

DUALITY IN STRING THEORY

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In this lecture we review some of the recent developments in string theory on an introductory and qualitative level. In particular we focus on $S - T - U$ dualities of toroidally compactified ten-dimensional string theories and outline the connection to M-theory. Dualities among string vacua with less supersymmetries in six and four space-time dimensions is discussed and the concept of F-theory is briefly presented.

1 Introduction

During the past two years string theory has seen spectacular progress; for the first time it has been possible to control a subset of the interaction in the strong coupling regime of string theory. This is due to the observation that many (if not all) of the perturbatively distinct string theories are related when all quantum corrections are taken into account.¹⁻⁴⁶ In particular it has been observed that often the strong coupling regime of one string theory can be mapped to the weak coupling regime of another, perturbatively different string theory. This situation is termed duality among string theories and it offers the compelling picture that the known perturbative string theories are merely different regions in the moduli space of one underlying theory termed ‘M-theory’. The purpose of this lecture is to review on an introductory level some of the recent advances in establishing such string dualities.^a However, this lecture does not cover all aspects of string duality but rather is meant to compliment the lectures by C. Bachas, M. Duff, A. Giveon and F. Quevedo at this meeting.

This lecture is organized as follows. In section 2 we discuss the ‘old’ perturbative string theory. In sections 3-5 we review T-, S- and U-dualities

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^aWe omit most of the technical details and refer the reader to the original literature. There are a number of nice review lectures⁴⁷⁻⁵¹ which also include a more extended list of references.

of toroidally compactified string theories. Section 6 introduces M-theory and its connection to ten-dimensional string theory. In sections 7 and 8 we discuss string dualities for compactifications with six (section 7) and four (section 8) space-time dimensions where some of the supercharges are broken.

2 Perturbative string theory

In string theory the fundamental objects are one-dimensional strings which, as they move in time, sweep out a two-dimensional worldsheet Σ . This worldsheet is embedded in some higher dimensional target space which is identified with a Minkowskian space-time. Particles in this target space appear as (massless) eigenmodes of the string and their scattering amplitudes are generalized by appropriate scattering amplitudes of strings. Strings can be open or closed, oriented or unoriented.^b String scattering amplitudes are built from a fundamental vertex; for closed strings this vertex is depicted in figure 1.^c It rep-

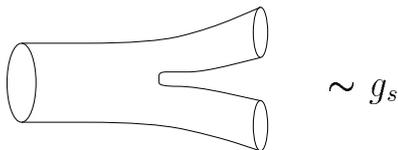


Figure 1: Fundamental string interaction

resents the splitting of a string or the joining of two strings and the strength of this interaction is governed by a dimensionless string coupling constant g_s . Out of the fundamental vertex one composes all possible closed string scattering amplitudes \mathcal{A} , for example the four-point amplitude shown in figure 2. The expansion in the topology of the Riemann surface (i.e. the number of holes in the surface) coincides with a power series expansion in the string coupling constant formally written as

$$\mathcal{A} = \sum_n g_s^{2n+2} \mathcal{A}^{(n)} , \quad (2.1)$$

where $\mathcal{A}^{(n)}$ is the scattering amplitude on a Riemann surface of genus n . The exponent $2n + 2$ of g_s is nothing but (minus) the Euler characteristic of a two dimensional surface with n handles and four boundaries corresponding to the two incoming and the two outgoing strings. In all string theories there is a

^bFor an introduction to string theory we refer to the literature.⁵²⁻⁵⁵

^cOpen string scattering is discussed in the lecture by C. Bachas.

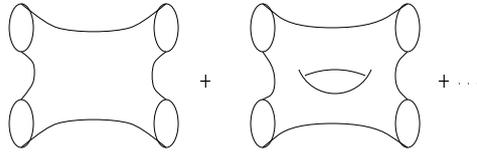


Figure 2: String perturbation theory. The order of g_s in the perturbative expansion is governed by the number of handles in the world sheet.

massless scalar field D called the dilaton which couples to the Gauss-Bonnet density of the world sheet. Therefore the vacuum expectation value (VEV) of the dilaton determines the size of the string coupling and one finds^{56, 52}

$$g_s = e^{\langle D \rangle} . \quad (2.2)$$

g_s is a free parameter since the dilaton is a flat direction (a modulus) of the effective potential. String perturbation theory is restricted to the region of parameter space (which is also called the moduli space) where $g_s < 1$ and the tree level amplitude (genus 0) is the dominant contribution with higher loop amplitudes suppressed by higher powers of g_s .

The interactions of the string are governed by a two-dimensional field theory on the world-sheet Σ . \mathcal{A} can be interpreted as an unitary scattering amplitude in the target space whenever this two-dimensional field theory is conformally invariant. This condition puts a restriction on the number of space-time dimensions d and the space-time spectrum. The known consistent string theories necessarily have $d \leq 10$ and they are particularly simple in the maximal possible dimension $d = 10$.^d There are only five consistent string theories in $d = 10$: the type IIA, the type IIB, the heterotic $E_8 \times E_8$, the heterotic $SO(32)$ and the type I $SO(32)$ string; their massless spectra are summarized in Table 1. All five theories are space-time supersymmetric. The type II theories have 32 supercharges Q and two gravitinos ψ_μ ; in type IIA they have opposite chirality while in type IIB they have the same chirality. This is often referred to as the non-chiral and chiral $N = 2$ supersymmetry in $d = 10$. The other three string theories have only half of the supercharges (16) or what is called $N = 1$ supersymmetry.^e The heterotic string theories have

^d An additional constraint arises from the requirement of modular invariance of one-loop amplitudes which results in an anomaly free spectrum of the corresponding ten-dimensional low energy effective theory.⁵⁷

^e In space-time dimensions other than 10 there are ambiguous definitions of what is meant by N . Therefore it is convenient to always just count the total number of supercharges and this is what we always do in this lecture.

Table 1: Consistent string theories in ten dimensions.

| | # of Q 's | # of ψ_μ 's | bosonic spectrum | |
|-------------------------------|-------------|--------------------|---|---|
| IIA | 32 | 2 | NS-NS | $g_{\mu\nu}, b_{\mu\nu}, D$ |
| | | | R-R | $A_\mu, C_{\mu\nu\rho}$ |
| IIB | 32 | 2 | NS-NS | $g_{\mu\nu}, b_{\mu\nu}, D$ |
| | | | R-R | $e_{\mu\nu\rho\sigma}^*, b'_{\mu\nu}, D'$ |
| heterotic $E_8 \times E_8$ | 16 | 1 | $g_{\mu\nu}, b_{\mu\nu}, D$ A_μ^a in adjoint of $E_8 \times E_8$ | |
| heterotic $SO(32)$ | 16 | 1 | $g_{\mu\nu}, b_{\mu\nu}, D$ A_μ^a in adjoint of $SO(32)$ | |
| type I $SO(32)$ | 16 | 1 | NS-NS | $g_{\mu\nu}, D$ |
| | | | open string | A_μ^a in adjoint of $SO(32)$ |
| | | | R-R | $b_{\mu\nu}$ |

non-Abelian gauge symmetries and thus vector bosons and their superpartners (gauginos) in the adjoint representation of the gauge group. In addition to the closed string theories there is one supersymmetric consistent theory (called type I) containing unoriented open and closed strings with $SO(32)$ Chan-Paton factors coupling to the ends of the open string.

The bosonic spectrum of type I and type II theories can appear in two distinct sectors (NS-NS or R-R) depending on the boundary conditions of the worldsheet fermions. Also note that the metric $g_{\mu\nu}$, the antisymmetric tensor $b_{\mu\nu}$ and the dilaton D are common to all five vacua.

3 T-duality

The ten-dimensional string theories can be compactified to obtain theories with a lower number of space-time dimensions d . The simplest of such compactifications are toroidal compactifications where the internal manifold is a n -dimensional torus T^n ($n = 10 - d$). Such compactifications leave all supercharges unbroken.

For simplicity we start by considering closed string theories with one compact dimension (or a S^1 -compactification). In this case there are nine space-time coordinates X^μ satisfying the boundary conditions^f

$$X^\mu(\sigma = 2\pi, \tau) = X^\mu(\sigma = 0, \tau), \quad (3.1)$$

^fThe indices μ, ν always denote the space-time directions, i.e. $\mu = 0, \dots, d - 1$.

and one internal coordinate Y which can wrap m times around the S^1 of radius R ,

$$Y(\sigma = 2\pi, \tau) = Y(\sigma = 0, \tau) + 2\pi m R . \quad (3.2)$$

The massless spectrum of the nine-dimensional theory includes the two Abelian Kaluza–Klein gauge bosons $g_{\mu 10}$ and $b_{\mu 10}$ leading to a gauge group $G = U(1)^2$. In addition there is a massless scalar field $g_{10 10}$ which is a flat direction of the effective potential and which is directly related to the radius R . The appearance of flat directions is a generic feature of string compactifications and such scalar fields are called moduli. For the case at hand this moduli space is one-dimensional and hence there is a one parameter family of inequivalent string vacua.^g The boundary condition (3.2) leads to a quantization of the internal momentum component p_{10} and a whole tower of massive Kaluza–Klein states labelled by an integer k . In addition there are also massive winding modes labelled by m and altogether one finds

$$M^2 = \frac{k^2}{R^2} + \frac{m^2 R^2}{4} + (N + \tilde{N} - 2) , \quad (3.3)$$

where N and \tilde{N} are the number operators of the left and right moving oscillator excitations. This spectrum is invariant under the exchange of R with $2/R$ if simultaneously the winding number m is exchanged with the momentum number k . This is the first duality – a Z_2 invariance of the spectrum, called *T-duality*.^h

This simple example already displays another generic feature of string compactifications – special points (or subspaces) in the moduli space where the gauge symmetry is enhanced. For fixed $R = \sqrt{2}$ there are four additional massless gauge bosons corresponding to $mk = \pm 1, N + \tilde{N} = 1$. These states combine with the two $U(1)$ gauge fields and enlarge the $U(1)^2$ gauge symmetry toⁱ

$$U(1) \times U(1) \longrightarrow SU(2) \times SU(2) . \quad (3.4)$$

The exact same generic features – Abelian Kaluza–Klein gauge bosons and a moduli space with non-Abelian enhancement on special subspaces – also occur for higher dimensional toroidal compactifications on a torus T^n . The massless states at a generic point in the moduli space include the Kaluza–Klein gauge bosons $g_{\mu i}$ and $b_{\mu i}$ of the gauge group $G = U(1)^{2n}$ and the toroidal

^gThe different solutions of a given string theory are often referred to as the vacuum states of that string theory or simply as the string vacua.

^hFor an extensive review about T-duality including further references we refer the reader to the literature.⁵⁸

ⁱNote that the enhancement of the gauge symmetry does not change the rank of G but only its size.

moduli g_{ij} , b_{ij} parameterizing a moduli space of inequivalent string vacua.^{*j*} This moduli space is found to be the n^2 -dimensional coset space⁵⁹

$$\mathcal{M} = \frac{SO(n, n)}{SO(n) \times SO(n)} \Big/ \Gamma_T, \quad (3.5)$$

where

$$\Gamma_T = SO(n, n, Z) \quad (3.6)$$

is the T-duality group relating equivalent string vacua.

In heterotic or type I theories there are n additional scalars A_i^a transforming in the adjoint representation of $E_8 \times E_8$ or $SO(32)$. However, only the $16 \cdot n$ scalars in the Cartan subalgebra are flat directions and their (generic) VEVs break the non-Abelian gauge symmetry to $U(1)^{16}$. Together with the toroidal moduli g_{ij} , b_{ij} they parameterize the $n(n+16)$ dimensional moduli space⁵⁹

$$\mathcal{M} = \frac{SO(n, n+16)}{SO(n) \times SO(n+16)} \Big/ \Gamma_T, \quad (3.7)$$

with the T-duality group

$$\Gamma_T = SO(n, n+16, Z). \quad (3.8)$$

Thus, in space-time dimensions less than ten there is a moduli space of inequivalent string vacua and at generic points in this moduli space the gauge group is $G = U(1)^{2n+16}$. On special subspaces of the moduli space there can be non-Abelian enhancement of the $U(1)^{16}$ factor, at most up to the original $E_8 \times E_8$ or $SO(32)$. Furthermore, a discrete T-duality group identifies (equivalent) string vacua.

It has also been shown that below ten dimensions the heterotic $E_8 \times E_8$ theory and the heterotic $SO(32)$ theory are continuously connected in the moduli space. That is, the two theories sit at different points of the same moduli space of one and the same heterotic string theory.⁶⁰ A very similar situation is found for type II theories below ten dimensions. Although the massless spectrum of type IIA and type IIB looks rather different in $d = 10$ it precisely matches in $d \leq 9$ as can be seen from table 2. Their low energy effective actions can also be shown to agree⁶¹ and furthermore, a careful analysis of the limits $R \rightarrow 0$ and $R \rightarrow \infty$ reveals a flip of the chiralities of the space-time fermions as is necessary to connect the non-chiral type IIA with the chiral Type IIB theory.^{62, 63} Thus, for $d \leq 9$ type IIA and type IIB also sit in the same moduli space and are T-dual to each other in that type IIA at a large

^{*j*}The indices i, j run over the internal dimensions, i.e. $i, j = 1, \dots, n = 10 - d$.

Table 2: Massless type II fields in nine dimensions

| | NS-NS | R-R |
|-----|---|---|
| IIA | $g_{\mu\nu}, b_{\mu\nu}, D, g_{\mu 10}, b_{\mu 10}$ | $A_\mu, A_{10}, C_{\mu\nu\rho}, C_{\mu\nu 10}$ |
| IIB | $g_{\mu\nu}, b_{\mu\nu}, D, g_{\mu 10}, b_{\mu 10}$ | $b'_{\mu 10}, D', C_{\mu\nu\rho 10}, b'_{\mu\nu}$ |

compactification radius is equivalent to type IIB at a small compactification radius and vice versa.

Finally, toroidally compactified type I theories naively do not have a T-duality symmetry. However, once extended objects – termed D-branes – are included as possible configurations type I theories also are T-dual.^{63–70} This exciting aspect of the recent developments is covered in the lecture of C. Bachas and elsewhere⁴⁹ and will not be discussed any further here.

4 S–duality

S–duality refers to the quantum equivalence of two theories A and B which are perturbatively distinct. Generically the strong coupling regime of A is mapped to the weak coupling regime of B and simultaneously the perturbative excitations of A are mapped to the non-perturbative excitations of the dual theory B and vice versa. Such a relation is of prime interest since it opens up the possibility to control the strong coupling regime of both theories.^k The concept of S–duality was first developed in four-dimensional $N = 4$ supersymmetric Yang–Mills theories which is a somewhat special case since in terms of the above terminology one has $B = A$ or in other words a selfduality.^{71–74, 7, 8} We start the discussion of S–duality with the $N = 4$ Yang–Mills theory⁷⁵ and then briefly turn to the S–duality between the heterotic $SO(32)$ and the type I string.

4.1 $N = 4$ Yang–Mills theories in four space-time dimensions

Extended supersymmetries are generated by N chiral charges Q_α^I and N anti-chiral charges $Q_{\dot{\beta}}^I$ ($I = 1, \dots, N$) which transform as Weyl spinors under the

^kThe T–duality discussed in the previous section also is an equivalence between string vacua but it already holds in perturbation theory. However, it also identifies different regions in the moduli space of string theories but these identifications do not involve the string coupling or equivalently the dilaton.

Lorentz group and satisfy the algebra^l

$$\left\{Q_{\alpha}^I, Q_{\beta}^J\right\} = 2\sigma_{\alpha\beta}^{\mu} P_{\mu} \delta^{IJ}, \quad \left\{Q_{\alpha}^I, Q_{\beta}^J\right\} = \epsilon_{\alpha\beta} Z^{IJ}. \quad (4.1)$$

For $N = 4$ the supersymmetric gauge multiplet contains a vector boson A_{μ} , four Weyl fermions χ_{α}^I and six real scalar fields $\phi^{[IJ]}$ all in the adjoint representation of a gauge group G . The scalar fields in the Cartan subalgebra of G are flat directions of the potential and thus at a generic point in their field space G is broken to its maximal Abelian subgroup $[U(1)]^{\text{rank}(G)}$. It has been shown that the field equations of such theories have solitonic solutions which can be interpreted as magnetic monopoles or more generally dyons – states which carry both electric charge q_e and magnetic charge q_m .^{77–79} Any two such states with charges (q_e^1, q_m^1) and (q_e^2, q_m^2) satisfy the Dirac-Zwanziger quantization condition^{80–82}

$$q_e^1 q_m^2 - q_e^2 q_m^1 = 2\pi k, \quad (4.2)$$

with k being integer.^m This condition is solved in terms of an elementary electric charge e by⁷⁴

$$q_e = n_e e - n_m e \frac{\theta}{2\pi}, \quad q_m = n_m \frac{4\pi}{e}, \quad (4.3)$$

where n_m, n_e are integers and θ is the coupling of the $F\tilde{F}$ term (the θ -angle). Furthermore, the mass of any dyonic state satisfies the BPS bound^{83, 84}

$$M \geq \langle\phi\rangle \sqrt{q_e^2 + q_m^2}. \quad (4.4)$$

For a $U(1)$ gauge theory the Maxwell equations in the vacuum are invariant under the exchange $\mathbf{E} \rightarrow -\mathbf{B}, \mathbf{B} \rightarrow \mathbf{E}$ (or equivalently under an exchange of the Bianchi identities with the field equation). In the presence of magnetic monopoles this symmetry might be extended to the full Maxwell equations including matter. Indeed, Montonen and Olive conjectured that there exists such symmetry if the electric and magnetic quantum numbers are interchanged and in addition the gauge coupling e is inverted according to⁷¹

$$e \rightarrow \frac{4\pi}{e}. \quad (4.5)$$

^lWe use the conventions $\alpha, \dot{\alpha} = 1, 2$ and thus the total number of supercharges is $4N$.⁷⁶

^mThis condition ensures that one test dyon with charges (q_e^1, q_m^1) moving in the field of another dyon with charges (q_e^2, q_m^2) does not feel an Aharonov-Bohm effect.

More generally this symmetry can be combined with the periodicity of the θ -angle $\theta \rightarrow \theta + 2\pi$. Together they generate the discrete group $SL(2, Z)$ acting on the complex coupling^{1, 7}

$$S := \frac{4\pi}{e^2} + i\frac{\theta}{2\pi} \quad (4.6)$$

according to

$$S \rightarrow \frac{aS - ib}{icS + d}, \quad ad - bc = 1, \quad (4.7)$$

where $a, d, b, c \in \mathbf{Z}$. In terms of this more general symmetry operation the quantum numbers (n_e, n_m) should be relabelled according to⁷

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}. \quad (4.8)$$

One can also check that the BPS bound (4.4) is invariant under the transformations (4.7),(4.8). This is of significance since the BPS-bound is implied by the $N = 4$ supersymmetry algebra and is believed to be an exact quantum formula (perturbatively and non-perturbatively)^{7,2}. To see this, one uses the $U(4)$ automorphism symmetry of the supersymmetry algebra to transform the matrix Z^{IJ} of eq. (4.1) into the form⁸⁵

$$Z^{IJ} = \begin{pmatrix} Z_1\epsilon & 0 \\ 0 & Z_2\epsilon \end{pmatrix}, \quad \text{where } \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.9)$$

The positivity of the superalgebra implies a bound on the mass of all states

$$M \geq |Z_{1,2}|. \quad (4.10)$$

The massive representations of the $N = 4$ algebra depend on eq. (4.10)⁸⁵. First, there is the ‘long’ massive multiplet with 256 states and maximal spin $s = 2$. The mass of this multiplet is strictly larger than any of the two central charges. The ‘middle’ multiplet contains 64 states of maximal spin $s = 3/2$ and its mass coincides with one of the two central charges. Finally, the ‘short’ multiplet contains 16 states of maximal spin $s = 1$ with a mass that saturates both central chargesⁿ

$$M = |Z_1| = |Z_2|. \quad (4.11)$$

The fact that these short multiplets have fewer degrees of freedom implies a non-renormalization theorem for their mass and therefore the BPS states can

ⁿThis is also the content of the massless representation; for example the Yang-Mills multiplet discussed at the beginning of this section has a total of 16 states.

be ‘followed’ into the strong coupling regime.^o This fact together with the $SL(2, Z)$ invariance of the BPS-bound (4.10) already supports the Montonen-Olive conjecture. However, a further highly non-trivial test of S-duality has been performed.⁷ Starting from a massive, short BPS-multiplet W^+ with charges $(n_e = 1, n_m = 0)$ S-duality predicts 16 dyonic multiplets with charges

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad (4.12)$$

where the constraint $ad - bc = 1$ implies that a and c have to be relatively primed. Indeed A. Sen found the predicted solitonic solutions for $(n_e = \text{odd}, n_m = 2)$ with exactly the right multiplicity.⁷ All further tests so far confirm S-duality as an exact symmetry in $N = 4$ Yang-Mills theories.^{11, 86, 87}

In order to discuss S-duality in string theory we have to ‘embed’ the previous discussion into $N = 4$ supergravity. The supergravity multiplet contains one graviton $g_{\mu\nu}$, four gravitinos $\psi_{\mu\alpha}^I$, six Abelian vector bosons $\gamma_\mu^{[IJ]}$ called graviphotons, four Weyl fermions χ_α^I , an antisymmetric tensor $b_{\mu\nu}$ and a real scalar D . In four space-time dimensions an antisymmetric tensor contains only one physical degree of freedom and is dual to a real scalar denoted by a . It can be combined with the dilaton into the complex scalar

$$\frac{S}{4\pi} = e^{-2D} + ia . \quad (4.13)$$

When coupled to a Yang-Mills vector multiplet the coupling S defined in eq. (4.6) is nothing but the VEV of this scalar S . The classical equations of motion are invariant under an $SL(2, R)$ acting on S as in eq. (4.7) with real coefficients but space-time instantons break this continuous symmetry to the discrete $SL(2, Z)$ symmetry.^{1, 2, 8}

$N = 4$ supergravity coupled to Yang-Mills gauge multiplets arises in the low energy limit of toroidally compactified heterotic string theories. The bosonic part of the massless spectrum is obtained by dimensional reduction of the ten-dimensional spectrum displayed in table 1. One finds the graviton $g_{\mu\nu}$, the vector bosons $g_{\mu i}, b_{\mu i}, A_\mu^a$ and the scalars $b_{\mu\nu} \sim a, D, g_{ij}, b_{ij}, A_i^a$ (as before Latin indices label compactified directions so for the case at hand we have $i, j = 1, \dots, 6$). The A_μ^a and A_i^a combine into a non-Abelian vector multiplet of either $E_8 \times E_8$ or $SO(32)$ while the remaining degrees of freedom form the gravitational multiplet and six Abelian vector multiplets. As we discussed in section 3, the scalars A_i^a in the Cartan subalgebra are flat directions whose

^oQuantum corrections changing BPS states into non-BPS states would imply that the number of degrees of freedom is changed by quantum corrections.

(generic) VEVs break the gauge group to $[U(1)]^{16}$. Thus at a generic point in field space there are 22 vector multiplets and one gravitational multiplet. The moduli space of the scalar fields is found to be^{59, 8}

$$\mathcal{M} = \frac{SO(6, 22)}{SO(6) \times SO(22)} \Big/ \Gamma_T \times \frac{SL(2, R)}{U(1)} \Big/ \Gamma_S \quad (4.14)$$

with the duality groups^p

$$\Gamma_T = SO(6, 22, Z) \ , \quad \Gamma_S = SL(2, Z) \ . \quad (4.15)$$

Γ_T only acts on the scalar moduli of the vector multiplets while Γ_S transforms the two moduli of the gravity multiplet. In general there is a yet larger duality group mixing all of the moduli termed U-duality⁴² this will be the topic of section 5.

4.2 S-duality between Type I and Heterotic $SO(32)$

In string theory a strong-weak coupling duality has been established between the type I theory and the heterotic $SO(32)$ theory^{14, 22, 23, 29}. First of all, from table 1 we immediately infer that the massless spectra of heterotic $SO(32)$ and type I strings are identical. However, the low energy effective actions are not the same in perturbation theory. For the heterotic string one has

$$S_{heterotic} \sim \int d^{10}x \sqrt{-g} e^{-2D} \left\{ R + 4(\partial D)^2 - \frac{1}{4} Tr F^2 - \frac{1}{12} H^2 \right\}, \quad (4.16)$$

where H is the field strength for the antisymmetric tensor field $H = db$. Type I string theory includes unoriented open and closed strings with $SO(32)$ Chan-Paton factors attached to the ends of the open string. The low energy effective action is made out of three contributions: the closed string NS-NS sector with the metric and the dilaton, the open string contribution resulting in the gauge kinetic term for the $SO(32)$ Yang-Mills fields, and the kinetic term for the antisymmetric tensor in the R-R sector

$$S_{type I} \sim \int d^{10}x \sqrt{-g} \left\{ e^{-2D} \left[R + 4(\partial D)^2 \right] - \frac{1}{4} e^{-D} Tr F^2 - \frac{1}{12} H^2 \right\}. \quad (4.17)$$

The different dilaton couplings arise because the second contribution comes from a disc diagram and the last contribution is a R-R term. Replacing

$$g_{\mu\nu} \rightarrow e^D g_{\mu\nu} \ , \quad D \rightarrow -D \quad (4.18)$$

^pOf course Γ_T is consistent with the general formula (3.8).

in the heterotic action (4.16) results in the type I action (4.17). This suggests the following relation between the couplings

$$g_{\text{het}} = e^{D_{\text{het}}} \sim g_{\text{I}}^{-1} = e^{-D_{\text{I}}} . \quad (4.19)$$

In particular the strong coupling limit of the heterotic theory is mapped to the weak coupling limit of the type I theory and vice versa.

Further evidence of this duality has been assembled. In the type I theory there are one-branes and five-branes⁸⁸ which can be mapped to the one- and five-branes^{3, 5, 89, 90} of the heterotic string. The effective D-brane action is given by⁹¹

$$S_{\text{I},p} \sim \frac{1}{g_{\text{I}}} \int d^{p+1} \eta \sqrt{-\det g_{\text{I},p+1}} , \quad (4.20)$$

where for simplicity we focus on vacua with constant dilaton and $H = F = 0$. The coordinates η parameterize the p -brane world volume and $\det g_{\text{I},p+1}$ is the determinant of the induced metric. Performing the duality transformation to the heterotic side (i.e. the inverse of (4.18)) gives

$$S_{\text{het},p} \sim g_{\text{het}}^{\frac{1-p}{2}} \int d^{p+1} \eta \sqrt{-\det g_{\text{het},p+1}} . \quad (4.21)$$

Thus, the one-brane of type I theory is mapped to a string with tension of order g_{het}^0 on the heterotic side. This suggests that the D-one-brane of type I can be identified with the heterotic string²⁹. The D-five-brane on the type I side corresponds to a five-brane with tension of order g_{het}^{-2} on the heterotic side which is the solitonic heterotic five-brane.

Finally, for compactifications (including Wilson lines) of both string theories to nine dimensions it has been observed that points of enhanced gauge symmetry on the heterotic side correspond to points where the perturbative description of type I theory breaks down²⁹.

5 U-duality

The concept of U-duality¹² was first discussed in $N = 8$ supergravity in $d = 4$ which has a total of 32 unbroken supercharges. This theory only has a gravitational multiplet containing the graviton $g_{\mu\nu}$, eight gravitinos $\psi_{\mu\alpha}$, 28 graviphotons γ_{μ} , 56 spin- $\frac{1}{2}$ Weyl fermions and 70 real scalar moduli ϕ . The 70 scalars parameterize the coset space⁹²

$$\mathcal{M}_{\phi} = \frac{E_{7,7}}{SU(8)} , \quad (5.1)$$

where $E_{7,7}$ is a specific non-compact version of E_7 . The equations of motion are invariant under $E_{7,7}(\mathbf{R})$ and the electric and magnetic charges of the 28 graviphotons combine into 28 complex central charges of the $N = 8$ superalgebra.

$N = 8$ supergravity arises as the low energy limit of type IIA (or equivalently type IIB) compactified on a six torus T^6 . As before the bosonic part of the massless spectrum can be obtained by dimensional reduction of the ten-dimensional spectrum listed in table 1. From the NS-NS sector of type IIA one obtains a graviton $g_{\mu\nu}$, 12 graviphotons $g_{\mu i}$, $b_{\mu i}$ and 38 scalars D , $b_{\mu\nu} \sim a$, g_{ij} , b_{ij} . In the R-R sector one finds 16 graviphotons A_μ , $C_{\mu ij}$ and 32 scalars A_i , C_{ijk} , $C_{\mu\nu i}$. The g_{ij} and b_{ij} are the moduli of the six torus which parameterize the moduli space given in eqs. (3.5), (3.6). Similarly, the dilaton-axion system spans a $SL(2, R)/U(1)$ coset divided by the S-duality group (4.15). So, altogether the NS-NS scalars live on the moduli space

$$\mathcal{M}_{\text{NS-NS}} = \frac{S(6,6)}{SO(6) \times SO(6)} \Big/ \Gamma_T \times \frac{SL(2, R)}{U(1)} \Big/ \Gamma_S. \quad (5.2)$$

It has been conjectured¹² that there is a much larger duality group called U-duality which contains $\Gamma_S \times \Gamma_T$ as its maximal non-compact subgroup but transforms all scalars – including the R-R scalars – into each other.⁹ For $N = 8$ supergravity this U-duality group is conjectured to be

$$\Gamma_U = E_{7,7}(Z) \quad (5.3)$$

so that globally the moduli space is

$$\mathcal{M} = \frac{E_{7,7}}{SU(8)} \Big/ \Gamma_U. \quad (5.4)$$

The conjectured presence of this much larger quantum symmetry Γ_U has been supported by a number of facts^{12, 14}. First of all the BPS-bound is found to be invariant under Γ_U . Furthermore, the twelve graviphotons from the NS-NS sector couple to ‘electrically’ charged states of the perturbative string spectrum. However, there can be no R-R charges in the perturbative spectrum and thus U-duality predicts the existence of non-perturbative states which carry R-R charge. Indeed, the solitonic solutions of the field equations come in representations of Γ_U and R-R charged states have been constructed as solitons and also as D-branes.

A similar analysis has been performed for all toroidal compactifications of type II string vacua and the result is listed in table 3¹². G_{SUGRA} denotes

⁹This implies that the R-R scalars are charged under the U-duality group.

Table 3: U-duality groups of toroidally compactified type II string theories.

| d | G_{SUGRA} | Γ_T | Γ_U |
|-----|----------------------------|---------------|----------------------------|
| 10 | $SL(2, R)$ | 1 | $SL(2, Z)$ |
| 9 | $SL(2, R) \times SO(1, 1)$ | Z_2 | $SL(2, Z) \times Z_2$ |
| 8 | $SL(3, R) \times SL(2, R)$ | $SO(2, 2, Z)$ | $SL(3, Z) \times SL(2, Z)$ |
| 7 | $SL(5, R)$ | $SO(3, 3, Z)$ | $SL(5, Z)$ |
| 6 | $SO(5, 5)$ | $SO(4, 4, Z)$ | $SO(5, 5, Z)$ |
| 5 | $E_{6,6}(R)$ | $SO(5, 5, Z)$ | $E_{6,6}(Z)$ |
| 4 | $E_{7,7}(R)$ | $SO(6, 6, Z)$ | $E_{7,7}(Z)$ |

the non-compact symmetry group of the field equations in the corresponding supergravity theories.⁹³ Using U-duality Witten was able to give a complete picture of all possible strong coupling limits of type II vacua which further supports the validity of the conjecture.¹⁴

For $d = 10$ the situation is somewhat special since type IIA and type IIB are perturbatively distinct theories. The $d = 10$ entry in table 3 refers to the type IIB string which has a $SL(2, Z)$ acting on its two scalars D, D' .⁹⁴ The ten-dimensional type IIA theory is discussed in the next section.

6 M-theory

The discussion of the T-S-U-dualities and the resulting interrelation between different string vacua led to the conjecture that there is one underlying theory – termed M-theory – with the perturbatively distinct string theories being just the various weak coupling limits in the moduli space of this fundamental M-theory. In fact one more region of this moduli space has been identified namely 11-dimensional supergravity.¹⁴ Supergravity in 11 space-time dimensions is somewhat special since it is the highest possible dimension if one requires that there be no massless state with spin higher than two.⁹⁵ The field content of this theory contains the graviton $g_{\mu\nu}$, one gravitino ψ_μ and an antisymmetric three-form $C_{\mu\nu\rho}$.⁹⁶

6.1 Type IIA and M-theory

Type IIA supergravity can be constructed by simple dimensional reduction of 11 dimensional supergravity.^{97–99} The Kaluza–Klein modes $g_{\mu 11}$ and the three-form $C_{\mu\nu\rho}$ give the R-R vector A_μ and the R-R three-form $C_{\mu\nu\rho}$, respectively. The antisymmetric tensor $b_{\mu\nu}$ is obtained from $C_{\mu\nu 11}$ whereas the

dilaton is related to the compactification radius specified by g_{1111} . Comparing the low energy effective actions of the two theories similar to the analysis of section 4.2 shows that the ten-dimensional type IIA string coupling g_A and the compactification radius R are related by¹⁴

$$R = g_A^{\frac{3}{2}}. \quad (6.1)$$

Furthermore, the 11th component of the momentum operator P_{11} appears as the central charge of the type IIA superalgebra and the Kaluza–Klein spectrum of the compactified 11 dimensional theory corresponds to a tower of BPS-saturated states with masses

$$M_n \sim \frac{n}{g_A}, \quad (6.2)$$

where n refers to the n th Fourier mode in the Kaluza–Klein expansion. In string perturbation theory g_A is taken to be small and hence the BPS states are very heavy and decouple. However, when the string coupling gets large the compactification radius R also is large and the BPS states become light. Hence, in the strong coupling limit type IIA theory effectively decompactifies and the light Kaluza–Klein spectrum of 11-dimensional supergravity is nothing but a tower of type IIA BPS-states. Another way to put this is the statement that M-theory compactified on a circle S^1 is type IIA string theory where the radius of this circle coincides with the type IIA string coupling constants according to eq. (6.1).

6.2 Heterotic $E_8 \times E_8$ string from M-theory

The strong coupling limit of the heterotic $E_8 \times E_8$ string theory is also related to 11-dimensional supergravity but this time not compactified on circle but rather on a Z_2 orbifold of the circle, i.e. S^1/Z_2 .³⁰ The space coordinate x^{11} is odd under the action of Z_2 and hence the three-form $C_{\mu\nu\rho}$ as well as $g_{\mu 11}$ are also odd. The Z_2 invariant spectrum in $d = 10$ consists of the metric $g_{\mu\nu}$, the antisymmetric tensor $C_{\mu\nu 11}$ and the scalar $g_{11 11}$. Up to the gauge degrees of freedom this is precisely the massless spectrum of the ten-dimensional heterotic string. The $E_8 \times E_8$ Yang-Mills fields arise in the twisted sector of the orbifold compactification which so far cannot be computed directly in M-theory but is deduced in an indirect manner. The Z_2 truncation of 11 dimensional supergravity is inconsistent in that it gives rise to gravitational anomalies.¹⁰⁰ In order to cancel such anomalies non-Abelian gauge fields have to be present in order to employ a Green-Schwarz mechanism.¹⁰¹ Such additional states can only appear in the twisted sectors of the orbifold theory which are located at

the orbifold fixed points $x^{11} = 0, x^{11} = \pi$. However, due to the Z_2 symmetry these two ten-dimensional hyperplanes have to contribute equally to the anomaly. This can only be achieved for a gauge group which is a product of two factors and thus $E_8 \times E_8$ with one E_8 factor on each hyperplane is the only consistent candidate for such a theory.³⁰

Exactly as in the type IIA case one has $R = g_{\text{het}}^{\frac{3}{2}}$ and thus a small compactification radius gives the weakly coupled heterotic string whereas the strongly coupled heterotic string is equivalent to decompactified 11-dimensional supergravity. Using the previous terminology the heterotic $E_8 \times E_8$ string theory can be viewed as M-theory compactified on S_1/Z_2 .

Let us summarize the situation so far. In ten dimensions the heterotic $SO(32)$ and the type I string theories are S-dual, that is they are quantum equivalent and merely two different perturbative expansions of the same quantum theory. The type IIB theory is selfdual with a strong coupling limit governed by $SL(2, Z)$. The strong coupling limits of type IIA and the heterotic $E_8 \times E_8$ theory are 11-dimensional supergravity and they can be viewed as circle or orbifold compactifications of M-theory.

In section 3 we recalled that below $d = 10$ the type IIA and type IIB theory are in the same moduli space and similarly the heterotic $E_8 \times E_8$ and $SO(32)$ theories are in the same moduli space. Thus, if one also treats the radii of the toroidal compactifications as parameters there now is an intriguing relation between all five perturbative string theories and it has been conjectured that all of them are just different weak coupling limits in the moduli space of one underlying (fundamental) theory called M-theory.⁵¹

7 String compactifications with broken supercharges in six space-time dimensions

All string vacua considered so far have either 32 or 16 supercharges and they are toroidal compactifications of ten-dimensional $N = 2$ or $N = 1$ supergravities which leave all supercharges intact. In lower space-time dimensions it is possible to consider compactifications which break some of the supercharges. Such compact manifolds which at the same time preserve the consistency of the field equations of string theory are known as Calabi–Yau manifolds.⁵²

7.1 The K3 surface

A Calabi-Yau manifold Y is a Ricci-flat Kähler manifold of vanishing first Chern class with holonomy group $SU(n)$ where n is the complex dimension of Y . A (complex) one dimensional Calabi-Yau manifold is topologically always

a torus. In two complex dimensions all Calabi-Yau manifolds are topologically equivalent to the $K3$ surface.¹⁰² The moduli space of non-trivial metric deformations on a $K3$ is 58-dimensional and given by the coset space

$$\mathcal{M} = R^+ \times \frac{SO(3, 19)}{SO(3) \times SO(19)} \Big/ SO(3, 19, Z) , \quad (7.1)$$

where the second factor is the Teichmüller space for Einstein metrics of volume one on a $K3$ surface and the first factor is associated with the size of the $K3$.¹⁰²

In addition to the metric deformations there are non-trivial two-forms on $K3$ which are related to the antisymmetric tensor of string theory. On any Kähler manifold, the differential k -forms can be decomposed into (p, q) -forms with p holomorphic and q antiholomorphic differentials. The harmonic (p, q) -forms form the cohomology groups $H^{p,q}$ of dimension $h^{p,q}$ and for $K3$ one has

$$\begin{array}{ccccccc} & & h^{0,0} & & & & 1 \\ & h^{1,0} & & h^{0,1} & & 0 & 0 \\ h^{2,0} & & h^{1,1} & & h^{0,2} & = & 1 & 20 & 1 \\ & h^{2,1} & & h^{1,2} & & 0 & 0 \\ & & h^{2,2} & & & & 1 \end{array} . \quad (7.2)$$

Thus there are 22 harmonic $p + q = 2$ -forms which represent the non-trivial deformations of the antisymmetric tensor b_{ij} . Together with the 58 deformations of the metric they form the 80-dimensional moduli space of $K3$ string compactifications^{103, 104}

$$\mathcal{M} = \frac{SO(4, 20)}{SO(4) \times SO(20)} \Big/ SO(4, 20, Z) . \quad (7.3)$$

For later reference let us also record that the Euler number of $K3$ is found to be

$$\chi = \sum_{p,q} (-)^{p+q} h^{p,q} = 24. \quad (7.4)$$

7.2 Supergravities on $K3$

Whenever the space-time dimension obeys $d = 4k + 2$ (k being an integer) the left and right handed spinors are independent and as a consequence one can have chiral supersymmetries. In $d = 10$ the irreducible spinor representations are Majorana-Weyl and the $N = 2$ supersymmetry either has two gravitinos of the same chirality (type IIB) or of opposite chirality (type IIA). In $d = 6$ the irreducible spinor representations are symplectic Majorana (or pseudo-real) and again one can have chiral and non-chiral supersymmetries. The toroidally

compactified type II string corresponds in $d = 6$ to a non-chiral supergravity with 16 chiral and 16 anti-chiral supercharges. Such theories have two chiral and two anti-chiral gravitinos and are often denoted as $(2, 2)$ supergravity.

Compactification on $K3$ breaks half of the supercharges and one finds the two possibilities of having either 16 chiral supercharges (two chiral gravitinos) or 8 chiral plus 8 anti-chiral supercharges (one chiral and one anti-chiral gravitino). The first possibility arises when type IIB is compactified on $K3$ and thus this supergravity is also called type IIB or $(2, 0)$. The second case occurs as a compactification of type IIA on $K3$ is denoted type IIA or $(1, 1)$ supergravity. Finally, compactification of the heterotic or type I string theory on $K3$ only leaves 8 supercharges (one gravitino) and is called $(1, 0)$ supergravity.

7.3 Type IIB compactified on $K3$

The chiral $(2, 0)$ supergravity has a gravitational multiplet containing as bosonic components the graviton $g_{\mu\nu}$ and five antisymmetric tensor fields $b_{\mu\nu}^+$ whose field strength is selfdual. The only other possible multiplet in this supergravity is a tensor multiplet containing an antisymmetric tensor $b_{\mu\nu}^-$ with an anti-selfdual field strength and five scalars ϕ . Compactifying the type IIB string on $K3$ leads to the massless modes $g_{\mu\nu}, b_{\mu\nu}, D, g_{ij}, b_{ij}$ from the NS-NS sector and $b'_{\mu\nu}, D', b'_{ij}, C_{\mu\nu\rho\sigma}, C_{\mu\nu ij}$ from the R-R sector where g_{ij} denote the 58 zero modes of the metric on $K3$ and the b_{ij}, b'_{ij} each are 22 harmonic two-forms. In $d = 6$ a four form has only one physical degree of freedom and is dual to a real scalar $C_{\mu\nu\rho\sigma} \sim a$. Furthermore, the 22 antisymmetric tensor fields $C_{\mu\nu ij}$ can be decomposed into three selfdual and 19 anti-selfdual tensors corresponding to an analogous decomposition of the 22 two-forms on $K3$.¹⁰² So altogether there are 81 NS-NS and 24 R-R scalars, 5 selfdual and 21 anti-selfdual tensors which altogether combine into one gravitational and 21 tensor multiplets. This is precisely the combination of an anomaly free type IIB spectrum.¹⁰⁵ The scalars parameterize the 105-dimensional moduli space^{106, 17, 32}

$$\mathcal{M} = \frac{SO(5, 21)}{SO(5) \times SO(21)} \Big/ \Gamma_U, \quad (7.5)$$

with a U-duality group $\Gamma_U = SL(5, 21, Z)$. One of the strong coupling limits of this theory is again 11-dimensional supergravity,^{33, 34} this time compactified on T^5/Z_2 .^r

^rCompactifying 11-dimensional supergravity on an orbifold T^5/Z_2 gives chiral $(2, 0)$ supergravity with one gravity multiplet and five tensor multiplets from the untwisted sector. The twisted sector is again inferred by anomaly cancellation and provides 16 further tensor multiplets. The weakly coupled type IIB theory on $K3$ corresponds to a ‘smashed’ T^5/Z_2

7.4 Type IIA compactified on $K3$

The non-chiral type IIA (1, 1) supergravity has a gravitational multiplet as well as vector multiplets. The bosonic fields in the gravity multiplet are the graviton $g_{\mu\nu}$, an antisymmetric tensor $b_{\mu\nu}$, four graviphotons γ_μ and one real scalar D . The vector multiplets contain a vector boson A_μ and four scalars.

Compactification of the type IIA string on $K3$ results in the following bosonic spectrum: the graviton $g_{\mu\nu}$, an antisymmetric tensor $b_{\mu\nu}$, the dilaton D , 58 scalars g_{ij} and 22 scalars b_{ij} all from the NS-NS sector. In the R-R sector there is one vector A_μ , an additional vector arising as the dual of the three-form $C_{\mu\nu\rho} \sim A'_\mu$ and 22 vectors $C_{\mu ij}$. These fields combine into one gravity multiplet and 20 vector multiplets. The 81 scalars, (all of them come from the NS-NS sector since $h_{p,q} = 0$ for $p+q$ odd), parameterize the moduli space^{103, 104}

$$\mathcal{M} = R^+ \times \frac{SO(4, 20)}{SO(4) \times SO(20)} / \Gamma_T \quad (7.6)$$

where the T-duality group is $\Gamma_T = SO(4, 20, Z)$.

7.5 Heterotic strings on T^4

Before we discuss $K3$ compactifications of the heterotic string we need to pause and first reconsider toroidal compactification of the heterotic string. Such string vacua also have 16 supercharges and the low energy limit is the non-chiral type IIA or (1, 1) supergravity. The bosonic spectrum is again obtained by dimensional reduction from the ten-dimensional heterotic spectrum of table 1. At a generic point in the moduli space (where the non-Abelian gauge symmetry is broken to $[U(1)]^{16}$) one has the graviton $g_{\mu\nu}$, an antisymmetric tensor $b_{\mu\nu}$, the dilaton D , 80 scalars g_{ij}, b_{ij}, A_i^a and 24 vectors $A_\mu^a, g_{\mu j}, b_{\mu j}$. Together they form one gravitational multiplet and 20 vector multiplets exactly as for type IIA compactified on $K3$. Furthermore, from eqs. (3.7), (7.6) we learn that also the moduli spaces of the two string compactifications coincide.⁸ This led to the conjecture that type IIA on $K3$ and the heterotic string on T^4 are possibly quantum equivalent or S-dual to each other.^{9-12, 13, 14}

The effective actions of the two perturbative theories can be compared using the methods of section 4.2. They turn out to agree if one identifies¹⁴

$$D_{\text{het}} = -D_A ,$$

where the 32 fixed points degenerate into 16 pairs and the 16 tensor multiplets are equally distributed among those pairs.

⁸In addition, the existence of a solitonic string suggested that there is a description in terms of elementary five-branes.³⁻⁶ In six dimensions this implies a string/string duality.⁹

$$\begin{aligned} H_{\text{het}} &= e^{-2D_A} * H_A , \\ (g_{\text{het}})_{\mu\nu} &= e^{-2D_A} (g_A)_{\mu\nu} , \end{aligned} \tag{7.7}$$

where $H = db$ is the field strength of the antisymmetric tensor and $*H$ is its Poincare dual. The first equation in (7.7) again implies a strong-weak coupling relation while the second is the equivalent of an electric-magnetic duality. Further evidence for this S-duality arises from the observation that the zero modes in a solitonic string background of the type IIA theory compactified on $K3$ have the same structure as the Kaluza-Klein modes of the heterotic string compactified on T^4 .^{15, 16}

However, there is an immediate puzzle. We know that on special subspaces of the moduli space of the toroidally compactified heterotic strings the gauge symmetry can be enhanced to non-Abelian gauge groups. At face value it seems impossible to obtain such a gauge symmetry enhancement in the type IIA theory where all vectors are in the R-R sector with no massless charged states possible. This question leads to another important topic in string dualities – the study of singularities or rather singular couplings in the moduli space.

7.6 Singularities in the moduli space

Singularities in the moduli space of string vacua can in principle be of two different origins. First, the singularities might be an artifact of perturbation theory and smoothed out once quantum corrections are taken into account. The other possibility is that a singularity is physical and signals the breakdown of some approximation. The prime example for this latter case is the appearance of additional massless degrees of freedom on a subspace of the moduli space.^t A well known example occurs in four-dimensional non-Abelian (supersymmetric) gauge theories with light and heavy degrees of freedom. Below the threshold scale M of the heavy states the gauge couplings of the light modes obey

$$g_{\text{low}}^{-2} = g_{\text{high}}^{-2} + c \log M, \tag{7.8}$$

where c is some model dependent constant and g_{high} is the coupling above the heavy threshold. In string theory the mass often is a function of the moduli $M(\phi)$ with a zero somewhere in the moduli space $M(\phi_0) = 0$. At

^tAlmost everywhere on the moduli space these degrees of freedom are heavy and thus have been integrated out of the effective theory. However, if their masses become small somewhere in the moduli space the approximation of integrating out such degrees of freedom is not valid on the entire moduli space. The breakdown of this approximation manifests itself as a singularity in some of the couplings of the effective theory.

such points the gauge coupling g_{low}^{-2} becomes singular due to the inappropriate approximation of integrating out the heavy states.^u

Indeed the $K3$ surface has orbifold singularities (following an A-D-E classification) whenever a two-cycle of the $K3$ shrinks to zero. A careful analysis shows that in $K3$ compactifications of type IIA the origin of these singularities can be interpreted as non-Abelian gauge bosons (of an A-D-E gauge group) becoming massless.^{14, 108} Thus a point of non-Abelian gauge enhancement on the heterotic side corresponds to an orbifold singularity on the type IIA side.^v

Although the original puzzle about the subspaces of enhanced gauge symmetries in type IIA is resolved the type IIB string compactified on $K3$ now raises a puzzle. Here the same singularities are present but the $(2, 0)$ supergravity does not have any vector multiplets and hence it is impossible to explain the singularities by additional massless gauge bosons. Instead, it has been argued that in type IIB the singularities are caused by extended objects – non-critical strings – becoming tensionless at the orbifold singularities of $K3$.²⁶ Those tensionless strings arise when a selfdual three brane of the ten-dimensional type IIB theory wraps around a vanishing two cycle of the $K3$.

7.7 The heterotic string compactified on $K3$

Compactifying the heterotic string on $K3$ only leaves 8 supercharges intact which is the minimal or $(1, 0)$ supergravity in $d = 6$. This theory has four distinct supermultiplets. The gravitational multiplet contains the graviton $g_{\mu\nu}$ and an anti-selfdual two form $b_{\mu\nu}^-$ as bosonic components. The tensor multiplet has a selfdual two-form $b_{\mu\nu}^+$ and a scalar D , the vector multiplet features only a vector A_μ and finally the hyper multiplet contains four scalars q as bosonic components. This supergravity is chiral and thus gauge and gravitational anomaly cancellation imposes constraints on the allowed spectrum. The anomaly can be characterized by the anomaly eight-form I_8 which is exact ($dI_8 = 0$) and invariant under general coordinate and gauge transformations. I_8 can be written as an exterior derivative of a seven-form $I_8 = dI_7$ whose variation obeys $\delta I_7 = dI'_6$ with I'_6 being the anomaly ($\delta S = \int I'_6$). The generic form^w of I_8 is given by¹⁰⁰

$$I_8 = \alpha \text{tr} R^4 + \beta (\text{tr} R^2)^2 + \gamma \text{tr} R^2 \text{tr} F^2 + \delta (\text{tr} F^2)^2, \quad (7.9)$$

^uQuantum mechanically the number and position of the singularities and the interpretation of the fields becoming massless at the singularities can be quite different to the classical picture.¹⁰⁷

^vThis phenomenon can also be viewed as the type IIA two-brane wrapping around the 2-cycles of the $K3$.¹⁰⁹

^wWe exclude Abelian gauge factors for simplicity.

where R is the curvature two-form, F is the Yang-Mills two-form and α, \dots, δ are real coefficients which depend on the spectrum of the theory. The anomaly can only be cancelled if α vanishes. One finds^{110–112}

$$\alpha = n_H - n_V + 29n_T - 273 \stackrel{!}{=} 0, \quad (7.10)$$

where n_H, n_V and n_T are the numbers of hyper, vector and tensor multiplets, respectively. In the perturbative spectrum of the heterotic string there is only one tensor multiplet (from $b_{\mu\nu}$) and hence anomaly cancellation demands

$$n_H - n_V = 244. \quad (7.11)$$

The remaining anomaly eight form has to factorize in order to employ a Green-Schwarz mechanism. More precisely, one needs $I_8 \sim X_4 \wedge \tilde{X}_4$ where

$$X_4 = \text{tr}R^2 - \sum_a v_a (\text{tr}F^2)_a, \quad \tilde{X}_4 = \text{tr}R^2 - \sum_a \tilde{v}_a (\text{tr}F^2)_a. \quad (7.12)$$

The index a labels the factors G_a of the gauge group $G = \otimes_a G_a$ and v_a, \tilde{v}_a are constants which depend on the massless spectrum.^{110–112} In the Green-Schwarz mechanism one defines a modified field strength H for the antisymmetric tensor

$$H = db + \omega^L - \sum_a v_a \omega_a^{YM}, \quad (7.13)$$

such that $dH = X_4$. (ω^L is a Lorentz-Chern-Simons term and ω_a^{YM} is the Yang-Mills Chern-Simons term.) Adding to the action the Green-Schwarz counterterm

$$\int_{R^6} b \wedge \tilde{X}_4 \quad (7.14)$$

finally results in a complete anomaly cancellation.¹¹⁰

In order to ensure a globally well-defined H on the compact $K3$ the integral $\int_{K3} dH$ has to vanish. Using eqs. (7.12), (7.13) this implies

$$\sum_a n_a \equiv \sum_a \int_{K3} (\text{tr}F^2)_a = \int_{K3} \text{tr}R^2 = 24, \quad (7.15)$$

where the last equation used the fact that 24 is the Euler number of $K3$. From eq. (7.15) we learn that the heterotic string compactified on $K3$ necessarily needs a non-vanishing instanton number n_a .

As a first example let us consider the $E_8 \times E_8$ theory with instantons only in one of the E_8 factors breaking that E_8 completely and leaving the other E_8

factor unbroken. The dimension of the moduli space of instantons on $K3$ in a group G with dual Coxeter number h and instanton number n is given by

$$\dim \mathcal{M}_n = 4(nh - \dim(G)) . \quad (7.16)$$

For the case at hand we have $\dim(G) = \dim(E_8) = 248$, $h = 30$, $n = 24$ and thus $\dim \mathcal{M}_{24} = 4 \cdot 472$. As we discussed at the beginning of this section, hypermultiplets contain four real scalars each, and thus the instanton moduli space is parameterized by 472 hypermultiplets. The heterotic string vacua have an additional 20 hypermultiplets which host the 80 moduli of $K3$. Thus, altogether one has

$$n_H = 472 + 20 = 492 , \quad n_V = \dim(E_8) = 248 , \quad (7.17)$$

which indeed satisfies eq. (7.11).^a

As a second example we consider again the $E_8 \times E_8$ heterotic string but this time with n_1 instantons in one E_8 and n_2 instantons in the other E_8 factor. Furthermore, for $n_a \geq 10$ the gauge group is completely broken at a generic point in the moduli space and one has $\dim \mathcal{M} = 4(720 - 2 \cdot 248) = 4 \cdot 224$. Thus, $n_H = 224 + 20 = 244$, $n_V = 0$ in agreement with (7.11). At special points in the moduli space an enhanced gauge group can open up which is at most $E_7 \times E_7$ when all instantons sit in $SU(2) \times SU(2)$.

Finally, for $n_1 = 8, n_2 = 16$ the gauge group is $SO(8)$ at a generic point in the moduli space and can be enhanced to $SO(28) \times SU(2)$ at a special point. It has been shown that this $E_8 \times E_8$ heterotic vacuum is quantum equivalent to the compactification of the $SO(32)$ heterotic string on $K3$.^{28, 37, 40} Thus, also for $K3$ compactifications the two heterotic string theories sit in one and the same moduli space.

In all of these examples the kinetic energy of the gauge fields is strongly constrained by supersymmetry. The dilaton is a member of the tensor multiplet which also contains the (selfdual) part of the antisymmetric tensor. Its couplings to the vector multiplets are fixed by the Chern–Simons interactions (7.13) which in turn also fixes the interactions of the dilaton with the gauge fields¹¹³

$$\mathcal{L} \sim \sqrt{g} \sum_a (v_a e^{-2D} + \tilde{v}_a) (tr F_{\mu\nu} F^{\mu\nu})_a . \quad (7.18)$$

For the second example discussed above one finds the coefficients^{111, 112}

$$v_1 = v_2 = \frac{1}{6} , \quad \tilde{v}_1 = \frac{(n_1 - 12)}{6} , \quad \tilde{v}_2 = \frac{(n_2 - 12)}{6} . \quad (7.19)$$

^aBy aligning some of the 24 instantons, i.e. going to special points of the moduli space, the gauge group can be enhanced up to at most $E_8 \times E_7$ when all 24 instantons sit in an $SU(2)$.

For $n_1 \neq n_2$ one of the \tilde{v} is necessarily negative and hence the gauge kinetic term (7.18) changes sign at

$$e^{-2D} = -\frac{\tilde{v}_1}{v_1} = 12 - n_1, \quad (7.20)$$

(where we have arbitrarily chosen n_1 to be less than 12). Eq. (7.20) implies a singularity at strong coupling. Similar to the case of the type IIB string compactified on $K3$ it has been argued that this singularity originates from a non-critical string with a tension controlled by the dilaton, becoming tensionless^{39, 114}

Apart from this strong coupling singularity also the instanton moduli space typically develops singularities when the size ρ of an instanton approaches zero^{89, 26}. These singularities necessarily are weak coupling singularities since the dilaton resides in the tensor multiplet which can have no gauge neutral couplings with the moduli in hypermultiplets and thus at $\rho \rightarrow 0$ the dilaton can be arbitrarily weak. It has been shown that the conformal field theory description nevertheless breaks down and the space-time geometry develops a semi-infinite tube with the dilaton becoming large ‘further down the tube’⁸⁹. For the $SO(32)$ heterotic string Witten gave convincing evidence that at the locus of the collapsing instanton additional vector bosons become massless corresponding to a non-perturbative enhancement of the gauge group¹¹⁵. For a single instanton the non-perturbative gauge group is $SU(2)$ while k instantons collapsing at distinct points of the instanton moduli space enlarge the perturbative gauge group to $G = SO(32) \times SU(2)^k$ with k additional hypermultiplets in the $(\mathbf{32}, \mathbf{2})$ representation of G . k instantons collapsing to the same point yield $G = SO(32) \times Sp(k)$ with one additional hypermultiplet in the $(\mathbf{32}, \mathbf{2k})$ representation of G . Thus, for 24 instantons the maximal gauge group possible is $G = SO(32) \times Sp(24)$. The enhancement factor $Sp(24)$ is completely invisible in string perturbation theory and it violates the perturbative bound on the rank of G set by the central charge of the underlying CFT.

Further evidence of this picture arises from the heterotic – type I S–duality which relates a small heterotic instanton to a D-five-brane carrying $SU(2)$ Chan–Paton factors. Whenever k of such D-branes sit on top of each other the gauge symmetry is enhanced to $Sp(k)$ ¹¹⁵.

For the $E_8 \times E_8$ heterotic string a similar mechanism is inconsistent. Instead it has been argued that at the locus of the collapsing instanton an entire non-critical string becomes tensionless^{39, 116}. One of the massless modes of this string is an additional tensor multiplet containing a real scalar as its lowest component. This scalar can be viewed as parameterizing a new (Higgs) branch in the moduli space. However, consistency requires eq. (7.10) to be satisfied

and thus on the new branch there necessarily is a different spectrum. For example for $n_T = 2, n_V = 0$ one has $n_H = 244 - 29 = 215$. This branch of moduli space is again invisible in heterotic perturbation theory where only $n_T = 1$ is possible. However, the non-perturbative properties of these heterotic vacua have been captured by constructing appropriate weakly coupled dual vacua. These constructions go under the name of F-theory and therefore we pause for a moment in order to have a brief look at F-theory.^{36, 37}

7.8 F-theory

The type IIB theory in ten space-time dimensions has an $SL(2, Z)$ quantum symmetry acting as in eq. (4.7) on the complex

$$\tau = e^{-2D} + iD', \quad (7.21)$$

where D and D' are the two scalar fields of type IIB theory (c.f. table 1). This fact led Vafa to propose that the type IIB string could be viewed as the toroidal compactification of a twelve-dimensional theory, called F-theory, where τ is the complex structure modulus of the two-torus T^2 and the Kähler class modulus is frozen.³⁶ Apart from having a geometrical interpretation of the $SL(2, Z)$ symmetry this proposal led to the construction of new, non-perturbative string vacua in lower space-time dimensions. In order to preserve the $SL(2, Z)$ quantum symmetry the compactification manifold cannot be arbitrary but has to be what is called an elliptic fibration. That is, the manifold is locally a fibre bundle with a two-torus T^2 over some base B but with a finite number of singular points. As a consequence non-trivial closed loops on B can induce a non-trivial $SL(2, Z)$ transformation of the fibre. This implies that the dilaton is not constant on the compactification manifold but can ‘jump’ by an $SL(2, Z)$ transformation.¹¹⁷ It is precisely this fact which results in non-trivial (non-perturbative) string vacua inaccessible in string perturbation theory.

For example F-theory compactified on an elliptic $K3$ yields an 8-dimensional vacuum with 16 supercharges which is quantum equivalent to the heterotic string compactified on T^2 .^{36, 41} On the other hand F-theory compactified on an elliptic Calabi–Yau threefold has 8 unbroken supercharges and is quantum equivalent to the heterotic string compactified on $K3$.³⁷ In fact there is a beautiful correspondence between the heterotic vacua labelled by the instanton numbers (n_1, n_2) and elliptically fibred Calabi–Yau manifolds with the base being the Hirzebruch surfaces \mathbb{F}_{n_2-12} (we have again chosen $n_2 \geq n_1$).³⁷ These F-theory vacua capture the non-perturbative physics of the heterotic string including the possibility of additional tensor multiplets, the transitions between the various branches of moduli space and subspaces of symmetry enhancement.^{37, 38, 44}

7.9 Heterotic – heterotic duality

The string–string duality⁹ in $d = 6$ which (in part) is responsible for the S-duality of the heterotic string on T^4 and the type IIA on $K3$ also has a somewhat surprising manifestation in $K3$ compactifications of the heterotic string. For $n_1 = n_2 = 12$ eq. (7.20) reveals that there is no strong coupling singularity so that this class of heterotic vacua is well defined for all values of the dilaton. Furthermore, it has been shown that there is a selfduality among the $n_1 = n_2 = 12$ heterotic vacua.^{35, 37, 38} More precisely, the theory is invariant under

$$\begin{aligned} D &\rightarrow -D, & g_{\mu\nu} &\rightarrow e^{-D} g_{\mu\nu} \\ H &\rightarrow e^{-D} * H, & X_4 &\leftrightarrow \tilde{X}_4 \end{aligned} \quad (7.22)$$

if in addition perturbative gauge fields are replaced by non-perturbative gauge fields and the hypermultiplet moduli spaces are mapped non-trivially onto each other. Curiously, this duality requires the existence of non-perturbative gauge fields with exactly the properties discussed in section 7.7 for $SO(32)$ heterotic strings. This posed a slight puzzle since for $E_8 \times E_8$ vacua the singularity of small instantons is caused by a non-critical string turning tensionless rather than additional massless gauge bosons. However, by mapping the $n_1 = n_2 = 12$ heterotic vacuum to a particular type I vacuum⁸⁸ which indeed does have non-perturbative gauge fields this issue has been resolved and the gauge fields are shown to exist also for this class of heterotic vacua.⁴³

8 String compactifications with broken supercharges in four space-time dimensions

In this last section we focus on string compactifications with 8 unbroken supercharges in $d = 4$ or in other words on string vacua with $N = 2$ supergravity. Such vacua arise either from the type II string compactified on a Calabi–Yau threefold or from the heterotic string compactified on $K3 \times T^2$.^b Also these two classes of string vacua are believed to be quantum equivalent.^{18, 19}

8.1 $N = 2$ supergravity

In $N = 2$ the gravitational multiplet contains the graviton $g_{\mu\nu}$, two gravitini $\psi_{\mu\alpha}^I$ and an Abelian graviphoton γ_μ . One also has vector multiplets V with a

^bSuch vacua and an expanded treatment of the content of this section including a more complete list of references can be found in a recent review.¹¹⁸

gauge field A_μ , two gauginos λ_α^I and a complex scalar ϕ as well as hypermultiplets with two Weyl spinors χ_α^I and four real scalars q^{IJ} . In addition there are three distinct multiplets containing an antisymmetric tensor; the vector-tensor multiplet VT features an Abelian gauge field A_μ , an antisymmetric tensor $b_{\mu\nu}$, two Weyl fermions χ_α^I and one real scalar D . The tensor multiplet T has an antisymmetric tensor $b_{\mu\nu}$, two Weyl spinors χ_α^I , a complex scalar ϕ and a real scalar D . Finally the double tensor multiplet Π contains two antisymmetric tensors $b_{\mu\nu}, b'_{\mu\nu}$, two Weyl spinors χ_α^I , and two real scalars D and D' . In four space-time dimensions an antisymmetric tensor only contains one physical degree of freedom and is dual to a real scalar $a(x)$ via $\epsilon^{\mu\nu\rho\sigma}\partial_\mu b_{\nu\rho} \sim \partial^\sigma a$. This duality can be elevated to a duality between entire supermultiplets and one finds that the vector-tensor multiplet is dual to a vector multiplet while the tensor multiplet and the double tensor multiplet are both dual to a hypermultiplet. Thus, the low energy effective theory can be described entirely in terms of only the gravitational multiplet, vector- and hypermultiplets. In particular, the moduli of string compactifications appear either in vector- or hypermultiplets. Supersymmetry prohibits gauge neutral interactions between vector and hypermultiplets and therefore the moduli space locally has to be a direct product¹¹⁹

$$\mathcal{M} = \mathcal{M}_H \times \mathcal{M}_V, \quad (8.1)$$

where \mathcal{M}_H is the (quaternionic) moduli space parameterized by the scalars of the hypermultiplets and \mathcal{M}_V is the moduli space spanned by the scalars in the vector multiplets.

In the heterotic string the dilaton is a member of a vector-tensor multiplet or equivalently a dual vector multiplet whereas in type IIA (IIB) the dilaton sits in a tensor (double tensor) multiplet or equivalently in the dual hypermultiplet. This assignment of the dilaton together with the fact that the dilaton organizes the string perturbation theory immediately leads to two non-renormalization theorems. In type II compactifications the \mathcal{M}_V component of the moduli space cannot receive any perturbative or non-perturbative corrections and the string tree level result is exact. Conversely, in heterotic compactifications the \mathcal{M}_H component suffers no quantum corrections.^{120, 109}

Supersymmetry also dictates the local geometry of these moduli spaces; \mathcal{M}_H is a quaternionic manifold¹²¹ while \mathcal{M}_V is special Kähler.¹²² Because of its technical simplicity most investigations so far focussed on the special Kähler manifold \mathcal{M}_V . A special Kähler manifold is a Kähler manifold with the metric $G_{i\bar{j}}$ expressed in terms of a Kähler potential K ,

$$G_{i\bar{j}} = \frac{\partial}{\partial\phi^i} \frac{\partial}{\partial\bar{\phi}^{\bar{j}}} K(\phi, \bar{\phi}), \quad (8.2)$$

where ϕ^i are the complex scalars in the vector multiplets. The term ‘special’ refers to the fact that the Kähler potential satisfies the additional constraint

$$K = -\log \left[2 (\mathcal{F} + \bar{\mathcal{F}}) - (\phi^i - \bar{\phi}^{\bar{i}}) (\mathcal{F}_i - \bar{\mathcal{F}}_{\bar{i}}) \right], \quad \mathcal{F}_i \equiv \frac{\partial \mathcal{F}}{\partial \phi^i}. \quad (8.3)$$

That is, K is determined by a single holomorphic function $\mathcal{F}(\phi)$ termed prepotential. The same prepotential also determines all the gauge couplings of the vector multiplets so that all couplings of low energy effective Lagrangian are expressed in terms of the holomorphic $\mathcal{F}(\phi)$.^c

8.2 The heterotic string compactified on $K3 \times T^2$

Compactifying a six-dimensional $(1, 0)$ supergravity on a two-torus T^2 results in $N = 2$ supergravity in $d = 4$ or equivalently the ten-dimensional heterotic string compactified on $K3 \times T^2$ has $N = 2$ supergravity in $d = 4$. The massless perturbative spectrum consists of the dilaton multiplet which is a vector-tensor or dual vector multiplet denoted by S . In addition, there are two Abelian vector multiplets T and U which contain the toroidal moduli of T^2 as well as further model-dependent Abelian or possibly non-Abelian vector multiplets. The total perturbative gauge group is

$$G = G' \times U(1)_S \times U(1)_T \times U(1)_U \times U(1)_\gamma, \quad (8.4)$$

where G' refers to the additional Abelian or non-Abelian part of the gauge group and $U(1)_\gamma$ corresponds to the graviphoton. Furthermore, there are charged as well as neutral hypermultiplets but their interaction will be of no concern here. However, as we learned in the previous sections at special points in the hypermultiplet moduli space there might be a non-perturbative enhancement of this gauge group.

The prepotential for this class of vacua is found to be^{120, 124}

$$\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \mathcal{F}^{(NP)}, \quad (8.5)$$

where $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ denote the tree level and one-loop contributions, respectively while $\mathcal{F}^{(NP)}$ denotes the a priori unknown non-perturbative corrections. The absence of any perturbative contributions beyond one-loop is a consequence of the $N = 2$ non-renormalization theorem.¹¹⁸

For all heterotic vacua the tree level contribution is found to be¹²⁵

$$\mathcal{F}^{(0)} = -S (TU - \phi^i \phi^{\bar{i}}), \quad (8.6)$$

^cThe precise expression is of no concern here but can be found in the literature.¹²³

where the ϕ^i are the vector multiplet moduli of the factor G' . The one loop contribution $\mathcal{F}^{(1)}$ is model dependent but does not depend on the dilaton S . It is strongly constrained by the T-duality of the two-torus T^2 and has been computed for particular classes of heterotic vacua.^{120, 124, 126–129}

8.3 Type II vacua compactified on Calabi-Yau threefolds

Calabi-Yau threefolds are Calabi-Yau manifolds of complex dimension three and holonomy group $SU(3)$.^{52, 130} For threefolds the hodge diamond is given by

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & 0 & \\
 & & 0 & & 0 & \\
 & 0 & h^{1,1} & & 0 & \\
 1 & h^{1,2} & & h^{1,2} & 1 & , \\
 & 0 & h^{1,1} & & 0 & \\
 & & 0 & & 0 & \\
 & & & & & 1
 \end{array} \quad (8.7)$$

where $h^{1,1}$ and $h^{1,2}$ are arbitrary.

It is believed that most (if not all) Calabi-Yau threefolds have an associated mirror manifold \tilde{Y} with reversed Hodge numbers, i.e. $h^{1,1}(\tilde{Y}) = h^{1,2}(Y)$ and $h^{1,2}(\tilde{Y}) = h^{1,1}(Y)$.¹³⁰ As a consequence type IIA theory compactified on Y is perturbatively equivalent to type IIB theory compactified on the mirror \tilde{Y} .

The non-trivial $(1, 1)$ and $(1, 2)$ forms on the threefold correspond to massless modes in string theory. In particular, for type IIA one has $n_V = h^{1,1}$ and $n_H = h^{1,2} + 1$ with the extra hypermultiplet being the type II dilaton.^d As we already argued in section 8.1 this fact renders the type II vector multiplet moduli space exact with no perturbative or non-perturbative corrections. The prepotential is entirely determined at the string tree level and in the limit of large Calabi-Yau manifolds it obeys¹³⁰

$$\mathcal{F} = -\frac{i}{6} d_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma + \text{worldsheet instantons} , \quad (8.8)$$

where $t^\alpha, \alpha = 1, \dots, h^{1,1}$ denote the scalar moduli of the vector multiplets and $d_{\alpha\beta\gamma}$ are the intersection numbers. The contribution of worldsheet instantons vanishes in the large volume limit $t_\alpha \rightarrow \infty$.

^dIn type IIB this assignment is exactly reversed and one has instead $n_V = h^{1,2}$ and $n_H = h^{1,1} + 1$.

The perturbative gauge group of type IIA compactifications is the Abelian group

$$G = U(1)^{h^{1,1}+1} \quad (8.9)$$

where the additional $U(1)$ corresponds to the graviphoton.

8.4 Heterotic – type II duality

It has been conjectured that heterotic strings compactified on $K3 \times T^2$ and type II strings compactified on Calabi-Yau threefolds are quantum equivalent.^{18, 19} A necessary condition for this duality to be true is the agreement of the two prepotentials. This immediately implies that the heterotic dilaton S is not mapped to the type II dilaton but rather to a modulus of the Calabi-Yau threefold, say t^S . Thus, the heterotic-type II duality is slightly different than the dualities considered so far in that the strong coupling regime is not mapped to the weak coupling regime of the dual theory but rather to an other perturbatively accessible region of the moduli space.

The heterotic-type II duality has been supported by a number of explicit pairs of string vacua where the type II prepotential and also the perturbative heterotic $\mathcal{F}^{(0)} + \mathcal{F}^{(1)}$ is known.^{18-21, 28, 42, 45} The matching of these functions is a highly non-trivial check and furthermore ‘predicts’ the non-perturbative $\mathcal{F}^{(NP)}$ of the heterotic vacuum. In fact there is a well-defined subclass of Calabi-Yau threefolds known as $K3$ -fibrations^e which do satisfy all necessary requirements for being the dual of perturbative heterotic vacua.^{21, 131} In particular, this class of type II vacua naturally explains the perturbative bound on the size of the heterotic gauge group and also captures the possibility of non-perturbative enhancement of G .

In $N = 2$ supergravity there are also certain higher derivate couplings which are governed by holomorphic functions. These couplings are of the form

$$g_n^{-2}(\phi) R^2 F^{2n-2} \quad , \quad (8.10)$$

where R is the Riemann curvature and F the field strength of the graviphoton. The heterotic-type II duality also requires these couplings to coincide for a given dual pair. Indeed this has been verified for specific examples providing yet another non-trivial test of the duality.^{20, 25, 46}

9 Conclusion

It is always difficult to conclude a subject which is currently ‘in full swing’. In fact after the presentation of this lecture many more exciting advances have

^eHere the base is a $\mathbb{C}P_1$ and the fibre a $K3$.

taken place which cannot be covered here. All tests of duality have so far succeeded in one way or another so that one is tempted to believe that there is some truth in this story.

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