Tomita-Takesaki theory: mathematical aspects

SEMINAR ON MATHEMATICAL ASPECTS OF THEORETICAL PHYSICS

II. INSTITUT FÜR THEORETISCHE PHYSIK, HAMBURG

SOMMERSEMESTER 2012

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30.April 2012

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CHAPTER 1

Von Neumann algebras

1.1 Definition of the Neumann algebra

A von Neumann algebra or W*-algebra is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ that is unital and closed in the weak operator topology. This algebra is needed for the Tomita-Takesaki theory. That is the reason why important definitions and properties will be introduced.

Recall. *A* ***seminorm*** $\nu(\cdot)$ *is a function from a vector space* V *to the real numbers satisfying following properties,* $\forall\ x, y \in V$ *and all scalars* c*:*

- $\nu(x) \geq 0$ *non-negativity* *(*$\nu(x) = 0$ *does not imply that* $x = 0$*, ohterwise* ***norm****)*
- $\nu(c \cdot x) = |c|\, \nu(x)$ *homogeneity*
- $\nu(x + y) \leq \nu(x) + \nu(y)$ *triangle inequality*

Definition 1.1. *(Topology induced by seminorm)*
Define 4 standard topologies on the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ [1]:

- *The **uniform topology** on $\mathcal{B}(\mathcal{H})$ is defined in terms of a single norm:*

$$\|A\| = sup\{\|A\psi\| : \psi \in \mathcal{H}, \|\psi\| \leq 1\},$$

with a given vector norm on $\mathcal{H}$. Hence, an operator A is a limit point of the sequence $(A_i)_{i\in\mathbb{N}}$ if and only if $(\|A_i - A\|)_{i\in\mathbb{N}}$ converges to 0.

- *The **weak topology** on $\mathcal{B}(\mathcal{H})$ is defined in terms of the family $\{\nu_{\phi,\psi} : \phi, \psi \in \mathcal{H}\}$ of seminorms where*

$$\nu_{\phi,\psi}(A) = \langle \phi \mid A\psi \rangle.$$

This topology (defined by a system of seminorms) is not first countable. So the closure is the set of limit points of "generalized sequences" (nets). Thus, a net of sequences of operators $(A_i)_{i\in\mathcal{I}}$ converges weakly if all matrix elements $\langle \phi \mid A_i\psi \rangle_{i\in\mathcal{I}}$ between arbitrary state vectors converge.

- *The **strong topology** on $\mathcal{B}(\mathcal{H})$ is defined in terms of the family $\{\nu_\psi : \psi \in \mathcal{H}\}$ of seminorms where*

$$\nu_\psi(A) = \|A\psi\|.$$

Thus, a net $(A_i)_{i\in\mathcal{I}}$ converges strongly to A if and only if $(\nu_\psi(A_i))_{i\in\mathcal{I}}$ converges to $\nu_\psi(A)$ for all $\psi \in \mathcal{H}$.

- *The **ultraweak topology** on $\mathcal{B}(\mathcal{H})$ is defined in terms of the family $\{\nu_\rho : \rho \in \mathcal{T}(\mathcal{H})\}$ where $\mathcal{T}(\mathcal{H})$ is the set of positive, trace 1 operators on $\mathcal{H}$ ("density operators") and*

$$\nu_\rho(A) = Tr(\rho A).$$

Thus, a net $(A_i)_{i\in\mathcal{I}}$ converges ultraweakly to A just in case $(Tr(\rho A_i))_{i\in\mathcal{I}}$ converges to $Tr(\rho A)$ for all $\rho \in \mathcal{T}(\mathcal{H})$.

Recall. *For any subset $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, is the* ***commutant***
$\mathcal{M}' = \{x \in \mathcal{B}(\mathcal{H}) : mx = xm \; \forall m \in \mathcal{M}\}$ *the set of all bounded operators on Hilbert space $\mathcal{H}$ commuting with elements of $\mathcal{M}$.*

If $\mathcal{M}$ is selfadjoint, i.e. $\mathcal{M}$ is a *-subalgebra, then $\mathcal{M}'$ is a C*-algebra of $\mathcal{B}(\mathcal{H})$ that is closed.

Definition 1.2. *A von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$, such that*

$$\mathcal{M} = \mathcal{M}''$$

Theorem 1.1. *(Bicommutant theorem)*
*Let $\mathcal{M}$ be a unital *-algebra of $\mathcal{B}(\mathcal{H})$, then following conditions are equivalent:*

- $\mathcal{M} = \mathcal{M}''$
- *$\mathcal{M}$ is closed on the weak operator topology*
- *$\mathcal{M}$ is closed on the strong operator topology*

Corollary. *Let $\mathcal{A}$ be a unital *-algebra of $\mathcal{B}(\mathcal{H})$, then it follows that if $\mathcal{M} = \mathcal{A}''$ a von Neumann algebra, i.e. $\mathcal{M} = \mathcal{M}''$, then $\mathcal{A}^{iv} \subseteq \mathcal{A}''$.*

1.2 Some facts

In the following we state some interesting properties for a von Neumann algebra $\mathcal{M}$.

- If $A \in \mathcal{M}$ is a selfadjoint operator and commutes with some $O \in \mathcal{B}(\mathcal{H})$, such that $AO = OA$, then O commutes also with all spectral projections P_i of A with $A = \sum_i \lambda_i P_i$, where $\lambda_i \in \mathbb{R}$, such that $OP_i = P_i O$, then $P_i \in \mathcal{M}$, i.e. P_i lies also in $\mathcal{M}$. Furthermore the span of all spectral projections is dense in $\mathcal{M}$.
- In order to understand the second property let us introduce the **polar decomposition** (from O. Bratteli D. W. Robinson [2])
  - The general polar decomposition represents each closed, densely defined operator A on a Hilbert space as a product $A = U(A^*A)^{1/2}$ of a partial

isometry U and a positive selfadjoint operator $|A| = (A^*A)^{1/2}$. The square root $(A^*A)^{1/2}$ is well defined, because $A^*A > 0$ positive.

– (Polar decomposition for bounded operators on a Hilbert space)
Let $A \in \mathcal{B}(\mathcal{H})$ and $|A| = (A^*A)^{1/2}$. Now define an operator U on $\mathcal{D} = \{|A|\,\psi\}$ with $\psi \in \mathcal{H}$ by

$$U\,|A|\,\psi = A\psi.$$

This defines a linear operator, because $|A|\,\psi = 0$ implies $A\psi = 0$ ($0 = \||A|\,\psi\| = \langle |A|\,\psi \mid |A|\,\psi\rangle^{1/2} = \langle (A^*A)^{1/2}\psi \mid (A^*A)^{1/2}\psi\rangle^{1/2} = \langle A\psi \mid A\psi\rangle^{1/2} = \|A\psi\| \to A\psi = 0$). Furthermore, U is isometric on $\mathcal{D}$ because $\|U\,|A|\,\psi\| = \|A\psi\| = \||A|\,\psi\|$ holds and it is partial isometric on $\mathcal{H}$ by setting it equal to zero on $\mathcal{D}$ and extending by linearity. This provides the polar decomposition of $A = U\,|A|$.

For an oparator $A \in$ unital C*-algebra, one can decompose the operator A (without proof) such that $A = \sum_{i=1}^{4} a_i U_i$, with $a_i \in \mathbb{C}$ and U_i unitary elements. Then for a bounded operator on the Hilbert space $A \in \mathcal{B}(\mathcal{H})$, $A \in \mathcal{M}$ if and only if for all unitary elements $V \in \mathcal{M}'$ we have $VAV^* = A$. Hence, if $A = U|A|$ is the polar decomposition of $A \in \mathcal{M}$, U and $|A|$ lie also in $\mathcal{M}$.
PROOF. $A \in \mathcal{B}(\mathcal{H})$: $A \in \mathcal{M} \Leftrightarrow \exists V \in \mathcal{M}'$ s.t. $VAV^* = A$, then $U\,|A| = A = VAV^* = VU\,|A|\,V^* = VUV^*V\,|A|\,V^* \quad \Rightarrow \quad U, |A| \in \mathcal{M}$.

CHAPTER 2

Tomita-Takesaki theory

First of all, we will introduce and recall some definitions, which we will need for the Tomita-Takesaki theory. After that, the modular structure implied by the Tomita-Takesaki theory will be defined and in the end, the relation to the KMS-states will be demonstrated.

2.1 Preliminaries

Definition 2.1. *A vector Ω is called* ***separating*** *for a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ if for any $A \in \mathcal{M}$, $A\Omega = 0$ for all $\Omega \in \mathcal{H}$ implies $A = 0$.*

Definition 2.2. *A vector Ω, $\Omega \in \mathcal{H}$, is called a* ***cyclic*** *vector for $\mathcal{M}$ being a *-subalgebra of all bounded operators on Hilbert space, then $\mathbf{span}(\mathcal{M}\Omega) \overset{dense}{\subset} \mathcal{H}$.*

The GNS-vacuum of a state over some *-algebra $\mathcal{M}$ is always cyclic.
One can show that there exists a dual relation for a **cyclic** vector in $\mathcal{M}$ and a **separating** vector in the commutant $\mathcal{M}'$.

Remark 2.1. *Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ and $\Omega \in \mathcal{H}$, then following statements are equivalent:*

- *Ω is cyclic for $\mathcal{M}$*
- *Ω is separating for $\mathcal{M}'$*

PROOF.

- (1)→ (2): Ω is cyclic for $\mathcal{M}$. Choose $A' \in \mathcal{M}'$ such that $A'\Omega = 0$. Then for any $B \in \mathcal{M}$, $A'B\Omega = BA'\Omega = 0$, hence $\{B\Omega, \quad B \in \mathcal{M}\} \overset{dense}{\subset} \mathcal{H}$ and $A' = 0$ on $\mathcal{H}$.

- (2)→ (1): The Hilbert space is $\mathcal{H} = \mathcal{M}\Omega \oplus (\mathcal{M}\Omega)^{\perp} \equiv m \oplus m_{\perp}$. Let $P = P_m$ be the projector on the non-orthogonal elements of $\mathcal{H}$, hence $P_m \in \mathcal{M}'$. For Ω separating for $\mathcal{M}'$, $(\mathbb{1} - P_m)\Omega = 0$ with $P_m\Omega = \Omega$, since $\mathbb{1} \in \mathcal{M}$ it follows $\mathbb{1} - P_m = 0$ and $\mathcal{M}\Omega \overset{dense}{\subset} \mathcal{H}$.

For the next proposition the following Definitions are needed.

Definition 2.3. *Let $\mathcal{M}$ be a von Neumann algebra. An operator $A \in \mathcal{M}$ is called **positive** if there exists a $B \in \mathcal{M}$ with $A = B^*B$; one writes $A \geq 0$. Let $\mathcal{M}_+$ be the Definition of a von Neumann algebra containing **positive** elements.*

Definition 2.4. *(Normal state)*
*A **normal** state of a von Neumann algebra $\mathcal{M}$ is an ultraweakly continuous state.*

Proposition 2.1. *Let $\mathcal{M}$ be a von Neumann algebra on Hilbert space $\mathcal{H}$, then the statements*

- *For ω normal: ω **faithful** normal state, i.e. $\omega(A) > 0$ for all positive nonzero elements $A \in \mathcal{M}_+$*

- *GNS-construction w.r.t. $(\mathcal{M}, \omega)$ yields a separating and cyclic vector*

are equivalent.

It means that if one has a von Neumann algebra on a Hilbert space, then it is sufficient to demand a faithful normal state to get a representation with a separating and cyclic vector. Also, the other way round is true, i.e. by performing a GNS-construction one gets a faithful normal state automatically.

2.2 Modular structure and the Tomita-Takesaki theorem

In order to construct the modular theory within the Tomita-Takesaki theory let us first of all define an anti-linear operator on the Hilbert space $\mathcal{H}$.

Definition 2.5. *(Anti-linear oparators S and F)*
*Let Ω be a separating and cyclic vector for the von Neumann algebra $\mathcal{M}$, then Ω is also separating and cyclic for the commutant $\mathcal{M}'$ (recall **Remark 2.1.**). Define the operators:*

- S_0: $\quad S_0 A\Omega = A^*\Omega \quad$ *for $A \in \mathcal{M}$, well defined on the domain $D(S_0) = \mathcal{M}\Omega$*
- F_0: $\quad F_0 A'\Omega = A'^*\Omega \quad$ *for $A' \in \mathcal{M}'$, well defined on the domain $D(F_0) = \mathcal{M}'\Omega$*
- *S_0 and F_0 are closable and:*

$$S_0^* = \overline{F_0} \equiv F$$
$$F_0^* = \overline{S_0} \equiv S$$

- *polar decomposition of S:*

$$S = J\Delta^{1/2},$$

  where

  - *Δ is a positive, selfadjoint operator (modular oparator associated with the pair $(\mathcal{M}, \Omega)$)*
  - *J is a anti-linear operator (modular conjugation)*

For the anti-linear operators defined above following relations are true:

Proposition 2.2. *(Relations)*

- $\Delta^{-1/2} = J\Delta^{1/2}J$
- $\Delta = FS$
- $\Delta^{-1} = SF$
- $J^2 = \mathbb{1}$
- $J = J^*$

PROOF. From the **Definition 2.3.** we have $S^* = F$, $S = J\Delta^{1/2}$, $S = S^{-1}$ and $J^{-1} = J^*$.

1. From $S = S^{-1} \Rightarrow J\Delta^{1/2} = \Delta^{-1/2}J^* \quad | \cdot J$ from right
$\Rightarrow \quad J\Delta^{1/2}J = \Delta^{-1/2}J^*J = \Delta^{-1/2}$

2. $S^*S = FS = \Delta^{1/2}J^*J\Delta^{1/2} = \Delta^{1/2}\mathbb{1}\Delta^{1/2} = \Delta$

3. $SS^* = SF = J\Delta^{1/2}\Delta^{1/2}J^* = J\Delta^1J^* = \mathbb{1}\Delta^{-1} = \Delta^{-1}$

4. From $S = S^{-1} \Rightarrow J\Delta^{1/2} = \Delta^{-1/2}J^* \quad | \cdot J$ from left
$\Rightarrow \quad J^2\Delta^{1/2} = J\Delta^{-1/2}J^*$. $J\Delta^{-1/2}J^*$ is a positive operator and because of the uniqueness of the polar decomposition it follows that $J^2 = \mathbb{1}$

5. $JJ = \mathbb{1} = J^*J \Rightarrow J^* = J$

Now we can formulate the Tomita-Takesaki theorem. For the proof, please refer to the book of O. Bratteli and D. Robinson [2].

Theorem 2.1. *(Tomita-Takesaki theorem)*
Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω, Δ the associated modular operator and J the associated conjugation operator,
then

- $J\mathcal{M}J = \mathcal{M}'$ *and*
- $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \qquad \forall t \in \mathbb{R}$

Note that the unitary operators Δ^{it} are well defined by the Borel functional calculus.

2.3 The modular automorphism group and KMS-states

Definition 2.6. *The unitaries Δ^{it}, $t \in \mathbb{R}$, induce a one parameter automorphism group $\{\sigma_t\}$ of $\mathcal{M}$ by*

$$\sigma_t(A) = \Delta^{-it/\beta}A\Delta^{it/\beta}, \qquad A \in \mathcal{M}, t \in \mathbb{R}.$$

Recall. *(KMS-condition)*

Let $\mathcal{M}$ be a von Neumann Algebra and $\{\alpha_t | t \in \mathbb{R}\}$ a one parameter group of automorphisms of $\mathcal{M}$, then the state ω on $\mathcal{M}$ satisfies the KMS-condition at β, $0 < \beta < \infty$, (where $\beta = \frac{1}{k_B T}$ denotes the inverse temperature) w.r.t. $\{\alpha_t\}$ if for any $A, B \in \mathcal{M}$ the mapping $t \longrightarrow \omega(A\alpha_t(B))$ has a analytical continuation for $0 < \Im t < \beta$ such that

$$\omega(\alpha_t(A)B) = \omega(B\alpha_{t+i\beta}(A)), \quad \forall t \in \mathbb{R}.$$

Theorem 2.2. *Every faithful normal state satisfies the KMS-condition (modular condition) w.r.t. the corresponding automorphism group, then $\alpha_t(A) = \sigma_t(A)$.*

Let $\mathcal{M}$ be a von Neumann algebra and ω a faithful normal state on $\mathcal{M}$, then one can find a cyclic and separating representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ (by GNS-construction, therefore recall **Proposition 2.1.**). For $A, B \in \mathcal{M}$

$$\begin{aligned}
\omega(B\sigma_{i\beta}(A)) = & \\
&= \langle \Omega \mid B\sigma_{i\beta}(A)\Omega\rangle_\omega \overset{\Delta^{-1}|\Omega\rangle = |\Omega\rangle}{=} \langle \Omega \mid B\Delta A\Delta^{-1}\Omega\rangle_\omega = \langle \Omega \mid B\Delta A\Omega\rangle_\omega = \\
&= \langle B^*\Omega \mid \Delta^{1/2}\Delta^{1/2}A\Omega\rangle_\omega = \langle JB\Omega \mid JA^*\Omega\rangle_\omega = \langle \Omega \mid AB\Omega\rangle_\omega = \\
&= \omega(AB)_\omega \\
&= \omega(\sigma_{t=0}(A)B)_\omega \qquad \text{KMS-CONDITION}
\end{aligned} \tag{2.3.1}$$

The analytic continuation of $\omega(\sigma_t(A))$ is periodic in imaginary time directions and bounded in real time directions, hence a bounded entire function. Therefore it is constandt by Liouville's theorem.

It is quite remarkable that there is a strong connection between the Tomita-Takesaki theory and the theory of equilibrium states in quantum statistical physics. The former is defined by topological notions on v. Neumann algebras whereas the latter notions come from the KMS-condition and both are connected via their modular structure.
Again, for every faithful normal state ω , the GNS-construction yields a separating vector Ω, hence Ω is also KMS-state w.r.t. the modular group σ_t.
Having a KMS state ω on a C*-algebra $\mathcal{A}$, one gets a von Neumann algebra $\mathcal{M}$ by $\pi_\omega(\mathcal{A})'' = \mathcal{M}$ and a cyclic and separating vector Ω_ω for $\mathcal{M}$.

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