^{(D-2)/2}\, \mathrm{e}^{2\langle\phi\rangle}\,, \tag{112}$$

with λ given in the Einstein frame.

Note that the parameter κ_D^2 is not determined by (112) in agreement with our previous arguments that it is intrinsically undetermined. There are basically two ways to proceed: First one may *choose* the constant κ_D^2 to be Newton's constant. This implies that one has to expand the metric around $g_{\mu\nu} = \eta_{\mu\nu}$ in the Einstein frame, so that $\lambda = 1$. This is a convenient setting, which is most commonly used (see, e.g. [39]) and which leads to $\kappa_D^2 = (\alpha')^{(D-2)/2} g_s^2$. However, this choice implies that a coupling constant (κ_D^2) in the effective Lagrangian depends on a parameter (g_s) that arises as the vacuum-expectation value of the dilaton. Alternatively one could insist that any dependence on g_s only arises as a result of the explicit couplings of the dilaton field in the effective Lagrangian. Or in other words, no *parameters* of the effective action are chosen to explicitly depend on g_s. In the Einstein frame this requires to expand the metric around $g_{\mu\nu} = \lambda\, \eta_{\mu\nu}$ with $\lambda^{(D-2)/4} = g_s$, while in the string frame an expansion around $g_{\mu\nu} = \eta_{\mu\nu}$ is necessary. In both frames one obtains $\kappa_D^2 = \alpha'^{(D-2)/2}$ with no explicit dependence on g_s. In the string frame, this effective action coincide with the one obtained by the σ-model approach.[35] In the Einstein frame this choice is somewhat awkward and rarely used.

As a further illustration of the two different parameter choices let us consider higher order gravitational interactions which generically arise in string theory. For example, the string calculations (still to leading order in g_s) of the S-matrix of graviton-graviton scattering give rise to contributions

[35] Observe that the σ-model approach does not insist on a particular ground-state value for the metric. Since it derives the effective action by integration of the field equations, it determines the Lagrangian only up to an overall constant.

that require an effective interaction quartic in the Riemann tensor

$$\mathcal{L}_{\rm eff} = \frac{1}{\kappa_D^2}\sqrt{-g}\,{\rm e}^{-2\phi}\Big[R + A\,\alpha'^3\Big(R_{abcd}R^{cdef}R_{efgh}R^{ghab} + \cdots\Big)\Big]\,. \qquad (113)$$

We have displayed $\mathcal{L}_{\rm eff}$ is the string frame and the higher-order terms depend on a dimensionless constant A which is independent of $g_{\rm s}$. In the Einstein frame the dilaton factor in front of the Ricci scalar is removed by a Weyl transformation which also changes the coupling in front of the R^4-terms into a factor $\exp(-12\phi/(D-2))$. Expanding the metric in the Einstein frame around $g_{\mu\nu} = \eta_{\mu\nu}$ and comparing the relevant S-matrix elements to the string calculation, one finds that $\kappa_D^2 = (\alpha')^{(D-2)/2}\, g_{\rm s}^2$ while $A = g_{\rm s}^{12/(D-2)}$ [39].[36] Again, the dependence on $g_{\rm s}$ cannot be tied to the presence of a dilaton interaction in neither one of the two terms. However, if one expands the Einstein metric around $g_{\mu\nu} = g_{\rm s}^{-4/(D-2)}\,\eta_{\mu\nu}$ (or equivalently, the metric in the string frame around $g_{\mu\nu} = \eta_{\mu\nu}$) one finds $\kappa_D^2 = (\alpha')^{(D-2)/2}$ and $A =$ constant – both couplings independent of $g_{\rm s}$. This form of the parameters is also obtained in the σ-model approach where the R^4-term arises as a 4-loop counterterm [105].

The final point of this appendix concerns the dilaton in arbitrary spacetime dimensions. It can be defined as the field in the σ-model action (63) taken in D dimensions. The corresponding vertex operator is composed only out of operators of the spacetime sector of the conformal field theory (CFT) and no operators in the 'internal CFT'. Let us denote the dilaton defined in this manner by $\phi^{(D)}$. This definition has the virtue that $\phi^{(D)}$ is invariant under T-duality transformations of the D-dimensional theory which originates from the existence of equivalence classes of the internal CFT. The same is true for the graviton and the antisymmetric tensor, whose vertex operators are similarly composed solely out of the spacetime sector of the CFT.

Compactification of the low-energy effective actions relates the dilatons of different dimensions by a volume-dependent factor of the compactification manifold Y and the metric associated with the compactified dimensions. More precisely, starting in $D = 10$ one finds (in the string frame)

$$\frac{1}{\kappa_{10}^2}\,{\rm e}^{-2\phi^{(10)}}\,V_n\,\sqrt{\det g_n} = \frac{1}{\kappa_{10-n}^2}\,{\rm e}^{-2\phi^{(10-n)}}\;, \qquad (114)$$

where n is the dimension of Y, V_n is the volume of the n-dimensional compactified coordinates and g_n is the metric associated with the compactified

[36] Note that the relation with the dilaton field used in the second reference of [39] is given by $\phi - \langle\phi\rangle = \sqrt{2}\kappa D$ (in 10 spacetime dimensions).

dimensions.[37] The latter is directly related to the vacuum-expectation value of certain moduli fields. Furthermore, the space-time part of the metric (in the string frame) is left unchanged in the parametrization used in (114) and the D-dimensional quantities are defined by

$$\frac{1}{\kappa_{10-n}^2} = \frac{V_n}{\kappa_{10}^2}\,, \qquad \phi^{(10-n)} = \phi^{(10)} - \tfrac{1}{4}\log\det g_n\ . \tag{115}$$

The D-dimensional string metric, the dilaton and the antisymmetric tensor are invariant under T-duality. For the perturbative dualities one can demonstrate this fact by performing a dimensional reduction on the 10-dimensional supergravity field theory (for instance on the Lagrangian (38)) from 10 to $10-n$ dimensions. Using the arguments of section 2.5 one establishes the existence of a rank-n group of invariances that leaves the string metric, the dilaton (defined according to (115)) and the antisymmetric tensor field invariant. However, the original 10-dimensional dilaton $\phi^{(10)}$ transforms under these symmetries.

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[37]To be precise, V_n is the (higher-dimensional) analog of the length L introduced in section 2.5. The geodesic volume which generalizes R_{11} in (44) is instead proportional to $V_n\sqrt{\det g_n}$.

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