Superselection Sectors

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CHAPTER I

Introduction

1. Superposition Principle and Superselection Sectors

The wave properties of quantum mechanical states manifest themselves in the possibility of coherent superposition: If ψ_1 and ψ_2 are the wave functions of two states, then the superposition principle states, that also every linear combination

$$\psi = \alpha\psi_1 + \beta\psi_2$$

is a wave function of a quantum mechanical state. For classical waves, which are solutions of a linear wave equation, this principle has an obvious interpretation. Its assertion for quantum mechanics is not so easy to understand. For as is well known the value of the wave function at a point has no direct physical meaning, and multiplication of the wave function with a complex number does not change the physical state. Hence the superposition given above can not be described by states; if we replace e.g. ψ_1 by $e^{i\varphi}\psi_1$, $\varphi \in \mathbb{R}$, then

$$\psi' = \alpha e^{i\varphi}\psi_1 + \beta\psi_2$$

in general describes a different state than ψ. The observability of the relative phase $e^{i\varphi}$ in superpositions is *the* characteristic of quantum theory.

Through their ability of interference quantum mechanical states behave completely differently from classical states, where a system is at any time exactly at one point of phase space, or, upon incomplete knowledge of the state, with a certain probability distributed in phase space. Hence quantum theory, which in principle is also valid for macroscopic systems, gives statements that are in complete opposition to classical physics. A drastic example is Schrödinger's cat.

Actually quantum mechanical states cannot always be superposed coherently, i.e. there are cases, in which the relative phase of two wave functions is not observable. The example from which Wick, Wightman and Wigner made this observation, is a superposition of a spin-$\frac{1}{2}$ particle state with a spin-0 particle state

$$\psi = \alpha\psi_1 + \beta\psi_2 \quad .$$

A rotation by 2π changes ψ_1 into $-\psi_1$, ψ_2 into ψ_2 and therefore ψ into

$$\psi' = -\alpha\psi_1 + \beta\psi_2 \quad .$$

For all experiments ψ' has the same properties as ψ. Thus if A is an observable, then $(\forall \alpha, \beta)$

$$\begin{aligned}(\psi, A\psi) &= (\psi', A\psi') \\ &= |\alpha|^2(\psi_1, A\psi_1) + |\beta|^2(\psi_2, A\psi_2) \pm \left(\bar{\alpha}\beta(\psi_1, A\psi_2) + \alpha\bar{\beta}(\psi_2, A\psi_1)\right)\end{aligned}$$

must hold for either sign and hence $(\psi_1, A\psi_2) = 0$. In the state determined by ψ the observables have the same expectation values as in the state that is described by the density matrix

$$\rho = |\alpha|^2 |\psi_1\rangle\langle\psi_1| + |\beta|^2 |\psi_2\rangle\langle\psi_2| \quad .$$

The impossibility to superpose such states coherently is called a superselection rule. The existence of superselection rules leads to the following picture of the quantum mechanical state space: The Hilbert space of quantum mechanical state vectors is a direct sum of orthogonal subspaces

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i \quad .$$

Every vector ϕ has a unique decomposition

$$\phi = \sum \phi_i, \qquad \phi_i \in \mathcal{H}_i \quad .$$

Calling A_i the restriction of A to $\mathcal{H}_i$,

$$A_i = A\Big|_{\mathcal{H}_i}$$

and writing $\phi = \sum \phi_i$ as a column vector with components in $\mathcal{H}_i$, A is written as the diagonal matrix

$$A = \begin{pmatrix} A_1 & & & \\ & \ddots & & 0 \\ & & A_i & \\ & 0 & & \ddots \end{pmatrix} \quad .$$

Coherent superpositions are only possible within one subspace $\mathcal{H}_i$. A set of states, for which the superposition principle is valid without restriction, is called a **superselection sector**.

2. Algebraic Formulation of Quantum Physics

In order to understand the occurrence of superselection sectors, we take a look at the structure of the algebra $\mathfrak{A}$ generated by the observables. This algebra consists of the operators

$$A = \begin{pmatrix} A_1 & & & \\ & \ddots & & 0 \\ & & A_i & \\ & 0 & & \ddots \end{pmatrix}$$

with $A_i \in \mathcal{B}(\mathcal{H}_i)$ (set of bounded operators in $\mathcal{H}_i$) and $\sup_i \|A_i\| < \infty$. It possesses the center

$$\mathcal{Z}(\mathfrak{A}) = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots \lambda_i & \\ 0 & & \ddots \end{pmatrix} \middle| \lambda_i \in \mathbb{C} \right\} \quad .$$

The states in a joint eigenspace of the center are exactly the ones constituting a superselection sector.

As an example we consider a system, in which the operators L_i of angular momentum generate the algebra of observables. This algebra contains as the center the multiples of $\vec{L}^2$. Hence the sectors are distinguished through the angular momentum quantum number l.

If one views the algebraic structure of the observables as the defining property of a physical system, then one needs a definition of states, which does not make use of the realization of the observables as operators on a Hilbert space. A state assigns to each observable a probability distribution of results of the measurement . If $a_1, \ldots, a_n$ are the spectral values of A, then the state ω yields the the probabilities w_i for the occurrence of the value a_i as a result of the measurement. ω is conveniently characterized by the expectation values of all observables

$$\omega \longleftrightarrow \omega(A) = \sum w_i a_i \quad , \qquad A \in \mathfrak{A} \quad .$$

Thus, ω defines a linear functional on $\mathfrak{A}$ with the following properties

(i) $\omega(\mathbb{1}) = 1$ (normalization)
(ii) $\omega(A^*A) \geq 0 \quad \forall A \in \mathfrak{A}$ (positivity)

Here we used that $\mathfrak{A}$ is a $*$-algebra (the $*$-invariant elements are the observables) and contains a unit. If $\mathfrak{A}$ is a matrix algebra over $\mathbb{C}$, then A^* is the adjoint matrix to A,

$$(A^*)_{ik} = \overline{A_{ki}} \quad .$$

If $\mathfrak{A}$ is an operator algebra on a Hilbert space, then A^* is the adjoint operator to A.

For a self-adjoint element $A = A^*$ of $\mathfrak{A}$ the probability distribution can be reconstructed from the expectation values. Namely, the probability measure is determined by its moments

$$\int d\mu_{\omega,A}(\lambda)\lambda^n = \omega(A^n) \quad , \qquad \|A\| < \infty \quad .$$

The spectrum of $A \in \mathfrak{A}$ can also be directly characterized algebraically. It is the set of $\lambda \in \mathbb{C}$, such that $(A - \lambda\mathbb{1})$ possesses no inverse in $\mathfrak{A}$. Let e.g. $A^2 = \mathbb{1}$, then

$$\begin{aligned}(A - \lambda\mathbb{1})(A - \mu\mathbb{1}) &= A^2 - (\lambda + \mu)A + \lambda\mu\mathbb{1} \\ &= (1 + \lambda\mu)\mathbb{1} - (\lambda + \mu)A \quad ,\end{aligned}$$

i.e. $(1 - \lambda^2)^{-1}(A + \lambda\mathbb{1})$ for $\lambda \neq \pm 1$ is an inverse of $(A - \lambda\mathbb{1})$. Hence, A's only possible spectral values are ± 1.

Now let $\omega(A) = \alpha$, then for the probabilities of the two possible results of a measurement holds

$$\begin{aligned} w(+1) + w(-1) &= 1 \\ w(+1) - w(-1) &= \alpha \quad , \end{aligned}$$

and thus $w(+1) = \frac{1}{2}(1+\alpha)$, $w(-1) = \frac{1}{2}(1-\alpha)$ are determined.

The algebraic description also applies to classical physics. Let $\mathfrak{A}$ be the algebra of complex valued functions on the phase space of a mechanical system. Let $f \in \mathfrak{A}$ with supp$f \subset K$, where K is a compact region. Let $\lambda \notin f(K)$, then the function $f_\lambda(x) = (f(x) - \lambda)^{-1}$ is continuous, i.e. $f - \lambda\mathbb{1}$ possesses an inverse, and the spectrum of f lies in $f(K)$. Normed positive linear functionals on $\mathfrak{A}$ are Radon measures, which are connected to probability measures in the usual way,

$$\mu(f) = \int f d\mu \quad .$$

If $\mathfrak{A}$ is an operator algebra on a Hilbert space $\mathcal{H}$, then every vector $\psi \in \mathcal{H}$ with $\psi \neq 0$ defines a state through

$$\omega_\psi(A) = \frac{(\psi, A\psi)}{\|\psi\|^2} \quad .$$

One obtains further states from density matrices in $\mathcal{H}$. Let $\rho \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $\rho \geq 0$ and Tr$\rho = 1$. Then one defines

$$\omega_\rho(A) = \mathrm{Tr}\rho A \quad .$$

For $\|A\| \leq \infty$ the operator ρA is a trace class operator, ω_ρ is thus well-defined. It is obviously linear and normalized. The positivity can e.g. be recognized by diagonalizing ρ,

$$\rho = \sum \rho_i |\psi_i\rangle\langle\psi_i|$$

with $(\psi_i, \psi_j) = \delta_{ij}$, $(\psi_i)_{i\in\mathbb{N}}$ an orthonormal basis of $\mathcal{H}$, $\rho_i \geq 0$, $\sum \rho_i = 1$:

$$\begin{aligned} \mathrm{Tr}\rho A^*A &= \sum_i (\psi_i, \rho A^*A, \psi_i) \\ &= \sum_i \rho_i \|A\psi_i\|^2 \geq 0 \quad . \end{aligned}$$

Characteristic for the algebraic concept of state is the unified description of pure and mixed states.

The set of states forms a convex set. If ω_1 and ω_2 are states, then so are $\lambda\omega_1 + (1 - \lambda)\omega_2$, $\lambda \in (0, 1)$. In physical terms one should think of the mixture $\lambda\omega_1 + (1 - \lambda)\omega_2$ as a state in which the state ω_1 is present with a probability λ and the state ω_2 with a probability $1 - \lambda$. Now, pure states are those, which cannot be decomposed into convex combinations of other states.

In order to avoid complications we will first confine ourselves to the finite dimensional case. Then our algebra is isomorphic to a multi-matrix algebra,

$$\mathfrak{A} = \bigoplus_i M_{n_i}(\mathbb{C}) \quad , \qquad n_i \in \mathbb{N},\, i = 1, \ldots, k \quad ,$$

where $M_{n_i}(\mathbb{C})$ is the set of $(n_i \times n_i)$-matrices with complex elements.

Let us take a look at the case $n_i = 1,\ \forall i$ at first. Then $\mathfrak{A}$ is Abelian, $\mathfrak{A} = \mathcal{Z}(\mathfrak{A})$. The states are of the form

$$\omega(A) = \sum \omega_i A_i$$

with $A_i \in \mathbb{C},\ \omega_i \geq 0,\ \sum \omega_i = 1$. There are k pure states, each of which constitutes a superselection sector.

Next we consider the case $k = 1,\ n_1 = 2$. In this case the states are of the form

$$\omega(A) = \frac{1}{2}\mathrm{Tr}(\mathbb{1} + \vec{n}\vec{\sigma})A$$

with the Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and a vector $\vec{n} \in \mathbb{R}^3,\ |\vec{n}| \leq 1$. ($(\mathbb{1} + \vec{n}\vec{\sigma}),\ \vec{n} \in \mathbb{R}^3$ is the general form of a hermitian 2×2-matrix with trace 1. Its determinant is $\frac{1}{4}(1 - \vec{n}^2)$. Hence, positivity requires $|\vec{n}| \leq 1$.) Thus, the set of states is, as a convex set, isomorphic to the unit ball $B^3 \subset \mathbb{R}^3$. The marginal points of the ball correspond to the pure states.

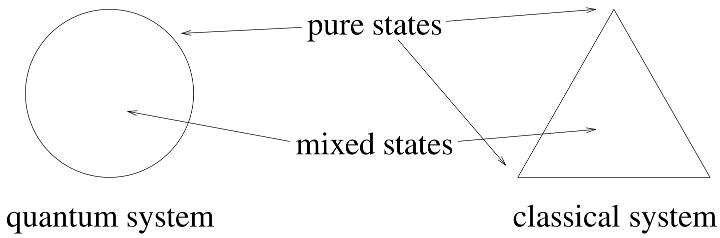

FIGURE I.1. State spaces

The case $n_1 > 2$ yields a more complicated picture. The state space contains so-called **faces**, i.e. subsets of a convex set, that are stable under convex decomposition. These faces are characterized by the kernels (null spaces) of the density matrices.

In the general case $k \geq 1,\ n_i \geq 1$ the state space has the form

$$\omega \longrightarrow (\rho_1, \ldots, \rho_n)$$

with $\rho_i \in M_{n_i}(\mathbb{C}),\ \rho_i \geq 0,\ \sum \mathrm{Tr}\rho_i = 1$. For the observables of the center of $\mathfrak{A}$

$$\mathcal{Z}(\mathfrak{A}) = \left\{ \begin{pmatrix} \lambda_1 \mathbb{1}_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \mathbb{1}_n \end{pmatrix},\ \lambda_i \in \mathbb{C} \right\}$$

there holds

$$\omega(A) = \sum (\mathrm{Tr}\rho_i)\lambda_i \quad .$$

Thus, for these observables the state space is classical. The quantum nature of the state space does not become visible before non-diagonal observables are taken into account. Thus, the occurrence of superselection sectors reflects a classical feature of the system.

It is an interesting fact that such classical features of the state space can arise in infinite quantum systems, even if the algebra of observables does not possess a nontrivial center.

As a simple example we examine a system of spins. Let at every point $x \in \mathbb{Z}$ reside a spin-$\frac{1}{2}$ degree of freedom, which is described by the Pauli matrices $\sigma_i(x)$ with $[\sigma_i(x), \sigma_j(y)] = 0$ for $x \neq y$.

At first we convince ourselves of the fact that the center of the algebra $\mathfrak{A}$ generated by $\sigma_i(x),\ x \in \mathbb{Z},\ i = 1,2,3$ is trivial. Suppose $A \in \mathcal{Z}(\mathfrak{A})$. Then for every $x \in \mathbb{Z}$, A is of the form

$$A = \sum_i \sigma_i(x) A_i + A_0 \quad ,$$

where $A_i,\ i = 0,\dots,3$ are finite sums of finite products of $\sigma_j(y),\ j = 1,2,3,\ y \in \mathbb{Z} \setminus \{x\}$. Since $A \in \mathcal{Z}(\mathfrak{A})$, it follows $[A, \sigma_j(x)] = 0$, thus

$$\epsilon_{ijk} \sigma_k(x) A_i = 0 \quad ,$$

i.e., e.g. $\sigma_1 A_3 - \sigma_3 A_1 = 0$. The commutator with σ_1 yields $\sigma_2 A_1 = 0$. Multiplication with σ_2 yields $A_1 = 0$. Thus, $A_i = 0,\ i \neq 0$. Iteration of the argument gives, that A is a multiple of $\mathbb{1}$.

Now we realize this algebra as an operator algebra on the Hilbert space of sequences

$$\mathcal{H} = \Big\{ \sum \lambda_s |s\rangle,\ \sum |\lambda_s|^2 < \infty,\ s : \mathbb{Z} \to \{\pm 1\} \Big\}$$

through

$$\begin{aligned} \sigma_3(x)|s\rangle &= s(x)|s\rangle \\ \sigma_1(x)|s\rangle &= |s'\rangle \\ \sigma_2(x)|s\rangle &= i s(x)|s'\rangle \quad , \end{aligned}$$

$s'(x) = -s(x),\ s'(y) = s(y),\ y \neq x$. $\mathcal{H}$ is a Hilbert space with a non-countable basis ($\dim \mathcal{H} = \#(\mathbb{R})$). The algebra $\mathfrak{A}$ of observables on the contrary possesses the countable basis

$$\Big\{ \sigma_{j_1}(x_1) \dots \sigma_{j_k}(x_k),\ x_1 < \dots < x_k,\ j_1, \dots, j_k \in \{1,2,3\},\ k \in \mathbb{N}_0 \Big\}$$

(the product of no factors is defined as $\mathbb{1}$). It is therefore obvious, that the spin algebra cannot transform arbitrary vectors into each other. There rather holds

$$\langle s'|A|s\rangle = 0 \quad , \qquad \forall A \in \mathfrak{A}$$

if $s'(x) \neq s(x)$ for infinitely many $x \in \mathbb{Z}$. Hence, $\mathcal{H}$ decomposes into a non-countable sum of subspaces, that are each invariant under $\mathfrak{A}$,

$$\mathcal{H} = \bigoplus_{[s]} \mathcal{H}_{[s]}$$

with

$$\mathcal{H}_{[s]} = \overline{\mathfrak{A}|s\rangle} = \Big\{ \sum_{s' \in [s]} \lambda_{s'} |s'\rangle \in \mathcal{H} \Big\}$$

and

$$[s] = \Big\{ s' : \mathbb{Z} \to \{\pm 1\},\ s'(x) \neq s(x) \text{ only for finitely many } x \in \mathbb{Z} \Big\} \quad .$$

We consider the two subspaces $\mathcal{H}_{[s_\pm]}$ with $s_\pm(x) = \pm 1,\ \forall x \in \mathbb{Z}$ and show, that the states are not able to interfere. For that purpose we consider the sequence of observables

$$M_n = \frac{1}{2n+1} \sum_{x=-n}^{n} \sigma_3(x) \quad , \qquad n \in \mathbb{N} \quad .$$

There holds

(1) $[M_n, A] \to 0, \forall A \in \mathfrak{A}$
(2) $M_n \phi_\pm \to \pm\phi_\pm,\ \phi_\pm \in \mathcal{H}_{[s_\pm]}$

and with it

$$(\phi_+, A\phi_-) = \lim(\phi_+, M_n A\phi_-) = \lim(\phi_+, AM_n\phi_-) = -(\phi_+, A\phi_-) \quad ,$$

i.e. $(\phi_+, A\phi_-) = 0$. This consequence holds for every simultaneous realization of the states ϕ_+, ϕ_- as Hilbert space vectors.

3. Operator Algebras

This section is intended to present the most important mathematical properties of operator algebras. For a more detailed treatment refer to the relevant textbooks and the lectures **Operator Algebras and Local Quantum Physics** by Detlev Buchholz.

DEFINITION. A C^*-Algebra $\mathfrak{A}$ is a complex (associative) algebra with an involution $A \to A^*$ and a norm $\|\cdot\|$, such that $\forall A, B \in \mathfrak{A}$

$$\|AB\| \leq \|A\|\,\|B\|$$
$$\|A^*\| = \|A\|$$
$$\|A^*A\| = \|A\|^2$$

and such that $\mathfrak{A}$ is complete with respect to this norm.

Interesting is the fact that on a C^*-algebra there exists only one norm fulfilling the stated properties. For some special operators this is easily seen.

Let $\mathbb{1} \in \mathfrak{A}$, $\mathfrak{A} \neq \{\mathbb{0}\}$ be the unit ($\mathbb{1}A = A\mathbb{1} = A$, $\mathbb{1}^* = \mathbb{1}$), then

$$\|\mathbb{1}\| = \|\mathbb{1}\mathbb{1}\| = \|\mathbb{1}\|^2 \quad \Rightarrow \quad \|\mathbb{1}\| = 1 \quad .$$

Let U be an isometry, i.e. $U^*U = \mathbb{1}$, then it follows that

$$\|U\|^2 = \|U^*U\| = \|\mathbb{1}\| = 1 \quad \Rightarrow \quad \|U\| = 1 \quad .$$

Let P be a projection, i.e. $P = P^2 = P^*$ (orthogonal projection). It follows that

$$\|P\| = \|P^2\| = \|P^*P\| = \|P\|^2 \quad .$$

Thus, $\|P\| = 1$ for $P \neq 0$. In general in a C^*-algebra with $\mathbb{1}$ there holds

$$\|A\| = \inf\left\{\rho \in \mathbb{R}_+,\ A^*A - r^2\,\mathbb{1} \text{ is invertible in } \mathfrak{A}\ \forall r > \rho\right\} \quad .$$

The norm is thus an intrinsic property of the algebra, in particular every isomorphism is isometric, i.e. norm conserving. Furthermore is the inverse of $A^*A - r^2\,\mathbb{1}$, if it exists, contained in the C^*-subalgebra of $\mathfrak{A}$ generated by A^*A and $\mathbb{1}$. Embeddings of C^*-algebras in larger C^*-algebras are thus isometric, too. Homomorphisms are automatically contracting, i.e. if $\phi : \mathfrak{A} \to B$ is a ($*$-algebra) homomorphism, then $\|\phi(A)\| \leq \|A\|$. If $(\mathfrak{A}_1, \|\cdot\|_1)$ is a Banach $*$-algebra, i.e. all the properties of a C^*-algebra except for $\|A^*A\|_1 = \|A\|_1^2$ are fulfilled, and $\phi : \mathfrak{A}_1 \to \mathfrak{A}$ a homomorphism into a C^*-algebra, then

$$\|\phi(A)\| \leq \|A\|_1 \quad .$$

WARNING. The uniqueness holds only for norms on the whole (complete) algebra. On an arbitrary $*$-algebra there may be more than one norm (or none) with the demanded properties.

EXAMPLES.

(i) Let X be a compact Hausdorff space, and let $C(X)$ be the $*$-algebra of continuous functions $f : X \to \mathbb{C}$ with $f^*(x) = \overline{f(x)}$. $(|f(x)|^2 - r^2)^{-1}$ is continuous on X, $\forall r > \rho$ if and only if $\rho \geq \sup_{x \in X} |f(x)|$. Thus,

$$\|f\| = \sup_{x \in X} |f(x)| \quad .$$

(ii) Let $\mathfrak{A} = M_n(\mathbb{C})$. $A^*A - r^2\,\mathbb{1}$ is invertible, $A \in \mathfrak{A}$, if r^2 is no eigenvalue of A^*A. Thus,

$$\|A\| = \rho \quad ,$$

where ρ^2 is the largest eigenvalue of A^*A. This norm coincides with

$$\|A\| = \sup_{\substack{\phi \in \mathbb{C}^n \\ \|\phi\| = 1}} \|A\phi\| \quad ,$$

where $\mathbb{C}^n$ was turned into a Hilbert space by $\|\phi\|^2 = \sum_{n=1}^{n} |\phi_n|^2$.

(iii) Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{A}$ a norm-closed, self-adjoint algebra of bounded Hilbert space operators. Then

$$\|A\| = \sup_{\substack{\phi \in \mathcal{H} \\ \|\phi\| = 1}} \|A\phi\| \quad .$$

(iv) Let $\mathfrak{A}_0$ be the algebra of observables of the spin chain. $\mathfrak{A}_0$ possesses a unique C^*-norm, since every element A^*A, $A \in \mathfrak{A}_0$ generates a subalgebra of a finite dimensional matrix algebra, on which the norm is uniquely determined.

(v) Let $\mathcal{K}$ be a Hilbert space and Γ an anti-unitary involution on $\mathcal{K}$, i.e.

$$\begin{aligned} \Gamma(\lambda f + \mu g) &= \overline{\lambda}\Gamma f + \overline{\mu}\Gamma g \quad , \qquad f, g \in \mathcal{K},\ \lambda, \mu \in \mathbb{C} \\ (\Gamma f, \Gamma g) &= (g, f) \\ \Gamma^2 &= 1 \quad . \end{aligned}$$

We consider the $*$-algebra with $\mathbb{1}$ generated by the symbols $B(f)$, $f \in \mathcal{K}$ with the relations

$$\begin{aligned} f &\to B(f) \text{ linear} \\ B(f)^* &= B(\Gamma f) \\ \{B(f), B(g)\} &= (\Gamma f, g)\,\mathbb{1} \quad . \end{aligned}$$

For every $2n$-dimensionald Γ-invariant subspace of $\mathcal{K}$ the corresponding subalgebra is isomorphic to $M_{2^n}(\mathbb{C})$. Hence, again the C^*-norm is unique. It holds e.g. for f with $\Gamma f = f$

$$\|B(f)\| = \|B(f)^*B(f)\|^{\frac{1}{2}} = \|B(\Gamma f)B(f)\|^{\frac{1}{2}} = \|B(f)^2\|^{\frac{1}{2}}$$
$$= \|\frac{1}{2}\{B(f), B(f)\}\|^{\frac{1}{2}} = \|\frac{1}{2}(\Gamma f, f)\|^{\frac{1}{2}} = (\frac{1}{2}\|f^2\|)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\|f\| \quad .$$

This C^*-algebra is called the **CAR-algebra** ('canonical anticommutation relation') to $(\mathcal{K}, \Gamma)$, notation CAR$(\mathcal{K}, \Gamma)$. It is important for the description of fermions.

(vi) Let L be a real vector space with a non-degenerate symplectic form σ, i.e.

$$\begin{aligned} \sigma :\ L \times L &\to \mathbb{R} \\ \sigma(f, g) &= -\sigma(g, f) \\ \sigma(\lambda f + \mu g, h) &= \lambda\sigma(f, h) + \mu\sigma(g, h) \\ \sigma(f, g) = 0\, \forall g \quad &\Rightarrow \quad f = 0 \quad . \end{aligned}$$

We consider the $*$-algebra generated by the symbols $W(f)$ with the relations

$$\begin{aligned} W(f)W(g) &= e^{i\sigma(f,g)}W(f + g) \\ W(f)^* &= W(-f) \quad . \end{aligned}$$

This algebra possesses $W(0)$ as the unit. On this algebra there is a unique C^*-norm (cf. Bratteli, Robinson). The algebra thus defined is denoted as the **Weyl algebra** $\mathcal{W}(L, \sigma)$.

(vii) We consider the $*$-algebra generated by n elements ψ_i, $i = 1, \ldots, n$ with the relations

$$\begin{aligned} \psi_j^*\psi_i &= \delta_{ij} \\ \sum_{i=1}^{n} \psi_i\psi_i^* &= 1 \quad . \end{aligned}$$

This algebra possesses a unique C^*-norm (Cuntz, DR). The generated C^*-algebra is called the **Cuntz algebra** $\mathcal{O}_n$. It plays an important role in the Doplicher-Roberts theory of group duals.

(viii) Let $\mathfrak{A}_0$ be the $*$-algebra with $\mathbb{1}$ that is generated by a self-adjoint element A. This algebra possesses a lot of C^*-norms. The general element of the algebra is a polynomial in A, $p_n(A)$. Every compact region $K \subset \mathbb{R}$ defines by

$$\|p_n(A)\|_K = \sup_{x \in K} |p_n(x)|$$

a C^*-norm on $\mathfrak{A}_0$. The completion with respect to the norm $\|\cdot\|_K$ yields the algebra $C(K)$ of continuous functions on K.

Concerning the structure of C^*-algebras the following two theorems hold:

THEOREM 1. *Every commutative C^*-algebra with $\mathbb{1}$ is isomorphic to the algebra of continuous functions on a compact Hausdorff space.*

THEOREM 2. *Every C^*-algebra is isomorphic to a norm-closed, self-adjoint algebra of operators on a (not necessarily separable) Hilbert space.*

If we apply Theorem 1 to the C^*-algebra generated by a self-adjoint element A, then the mentioned Hausdorff space is the spectrum of A,

$$\mathrm{sp}(A) = \left\{ \lambda \in \mathbb{C} \,\middle|\, A - \lambda \mathbb{1} \text{ not invertible in } \mathfrak{A} \right\} \quad .$$

The spectrum is a closed subspace of the interval $[-\|A\|, \|A\|]$. The element corresponding to the continuous function f is denoted by $f(A)$. Here A corresponds to the identical function $\mathrm{sp}(A) \ni x \mapsto x$, to every polynomial corresponds the corresponding polynomial in A, and $f(A)$ can be obtained as the limit of the sequence

$$f(A) = \lim_{n\to\infty} p_n(A) \quad ,$$

if the sequence of polynomials p_n on $\mathrm{sp}(A)$ converges uniformly to f.

Now we take a look at representations and states. A representation of a C^*-algebra $\mathfrak{A}$ is a homomorphism π from $\mathfrak{A}$ into the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, such that

$$\pi(A)^* = \pi(A^*) \quad .$$

Here $\pi(A)^*$ is the adjoint operator to $\pi(A)$. According to the mentioned properties of the norm, representations are automatically continuous,

$$\|\pi(A)\| \leq \|A\| \quad ,$$

and they are isometric, if they are faithful (i.e. if their kernel $\ker = \{A \in \mathfrak{A}, \pi(A) = 0\} = \{\mathbb{0}\}$).

Connected to a representation is a family of states

$$S_\pi(\mathfrak{A}) = \left\{ \omega \in S(\mathfrak{A}) \,\middle|\, \exists \rho \in \mathcal{B}(\mathcal{H}),\ \rho \geq 0, \mathrm{Tr}\rho = 1 \text{ with } \omega(A) = \mathrm{Tr}\rho\pi(A) \right\} \quad .$$

Here $S(\mathfrak{A})$ is the set of states of $\mathfrak{A}$, i.e. the set of linear functionals on $\mathfrak{A}$ satisfying the two conditions $\omega(A^*A) \geq 0,\ A \in \mathfrak{A}$ and $\omega(\mathbb{1}) = 1$. States are hermitian,

$$\omega(A^*) = \overline{\omega(A)} \quad ,$$

which follows from

$$\begin{aligned} 0 &\leq \omega\Big((\lambda\mathbb{1} + A)^*(\lambda\mathbb{1} + A)\Big) \\ &= |\lambda|^2 + \overline{\lambda}\omega(A) + \lambda\omega(A^*) + \omega(A^*A) \qquad \forall \lambda \in \mathbb{C} \quad . \end{aligned}$$

If we set $\lambda = \omega(A)$, then

$$|\omega(A)| \leq \omega(A^*A)^{\frac{1}{2}} \quad .$$

From this now follows that states are automatically continuous. For $\|A^*A\|\mathbb{1} - A^*A$ is a positive continuous function of A^*A and hence possesses a representation

$$\|A^*A\|\mathbb{1} - A^*A = C^*C$$

for some $C \in \mathfrak{A}$ (e.g. $C = (\|A^*A\|\mathbb{1} - A^*A)^{\frac{1}{2}}$). Thus,

$$0 \leq \omega(C^*C) = \|A^*A\|\omega(\mathbb{1}) - \omega(A^*A) \quad ,$$

thus,

$$|\omega(A)| \leq \omega(A^*A)^{\frac{1}{2}} \leq \|A^*A\|^{\frac{1}{2}} = \|A\| \quad ,$$

i.e.

$$\|\omega\| = \sup_{\substack{A \in \mathfrak{A} \\ \|A\| = 1}} \|\omega(A)\| = 1 \quad .$$

On $S(\mathfrak{A})$ can be defined a metric by

$$d(\omega_1, \omega_2) = \|\omega_1 - \omega_2\| \quad .$$

$S(\mathfrak{A})$ is a norm-closed subset of $\mathfrak{A}^*$, the dual space of $\mathfrak{A}$.

$S_\pi(\mathfrak{A})$ is called a **folium**, i.e. a set S of states with the property

$$\sum_i \omega(A_i^* A_i) = 1 \quad \Rightarrow \quad A \mapsto \sum_i \omega(A_i^* A A_i) \in S \quad ,$$

where $\omega \in S$ and $A_1, \dots, A_n \in \mathfrak{A}$. $S_\pi(\mathfrak{A})$ is a closed (with respect to the metric d) subset of $S(\mathfrak{A})$.

Intertwiners play an important role in comparing two representations. Let π_1 and π_2 be two representations of $\mathfrak{A}$ in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We call bounded operators $T : \mathcal{H}_1 \to \mathcal{H}_2$ with the property

$$T\pi_1(A) = \pi_2(A)T \quad , \qquad \forall A \in \mathfrak{A}$$

intertwiners from π_1 to π_2. Let the set of intertwiners from π_1 to π_2 be denoted by (π_1, π_2). There holds

$$\begin{aligned} T \in (\pi_1, \pi_2) \quad &\Rightarrow \quad T^* \in (\pi_2, \pi_1) \\ T \in (\pi_1, \pi_2),\ S \in (\pi_2, \pi_3) \quad &\Rightarrow \quad ST \in (\pi_1, \pi_3) \end{aligned}$$

and hence

$$T \in (\pi_1, \pi_2) \quad \Rightarrow \quad T^*T \in (\pi_1, \pi_1) \quad .$$

We now introduce the following concepts:

(i) A representation π is called **irreducible**, if no subspace of $\mathcal{H}$, apart from $\{\mathbb{0}\}$ and $\mathcal{H}$, exists that is invariant under $\pi(\mathfrak{A})$. (It suffices to demand that there exists no closed invariant subspace.) For irreducible representations Schur's Lemma holds: $(\pi, \pi) = \mathbb{C}\mathbb{1}$.

(ii) Two representations π_1 and π_2 are called **unitarily equivalent**, $\pi_1 \simeq \pi_2$, if there exists a unitary ($U^*U = UU^* = \mathbb{1}$) intertwiner $U \in (\pi_1, \pi_2)$.

(iii) Two representations π_1 and π_2 are called **disjoint**, $\pi_1 \between \pi_2$, if no intertwiner apart from $\mathbb{0}$ exists, $(\pi_1, \pi_2) = \{\mathbb{0}\}$. There holds:

$$\begin{aligned} \pi_1, \pi_2 \text{ irreducible:} \qquad \pi_1 \not\simeq \pi_2 \quad &\Leftrightarrow \quad \pi_1 \between \pi_2 \\ \pi_1 \between \pi_2 \quad &\Leftrightarrow \quad \mathrm{dist}(S_{\pi_1}(\mathfrak{A}), S_{\pi_2}(\mathfrak{A})) = 2 \quad . \end{aligned}$$

(iv) **Quasiequivalence**:

$$\begin{aligned} \pi_1 \approx \pi_2 \quad &\Leftrightarrow \quad S_{\pi_1}(\mathfrak{A}) = S_{\pi_2}(\mathfrak{A}) \\ \pi_1, \pi_2 \text{ irreducible :} \qquad \pi_1 \approx \pi_2 \quad &\Leftrightarrow \quad \pi_1 \simeq \pi_2 \quad . \end{aligned}$$

(v) π is called **factorial** if the center of algebra of selfintertwiners is trivial, $\mathcal{Z}(\pi,\pi) = \mathbb{C}\mathbb{1}$. In this case π is quasiequivalent to all its subrepresentations, and $S_\pi(\mathfrak{A})$ does not contain a closed subfolium. Two factorial representations are either quasiequivalent or disjoint.

Besides the norm topology there exists the $*$-weak topology on the state space. It is generated by the semi-norms

$$\|\varphi\|_A = |\varphi(A)| \quad , \qquad A \in \mathfrak{A},\ \varphi \in \mathfrak{A}^*$$

on the space $\mathfrak{A}^*$ of continuous linear functionals on $\mathfrak{A}$. The following important proposition holds:

FELL'S THEOREM. *Let π be a faithful representation of $\mathfrak{A}$. Then $S_\pi(\mathfrak{A})$ is $*$-weakly dense in $S(\mathfrak{A})$.*

This theorem explains 'in principle', why and in what sense states of a folium can approximate an arbitrary given state. Yet, in the application of the theorem one has to pay attention that in a physical situation in general not all states of a folium are available, but only the ones that can be prepared at 'finite expense'.

We have seen that density matrices describe states in representations. In fact one obtains all states in this way. There even holds the following proposition:

THEOREM (GNS-CONSTRUCTION). *Let ω be a state of a C^*-algebra $\mathfrak{A}$. Then there exists a Hilbert space $\mathcal{H}$, a vector $\Omega \in \mathcal{H}$, and a representation π of $\mathfrak{A}$ in $\mathcal{H}$, such that*

(i) $(\Omega, \pi(A)\Omega) = \omega(A)$, $\forall A \in \mathfrak{A}$,

(ii) *$\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$ (i.e. Ω is* **cyclic** *for $\pi(\mathfrak{A})$).*

$(\mathcal{H}, \pi, \Omega)$ is called **GNS-triple** *to ω. Let $(\mathcal{H}', \pi', \Omega')$ be another triple fulfilling (i) and (ii). Then there exists a unitary operator $U \in (\pi, \pi')$ with $U\Omega = \Omega'$. (One says, the GNS-triple is unique (up to unitary equivalence)).*

PROOF. We define on $\mathfrak{A}$ the sesquilinear form $(A, B)_\omega = \omega(A^*B)$. $(\,\cdot\,,\,\cdot\,)_\omega$ is a semidefinite scalar product, i.e.

(i) linear in the right, antilinear in the left argument (⚠In the mathematical literature mostly the opposite.),

(ii) hermitian

$$(B, A)_\omega = \omega(B^*A) = \overline{\omega(A^*B)} = \overline{(A, B)_\omega} \quad ,$$

(iii) positive semidefinite.

In particular the **Schwarz inequality** holds

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B) \quad .$$

Hence the set $N_\omega = \{A \in \mathfrak{A} \,|\, \omega(A^*A) = 0\}$ is a subspace of $\mathfrak{A}$. N_ω is even a left ideal, i.e. $A \in N_\omega,\ B \in \mathfrak{A} \ \Rightarrow\ BA \in N_\omega$, since (with $C = ((BA)^*B)^*$)

$$\omega((BA)^*BA) = \omega(C^*A) \leq \omega(C^*C)^{\frac{1}{2}}\omega(A^*A)^{\frac{1}{2}} = 0$$

because of $\omega(A^*A) = 0$. We now consider the space $\mathcal{H}_0 = \mathfrak{A}/N_\omega$ with the canonical map

$$\begin{cases} \mathfrak{A} \ \rightarrow\ \mathcal{H}_0 \\ A \ \mapsto\ \hat{A} = A + N_\omega \end{cases} \quad .$$

$\mathcal{H}_0$ is a pre-Hilbert space with positiv definite scalar product

$$(\hat{A}, \hat{B}) = (A, B)_\omega \quad .$$

Its completion is a Hilbert space $\mathcal{H}_0 = L^2(\mathfrak{A}, \omega)$. On $\mathcal{H}_0$ a representation can be defined through left multiplication,

$$\pi_0(A)\hat{B} = \widehat{AB} \quad .$$

π_0 is well-defined, for N_ω is a left ideal. In addition $\pi_0(A)$ is bounded. Namely,

$$\|\pi_0(A)\hat{B}\|^2 = \|\widehat{AB}\|^2 = \omega((AB)^*AB) = \omega(B^*A^*AB) \quad .$$

Since $\|A^*A\|\,\mathbb{1} - A^*A = C^*C$ for a $C \in \mathfrak{A}$, from $\omega(B^*C^*CB) \geq 0$ follows

$$\|\pi_0(A)\hat{B}\|^2 = \|A\|^2\|B\|^2 \quad ,$$

i.e. $\|\pi_0(A)\| \leq \|A\|$. Hence, $\pi_0(A)$ can be uniquely continued to a bounded operator $\pi(A)$ on $\mathcal{H}$. That π is a $*$-representation follows from

$$(\hat{C}, \pi(A^*)\hat{B}) = \omega(C^*A^*B) = \omega((AC)^*B) = (\pi(A)\hat{C}, \hat{B}) \quad .$$

We set $\Omega = \hat{\mathbb{1}}$ and obtain a GNS-triple $(\mathcal{H}, \pi, \Omega)$. The uniqueness of the GNS-triple emerges from the unitary intertwiner $U \in (\pi, \pi')$,

$$U\pi(A)\Omega = \pi'(A)\Omega' \quad , \qquad A \in \mathfrak{A} \quad .$$

U is densely defined (since Ω is cyclic), isometric

$$\begin{aligned} \|U\pi(A)\Omega\|^2 = \|\pi'(A)\Omega'\|^2 &= (\Omega', \pi'(A^*A)\Omega') = \omega(A^*A) \\ &= (\Omega, \pi(A^*A)\Omega) = \|\pi(A)\Omega\|^2 \quad , \end{aligned}$$

and onto (since Ω' is cyclic), thus unitary. The intertwining properties follow from

$$U\pi(A)\pi(B)\Omega = U\pi(AB)\Omega = \pi'(AB)\Omega' = \pi'(A)\pi'(B)\Omega' = \pi'(A)U\pi(B)\Omega$$

on $\pi(\mathfrak{A})\omega = \mathcal{H}_0$ and on $\mathcal{H}$ by continuity. □

Another mathematical concept we need is the von Neumann algebra or W^*-algebra. Von Neumann algebras can be introduced in different ways. The most direct method starts from a $*$-algebra of Hilbert space operators. On the algebra $\mathcal{B}(\mathcal{H})$ there exist besides the norm topology a whole lot of further topologies.

(i) The **weak** operator topology. This is generated by the semi-norms

$$\|A\|_{\phi,\psi} = |(\phi, A\psi)| \quad , \qquad \phi, \psi \in \mathcal{H} \quad .$$

(ii) The **strong** operator topology. It is generated by the semi-norms

$$\|A\|_\phi = \|A\phi\| \quad , \qquad \phi \in \mathcal{H} \quad .$$

(iii) The **ultraweak** (or σ-) operator topology. It is generated by the semi-norms

$$\|A\|_{\sigma,\rho} = |\mathrm{Tr}\rho A| \quad , \qquad \rho \text{ density matrix in } \mathcal{H} \quad .$$

(iv) The **ultrastrong** (or s-) operator topology, generated by the semi-norms

$$\|A\|_{s,\rho} = (\mathrm{Tr}\rho A^*A)^{\frac{1}{2}} \quad , \qquad \rho \text{ density matrix} \quad .$$

(v) The $*$-**ultrastrong** (or s^*-) operator topology, generated by the semi-norms

$$\|A\|_{s^*,\rho} = (\|A\|^2_{s,\rho} + \|A^*\|^2_{s,\rho})^{\frac{1}{2}} \quad , \qquad \rho \text{ density matrix} \quad .$$

If $\mathcal{H}$ is infinite-dimensional, then none of these topologies satisfies the axioms of countability. Therefore they cannot be characterized by convergent sequences, one needs convergent nets, instead. Comparison of the topologies yields

$$\begin{array}{ccccccc} \text{weak} & < & \text{strong} & < & \text{strong}^* & & \\ \wedge & & \wedge & & \wedge & & \\ \sigma & < & s & < & s^* & < & \text{norm} \end{array} \quad .$$

One can now close a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ in either of these topologies. Surprisingly one obtains for all topologies between weak and s^* the same closure. This is again a $*$-algebra. A $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ (closed in this topology) is called **von Neumann algebra** or W^***-algebra**.

Von Neumann algebras always have a unit. Yet, this does not necessarily have to be the unit operator on the Hilbert space, but may be a projection. If one speaks of a von Neumann algebra on a Hilbert space though, then it is meant that the $\mathbb{1}$ of the algebra is the $\mathbb{1}$ operator, too. Thus if necessary, $\mathcal{H}$ is replaced by $P\mathcal{H}$.

Next an algebraic definition of von Neumann algebras will be given. Let $\mathfrak{A}$ be a $*$-algebra of operators on a Hilbert space $\mathcal{H}$ containing the $\mathbb{1}$ operator. Then von Neumann's bicommutant theorem holds.

THEOREM. *The von Neumann algebra generated by* $\mathfrak{A}$ *is*

$$\mathfrak{A}'' = \Big\{A \in \mathcal{B}(\mathcal{H}) \Big| \forall B \in \mathcal{B}(\mathcal{H}) \text{ with } [B, A_0] = 0 \,\forall A_0 \in \mathfrak{A} \text{ holds } [B, A] = 0\Big\} \quad .$$

For a set $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$

$$\mathcal{M}' = \Big\{B \in \mathcal{B}(\mathcal{H}) \Big| [B, C] = 0 \,\forall C \in \mathcal{M}\Big\}$$

is called the **commutant** of $\mathcal{M}$. $\mathfrak{A}''$ is thus the commutant of the commutant (the **bicommutant**) of $\mathfrak{A}$.

The characterization of von Neumann algebras through the bicommutant theorem uses the embedding of the algebra in $\mathcal{B}(\mathcal{H})$. As a matter of fact it is possible to characterize von Neumann algebras intrinsically. For that purpose one notes that the σ-, s- and s^*-topologies are described by density matrices, thus by states of a folium. Von Neumann algebras $\mathcal{N}$ can now also be characterized by the fact that they are C^*-algebras with a distinguished (closed) folium, the folium of normal states $S_{\mathrm{n}}(\mathcal{N})$. This folium satisfies the conditions:

(i) $A, B \in \mathcal{N}$, $\omega(A) = \omega(B) \,\forall \omega \in S_{\mathrm{n}}(\mathcal{N}) \;\Rightarrow\; A = B$.

(ii) Let $f:\; S_{\mathrm{n}}(\mathcal{N}) \to \mathbb{C}$ be a bounded function with

$$f(\lambda\omega_1 + (1-\lambda)\omega_2) = \lambda f(\omega_1) + (1-\lambda) f(\omega_2) \quad ,$$

where $\lambda \in [0, 1]$ and $\omega_{1,2} \in S_{\mathrm{n}}(\mathcal{N})$. Then there exists a $A \in \mathcal{N}$ with

$$f(\omega) = \omega(A) \quad .$$

REMARKS.

(i) One can describe these conditions also in the way, that $\mathcal{N}$ is the space of continuous linear functionals on the subspace $\mathcal{N}_*$ of $\mathcal{N}^*$ generated by $S_{\rm n}(\mathcal{N})$, $\mathcal{N} = (\mathcal{N}_*)^*$. One calls $\mathcal{N}_*$ the **predual** of $\mathcal{N}$.

(ii) Whether a state on a von Neumann algebra $\mathcal{N}$ is normal, can be decided from the following property. Let

$$A_\lambda \in \mathcal{N}_+ = \left\{A \in \mathcal{N} \,\middle|\, \exists B \in \mathcal{N} \text{ with } B^*B = A\right\}$$

be a monotoneously increasing net, i.e.

$$A_\lambda \geq A_{\lambda'} \text{ for } \lambda \geq \lambda'$$

with an upper bound $c\mathbb{1}$. By

$$f(\omega) = \sup_\lambda \omega(A_\lambda)$$

a bounded linear function on $S_{\rm n}(\mathcal{N})$ is defined. Hence, there exists $A \in \mathcal{N}$ with $f(\omega) = \omega(A)$. A is per constructionem the lowest upper bound of the set (A_λ)

$$A = \sup A_\lambda \quad .$$

A state ω on $\mathcal{N}$ is called **normal** (is an element of $S_{\rm n}(\mathcal{N})$), if for all bounded monotoneously increasing nets

$$\sup \omega(A_\lambda) = \omega(\sup A_\lambda) \quad .$$

(iii) Singular (i.e. not normal) states on von Neumann algebras occur mostly in the following context: Let $\mathfrak{A} \subset \mathcal{N}$ be a σ-dense $*$-subalgebra of $\mathcal{N}$, and let ω be a state on $\mathfrak{A}$. Due to the Hahn-Banach theorem (which in its turn is based upon Zorn's lemma) ω possesses a continuation to a state on $\mathcal{N}$. A normal continuation yet exists only, if ω is σ-continuous.

(iv) $*$-isomorphisms of von Neumann algebras are automatically continuous with respect to the σ-, s- and s^*-topologies.

EXAMPLES.

(i) Let $\mathfrak{A} \in C([0,1])$ be the C^*-algebra of continuous complex-valued functions on the interval $[0,1]$. The Lebesgue measure defines by

$$\lambda(f) = \int dx f(x)$$

a state on $\mathfrak{A}$. Other states in the folium of λ are of the form

$$\lambda_g(f) = \int dx g(x) f(x)$$

with $g \in \mathfrak{A}$, $g \geq 0$ and $\int dx g(x) = 1$. The distance between two states of this form is

$$d(\lambda_{g_1}, \lambda_{g_2}) = \sup_{f, \|f\| \leq 1} \left| \int dx (g_1(x) - g_2(x)) f(x) \right|$$
$$= \int dx |g_1(x) - g_2(x)| = \|g_1 - g_2\|_1 \quad .$$

The closure of the folium therefore contains all states of the form λ_g with $g \in L^1([0,1])$, $g \geq 0$ and $\int dx g(x) = 1$, i.e. all Lebesgue absolute continuous probability measures. The von Neumann algebra corresponding to this folium is $L^\infty([0,1])$, i.e. the algebra of all essentially bounded measurable functions, modulo the functions, that differ from zero only on a set of measure null.

(ii) Let $\mathcal{K}_1$ be the algebra of operators $\lambda\mathbb{1}+A$, $\lambda \in \mathbb{C}$, A compact, on a Hilbert space $\mathcal{H}$. $\mathcal{K}_1$ is generated (as a closed linear space) by $\mathbb{1}$ and the rank-1 operators $|\phi\rangle\langle\psi|$. Hence, a state $\omega \in S(\mathcal{K}_1)$ is fixed by its values $\omega(|\phi\rangle\langle\psi|)$ for all $\phi, \psi \in \mathcal{H}$. We now define an operator ρ on $\mathcal{H}$ by its matrix elements

$$(\phi, \rho\psi) = \omega(|\psi\rangle\langle\phi|) \quad .$$

One easily shows (with $|\omega(A)| \leq \|A\|$ and $\||\psi\rangle\langle\phi|\| = \||\psi\rangle\langle\phi|\phi\rangle\langle\psi|\|^{\frac{1}{2}} = \|\phi\|\,\|\psi\|$), that ρ is bounded. Positivity of ρ follows from

$$(\phi, \rho\phi) = \omega(|\phi\rangle\langle\phi|) \geq 0 \text{ because of } |\phi\rangle\langle\phi| \geq 0 \quad .$$

ρ is a trace class operator, because for every orthonormal system $(\phi_n) \in \mathcal{H}$ holds

$$\sum_{n=1}^{N}(\phi_n, \rho\phi_n) = \omega(\sum_{n=1}^{N} |\phi_n\rangle\langle\phi_n|) \leq 1 \quad .$$

Hence, a state ω is characterized by a positive trace class operator ρ with $\mathrm{Tr}\rho \leq 1$,

$$\omega(\lambda\mathbb{1} + A) = \lambda + \mathrm{Tr}\rho A \quad , \qquad A \text{ compact} \quad ,$$

and possesses the decomposition into disjoint minimal folia

$$\omega = (1 - \mathrm{Tr}\rho)\omega_\infty + \mathrm{Tr}\rho\frac{\mathrm{Tr}(\rho\,\cdot)}{\mathrm{Tr}\rho}$$

with $\omega_\infty(\lambda\mathbb{1} + A) = \lambda,\ A$ compact.

Consider the folium with $\mathrm{Tr}\rho = 1$. Let f be a linear functional on the trace class operators, that is bounded on the density matrices. Then

$$f(|\phi\rangle\langle\phi|) = (\phi, A\phi)$$

defines a bounded linear operator A on $\mathcal{H}$. Conversely, every $A \in \mathcal{B}(\mathcal{H})$ defines a linear functional on the trace class operators. We therefore see, that the density matrices form the folium of normal states on the von Neumann algebra $\mathcal{B}(\mathcal{H})$.

4. Process of Measurement

The occurrence of superselection sectors gives an answer to the problem of the quantum mechanical description of the process of measurement. According to the formulation of von Neumann the process of measurement consists of a reduction of the wave packet: Let e.g. P be a projection, then the measurement of P effects a transformation of the pure state ψ into the mixture

$$|P\psi\rangle\langle P\psi| + |(1-P)\psi\rangle\langle(1-P)\psi| \quad ,$$

in which the two components $P\psi$ and $(1-P)\psi$ occur with the probabilities $\|P\psi\|^2$ and $\|(1-P)\psi\|^2$. Since a unitary time evolution could never yield this result, this was taken as an additional interaction, what of course is very dissatisfying.

However, even if one finds a mechanism turning pure states into mixtures, e.g. coupling of the system to a second system and subsequent separation, then the problem still remains, that a mixture has a lot of decompositions into pure states, so reading off $\|P\psi\|^2$ is not obvious (there is only one decomposition into orthogonal states, however it seems to be a little artificial to claim, that this happens in the process of measurement).

A much better interpretation has been developed by Hepp (see Landsman for a current overview). According to him the essential point is, that the system to be observed either possesses superselection sectors by itself or will be coupled to such a system, and that in the limit of large evolution times the state turns into a mixture of disjoint states. The observation of the intensity of the single components then is a measurement of a central observable, that is not disturbed by quantum physical interference.

As a first example we consider a non-relativistic particle, whose pure states are described by $\psi \in L^2(\mathbb{R}^3)$, $\|\psi\| = 1$. As the algebra of observables we consider $\mathcal{K}_1$. We want to measure the transition probability to a state described by $\phi \in L^2(\mathbb{R}^3)$, $\|\psi\| = 1$. For that purpose we choose a Hamilton operator H, that possesses ϕ as the only bound state and has on the orthogonal complement of ϕ a Lebesgue absolute continuous spectrum. The time evolution on $\mathcal{K}_1$ is given by the automorphism

$$\alpha_t(\,\cdot\,) = e^{iHt} \cdot e^{-iHt} .$$

We obtain for $A = |\phi_1\rangle\langle\psi_1|$, $\phi_1, \psi_1 \in \mathcal{H}$, $\omega = \omega_\psi = (\psi, \cdot\,\psi)$,

$$\omega_\psi \alpha_t(A) = \omega_\psi \alpha_t(|\phi_1\rangle\langle\psi_1|) = (\psi, \alpha_t(|\phi_1\rangle\langle\psi_1|)\psi) = (\psi, e^{iHt}\phi_1)\,(\psi_1, e^{-iHt}\psi) \quad .$$

Let $\psi = (\phi, \psi)\phi + \psi^\perp$. There holds

$$(\psi^\perp, e^{iHt}\phi_1) = \int dE e^{iEt}(\psi^\perp(E), \phi_1(E)) \to 0 \quad , \qquad t \to \infty \quad ,$$

since $E \mapsto (\psi^\perp(E), \phi_1(E)) \in L^1(\mathbb{R})$ (Riemann-Lebesgue lemma). Thus,

$$\omega_\psi \alpha_t(A) \to |(\phi, \psi)|^2 (\phi, A\phi) \quad , \qquad t \to \infty$$

for A compact and

$$\omega_\psi \alpha_t \to |(\phi, \psi)|^2 \omega_\phi + (1 - |(\phi, \psi)|^2)\omega_\infty \quad , \qquad t \to \infty \quad .$$

Thus, $\omega_\psi \alpha_t$ decomposes asymptotically into two disjoint components ω_ϕ and ω_∞, and the intensity of the component ω_ϕ is given by the transition probability $|(\phi, \psi)|^2$.

Next we will take a look at the following example. Let ψ_α, $\alpha = 1, 2$ be the massive free Dirac field in two dimensions.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

is a solution of the Dirac equation

$$(im + \gamma^\mu \partial_\mu)\psi = 0 \quad , \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad , \qquad (\gamma^0)^* = \gamma^0,\, (\gamma^1)^* = -\gamma^1$$

in the sense of operator valued distributions. One considers the operators

$$\psi(f) = \sum_{\alpha,\beta} \int d^2x \overline{f_\alpha(x)} \psi_\beta(x) (\gamma^0)_{\alpha\beta} \quad , \qquad f \in \mathcal{D}(\mathbb{R}^2, \mathbb{C}^2)$$

and demands

(1) $$\psi((\gamma_\mu \partial^\mu + im)f) = 0 \quad , \qquad \forall f \in \mathcal{D}(\mathbb{R}^2, \mathbb{C}^2) \quad .$$

On the spacelike hyperplanes the commutation relations take a particularly simple form. For $f, g \in \mathcal{D}(\mathbb{R}, \mathbb{C}^2)$,

$$\{\psi(f \otimes \delta_t), \psi(g \otimes \delta_t)\} = 0$$

$$\{\psi(f \otimes \delta_t), \psi(g \otimes \delta_t)^*\} = (f, g) \equiv \sum_{\alpha=1}^{2} \int dx \overline{f_\alpha(x)} g_\alpha(x) \quad .$$

(That ψ is well-defined on $f \otimes \delta_t$, too, follows from (1) and properties of the Dirac equation). The Fock space is the GNS Hilbert space to the state

$$\omega(\psi(f_1) \cdots \psi(f_n)\, \psi(g_m)^* \cdots \psi(g_1)^*) = \delta_{nm} \det \Big(S_+(f_i, g_j) \Big) \quad , \qquad i, j = 1, \ldots, n \quad .$$

Here,

$$S_+(f, g) = \sum_{\alpha,\beta} \int d^2x d^2y \overline{f_\alpha(x)} S_+(x - y)_{\alpha\beta} g_\beta(y)$$

and

$$\begin{aligned} S_+(x) &= i\gamma^0(\gamma^\mu \partial_\mu - im)\Delta_+(x) \\ &= i\gamma^0(\gamma^\mu \partial_\mu - im) \int_{-\infty}^{+\infty} \frac{dp}{2\sqrt{m^2 + p^2}} e^{-i(\sqrt{m^2+p^2}x^0 - px^1)} (2\pi)^{-3} \quad . \end{aligned}$$

Because of the anticommutation relations the Dirac field cannot be interpreted as an observable itself, but combinations, which are invariant under the global gauge transformation $\psi \to e^{i\alpha}\psi$, $\alpha \in \mathbb{R}$. Let $\mathfrak{A}$ be the C^*-algebra, that is generated by operators of the form $\psi(f)^*\psi(g)$. We consider now in Fock space the expression

$$q(x) = \int_{-\infty}^{x^1} dy^1 j_0(x^0, y^1) \quad , \qquad j_0(x) = \sum_\alpha :\psi_\alpha^* \psi_\alpha(x): \quad .$$

j_0 is the charge density of the free Dirac field, the double dots stand for the subtraction of the (divergent) vacuum expectation value. Formally,

$$\begin{aligned} \partial_1 q(x) &= j_0(x) \\ \partial_0 q(x) &= \int_{-\infty}^{x^1} dy^1 \partial_0 j_0(x^0, y^1) = -\int_{-\infty}^{x^1} dy^1 \partial_1 j_1(x^0, y^1) = -j_1(x) \end{aligned}$$

with $j_1(x) =: \psi^*(x)\gamma^0\gamma^1\psi(x):$. Furthermore

$$\Box q(x) = (\partial_0^2 - \partial_1^2) q(x) = -\partial_0 j_1 - \partial_1 j_0 = -2im :\psi^* \gamma^0 \gamma^5 \psi: \quad .$$

$:\psi^*\gamma^0\gamma^5\psi:(x)$ can now be expressed as a function of $q(x)$,

$$-2im :\psi^*\gamma^0\gamma^5\psi: (x) = 2m^2 \vdots q(x)(\cos(2\pi q(x)) - 1)\vdots \quad ,$$

where

$$\vdots e^{2\pi i\lambda q(x)}\vdots = \frac{e^{2\pi i\lambda q(x)}}{\omega_0(e^{2\pi i\lambda q(x)})} \quad .$$

Thus, one can see that q is a solution of the Sine-Gordon equation with a certain coupling constant. In fact it is a special case of the connection between Sine-Gordon and massive Thirring model (see Coleman, Lehmann-Stehr, Schroer-Truong). q (respectively bounded operators constructed from it) can be understood as an observable, for it commutes in spacelike distances with all other observables and also with itself. Since the transformation $q(x) \to q(x) + \mathbb{1}$ conserves all algebraic relations (in particular the field equation for q), it represents a symmetry. However, this symmetry is spontaneously broken. Since the vacuum expectation value of q in Fock space vanishes,

$$\omega_0(q(x) + \mathbb{1}) = 1 = \omega_1(q(x)) \quad .$$

We thus obtain a family of vacua ω_n, $\omega_n(q(x)) = n$, $n \in \mathbb{Z}$. The corresponding GNS representations are disjoint. We now consider the one particle state

$$\omega_g = (\psi(g)\Omega, \pi(\,\cdot\,)\psi(g)\Omega)$$

with $g \in \mathcal{D}(\mathbb{R}^2, \mathbb{C}^2)$,

$$\|g\|^2 = (2\pi)^{-3} \int_{\mathbb{R}} \frac{dp}{\sqrt{p^2+m^2}} \sum_\alpha \left|\tilde{g}_\alpha(\sqrt{p^2+m^2}, p)\right|^2 = 1 \quad .$$

The following holds:

$$\omega_g(q(x)) \longrightarrow \begin{cases} 0 & x \to \text{left spacelike } \infty \\ -1 & x \to \text{right spacelike } \infty \end{cases} \quad .$$

Thus, this state interpolates between two vacua, it is a soliton (and can be identified with the soliton of the Sine-Gordon theory).

Now we consider the time evolution of this state. Let α_t be the time evolution induced by the Dirac equation. Then,

$$\omega_g \circ \alpha_t \underset{w^*}{\longrightarrow} \|P_+ g\|^2 \omega_0 + \|P_- g\|^2 \omega_1$$

with

$$\|P_\pm g\|^2 = \int_{\mathbb{R}_\pm} \frac{dp}{\sqrt{m^2+p^2}} \sum_\alpha \left|\tilde{g}_\alpha(\sqrt{m^2+p^2}, p)\right|^2 \quad .$$

We thus see, that in the limit $t \to \infty$ the one particle state decomposes into two disjoint components with intensities equal to the probability, that the particle is moving to the right or left, respectively.

CHAPTER II

Local Quantum Physics

1. Principle of Locality

One of the most important principles in physics is the principle of locality. It says that physical systems can only influence each other locally. Where actions over distances are observed, there has to be a mediating system, as e.g. the electric field for the Coulomb force between charges. The principle of locality allows to approximately isolate systems and strongly restricts the possible physical laws.

In quantum theory one can formulate the principle of locality in the following way. Let $\mathfrak{A}$ be the C^*-algebra of the quantum mechanical observables. $\mathfrak{A}$ possesses a norm-dense subalgebra $\mathfrak{A}_0$, whose self-adjoint elements can be interpreted as local measurements. One thereby associates to each $A \in \mathfrak{A}_0$ the set $L(A)$ of spacetime regions, in which A can be measured, such that

(i) $\mathcal{O} \in L(A)$, $\mathcal{O}_1 \supset \mathcal{O} \quad \Rightarrow \quad \mathcal{O}_1 \in L(A)$

(ii) If B is an element of the $*$-algebra generated by $A_1, \ldots, A_n$, then

$$L(B) \supset \bigcap_{i=1}^{n} L(A_i) \quad ,$$

i.e. if all A_i can be measured in one region, then so can B.

One can now consider for each spacetime region $\mathcal{O}$ the algebra

$$\mathfrak{A}(\mathcal{O}) = \Big\{ A \in \mathfrak{A}_0,\ \mathcal{O} \in L(A) \Big\} \quad .$$

$\mathfrak{A}(\mathcal{O})$ can be viewed as the algebra of observables of the subsystem connected to the region $\mathcal{O}$.

According to Haag and Kastler a quantum field theoretical model can be completely characterized by the map from regions $\mathcal{O}$ to operator algebras $\mathfrak{A}(\mathcal{O})$ on a Hilbert space $\mathcal{H}$, such that

$$\mathcal{O}_1 \subset \mathcal{O}_2 \quad \Rightarrow \quad \mathfrak{A}(\mathcal{O}_1) \hookrightarrow \mathfrak{A}(\mathcal{O}_2) \quad , \qquad \text{(unital embedding)} \quad .$$

It is sufficient to fix the **Haag-Kastler net** $\mathfrak{A} = (\mathfrak{A}(\mathcal{O}))_{\mathcal{O}}$ on regions of the form

$$\mathcal{O} = (V_+ + x) \cap (V_- + y) \quad , \qquad y - x \in V_+ \quad ,$$

the so-called **double cones**. Here

$$V_\pm = \Big\{ x \in \mathbb{R}^4 \,\Big|\, x^0 \gtrless \pm |\vec{x}| \Big\}$$

is the forward-, backward cone, respectively. Algebras for general regions $\mathcal{G}$ can now be introduced through the definition

$\mathfrak{A}(\mathcal{G}) =$ $*$-algebra (C^*-algebra, von Neumann algebra) generated by $\mathfrak{A}(\mathcal{O}),\ \mathcal{O} \in \mathcal{G}$

One often initially knows the algebras $\mathfrak{A}(\mathcal{O})$ only as abstract algebras. Then in addition one needs to have knowledge of the embedding homomorphisms

$$i_{\mathcal{O}_1\mathcal{O}_2} : \mathfrak{A}(\mathcal{O}_1) \to \mathfrak{A}(\mathcal{O}_2) \quad , \qquad \text{for } \mathcal{O}_1 \subset \mathcal{O}_2$$

with the compatibility condition $i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_1}$ for $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3$. This system then defines an abstract global algebra $\mathfrak{A}_\infty$, which is characterized by the following universality condition:

(i) $\exists$ embeddings $i_\mathcal{O} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}_\infty$, such that

$$i_{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_1} \quad , \qquad \mathcal{O}_1 \subset \mathcal{O}_2 \quad . \tag{2}$$

(ii) For every family of homomorphisms $\varphi_\mathcal{O} : \mathfrak{A}(\mathcal{O}) \to B$ with the compatibility condition (2) there is a homomorphism $\varphi : \mathfrak{A}_\infty \to B$ with $\varphi \circ i_\mathcal{O} = \varphi_\mathcal{O}$.

$\mathfrak{A}_\infty$ is generated by the elements $i_\mathcal{O}(A),\ A \in \mathfrak{A}(\mathcal{O}),\ \mathcal{O} \in \mathcal{K}$ with the relations

(i) $\mathfrak{A}(\mathcal{O}) \ni A \to i_\mathcal{O}(A)$ is a $\mathbb{1}$-preserving $*$-homomorphism

(ii) $i_{\mathcal{O}_2}(i_{\mathcal{O}_2\mathcal{O}_1}(A)) = i_{\mathcal{O}_1}(A),\ A \in \mathfrak{A}(\mathcal{O}_1),\ \mathcal{O}_1 \subset \mathcal{O}_2$

By this $\mathfrak{A}_\infty$ is uniquely fixed as a unital $*$-algebra.

Now we want to make use of the fact that the set $\mathcal{K}$ of double cones of Minkowski space is directed, i.e.

$$\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K} \quad \Rightarrow \quad \exists \mathcal{O} \in \mathcal{K} \text{ with } \mathcal{O}_1 \subset \mathcal{O},\ \mathcal{O}_2 \subset \mathcal{O} \quad .$$

For that reason,

$$\mathfrak{A}_\infty = \bigcup_\mathcal{O} i_\mathcal{O}(\mathfrak{A}(\mathcal{O})) \quad .$$

If now every local algebra $\mathfrak{A}(\mathcal{O})$ is a C^*-algebra, then $\mathfrak{A}_\infty$ possesses a unique C^*-norm, and one can introduce the completion

$$\mathfrak{A} = \overline{\bigcup_\mathcal{O} i_\mathcal{O}(\mathfrak{A}(\mathcal{O}))} \quad .$$

$\mathfrak{A}$ is called the **quasilocal** algebra. One talks about the C^*-inductive limit of the system $(\mathfrak{A}(\mathcal{O}))$.

The C^*-inductive limit has the property, that no new relations can arise. For let π be a representation of $\mathfrak{A}$, and $\pi(A) = 0$ for an $A \in \mathfrak{A}$. There exists a sequence $\mathcal{O}_n \in \mathcal{K},\ A_n \in \mathfrak{A}(\mathcal{O}_n)$ with $\|A_n - A\| \to 0$. Then, $\|\pi(A_n)\| \to 0$. Now,

$$\|\pi(A_n)\| = \inf \left\{ \|A_n - B_n\|,\ B_n \in \mathfrak{A}(\mathcal{O}_n),\ \pi(B_n) = 0 \right\} \quad .$$

Hence there exists a sequence $B_n \in \mathfrak{A}(\mathcal{O}_n),\ \pi(B_n) = 0$ with $\|A_n - B_n\| \to 0$, thus,

$$\|A_n - B_n\| \leq \|A - A_n\| + \|A - B_n\| \to 0 \quad ,$$

i.e. the kernel of π is the completion of the union of the kernels of $\pi|_{\mathfrak{A}(\mathcal{O})}$. If in particular $\pi|_{\mathfrak{A}(\mathcal{O})}$ is faithful for all $\mathcal{O}$, then π is faithful, too.

Let us consider as an example the spin algebra from the introductory chapter. We consider the set $\mathcal{M}$ of intervals $I = \{n \in \mathbb{Z},\ n_1 \leq n \leq n_2\},\ n_2 \geq n_1$ in $\mathbb{Z}$. To every interval the algebra $\mathfrak{A}(I)$ is associated, which is generated by the σ-matrices at the point $n \in I$. This algebra is isomorphic to $M_{2^{n_2-n_1}}(\mathbb{C})$. The embeddings $\mathfrak{A}(I) \hookrightarrow \mathfrak{A}(J)$ for $I \subset J$ are defined in a natural way. The C^*-inductive limit is the C^*-algebra of the spin chain we already considered. Since the local algebras $\mathfrak{A}(I)$ are simple (do not possess any nontrivial ideals), $\mathfrak{A}$ is simple, too, and all representations are faithful.

Let us now take a look at the structure of the state space. We assume that every local algebra $\mathfrak{A}(\mathcal{O})$ possesses a distinguished folium $S_{\mathrm{n}}(\mathfrak{A}(\mathcal{O}))$ of states (possibly states with finite energy density in quantum field theoretical examples). Equivalently, we can assume every local algebra to be a von Neumann algebra. We now consider the set $S_{\mathrm{ln}}(\mathfrak{A})$ of locally normal states of $\mathfrak{A}$, these are states $\omega \in S(\mathfrak{A})$ with the property $\omega|_{\mathfrak{A}(\mathcal{O})} \in S_{\mathrm{n}}(\mathfrak{A}(\mathcal{O}))$. Even if the local folia are minimal (as in the example of the spin algebra), $S_{\mathrm{ln}}(\mathfrak{A})$ possesses in general a lot of different minimal folia. In the example of the spin algebra one can choose for every $k \in \mathbb{Z}$ a vector $\vec{n}(k) \in B^3$, which describes on $\mathfrak{A}(\{k\})$ the state $\omega(k)$ with the density matrix $\rho(k) = \frac{1}{2}(\mathbb{1} + \vec{n}(k)\vec{\sigma}(k))$. Every such sequence defines a state on $\mathfrak{A}$. The folium generated by it is minimal (the state is primary, i.e. the GNS representation is factorial). Two such sequences $(\omega(k), \omega'(k))$ define disjoint folia if

$$\sum_k \|\omega(k) - \omega'(k)\|^2 = \sum_k |\vec{n}(k) - \vec{n}'(k)|^2 = \infty \quad ,$$

for the following estimate holds:

$$\|\omega - \omega'\| \geq 2(1 - e^{-\frac{1}{8}\sum \|\omega(k) - \omega'(k)\|^2}) \quad .$$

PROOF. We have

$$\begin{aligned} \|\omega(k) - \omega'(k)\| = \|\rho(k) - \rho'(k)\|_1 &\leq \|(\sqrt{\rho(k)} - \sqrt{\rho'(k)})(\sqrt{\rho(k)} + \sqrt{\rho'(k)})\|_1 \\ &\leq \|\sqrt{\rho(k)} - \sqrt{\rho'(k)}\|_2 \, \|\sqrt{\rho(k)} + \sqrt{\rho'(k)}\|_2 \\ &= 2\sqrt{1 - (\mathrm{Tr}\sqrt{\rho(k)}\sqrt{\rho'(k)})^2} \quad . \end{aligned}$$

Furthermore (for $\rho_N = \prod_{k=1}^N \rho(k)$),

$$-\sqrt{\rho_N} - \sqrt{\rho'_N} \leq \sqrt{\rho_N} - \sqrt{\rho'_N} \leq \sqrt{\rho_N} + \sqrt{\rho'_N}$$

and hence ($A = \sqrt{\rho_N} - \sqrt{\rho'_N}$, $B = \sqrt{\rho_N} + \sqrt{\rho'_N}$),

$$-1 \leq B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \leq 1 \quad .$$

(Here $B^{-\frac{1}{2}}$ is defined to null on the null space of B.) Thus,

$$\|\omega - \omega'\| \geq \|(\omega - \omega')(B^{-\frac{1}{2}} A B^{-\frac{1}{2}})\| = \mathrm{Tr}(\rho_N - \rho'_N) B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \quad .$$

$\rho_N - \rho'_N = \frac{1}{2}(AB + BA)$ and cyclicity of the trace yield

$$\|\omega - \omega'\| \geq \mathrm{Tr} A B^{-\frac{1}{2}} A B^{-\frac{1}{2}} = \mathrm{Tr}(B^{-\frac{1}{4}} A B^{-\frac{1}{4}})^* B^{-\frac{1}{4}} A B^{-\frac{1}{4}} = \mathrm{Tr} C^* C$$

with $C = B^{-\frac{1}{4}} A B^{-\frac{1}{4}}$. Because of

$$\mathrm{Tr}(C^*C + CC^* - C^2 - C^{*2}) = \mathrm{Tr}(C - C^*)^*(C - C^*) \geq 0$$

we have $\mathrm{Tr}C^*C = \frac{1}{2}\mathrm{Tr}(C^*C + CC^*) \geq \frac{1}{2}\mathrm{Tr}(C^2 + C^{*2})$. However,

$$\mathrm{Tr}C^2 = \mathrm{Tr}C^{*2} = \mathrm{Tr}A^2 = \|\sqrt{\rho_N} - \sqrt{\rho'_N}\|_2^2 = 2(1 - \mathrm{Tr}\sqrt{\rho_N}\sqrt{\rho'_N}) \quad ,$$

so that $\|\omega - \omega'\| \geq 2(1 - \mathrm{Tr}\sqrt{\rho_N}\sqrt{\rho'_N})$. Now,

$$\mathrm{Tr}\sqrt{\rho_N}\sqrt{\rho'_N} = \prod_{k=1}^{N} \mathrm{Tr}\sqrt{\rho(k)}\sqrt{\rho'(k)} \quad .$$

From the estimate for $\|\omega(k) - \omega'(k)\|$ follows

$$1 - (\mathrm{Tr}\sqrt{\rho(k)}\sqrt{\rho'(k)})^2 \geq \frac{1}{4}\|\omega(k) - \omega'(k)\|^2$$

or

$$\mathrm{Tr}\sqrt{\rho(k)}\sqrt{\rho'(k)} \leq \sqrt{1 - \frac{1}{4}\|\omega(k) - \omega'(k)\|^2} \leq e^{-\frac{1}{8}\|\omega(k)-\omega'(k)\|^2} \quad .$$

Thus,

$$\mathrm{Tr}\sqrt{\rho_N}\sqrt{\rho'_N} \leq e^{-\frac{1}{8}\sum_{k=1}^{N}\|\omega(k)-\omega'(k)\|^2}$$

and therefore

$$\|\omega - \omega'\| \geq 2(1 - e^{-\frac{1}{8}\sum_{k=1}^{N}\|\omega(k)-\omega'(k)\|^2})$$

for all N. □

A new situation occurs if the system of subsets, over which the local algebras are defined, is not directed, e.g. if one considers a theory on a globally hyperbolic Lorentz manifold $\Sigma \times \mathbb{R}$, Σ compact, and the algebras $\mathfrak{A}(\mathcal{K})$ for contractible regions in Σ are given. Then one can consider again the $*$-algebra generated by $i_{\mathcal{K}}$, $A \in \mathfrak{A}(\mathcal{K})$ with the mentioned relations. This however does not have the structure of a union of local algebras. For that reason there arise algebraic expressions, which cannot be calculated in the local algebras. Hence, the algebra can fulfill new relations. In particular there is no unique C^*-(semi)norm on this algebra. However, there is a uniquely determined maximal C^*-seminorm, and if there exists a family of faithful representations $\pi_{\mathcal{K}}$ of $\mathfrak{A}(\mathcal{K})$ in a Hilbert space $\mathcal{H}$ with $\pi_{\mathcal{K}} \circ i_{\mathcal{K}\mathcal{K}'} = \pi_{\mathcal{K}'}$ for $\mathcal{K}' \subset \mathcal{K}$, then this maximal C^*-seminorm coincides on the local algebras with the C^*-norm of the local algebras.

As an example we consider the CAR algebra over $L^2(S^1)$ with Γ as complex conjugation. For every proper interval I in S^1 wie define $\mathfrak{A}(I)$ as the even subalgebra of CAR$(L^2(I), \Gamma)$, i.e. the subalgebra, which is generated by products containing an even number of Fermi operators $B(f)$. If $I \subset J$, then $L^2(I)$ is in a natural way a subspace of $L^2(J)$. This defines an embedding $i_{JI} : \mathfrak{A}(I) \to \mathfrak{A}(J)$. The family of these embeddings satisfies the compatibility condition

$$i_{KJ} \circ i_{JI} = i_{KI} \quad , \qquad \text{for } I \subset J \subset K$$

and hence defines an algebra $\mathfrak{A}(S^1)$. The structure of $\mathfrak{A}(S^1)$ is easy to describe.

First, $\mathfrak{A}(I)$ is the algebra spanned by operators of the form

$$b_I(f, g) = 2B(f)B(g) \quad , \qquad f, g \in L^2_{\text{real}}(I),\ \Gamma f = f,\ \Gamma g = g$$

The canonical anticommutation relations for B,

$$\{B(f), B(g)\} = (f,g)\mathbb{1}$$

lead to the relations

(i) $f, g \to b_I(f,g)$ is real bilinear
(ii) $b_I(f,f) = \|f\|^2\,\mathbb{1}$
(iii) $b_I(f,g)b_I(g,h) = \|g\|^2 b_I(f,h)$

$\mathfrak{A}(S^1)$ now is the algebra generated by $b_I(f,g),\ f,g \in \frac{1}{2}(\mathbb{1}+\Gamma)L^2(I)$ for all I and the additional relation

(iv) $b_I(f,g) = b_J(f,g)$ for $I \subset J$.

Let now $I_1, I_2, J_\pm$ be intervals on S^1 with

$$I_1 \cup I_2 \subset J_\pm \qquad J_+ \cup J_- = S^1 \qquad I_1 \cap I_2 = \emptyset$$

and there is no interval I with $I_1 \cap I_2 \subset I \subset J_+ \cap J_-$. Let $f \in L^2(I_1)$, $g \in$

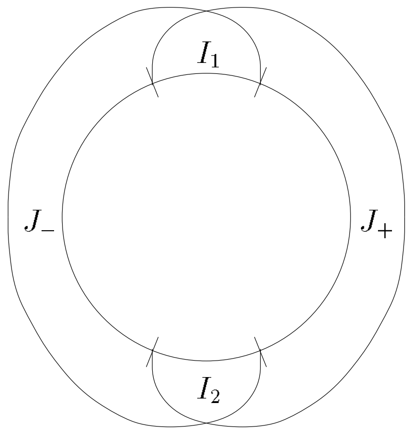

FIGURE II.2. Intervals on S^1

$L^2(I_2)$, $\Gamma f = f$, $\Gamma g = g$, $\|f\| = \|g\| = 1$. We consider the operator

$$Y = b_{J_+}(f,g)b_{J_-}(g,f)$$

Y does not depend on the choice of g, for let $g' \in L^2(I_2)$ with $\|g'\| = 1$ and $\Gamma g' = g'$. Then,

$$b_{J_+}(f',g') = b_{J_+}(f,g)b_{J_+}(g,g') = b_{J_+}(f,g)b_{I_2}(g,g')$$

and

$$b_{J_-}(g',f) = b_{I_2}(g',g)b_{J_-}(g,f) \quad .$$

With $b_{I_2}(g,g')b_{I_2}(g',g) = b_{I_2}(g,g) = \mathbb{1}$ follows

$$b_{J_+}(f,g')b_{J_-}(g',f) = Y \quad .$$

With $b_{J_+}(f,g) = -b_{J_+}(g,f)$ also follows the independence of f. Then, however, Y also becomes independent of the choice of the intervals, and we have

$$Y^2 = 1$$

and

$$[Y, b_I(f,g)] = 0 \quad , \qquad \forall f, g \, \forall I \quad .$$

Every element $A \in \mathfrak{A}(S^1)$ can be decomposed into

$$A = A_+ + A_- \quad , \qquad \text{with } A_\pm = \frac{1}{2}(\mathbb{1} + Y) \quad ,$$

and $\mathfrak{A}(S^1)$ possesses the corresponding decomposition

$$\mathfrak{A}(S^1) = \mathfrak{A}_+(S^1) \oplus \mathfrak{A}_-(S^1) \quad .$$

$\mathfrak{A}_+(S^1)$ is obviously isomorphic to the even part of CAR$(L^2(S^1), \Gamma)$, for $b_I(f,g)_+$ is independent of I. Namely, let I_1, I_2 be two intervals with $f, g \in L^2(I_1) \cap L^2(I_2)$. If there exists an interval with $I \supset I_1 \cup I_2$, then

$$b_{I_1}(f,g) = b_I(f,g) = b_{I_2}(f,g) \quad .$$

If there is no such interval, then $I_1 \cup I_2 = S^1$ and $I_1 \cap I_2$ has two components $I_\pm$. We decompose f and g into $f = f_+ + f_-$, $g = g_+ + g_-$, which each have support in one of the components, and obtain a corresponding decomposition of $b_{I_1}(f,g)$ and $b_{I_2}(f,g)$. We have

$$b_{I_1}(f_\pm, g_\pm) = b_{I_\pm}(f_\pm, g_\pm) = b_{I_2}(f_\pm, g_\pm)$$

and

$$b_{I_1}(f_\pm, g_\mp)_+ = [Y, b_{I_2}(f_\pm, g_\mp)]_+ = b_{I_2}(f_\pm, g_\mp)_+ \quad .$$

$\mathfrak{A}_+(S^1)$ corresponds to the so-called **Ramond sector** (fermions on S^1 with periodical boundary conditions). We now want to convince ourselves of the fact, that $\mathfrak{A}_-(S^1)$ corresponds to the **Neveu-Schwarz sector** (fermions on S^1 with antiperiodical boundary conditions). We consider the twofold covering $\tilde{S}^1$ of S^1 and the Hilbert space

$$L^2(\tilde{S}^1)_{\mathrm{a}} = \{f \in L^2(\tilde{S}^1),\; f(-\sqrt{z}) = -f(\sqrt{z})\} \quad .$$

(We have embedded $\tilde{S}^1$ into the Riemann surface of $\sqrt{z}$). Over this space we can again consider the even CAR algebra. It describes fermions on S^1 with antiperiodical boundary conditions. We choose a covering map $\tilde{S}^1 \to S^1$ by distinguishing a point z_0 in S^1. The inverse image of an interval $I \subset S^1$ is the union of two intervals $I^{(1)} \cup I^{(2)}$ on $\tilde{S}^1$. We now choose $I^{(1)} \subset (\sqrt{z_0}, -\sqrt{z_0})$ if $I \not\ni z_0$ and otherwise such that $I^{(1)} \ni (-\sqrt{z_0})$. Now to every $f \in L^2(I)$, $I \subset S^1$ will be associated a $\tilde{f}_I \in L^2(\tilde{S}^1)_{\mathrm{a}}$ by

$$\text{supp}\tilde{f}_I \subset I^{(1)} \cup I^{(2)} \quad , \qquad \tilde{f}(\sqrt{z}) = \begin{cases} f(z) & \sqrt{z} \in I^{(1)} \\ -f(z) & \sqrt{z} \in I^{(2)} \end{cases} \quad .$$

Through the identification

$$b_I(f,g) = 2B(\tilde{f}_I)B(\tilde{g}_I)$$

now an isomorphism is defined from CAR$(L^2(\tilde{S}^1)_{\mathrm{a}}, \Gamma)_{\text{even}}$ to $\mathfrak{A}_-(S^1)$.

This algebra $\mathfrak{A}(S^1) = \mathfrak{A}_+(S^1) + \mathfrak{A}_-(S^1)$ possesses as a unique C^*-norm

$$\|A\| = \sup\Big\{\|A_+\|, \|A_-\|\Big\} \quad ,$$

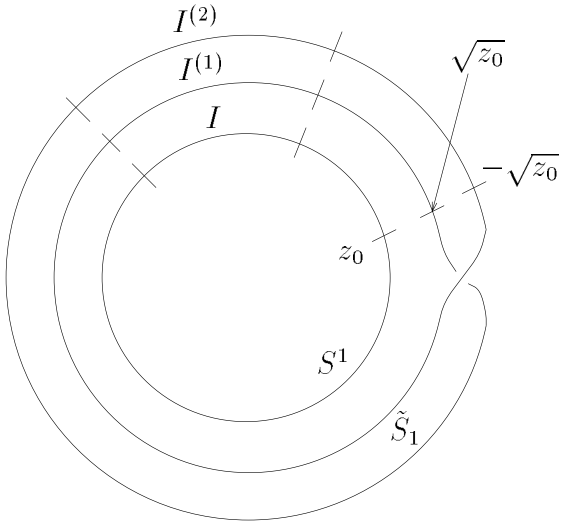

FIGURE II.3. Covering S^1

where the C^*-norms on the even CAR algebras $\mathfrak{A}_+(S^1)$ and $\mathfrak{A}_-(S^1)$ are unique.

2. Haag-Kastler axioms

The guiding principle in algebraic quantum field theory is that a quantum field theoretical model is defined by the system of the local algebras and the corresponding embeddings (local quantum physics). A model representing a relativistic theory of elementary particles should exhibit the following properties:

(1) If $\mathcal{O}_1$ is spacelike to $\mathcal{O}_2$ and $\mathcal{O}_1 \cup \mathcal{O}_2 \subset \mathcal{O}$, then in $\mathfrak{A}(\mathcal{O})$ holds

$$[A_1, A_2] = 0 \quad , \qquad \text{for } A_1 \in \mathfrak{A}(\mathcal{O}_1),\ A_2 \in \mathfrak{A}(\mathcal{O}_2) \quad .$$

This describes the relativistic causality. Operations in $\mathcal{O}_1$ (described by isometries $V \in \mathfrak{A}(\mathcal{O}_1)$) do not influence the results of measurement of observables in $\mathcal{O}_2$,

$$\omega(V^*AV) = \omega(A) \quad , \qquad A \in \mathfrak{A}(\mathcal{O}_2),\ V \in \mathfrak{A}(\mathcal{O}_1),\ V^*V = 1,\ \omega \in S(\mathfrak{A}) \quad .$$

This condition continues to make sense in general spacetimes with causal structure.

(2) Poincaré transformations $P_+^\uparrow \ni L = (a, \Lambda)$ are represented by automorphisms α_L of $\mathfrak{A}$, with

$$\alpha_L(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(L\mathcal{O}) \quad .$$

This condition can be transferred analogously to spacetimes with other symmetries. In a general curved spacetime this condition has no analogue.

(3) Stability: This condition has not found a really satisfying formulation. One for most cases sufficient formulation is the existence of a Poincaré invariant ground state ω_0 inducing a faithful GNS representation. Here a state ω_0 is a ground state (with respect to a 1-parameter group α_{te}, $e \in V_+$, $e^2 = 1$ of timelike translations) if for all $A, B \in \mathfrak{A}$ the function

$$\mathbb{R} \ni t \mapsto \omega_0(A\alpha_{te}(B))$$

is continuous and boundary value of a bounded analytic function on the upper half plane. Then, ω_0 is automatically invariant, $\omega_0 \circ \alpha_{te} = \omega_0$, since for $A = A^*$

$$\omega_0(\alpha_t(A)) = \omega_0(\mathbb{1}\alpha_t(A)) = \overline{\omega_0(\alpha_t(A))} \quad ,$$

i.e. $\omega_0\alpha_t$ is entire and bounded. In this case one defines in the GNS Hilbert space $\mathcal{H}_0$ to ω_0 a unitary time evolution operator by

$$U(t)\pi_0(A)\Omega_0 = \pi_0\alpha_{te}(A)\Omega_0 \quad .$$

The isometry of $U(t)$ follows from the invariance of ω_0, the invertibility directly from $U(-t)U(t) = \mathbb{1}$. By definition $t \mapsto U(t)$ is continuous in the weak operator topology and then also in the strong operator topology because of

$$\begin{aligned}\|U(t) - U(s)\phi\|^2 &= 2\Big(\|\phi\|^2 - \text{Re}(U(t)\phi, U(s)\phi)\Big)\\ &= 2\text{Re}(U(t)\phi, (U(t) - U(s))\phi)\end{aligned}$$

It follows that $U(t)$ is a strongly continuous unitary 1-parameter group. By the Stone theorem there is a unique self-adjoint operator H with

$$U(t) = e^{iHt} \quad .$$

Obviously $H\Omega_0 = 0$. Furthermore H possesses spectrum in $\mathbb{R}_+$, for if $f \in \mathcal{D}(\mathbb{R})$ with supp$f \subset \mathbb{R}_-$ with Fourier transform $\tilde{f}$, then

$$\begin{aligned}(\pi_0(A)\Omega, f(H)\pi_0(B)\Omega) &= \int dt \tilde{f}(t)(\pi_0(A)\Omega, U(t)\pi_0(B)\Omega)\\ &= \int dt \tilde{f}(t)\omega_0(A^*\alpha_{te}(B))\\ &= \int_\Gamma dz \tilde{f}(z) F_{A^*B}(z) \quad .\end{aligned}$$

Using the estimate

$$|\tilde{f}(z)|e^{-E_0 \text{Im} z}(|\text{Re} z|^2 + 1)^n < \int dE \Big|((1 - \frac{d^2}{dE^2})^n f)(E)\Big|$$

$E_0 = \sup\{E,\ E \in \text{supp} f\}$, and the path Γ of figure 2, we find $f(H) = 0$.

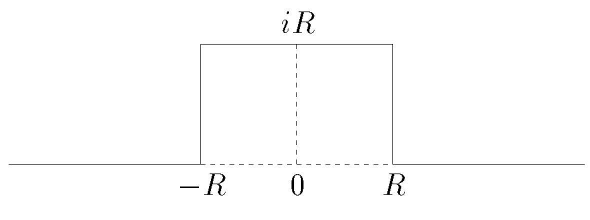

FIGURE II.4. The path Γ.

(4) Existence of a dynamical law.
If f is the solution of a hyperbolic differential equation, then $f(x)$ is determined by the values of f and its normal derivative on a space hypersurface C, possessing the property that every past directed (future directed) inextendible causal curve starting at x intersects the surface C.

In quantum field theory the exact formulation of a dynamical law is difficult (and has been achieved only in a few exceptional cases). In the algebraic frame one can however formulate the so-called **time slice axiom**:
Let $\mathcal{G}$ be a globally hyperbolic region, and let U be a neighborhood of a Cauchy surface of $\mathcal{G}$. Then,

$$\mathfrak{A}(\mathcal{G}) = \mathfrak{A}(U) \quad .$$

Curiously, the time slice axiom has not found any interesting applications. This is definitely due to the fact, that the characterization of dynamics in the framework of algebraic quantum field theory has not yet been achieved satisfactorily. The time slice axiom has as a consequence the additivity of the net,

$$\mathfrak{A}(\mathcal{O}) = \bigvee_\alpha \mathfrak{A}(\mathcal{O}_\alpha) \quad , \qquad \forall \text{ coverings } (\mathcal{O}_\alpha) \text{ of } \mathcal{O} \quad .$$

PROOF. Let $(\mathcal{O}_\alpha)$ be a covering and $\mathcal{C}$ be a Cauchy sequence. Let U be a neighborhood of $\mathcal{C}$ with the property, that every double cone in U lies in a double cone $\mathcal{O}_\alpha$. Then,

$$\mathfrak{A}(U) \subset \bigvee_\alpha \mathfrak{A}(\mathcal{O}_\alpha) \subset \mathfrak{A}(\mathcal{O}) \quad .\Box$$

3. Examples of Quantum Field Theories

The standard example for a quantum field theory is the free scalar hermitian field, which is characterized by the field equation

$$(\Box + m^2)\varphi = 0 \quad , \qquad \varphi = \varphi^* \quad .$$

In quantum field theory φ we interpret φ as an operator valued distribution, i.e. $\varphi(f)$ is an operator for every test function $f \in \mathcal{D}(\mathbb{R}^4)$. The field equation then means

$$\varphi((\Box + m^2)f) = 0 \quad , \qquad f \in \mathcal{D}(\mathbb{R}^4) \quad .$$

Poincaré transformations act through automorphisms

$$\alpha_{(\Lambda,a)}(\varphi(x)) = \varphi(\Lambda x + a) \quad ,$$

i.e.

$$\alpha_{(\Lambda,a)}(\varphi(f)) = \varphi(f_{(\Lambda,a)}) \quad , \qquad f_{(\Lambda,a)} = f(\Lambda^{-1}(x-a)) \quad .$$

We now consider the $*$-algebra $\mathfrak{A}_0$ generated by the symbols $\varphi(f)$, $f \in \mathcal{D}(\mathbb{R}^4)$ with the relations

$$f \mapsto \varphi(f) \text{ is linear.}$$
$$\varphi(f)^* = \varphi(\overline{f})$$
$$\varphi((\Box + m^2)f) = 0$$
$$[\varphi(f), \varphi(g)] = 0 \text{ if supp} f \times \text{supp} g \quad .$$

Let ω_0 be a Poincaré invariant ground state of $\mathfrak{A}_0$. ω_0 is characterized by the multilinear functionals

$$f_1, \ldots, f_n \mapsto \omega_0(\varphi(f_1) \cdots \varphi(f_n)) \quad .$$

We will assume that these functionals can be continued to tempered distributions W_n on $\mathcal{S}(\mathbb{R}^4)$.

The spectrum condition and Poincaré invariance yields the general form of the two-point function W_2,

$$W_2(x, y) = \int_{\overline{V}_+} d^4p \rho(p^2) e^{-ip(x-y)}$$

with a Lorentz invariant measure $\rho(p^2)d^4p$ on $\overline{V}_+$. Due to the field equation the measure has to be concentrated on the mass shell $p^2 = m^2$, i.e. $\rho(p^2) = c\delta(p^2 - m^2),\ c > 0$. Then,

$$\omega_0([\varphi(f), \varphi(g)]) = c \int d^4x d^4y f(x) g(y) \Delta(x - y)$$

with

$$\Delta(x) = -2(2\pi)^{-3} \int \frac{d^3\vec{p}}{2E(\vec{p})} \sin(E(\vec{p})x^0 - \vec{p}\vec{x}) \quad .$$

One now shows (Jost-Schroer-Pohlmeyer theorem, see Streater-Wightman) that the commutator $[\varphi(f), \varphi(g)]$ is a multiple of $\mathbb{1}$ if Ω is the unique ground state in $\mathcal{H}$.

For this purpose, one decomposes the field operator $\varphi(f)$ into creation and annihilation parts,

$$\varphi(f) = \varphi(f_+) + \varphi(f_-)$$

with $f_\pm \in \mathcal{S}(\mathbb{R}^4)$, $\mathrm{supp}\tilde{f}_\pm \subset V_\pm$,

$$\begin{aligned} \tilde{f}_+(p) = \tilde{f}(p) \quad , \qquad p^2 = m^2 \quad , \qquad p_0 = E(\vec{p}) \\ \tilde{f}_-(p) = \tilde{f}(p) \quad , \qquad p^2 = m^2 \quad , \qquad p_0 = -E(\vec{p}) \quad . \end{aligned}$$

Let Ω be the vector in the GNS representation of ω_0 inducing ω_0. We have $\varphi(f_-)\Omega = 0$ and therefore

$$[\varphi(f), \varphi(g)]\Omega = [\varphi(f_+), \varphi(g_+)]\Omega + \Big(\varphi(f_-)\varphi(g_+) - \varphi(g_-)\varphi(f_+)\Big)\Omega \quad .$$

The momentum transfer due to $\varphi(f_-)\varphi(g_+) - \varphi(g_-)\varphi(f_+)$ is contained in

$$\Gamma = \Big\{p - q,\ p^2 = q^2 = m^2,\ p_0, q_0 > 0\Big\} \subset \{0\} \cup \{p \,|\, p^2 < 0\} \quad .$$

From the spectrum condition follows

$$\Big(\varphi(f_-)\varphi(g_+) - \varphi(g_-)\varphi(f_+)\Big)\Omega = P_0\Big(\varphi(f_-)\varphi(g_+) - \varphi(g_-)\varphi(f_+)\Big)\Omega \quad ,$$

where P_0 is the projection on the ground state vectors. If Ω is the unique ground state vector, then

$$\Big(\varphi(f_-)\varphi(g_+) - \varphi(g_-)\varphi(f_+)\Big)\Omega = \Big(\Omega, [\varphi(f), \varphi(g)]\Omega\Big)\Omega \quad ,$$

and therefore

$$[\varphi(f_+), \varphi(g_+)]\Omega = (1 - P_0)[\varphi(f), \varphi(g)]\Omega \quad .$$

If we substitute f and g by their translates f_x, g_x, then we obtain on the left side the boundary value of a function analytic in $\mathrm{Im}x, \mathrm{Im}y \in V_+$ vanishing on the open set $\{(x, y), \mathrm{supp}f + x \times \mathrm{supp}g + y\}$. Thus, $[\varphi(f_+), \varphi(g_+)]\Omega = 0$ and

$$[\varphi(f), \varphi(g)]\Omega = P_0[\varphi(f), \varphi(g)]\Omega \quad .$$

According to the Reeh-Schlieder theorem (next chapter) $\mathfrak{A}_0(\mathcal{O})\Omega$ is dense in $\mathcal{H}$. This fact implies that if $A^*\Omega = 0$ for $A \in \mathfrak{A}_0(\mathcal{O})$, then for all $B \in \mathfrak{A}_0(\mathcal{O}_1)$, $\mathcal{O}_1 \subset \mathcal{O}'$ and for all $\phi \in \mathfrak{A}_0\Omega$

$$(B\Omega, A\phi) = (A^*B\Omega, \phi) = (BA^*\Omega, \phi) = 0 \quad ,$$

i.e. $A\phi = 0$, thus $A = 0$. We conclude that the commutator above coincides with its scalar value on Ω. Hence, the free scalar quantum field is up to the choice of the constant c uniquely determined. The value $c = 1$ can be fixed by the condition that the Hamilton operator in the limit of correspondence coincides with the classical energy.

The algebra $\pi(\mathfrak{A}_0)$ possesses apart from null no C^*-seminorm, since the operators $\pi_0(\varphi(f)) \neq 0$ are necessarily unbounded because of the canonical commutation relations. We hence substitute the algebra by the Weyl algebra $\mathcal{W}(L, \sigma)$ with

$$L = \mathcal{D}(\mathbb{R}^4, \mathbb{R})\big/\mathrm{image}(\Box + m^2)$$
$$\sigma(f, g) = \int f(x)\Delta(x - y)g(y) \quad .$$

σ is a non-degenerate symplectic form on L. The ground state is characterized by

$$\omega_0(W(f)) = e^{-\frac{1}{2}||f||^2}$$

with

$$||f||^2 = \int \frac{d^3\vec{p}}{2E(\vec{p})}|\tilde{f}(E(\vec{p}), \vec{p})|^2 \quad .$$

The corresponding GNS Hilbert space is the symmetric Fock space

$$\mathcal{H} = \bigoplus_{n=0}^{\infty}(\underbrace{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1}_{n})_{\text{symm}} \quad , \qquad \mathcal{H}_1 = L^2(\mathbb{R}^3, \frac{d^3\vec{p}}{2E(\vec{p})})$$

and

$$(W(f)\Omega)_n(\vec{p}_1, \ldots, \vec{p}_n) = \frac{i^n}{\sqrt{n!}}\tilde{f}(E(\vec{p}_1), \vec{p}_1) \cdots \tilde{f}(E(\vec{p}_n), \vec{p}_n)e^{-\frac{1}{2}||f||^2} \quad .$$

The described model satisfies the time slice axiom. The first observation is that additivity holds. For if $\mathcal{O} = \bigcup d_\alpha \mathcal{O}_\alpha$ is a covering of $\mathcal{O}$ and $f \in \mathcal{D}(\mathcal{O})$, then there exist $\alpha_1, \ldots \alpha_n$ such that $\mathrm{supp}f \subset \bigcup \mathcal{O}_{\alpha_i}$. Let $\varphi_i \in \mathcal{D}(\mathcal{O}_{\alpha_i})$ with $\Sigma\varphi_{\alpha_i} = 1$ on $\mathrm{supp}f$. Then,

$$W(f) = \prod_{j=1}^{n} W(f\varphi_j)e^{\frac{1}{2}\sum\limits_{j<k}\sigma(f\varphi_j, f\varphi_k)}$$

and therefore

$$W(f) \in \left\{\mathfrak{A}(\mathcal{O}_{\alpha_i}),\ i = 1 \ldots, n\right\}^{\text{generated algebra}} \quad .$$

The time slice axiom now follows from properties of the Klein-Gordon equation. Let $\mathcal{O}_1$ be a double cone in the future dependence region of U, and let $f \in \mathcal{D}(\mathcal{O}_1)$. Then $F = \Delta_{av} * f$ is a solution of the inhomogeneous Klein-Gordon equation $(\Box + m^2)F = f$ with supp$F \subset \mathcal{O}_1 + \overline{V_-}$. Let h be a C^∞-function with supp$h \subset U + \overline{V_+}$ and $h = 1$ on $(\mathcal{O}_1 + \overline{V_-}) \setminus (U + \overline{V_+})$. Set $g = (\Box + m^2)Fh$. Fh vanishes outside $(\mathcal{O}_1 + \overline{V_-}) \cap (U + \overline{V_+})$ and coincides in the complement of $U + \overline{V_+}$ with F. Hence, supp$g \subset (U + \overline{V_+}) \cap (U + \overline{V_-}) = U$ if U contains along with every two points connected by a causal curve also the curve. Now,

$$f - g = (\Box + m^2)F(1 - h) \quad ,$$

but supp$F(1 - h) \subset (\mathcal{O}_1 + \overline{V_-}) \cap (U + \overline{V_+})$ is compact, thus $W(f) = W(g)$, i.e. $\mathfrak{A}(U) \supset \mathfrak{A}(\mathcal{O}_1)$.

With this model one can obtain new models by tensor multiplication. For this purpose, one substitutes the symplectic spaces by direct sums

$$L = \bigoplus L_i$$

with symplectic form $\sigma(f, g) = \sum_i \sigma_i(f_i, g_i)$ and ground state

$$\omega_0(W(f)) = \prod_i \omega_a^{(i)}(W(f_i)) = e^{-\frac{1}{2}\|f\|^2} \quad , \qquad \|f\|^2 = \sum \|f_i\|_i^2 \quad .$$

If the masses of the fields are equal, then the theory possesses $O(n)$ as the symmetry group,

$$O(n) \ni g \mapsto \alpha_g,\ \alpha_g(W(f)) = W(gf) \quad , \qquad gf_i = \sum_j g_{ij} f_i \quad .$$

Now one can consider in the vacuum representation for every closed subgroup G of $O(n)$ the algebras

$$\mathfrak{A}_G(\mathcal{O}) = \left\{A \in \mathfrak{A}(\mathcal{O}),\ \alpha_g(A) = A,\ \forall g \in G\right\} \quad .$$

These algebras form the prototype of the algebras of observables in the theory of superselection sectors. We will examine, from which properties of the net the group G can be read off.

As a first example we consider in the case $n = 1$ the group $\mathbb{Z}_2$ with the automorphism

$$\gamma(W(f)) = W(-f) \quad .$$

The invariant algebra is generated by the operators

$$V(f) = \frac{1}{2}\Big(W(f) + W(-f)\Big)$$

with the relations

$$V(f) = V(-f)$$
$$V(f)^* = V(f)$$
$$V(f)V(g) = \frac{1}{2}\Big\{e^{-\frac{i}{2}\sigma(f,g)}V(f+g) + e^{\frac{i}{2}\sigma(f,g)}V(f-g)\Big\} \quad .$$

We now consider two spacelike separated double cones $\mathcal{O}_1$ and $\mathcal{O}_2$ and define the two algebras

$$\mathfrak{A}_{\text{min}}(\mathcal{O}_1 \cup \mathcal{O}_2) = \Big\{\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)\Big\}^{\text{generated algebra}}$$
$$= \Big\{ \sum_{\text{finite}} \lambda_{f,g}\Big(V(f+g) + V(f-g)\Big) \Big| \quad \text{supp} f \subset \mathcal{O}_1,\ \text{supp} g \subset \mathcal{O}_2,\ \lambda_{f,g} \in \mathbb{C}\Big\}$$
$$\mathfrak{A}_{\text{max}}(\mathcal{O}_1 \cup \mathcal{O}_2) = \mathfrak{A} \cap \bigcap_{\mathcal{O}_3 \subset \mathcal{O}_1' \cap \mathcal{O}_2'} \mathfrak{A}(\mathcal{O}_3)'$$
$$= \Big\{ \sum_{\text{finite}} \lambda_{f,g} V(f+g) \Big| \text{supp} f \subset \mathcal{O}_1,\ \text{supp} g \subset \mathcal{O}_2,\ \lambda_{f,g} \in \mathbb{C}\Big\} \quad .$$

$\mathfrak{A}_{\text{min}}$ and $\mathfrak{A}_{\text{max}}$ are different, for there exists an automorphism on $\mathfrak{A}_{\text{max}}$, which acts trivial on $\mathfrak{A}_{\text{min}}$,

$$\gamma_0(V(f+g)) = V(f-g) \quad .$$

Now one can construct a conditional expectation of $\mathfrak{A}_{\text{max}}$ on $\mathfrak{A}_{\text{min}}$,

$$E(A) = \frac{1}{2}(A + \gamma_0(A)) \quad .$$

E has the properties

(i) linear,
(ii) positive, $E(A) \geq 0$ for $A \geq 0$,
(iii) normalized, $E(\mathbb{1}) = 1$,
(iv) $E(A_1 B A_2) = A_1 E(B) A_2,\ A_i \in \mathfrak{A}_{\text{min}},\ B \in \mathfrak{A}_{\text{max}}$.

One now considers the inequality

$$E(A) \geq \lambda A \quad , \qquad A \geq 0,\ \lambda \geq 0$$

and defines $\text{ind}_E = \inf \lambda^{-1}$. ind_E corresponds to the Jones index. In our case obviously $\text{ind}_E \geq 2$. One easily shows, that actually $\text{ind}_E = 2$ holds. Here, 2 is the order of the group $\mathbb{Z}_2$.

As a further class of models we consider the spin-$\frac{1}{2}$ fields, which are characterized by the **Dirac** equation

$$(\gamma_\mu \partial^\mu + im)\psi = 0 \quad , \qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu},\ \gamma_0 = \gamma_0^*,\ \gamma_j = -\gamma_j^*,\ i = 1, 2, 3 \quad .$$

We consider the smeared field operators

$$\psi(f) = \int d^4x (f(x), \gamma_0 \psi(x)) \quad .$$

Let C be an antilinear operator in $\mathbb{C}^4$ with $C^2 = 1$ and $C\gamma_\mu C = -\gamma_\mu$. Then C anticommutes with the Dirac operator $D = \gamma_\mu \partial^\mu + im$ and we obtain on $\mathcal{K}_0 =$

$\mathcal{D}(\mathbb{R}^4, \mathbb{C}^4)\big/\text{image}(D)$ an involution Γ_0 with $(\Gamma_0 f)(x) = -C(f(x))$. The scalar product on $\mathcal{K}_0$ is defined by

$$(f,g) = \int d^4x \Big(f(x), \gamma_0(\gamma_\mu \partial^\mu + im)\Delta(x-y)g(y) \Big)$$
$$= \int d^3\vec{p} \sum_{\pm} \Big(\tilde{f}(\pm E(\vec{p}), \vec{p}), P_\pm(\vec{p})\tilde{g}(\pm E(\vec{p})) \Big)$$

with

$$P_\pm(\vec{p}) = \frac{\gamma_0(\gamma_\mu p^\mu + m)}{2E(\vec{p})} \quad , \qquad p_0 = \pm E(\vec{p}),\ P_+ + P_- = \mathbb{1},\ P_+ P_- = \mathbb{0} \quad .$$

If in particular $f(x^0, \vec{x}) = \delta(x^0 - t)g(\vec{x})$, then

$$(f,f) = \int d^3\vec{p} |\tilde{g}(\vec{p})|^2 = \int d^3x |g(\vec{x})|^2 \quad .$$

Γ_0 is an antiunitary involution on the Hilbert space $\mathcal{K}$ arising from the completion of $\mathcal{K}_0$.

A **Majorana** field χ satisfies the condition $C\chi(x) = \chi(x)$. One defines the corresponding CAR algebra CAR$(\mathcal{K}, \Gamma_0)$ by

$$B(f) = \chi(\Gamma_0 f) \quad .$$

We have

$$\chi(\Gamma_0 f)^* = \int d^4x \Big(-Cf(x), \gamma_0 \chi(x) \Big)^*$$
$$= \int d^4x \Big(C\chi(x), \gamma_0 f(x) \Big)$$
$$= \chi(f)$$

and therefore, as required

$$B(f)^* = B(\Gamma_0 f) \quad .$$

For a Dirac field one chooses the Hilbert space

$$\mathcal{K}_D = \mathcal{K} \oplus \mathcal{K}$$

with the involution $\Gamma(f,g) = (\Gamma_0 g, \Gamma_0 f)$ and identifies

$$B(f,g) = \psi(\Gamma_0 f) + \psi(g)^* \quad .$$

$\mathcal{K}_D$ decomposes into the two Γ-invariant subspaces

$$\mathcal{K}_1 = \Big\{ (f,f),\ f \in \mathcal{K}_0 \Big\} \quad , \qquad \mathcal{K}_2 = \Big\{ (if, -if),\ f \in \mathcal{K}_0 \Big\} \quad .$$

This decomposition corresponds to a decomposition of the Dirac field ψ in 2 anticommuting Majorana fields

$$\psi(f) = B(\Gamma_0 f, 0) = \frac{1}{2} B(\Gamma_0 f, \Gamma_0 f) + \frac{1}{2i} B(i\Gamma_0 f, -i\Gamma_0 f)$$
$$= \frac{1}{\sqrt{2}}\chi_1(f) + \frac{i}{\sqrt{2}}\chi_2(f) \quad .$$

If we couple n anticommuting Majorana fields, then we obtain

$$\mathcal{K}^{(n)} = \mathcal{K} \otimes \mathbb{C}^n \quad , \qquad \Gamma^{(n)}(f_1, \ldots, f_n) = (\Gamma_0 f_1, \ldots, \Gamma_0 f_n) \quad .$$

The symmetry group $O(n)$ again acts in a natural way. The Poincaré group acts by

$$U(A,a)\psi(x)U(A,a)^{-1} = S(A)^{-1}\psi(\Lambda(A)x + a) \quad .$$

Here $A \mapsto S(A)$ is a representation of SL$(2,\mathbb{C})$ in $\mathbb{C}^4$ equivalent to

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \quad .$$

The Lorentz transformation $\Lambda(A)$ belonging to A is defined by

$$\Lambda(A) = A\underset{\sim}{x}A^* \quad , \qquad \underset{\sim}{x} = x^0\,\mathbb{1} + \vec{x}\vec{\sigma} \quad .$$

$S(A)$ is defined by its commutation relation with the γ-matrices,

$$S(A)\gamma^\mu S(A)^{-1} = \Lambda(A)^\mu{}_\nu \gamma^\nu \quad .$$

In particular: $S(A^{-1}) = \gamma_0 S(A)^* \gamma_0$. A ground state is found in the following way. We have

$$\alpha_t B(f) = B(e^{iht}f) \quad , \qquad (hf)(x) = \frac{1}{i}\gamma_0(\gamma_k \partial^k + im)f$$

and $\Gamma_0 h = -h\Gamma_0$. For the projections $P_\pm$ on positive and negative energies,

$$\widetilde{P_\pm f}(p_0, \vec{p}) = P_\pm(\vec{p})\tilde{f}(\pm E(\vec{p}), \vec{p})$$

then holds

$$\Gamma_0 P_+ = P_- \Gamma_0 \quad .$$

A ground state is then characterized by

$$\omega_0(AB(f)) = \omega_0(AB(P_+ f)) \quad .$$

One finds

$$\omega_0(B(f)A) = \omega_0(B(P_- f)A)$$

and the recursive relation

$$\omega_0(B(f_1)\cdots B(f_n)) = \sum_{k=2}^{n} (-1)^k (f_1, P_+ f_k)\omega_0(B(f_2)\cdots \cancel{B(f_k)} \cdots B(f_n))$$

with the solution

$$\omega_0(B(f_1)\cdots B(f_n)) = 0 \quad , \qquad n \text{ odd}$$

$$\omega_0(B(f_1)\cdots B(f_{2n})) = \sum_{\substack{\sigma\,\in\, S_{2n} \\ \sigma(1) < \cdots < \sigma(n) \\ \sigma(i) < \sigma(i+n)}} \text{sign}(\sigma) \prod (f_{\sigma(i)}, P_+ f_{\sigma(n+i)})(-1)^{\frac{n(n+1)}{2}} \quad .$$

One for the theory of superselection sector interesting class are the chiral models in 2 dimensions. These are theories, whose field equation takes on the simple form

$$\alpha_{te} = \text{id} \quad , \qquad t \in \mathbb{R}$$

for a lightlike vector e. Double cones in 2 dimensions have the simple form

$$\mathcal{O} = \Big\{(x^0, x^1),\ x^0 + x^1 \in I,\ x^0 - x^1 \in J\Big\}$$

for two open intervals $I, J \subset \mathbb{R}$. Because of the chirality condition $\alpha_{te} = \mathrm{id}$ one of the two intervals can be shifted without changing the algebra of the double cone,

$$\mathfrak{A}_0(I \times \{J + x\}) = \mathfrak{A}_0(I \times J) \quad .$$

The chiral net is now defined by

$$\mathfrak{A}(I) = \bigcup_J \mathfrak{A}(I \times J) \quad .$$

If the original net is additive then $\mathfrak{A}(I \times J)$ is even independent from J. From the locality condition for $\mathfrak{A}_0$ follows

$$[\mathfrak{A}(I), \mathfrak{A}(J)] = 0 \quad , \qquad \text{for } I \cap J = \emptyset \quad .$$

At first we consider the free massless field in 2 dimensions. Let φ be a solution of the

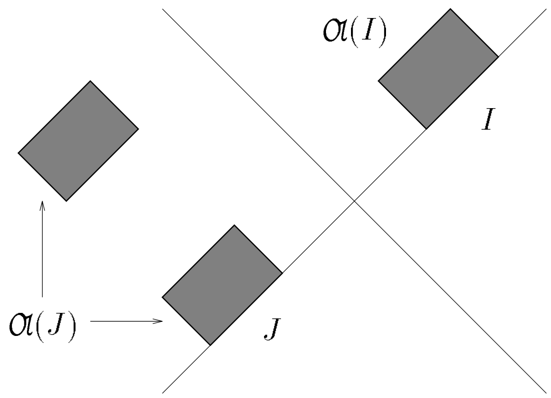

FIGURE II.5. Shifting one of the intervals

field equation $\Box\varphi = 0$. Then $j = (\partial_0 - \partial_1)\varphi$ satisfies the chirality condition

$$(\partial_0 + \partial_1) j = 0 \quad .$$

With $u = x^0 - x^1$ we have

$$(\Omega, j(u)j(u')\Omega) = \frac{1}{\pi}\int_0^\infty dp\, p e^{-ip(u-u')}$$

$$\lim_{\epsilon \uparrow 0} \frac{-1}{\pi(u - u' - i\epsilon)^2}$$

and $[j(u), j(u')] = 2i\delta'(u - u')$. The corresponding Weyl algebra is defined through the symplectic space $(\mathcal{D}(\mathbb{R}), \sigma)$ with

$$\sigma(f, g) = 2\int f dg \quad .$$

Note that σ is invariant under orientation preserving diffeomorphisms $\phi \in \mathrm{Diff}^+(\mathbb{R})$. The ground state is defined by

$$\omega_0(W(f)) = e^{-\frac{1}{2}||f||^2}$$

with

$$\|f\|^2 = 2 \int_0^\infty dp\, p |\tilde{f}(p)|^2 \quad .$$

One again obtains interesting other models by tensor multiplication and transition to subtheories. One obtains a simple class e.g. by restriction of the test functions to the ones of the form $P(\frac{d}{du})f,\ f \in \mathcal{D}(\mathbb{R})$.

4. Positive Energy Representations, Reeh-Schlieder Theorem, Borchers Property

The most important class of representations for elementary particle physics are the positive energy representations. A **positive energy representation** of a C^*-algebra $\mathfrak{A}$ with an automorphism group $(\alpha_t)_{t\in\mathbb{R}}$ is a representation π of $\mathfrak{A}$ in a Hilbert space $\mathcal{H}$ together with a strongly continuous unitary representation U of $\mathbb{R}$ in $\mathcal{H}$, such that

$$U(t)\pi(A)U(t)^{-1} = \pi\alpha_t(A)$$

and such that the self adjoint generator $H = \frac{1}{i}\frac{d}{dt}U(t)|_{t=0}$ of $(U(t))$ possesses a non-negative spectrum. In this case the Borchers-Arveson theorem holds:

THEOREM. *Let (π, U) be a positive energy representation of $(\mathfrak{A}, \alpha)$. Then there exists a strongly continuous group $V(t) \in \pi(\mathfrak{A})''$, such that (π, V) is a positive energy representation of $(\mathfrak{A}, \alpha)$.*

REMARK. The generator of V can thus be approximated by elements of $\pi(\mathfrak{A})$. In this sense it can be identified with the observable 'energy'. This justifies a posteriori the name 'positive energy representation'.

SKETCH OF PROOF(cf. Bratteli-Robinson I 3.2.46). Let $\mathcal{M} = \pi(\mathfrak{A})''$, with A also $U(t)AU(t)^{-1} \in \mathcal{M}$. Also let $A' \in \mathcal{M}' = \pi(\mathfrak{A})'$. Then for $B \in \mathfrak{A}$ holds

$$[U(t)^{-1}A'U(t), \pi(B)] = U(t)^{-1}[A', \pi\alpha_t(B)]U(t) = 0 \quad ,$$

thus $U(t)^{-1}A'U(t) \in \mathcal{M}'$. Hence $U(t)AU(t)^{-1}$ commutes with all elements of $\mathcal{M}'$. We define $\beta_t(A) = U(t)AU(t)^{-1}$ for $A \in \mathcal{M}$. $(\beta_t)_{t\in\mathbb{R}}$ is a 1-parameter group of automorphisms of $\mathcal{M}$. For test functions $f \in \mathcal{S}(\mathbb{R})$ we set

$$\beta_f(A) = \int dt\, f(t)\beta_t(A) \quad .$$

Here the integral on the right side is defined in the following way: Let $\phi, \psi \in \mathcal{H}$. Then

$$t \to (\phi, \beta_t(A)\psi) = A_{(\phi,\psi)}(t)$$

is a continuous function with $|A_{(\phi,\psi)}(t)| \leq \|\phi\|\,\|\psi\|\,\|A\|$. By

$$\phi, \psi \mapsto \int dt\, f(t) A_{(\phi,\psi)}(t)$$

a sesquilinear form is defined on $\mathcal{H}$, which is bounded in both entries and hence defines a bounded operator,

$$(\phi, \beta_f(A)\psi) = \int dt\, f(t)(\phi, \beta_t(A)\psi) \quad .$$

For $A' \in \mathcal{M}'$ holds

$$(A'^{*}\phi, \beta_f(A)\psi) = (\phi, \beta_f(A)A'\psi)$$

and therefore $\beta_f(A) \in \mathcal{M}$.

We now define the β-spectrum $\mathrm{sp}_\beta(A)$ of $A \in \mathcal{M}$ in the following way:

$$\mathrm{sp}_\beta(A) = \Big\{E \in \mathbb{R} \Big| \forall \text{ neighborhoods } U \text{ of } E \text{ there is a test function } f \in \mathcal{S}(\mathbb{R}) \text{ with supp}\tilde{f} \subset U \text{ and } \beta_f(A) \neq 0 \Big\} \quad .$$

Let $I \subset \mathbb{R}$ be an open interval. An $A \in \mathcal{M}$ with $\mathrm{sp}_\beta(A) \subset I$ can be provided in the following way. Let $g \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}\tilde{g} \subset I$, and let $A_0 \in \mathcal{M}$. We set $A = \beta_g(A_0)$. We want to show that $\mathrm{sp}_\beta(A) \subset I$. Let $E \notin I$. We must show that there is a neighborhood U of E, such that for all test functions $f \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}\tilde{f} \subset U$ the operator $\beta_f(A) = 0$. We choose U in a way that $U \cap \mathrm{supp} g \in \emptyset$. Now let $f \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}\tilde{f} \subset U$. Then for $\phi, \psi \in \mathcal{H}$:

$$\begin{aligned}(\phi, \beta_f(A)\psi) &= \int dt f(t)(\phi, \beta_t(A)\psi) \\ &= \int dt f(t)(U(T)^{-1}\phi, \beta_t(A_0)U(T)^{-1}\psi) \\ &= \int dt f(t)\Big\{ \int ds g(s)(\phi, \beta_{t+s}(A_0)\psi)\Big\} \\ &= \int dt ds f(t-s)g(s)(\phi, \beta_t(A_0)\psi) \quad .\end{aligned}$$

However,

$$\begin{aligned}\int ds f(t-s)g(s) &= \int ds \frac{1}{\sqrt{2\pi}} \int dE \tilde{f}(E) e^{i(t-s)E} g(s) \\ &= \int dE \tilde{f}(E)\tilde{g}(E)e^{itE} = 0 \quad ,\end{aligned}$$

for $\mathrm{supp}\tilde{f} \cap \mathrm{supp}\tilde{g} = 0$, thus $\beta_f(A) = 0$.

We now define the subspaces $\mathcal{M}_E$ of $\mathcal{M}$ with energy transfer $\geq E$:

$$\mathcal{M}_E = \Big\{A \in \mathcal{M}, \, \mathrm{sp}_\beta(A) \subset [E, \infty]\Big\} \quad .$$

We have $\mathcal{M}_E\mathcal{H} \subset \{\phi \in \mathcal{H}, \mathrm{sp}_U(\mathcal{H}) \subset [E, \infty]\}$ because of $\mathrm{sp}_U \in [0, \infty]$. Now let

$$\mathcal{H}_E = \bigcap_{E' < E} \overline{\mathcal{M}_E \mathcal{H}}$$

and P_E be the projection on $\mathcal{H}_E$. Obviously,

$$\begin{aligned} &P_E \to 0 \quad , \quad \text{for } E \to \infty \quad , \quad \text{since } H \text{ is positive} \\ &P_{E'} \to P_E \quad , \quad \text{for } E' \nearrow E \\ &P_E = \mathbb{1} \quad , \quad \text{for } E \leq 0 \quad , \quad \text{since } \mathbb{1} \in \mathcal{M}_E \text{ for } E < 0 \end{aligned}$$

Thus one can define a selfadjoint positive operator H_0 by

$$H_0 = \int E dP_E$$

and a unitary group by

$$V(t) = \int e^{iEt} dP_E \quad .$$

For $A' \in \mathcal{M}'$ holds

$$A'\mathcal{M}_E\mathcal{H} \subset \mathcal{M}_E\mathcal{H} \quad .$$

$\mathcal{M}_E\mathcal{H}$ and therefore also $\mathcal{H}_E$ are invarant subspaces under $\mathcal{M}'$, thus

$$A'P_E = P_E A' P_E \quad , \qquad \forall A' \in \mathcal{M}' \quad .$$

Since $\mathcal{M}'$ is a $*$-algebra and $P_E = P_E^*$, it also follows that

$$P_E A' = P_E A' P_E \quad ,$$

thus $[P_E, A'] = 0$, i.e. $P_E \in \mathcal{M}'' = \mathcal{M}$ and also $V(t) \in \mathcal{M}$.

It remains to show that $V(t)AV(-t) = \beta_t(A),\ A \in \mathcal{M}$. Let $\gamma_t(A) = V(t)AV(-t)$. One first shows

$$\text{sp}_\beta(A) \supset \text{sp}_\gamma(A) \quad , \qquad A \in \mathcal{M} \quad .$$

However, in this case for $f, g \in \mathcal{S}(\mathbb{R})$ with $\text{supp}\tilde{f} \cap \text{supp}\tilde{g} = \emptyset$ holds

$$\gamma_f \beta_g(A) = 0 \quad , \qquad A \in \mathcal{M} \quad .$$

The bounded continuous function

$$\mathbb{R}^2 \mapsto (\phi, \gamma_s \beta_t(A)\psi) = f(s,t)$$

thus has a Fourier transform with the property

$$\tilde{f}(E, E') = 0 \quad , \qquad \text{for } E \neq E' \quad ,$$

thus $\tilde{f}(E, E') = \tilde{F}(E)\delta(E - E')$. Thus f depends only on the sum $s + t$. Hence, we have

$$\gamma_{-t}\beta_t(A) = \gamma_{+0}\beta_0(A) = A \quad .\square$$

The multidimensional generalization holds, too: If U is a strongly continuous unitary representation of the translation group of Minkowski space with

$$U(x)\pi(A)U(x)^{-1} = \pi\alpha_x(A)$$
$$U(x) = e^{ixP} \quad , \qquad \text{sp}P \subset \overline{V_+} \quad ,$$

then one can choose $U(x) \in \pi(\mathfrak{A})''$.

Now let $\mathfrak{A}(\mathcal{O})$ be a local net with translation symmetry and a translation covariant positive energy representation. Let $\psi \in \mathcal{D}(e^{aP}),\ a \in V_+$ and let $\mathcal{G}$ be a region in Minkowski space with the following property: $\exists \mathcal{G}_0 \subset \mathcal{G}$ and a neighborhood $\mathcal{V}$ of zero in the translation group, such that:

(i) $\mathcal{G} \supset \mathcal{G}_0 + \mathcal{V}$
(ii) $\bigvee_x \mathfrak{A}(\mathcal{G}_0 + x) = \mathfrak{A}''$.

Then $\overline{\mathfrak{A}(\mathcal{G})\psi} = \overline{\mathfrak{A}\psi}$ (**Reeh-Schlieder theorem**).

PROOF. Let $\phi \perp \mathfrak{A}(\mathcal{G})\psi$. Then for $A_1, \ldots, A_n \in \mathcal{G}_0,\ x_1 \ldots, x_n \in \mathcal{V}$

$$\begin{aligned} 0 &= (\phi, \alpha_{x_1}(A_1) \cdots \alpha_{x_n}(A_n)\psi) \\ &= (\phi, U(x_1)A_1U(x_2 - x_1) \cdots U(x_n - x_{n-1})A_nU(-x_n)\psi) \\ &= \lim_{\substack{z_i \to x_i \\ \mathrm{Im}z_{j+1} - z_j\ (j=1,\ldots,n-1), \\ \mathrm{Im}z_1,\ a - \mathrm{Im}z_n \in V_+}} \underbrace{(\phi, e^{iz_1P}A_1e^{i(z_2-z_1)P}A_2 \cdots e^{i(z_n-z_{n-1})P}A_ne^{(-iz_n-a)P}e^{aP}\psi)}_{f(z_1,\ldots,z_n)} \quad . \end{aligned}$$

The function f is analytic in the region

$$\mathcal{T}_a^{(n)} = \Big\{(z_1, \ldots, z_n) \in \mathbb{C}^n,\ \mathrm{Im}z_1,\ \mathrm{Im}z_{j+1} - z_j\ (j = 1, \ldots, n-1),\ a - \mathrm{Im}z_n \in V_+$$

and possesses continuous limits for $z_1, \ldots, z_n \in \mathbb{R}^4$ vanishing on the open set $\underbrace{\mathcal{V} \times \cdots \mathcal{V}}_{n}$. Thus f is identically zero, and with it also the boundary values for all $x_1, \ldots, x_n \in \mathbb{R}^4$. It follows $\phi \perp \mathfrak{A}\psi$, i.e. $\mathfrak{A}(\mathcal{G})\psi$ is dense in $\overline{\mathfrak{A}\psi}$. □

Examples for regions $\mathcal{G}$:

(i) wedge regions of the form $\{|x^0| < x^1\}$,
(ii) forward light cones,
(iii) spacelike cones $S = a + \bigcup_{\lambda>0} \lambda\mathcal{O}$, $\mathcal{O} \subset \{0\}'$ double cone,
(iv) double cone $\mathcal{O}$, if weak additivity holds: $\bigvee \mathfrak{A}(\mathcal{O}_1 + x) = \mathfrak{A}''$ for a smaller double cone $\mathcal{O}_1$ with $\overline{\mathcal{O}_1} \subset \mathcal{O}$. Additivity implies weak additivity.

From the Reeh-Schlieder theorem follows together with locality that ψ is separating for all algebras $\mathfrak{A}(R)$ with $R \subset \mathcal{G}'$ if $\overline{\mathfrak{A}\psi} = \mathcal{H}$:

$$A\psi = 0,\ A \in \mathfrak{A}(R) \quad \Rightarrow \quad A = 0 \quad .$$

For from $A\psi = 0,\ A \in \mathfrak{A}(R)$ follows $AB\psi = 0\,\forall B \in \mathfrak{A}(G)$. A thus vanishes on a dense set and hence everywhere.

The last of the general theorems we want to consider is the so-called **Borchers property** of projection operators in local algebras. We consider a pair of regions $\mathcal{G}_0 \subset \mathcal{G}$ with $\mathcal{G}_0 + \mathcal{V} \subset \mathcal{G}$ for a neighborhood $\mathcal{V}$ of zero in the translation group. Let $E \in \mathfrak{A}(\mathcal{G}_0)$ be a projection.

THEOREM (BORCHERS PROPERTY, 'PROPERTY B'). *There exists an isometry* $V \in \mathfrak{A}(\mathcal{G})$ *with* $V^*V = E$.

PROOF. Let $\mathcal{O} \subset \mathcal{G} \cap \mathcal{G}_0'$ and let $\phi \in \mathcal{D}(e^{aP})$ for an $a \in V_+$. Then ϕ is cyclic for $\mathfrak{A}(\mathcal{O})'$. We will show later that $E\phi$ is separating for $\mathfrak{A}(\mathcal{G})'$.

To $E\phi$ separating there exists a ψ which is cyclic for $\mathfrak{A}(\mathcal{G})'$ and induces the same state

$$(E\phi, AE\phi) = (\psi, A\psi) \quad , \qquad A \in \mathfrak{A}(\mathcal{G})' \text{ (Sakai, theorem 2.7.9)} \quad .$$

We now define

$$VA\psi = AE\phi \quad , \qquad A \in \mathfrak{A}(\mathcal{G})' \quad .$$

V is an isometry. Furthermore for $B \in \mathfrak{A}(\mathcal{G})'$ holds

$$VBA\psi = BAE\phi = BVA\psi \quad ,$$

i.e. $V \in \mathfrak{A}(\mathcal{G})'' = \mathfrak{A}(\mathcal{G})$. □

The fact that $E\phi$ is separating follows from the next lemma.

LEMMA. *Let $\phi \in \mathcal{D}(e^{aH})$, $a > 0$, and let $E \in \mathfrak{A}(\mathcal{G}_0)$ be a projection. Then $E\phi$ is separating for $\mathfrak{A}(\mathcal{G})'$. ($\mathcal{G}_0, \mathcal{G}$ as in the theorem)*

PROOF. Let $A \in \mathfrak{A}(\mathcal{G})'$ with $AE\phi = 0$. ϕ is by definition cyclic on $\mathcal{G}_0, \mathcal{G}$ for $\mathfrak{A}(\mathcal{G})'' \cap \mathfrak{A}(\mathcal{G})'$, thus $AE = 0$ follows. Furthermore for an $\epsilon > 0$ holds

$$[A, \alpha_t(E)] = 0 \quad , \qquad \text{for } |t| < \epsilon \quad . \tag{3}$$

We consider the analytic functions ($\psi \in \mathcal{H}$)

$$F^{(n)}_{t_1,\dots,t_n}(z) = (e^{-aH}\psi, Ae^{izH}\alpha_{t_1}(E)\cdots\alpha_{t_n}e^{-izH}\phi) \quad , \qquad 0 < \text{Im} z < a$$
$$G^{(n)}_{t_1,\dots,t_n}(z) = (e^{-aH}\psi, e^{izH}\alpha_{t_1}(E)\cdots\alpha_{t_n}e^{-izH}A\phi) \quad , \qquad -a < \text{Im} z < 0$$

for $t_1, \dots, t_n \in (\delta, \delta)$, $0 < \delta < \frac{\epsilon}{2}$. Because of $E^2 = E$,

$$F^{(n)}_{0,\dots,0}(z) = F^{(1)}_0(z) \quad , \qquad G^{(n)}_{0,\dots,0}(z) = G^{(1)}_0(z) \quad .$$

The functions $F^{(n)}_{t_1,\dots,t_n}$ and $G^{(n)}_{t_1,\dots,t_n}$ possess continuous boundary values on the real axis, which because of (3) coincide on the interval $[-(\epsilon - \delta), \epsilon - \delta]$. Hence, they can be continued to a function $H^{(n)}_{t_1,\dots,t_n}(z)$ analytic in the region

$$\left\{ z \,\middle|\, 0 < |\text{Im} z| < a \text{ or } \text{Im} z = 0,\ |\text{Re} z| < \epsilon - \delta \right\} \quad .$$

This function vanishes because of $AE = 0$ and (3) at the points $z = -t_i$, $i = 1, \dots, n$. Inside the circles $|z| < \delta$ it thus possesses n zeros for values of t_i pairwise different. Yet, the number of zeros does not change in the limit $t_i \to 0$, since from the estimate (for $t_i \neq t_j$, $i \neq j$)

$$\left| \frac{H^{(n)}_{t_1,\dots,t_n}(z)}{\prod_{i=1}^n (z + t_i)} \right| \leq \|\psi\| \, \|e^{aH}\phi\| \, \|A\| (\epsilon - 2\delta)^{-n}$$

(maximum principle), thus

$$\left| H^{(n)}_{t_1,\dots,t_n}(z) \right| \leq \text{const.} \left(\frac{|z| + \delta}{\epsilon - 2\delta} \right)^n$$

and the continuity of $H^{(n)}_{t_1,\dots,t_n}(z)$ in $t_1, \dots, t_n$ follows (with $H^{(n)}_{0,\dots,0}(z) = H^{(1)}_0(z)$)

$$\left| H^{(1)}_0(z) \right| \leq \text{const.} \, |z|^n \quad ,$$

i.e. $H^{(1)}_0$ possesses a zero of the n-th order at $z = 0$. Since this is true for all n, $H^{(1)}_0$ vanishes identically. Thus,

$$Ae^{-aH} E e^{aH} \phi = 0 \quad .$$

Since the vector $Ae^{-aH}Ee^{aH}\phi$ is analytic for the energy, it is cyclic for $\mathfrak{A}(\mathcal{G}_0)$ and hence separating for $\mathfrak{A}(\mathcal{G})'$, thus $A = 0$ holds. □

An application of the Borchers property is the following

THEOREM. *Let $(\mathfrak{A}(\mathcal{O}))$ be a net of von Neumann algebras satisfying the Borchers property, i.e. for $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$ and projections $E \in \mathfrak{A}(\mathcal{O}_1)$, $E \neq 0$ there exists an isometry $V \in \mathfrak{A}(\mathcal{O}_2)$ with $V^*V = E$. Then $\mathfrak{A} = \overline{\bigcup \mathfrak{A}(\mathcal{O})}$ is a simple algebra (i.e. $\mathfrak{A}$ does not contain any closed ideals apart from $\{\mathbb{0}\}$ and $\mathfrak{A}$).*

PROOF. Let J be a closed ideal of $\mathfrak{A}$, $J \neq \{\mathbb{0}\}$. Then $J \cap \mathfrak{A}(\mathcal{O}) \neq \{\mathbb{0}\}$ for some $\mathcal{O}$ (compare 1). Let $B \in J \cap \mathfrak{A}(\mathcal{O})$, $B > 0$, and let E be the spectral projection of B to the interval $[\epsilon, \|B\|]$, $0 < \epsilon < \|B\|$. Then $E \neq 0$ and $B \geq \epsilon E$. According to the Borchers property, in $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{O}_1 \supset \overline{\mathcal{O}}$ there is an isometry V with $VV^* = E$. Thus,

$$V^*BV \geq \epsilon V^*EV = \epsilon V^*VV^*V = \epsilon \mathbb{1} \quad ,$$

i.e. V^*BV is invertible. Since with B also V^*BV belongs to J, $J = \mathfrak{A}$ follows. □

CHAPTER III

Global Gauge Symmetries

1. Field Algebra and Algebra of Observables

We want to take a look at the typical features of a theory with inner symmetries. Starting point is a net of operator algebras, whose elements cannot necessarily be interpreted as observables, as e.g. Fermi fields.

Let $(\mathcal{F}(\mathcal{O}))$ be a net of von Neumann algebras in a Hilbert space $\mathcal{H}$ indexed by the double cones $\mathcal{O}$ of Minkowski space. We call the algebras $\mathcal{F}(\mathcal{O})$ the local field algebras and $\mathcal{F} = \overline{\bigcup \mathcal{F}(\mathcal{O})}$ the field algebra. We make the following assumptions:

(i) $\bigcap_{\mathcal{O}} F(\mathcal{O})' = \mathbb{C}\mathbb{1}$ (irreducibility).

(ii) $\exists$ a strongly continuous unitary representation U of the covering group of the Poincaré group $P_+^\uparrow$ in $\mathcal{H}$ with

$$U(L)\mathcal{F}(\mathcal{O})U(L)^{-1} = \mathcal{F}(L\mathcal{O}) \quad .$$

The generators P_μ of the translations satisfy the spectrum condition

$$\text{sp} \subset \overline{V_+} \quad ,$$

and there exists a unit vector $\Omega \in \mathcal{H}$ with $U(L)\Omega = \Omega$ unique up to a phase factor. We call Ω the **vacuum vector** and denote by α_L the automorphism of $\mathcal{F}$ implemented by $U(L)$,

$$\alpha_L(F) = U(L)FU(L)^{-1} \quad .$$

(iii) $\exists$ a compact group G (the **gauge group**) and a strongly continuous faithful representation U of G in $\mathcal{H}$ with the properties

$$U(g)\mathcal{F}(\mathcal{O})U(g)^{-1} = \mathcal{F}(\mathcal{O})$$
$$U(g)\Omega = \Omega$$
$$U(g)U(L) = U(L)U(g) \quad .$$

The automorphism of $\mathcal{F}$ implemented by $U(g)$ is denoted by α_g. (The use of the same symbols U and α for Poincaré transformations and gauge transformations follows conventions and should not lead to confusion.)

(iv) $\exists \kappa \in G$ with $\kappa^2 = 1$, such that for spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ and even (odd) elements $F_\pm^{(i)} \in \mathcal{F}(\mathcal{O}_i)$, $i = 1, 2$, $\alpha_\kappa(F_\pm^{(i)}) = \pm F_\pm^{(i)}$ graded locality holds:

$$[F_+^{(1)}, F_+^{(2)}] = 0 = [F_+^{(1)}, F_-^{(2)}] = \{F_-^{(1)}, F_-^{(2)}\} \quad .$$

(v) The net $(\mathcal{F}(\mathcal{O}))$ is additive,

$$\mathcal{F}(\mathcal{O}) = \bigvee_\alpha \mathcal{F}(\mathcal{O}_\alpha) \quad , \qquad \mathcal{O} = \bigcup \mathcal{O}_\alpha \quad .$$

(vi) The net $(\mathcal{F}(\mathcal{O}))$ satisfies G-invariant **Haag duality**, i.e.

$$\bigvee_{\mathcal{O}_1 \subset \mathcal{O}'} \Big(\mathcal{F}(\mathcal{O}_1) \cap U(G)' \Big) = \mathcal{F}(\mathcal{O})' \cap U(G)' \quad .$$

The properties (i)-(vi) are abstracted from examples, as e.g. the free Fermi field with inner degrees of freedom. The duality property (vi) is violated for the free massive Dirac field in 2 dimensions. Since the fields are not necessarily observable, other assumptions e.g. about the spacelike commutation relations can be made. However, we will see (this is the content of the Doplicher-Roberts theorem) that a net of algebras of observables (under assumptions still to be specified) uniquely determines a net of field algebras with the stated properties. From a physical point of view these properties do not mean a restriction to generality. We now define observables as gauge invariant elements of the field algebra,

$$\mathfrak{A}(\mathcal{O}) = \Big\{ A \in \mathcal{F}(\mathcal{O}),\, \alpha_g(A) = A \,\forall g \in G \Big\} = \mathcal{F}(\mathcal{O}) \cap U(G)' \quad ,$$

and obtain a net of operator algebras in $\mathcal{H}$ with the following properties:

(ii)' $\alpha_L(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(L\mathcal{O})$ (Poincaré covariance),

(iv)' $\mathcal{O}_1 \subset \mathcal{O}_2' \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)'$ (Locality).

If $U(G)$ is nontrivial then the Hilbert space $\mathcal{H}$ decomposes into subspaces that are invariant under $\mathfrak{A}$, every projection in $U(G)''$ projects on such an invariant subspace. Since G is compact, $U(G)''$ is isomorphic to a direct sum of finite dimensional matrix algebras

$$U(G)'' \simeq \bigoplus_\sigma M_{d_\sigma}(\mathbb{C}) \quad ,$$

where σ runs through all the equivalence classes of irreducible representations of G occuring in U and d_σ is the dimension of a representation of the class σ. We have:

PROPOSITION.

(i) *The irreducible invariant subspaces of $\mathcal{H}$ correspond exactly to the minimal projections of $U(G)''$.*

(ii) *Two such representations are equivalent if and only if these projections in $U(G)''$ are equivalent, i.e. if they belong to the same full matrix algebra.*

PROOF. The proof of the proposition follows from the fact that $U(G)'' = \mathfrak{A}'$. In order to see this identity, we again consider the conditional expectation $m : \mathcal{F} \to \mathcal{A}$ defined by taking the mean over G:

$$m(F) = \int_G dg \alpha_g(F) \quad .$$

Here dg is the normalized Haar measure on G, and the integral is defined in the sense of matrix elements. We have:

(i) $m\alpha_g = \alpha_g m = m$,
(ii) m is σ-continuous,
(iii) $m(\mathcal{F}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$.

Because of (ii) m can be continued on $\mathcal{B}(\mathcal{H})$. From this follows

$$\mathfrak{A}'' = m(\mathcal{F}'') \quad , \qquad \text{because of } \sigma\text{-continuity}$$
$$F'' = \mathcal{B}(\mathcal{H}) \quad , \qquad \text{because of irreducibility of } \mathcal{F}$$

and thus $\mathfrak{A}'' = m(\mathcal{B}(\mathcal{H})) = U(G)'$, thus $\mathfrak{A}' = U(G)''$. □

The Hilbert space $\mathcal{H}$ hence decomposes in the following way:

$$\mathcal{H} = \bigoplus_\sigma \mathcal{H}_\sigma \times \mathcal{H}'_\sigma$$

with $\mathcal{H}_\sigma$ as the irreducible representation space of G with representation U_σ, $\mathcal{H}'_\sigma$ as the irreducible representation space of $\mathfrak{A}$ with representation π_σ and

$$A\Big|_{\mathcal{H}_\sigma \times \mathcal{H}'_\sigma} = \mathbb{1}_{\mathcal{H}_\sigma} \times \pi_\sigma(A) \quad , \qquad A \in \mathfrak{A}$$
$$U(g)\Big|_{\mathcal{H}_\sigma \times \mathcal{H}'_\sigma} = U_\sigma(g) \times \mathbb{1}_{\mathcal{H}'_\sigma} \quad , \qquad g \in G$$

The trivial representation corresponds to the subspace $\mathcal{H} = \overline{\mathfrak{A}\Omega}$, and π_0 with $\pi_0(A) = A|_{\mathcal{H}_0}$ is called the **vacuum representation**. Since Poincaré transformations and gauge transformations commute, every one of the representations π_σ is invariant under Poincaré transformations.

We now want to compare the different representations. Let E be a minimal projection in $U(G)''$ and $\pi_E(A) = A|_{E\mathcal{H}}$, $A \in \mathfrak{A}$. E has the form

$$E = \int dg U(g)\overline{(\phi, U_\sigma(g)\phi)} \quad , \qquad \phi \in \mathcal{H}_\sigma,\ \|\phi\| = 1 \quad .$$

We choose $F \in \mathcal{F}(\mathcal{O})$ with $EF\Omega \neq 0$. Such an operator exists, since Ω is cyclic for $\mathcal{F}(\mathcal{O})$ (Reeh-Schlieder theorem). We consider the operator $G:\ \mathcal{H}_0 \to E\mathcal{H}$,

$$G\phi = EF\phi \quad , \qquad \phi \in \mathcal{H}_0 \quad .$$

G is not zero and possesses the partial intertwining property

$$G\pi_0(A) = \pi_E(A)G \quad , \qquad A \in \mathfrak{A}(\mathcal{O}') \quad .$$

On $\mathfrak{A}(\mathcal{O}')$ the vectors $G\Omega \in \mathcal{H}_E$ and $|G|\Omega \in \mathcal{H}_0$ induce the same states. Because of the cyclicity of Ω for $\mathcal{F}(\mathcal{O})$ there exists a normdense set of states $\omega \in \mathcal{S}_{\pi_E}(\mathfrak{A})$ that coincide on $\mathfrak{A}(\mathcal{O}')$ with states from $\mathcal{S}_{\pi_0}(\mathfrak{A})$. Thus, since the states in a folium are norm-closed,

$$\mathcal{S}_{\pi_0}(\mathfrak{A}(\mathcal{O}')) \subset \mathcal{S}_{\pi_E}(\mathfrak{A}(\mathcal{O}')) \quad .$$

However, since $\mathcal{H}_E$ also contains vectors cyclic for $\mathcal{F}(\mathcal{O})$ (e.g. $\phi \in E\mathcal{H} \cap \mathcal{D}(e^{aP})$, $\phi \neq 0$), vice versa

$$\mathcal{S}_{\pi_E}(\mathfrak{A}(\mathcal{O}')) \subset)\mathcal{S}_{\pi_0}(\mathfrak{A}(\mathcal{O}')) \quad .$$

The state sets of the two representations are thus equal after the restriction to $\mathfrak{A}(\mathcal{O}')$, the restrictions are thus quasiequivalent. ϕ and Ω are cyclic and separating for $\mathfrak{A}(\mathcal{O}')$, thus from quasiequivalence follows unitary equivalence. We obtain the following

PROPOSITION. *All irreducible subrepresentations π_σ satisfy the condition*

$$\pi_\sigma\Big|_{\mathfrak{A}(\mathcal{O}')} = \pi_0\Big|_{\mathfrak{A}(\mathcal{O}')}$$

for all double cones $\mathcal{O}$.

REMARK. This property of representations provides the starting point of the DHR theory of superselection sectors.

2. Simple Sectors, Haag Duality, Localized Endomorphisms

Among the sectors π_σ a particular role is played by the ones belonging to one-dimensional representations σ of G. Let E_σ be the projection on $\mathcal{H}_\sigma \times \mathcal{H}'_\sigma$ ($E_\sigma \in Z(U(G)'') = Z(\mathfrak{A}''))$. σ is **one-dimensional** if and only if

$$U(g)E_\sigma = E_\sigma U(g) = \chi_\sigma(g)E \quad , \qquad \chi \text{ character of } \sigma \quad .$$

DEFINITION. A representation π of $\mathfrak{A}$ satisfies **Haag duality** if

$$\pi(\mathfrak{A}(\mathcal{O}'))'' = \pi(\mathfrak{A}(\mathcal{O}))' \cap \pi(\mathfrak{A})'' \quad .$$

REMARK. From locality follows $\pi(\mathfrak{A}(\mathcal{O}'))'' \subset \pi(\mathfrak{A}(\mathcal{O}))' \cap \pi(\mathfrak{A})''$, thus only the converse inclusion has to be shown.

PROPOSITION. *π_σ satisfies Haag duality if σ is one-dimensional.*

PROOF. Let σ be one-dimensional. Then $\pi_\sigma(A) = A|_{E_\sigma\mathcal{H}}$, $A \in \mathfrak{A}$. Let $A' \in \pi_\sigma(\mathfrak{A}(\mathcal{O}))'$. Then $E_\sigma A' E_\sigma \in \mathfrak{A}(\mathcal{O})'$. We have

$$\mathfrak{A}(\mathcal{O})' = (\mathcal{F}(\mathcal{O}) \cap U(G)')' = \mathcal{F}(\mathcal{O})' \vee U(G)'' \quad .$$

The von Neumann algebra $\mathcal{F}(\mathcal{O})' \vee U(G)''$ contains as a weakly dense subalgebra the algebra

$$\mathcal{M} = \Big\{ \sum_{g\in G} F_g U(g),\ F_g \in \mathcal{F}(\mathcal{O})',\ F_g = 0 \text{ apart from finitely many } g \in G \Big\} \quad .$$

We have $E_\sigma \mathcal{M} E_\sigma = E_\sigma \mathcal{F}(\mathcal{O}) E_\sigma$, and since $m \to E_\sigma m E_\sigma$ is weakly dense

$$E_\sigma \mathcal{F}(\mathcal{O})' \vee U(G)'' E_\sigma = E_\sigma \mathcal{F}(\mathcal{O})' E_\sigma \quad .$$

Thus,

$$E_\sigma A' E_\sigma \in E_\sigma \mathcal{F}(\mathcal{O})' E_\sigma = E_\sigma (\mathcal{F}(\mathcal{O})' \cap U(G)') E_\sigma = E_\sigma \mathfrak{A}(\mathcal{O}')'' E_\sigma \quad ,$$

i.e. $A' \in \pi_\sigma(\mathfrak{A}(\mathcal{O}'))''$. □

REMARK. The converse also holds: π_σ violates Haag duality $\Rightarrow$ σ multidimensional. Sectors satisfying Haag duality are called **simple** sectors. In particular, the vacuum sector is simple.

Now we come to the basic construction of DHR theory: Let π be a representation of $\mathfrak{A}$ satisfying the DHR criterion. Let $\mathcal{O}$ be a double cone, and let π on $\mathfrak{A}(\mathcal{O}')$ be unitary equivalent to π_0 with unitary operator $V : \mathcal{H}_0 \to \mathcal{H}_\pi$,

$$V\pi_0(A) = \pi(A)V \quad , \qquad A \in \mathfrak{A}(\mathcal{O}') \quad .$$

We now define a representation $\tilde{\pi}$ in $\mathcal{H}_0$ equivalent to π by

$$\tilde{\pi}(A) = V^{-1}\pi(A)V \quad , \qquad A \in \mathfrak{A} \quad .$$

Obviously, $\tilde{\pi}(A) = \pi_0(A)$ if $A \in \mathfrak{A}(\mathcal{O}')$.

PROPOSITION. *Let $\mathcal{O}_1 \supset \mathcal{O}$. Then $\tilde{\pi}(\mathfrak{A}(\mathcal{O}_1)) \subset \pi_0(\mathfrak{A}(\mathcal{O}_1))$. In particular $\tilde{\pi}(\mathfrak{A}) \subset \pi_0(\mathfrak{A})$.*

PROOF. Let $A \in \mathfrak{A}(\mathcal{O}_1)$ and $A' \in \mathfrak{A}(\mathcal{O}_1')$. Then because of $\mathfrak{A}(\mathcal{O}_1') \subset \mathfrak{A}(\mathcal{O}')$ and $\tilde{\pi} = \pi_0$ on $\mathfrak{A}(\mathcal{O}')$

$$[\pi_0(A'), \tilde{\pi}(A)] = \tilde{\pi}([A', A]) = 0 \quad ,$$

i.e. $\tilde{\pi}(A) \in \pi_0(\mathfrak{A}(\mathcal{O}_1'))' = \pi_0(\mathfrak{A}(\mathcal{O}_1))$ because of Haag duality. □

Since π_0 is faithful ($\mathfrak{A}$ is simple), there exists an endomorphism $\rho : \mathfrak{A} \to \mathfrak{A}$ with $\rho = \pi_0^{-1} \circ \tilde{\pi}$. ρ possesses the following properties:

(i) ρ is localized in $\mathcal{O}$, i.e. $\rho(A) = A$, $A \in \mathfrak{A}(\mathcal{O}')$.

(ii) ρ is transportable, i.e. $\forall \mathcal{O}_1, \mathcal{O}_2$ with $\mathcal{O}_2 \supset \mathcal{O}_1 \cup \mathcal{O} \quad \exists U \in \mathfrak{A}(\mathcal{O}_2)$ with

$$\mathrm{Ad}U \circ \rho(A) = A \quad , \qquad A \in \mathfrak{A}(\mathcal{O}_1') \quad .$$

(iii) $\rho(\mathfrak{A}(\mathcal{O}_1)) \subset \mathfrak{A}(\mathcal{O}_1)$, $\forall \mathcal{O}_1 \supset \mathcal{O}$.

The first property is an immediate consequence of the definition. In order to see the second property we use a unitary operator $V_1 : \mathcal{H}_0 \to \mathcal{H}_\pi$ with the property

$$V_1\pi_0(A) = \pi(A)V_1 \quad , \qquad A \in \mathfrak{A}(\mathcal{O}_1') \quad .$$

Then $V_1^*V \in \pi_0(\mathfrak{A}(\mathcal{O}_1'))' = \pi_0(\mathfrak{A}(\mathcal{O}_2))$, and $U = \pi_0^{-1}(V_1^*V)$ satisfies the mentioned condition. The third property is an immediate consequence of the proposition.

DEFINITION.

$$\Delta(\mathcal{O}) = \Big\{\rho \in \mathrm{End}(\mathfrak{A}) \,\Big|\, \rho(A) = A,\, A \in \mathfrak{A}(\mathcal{O}') \wedge$$
$$\forall \mathcal{O}_1\, \exists U_1 \in \mathfrak{A} \text{ with } \mathrm{Ad}U_1 \circ \rho(A) = A,\, A \in \mathfrak{A}(\mathcal{O}_1')\Big\}$$

$$\Delta = \bigcup_{\mathcal{O}} \Delta(\mathcal{O}) \quad .$$

PROPOSITION. *$\rho(\mathfrak{A}) = \mathfrak{A}$, i.e. ρ automorphism $\quad\Leftrightarrow\quad \pi_0 \circ \rho$ satisfies Haag duality.*

PROOF. Let $\rho \in \Delta(\mathcal{O})$. Then $\rho(\mathfrak{A}(\mathcal{O}')) = \mathfrak{A}(\mathcal{O}')$. First, assume $\rho(\mathfrak{A}) = \mathfrak{A}$. Then ρ is invertible. Since ρ^{-1} acts trivially on $\mathfrak{A}(\mathcal{O}')$, according to (iii)

$$\rho^{-1}(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O}) \quad ,$$

thus $\mathfrak{A}(\mathcal{O}) \subset \rho(\mathfrak{A}(\mathcal{O}))$. It follows that

$$\pi_0 \circ \rho(\mathfrak{A}(\mathcal{O})) \supset \pi_0(\mathfrak{A}(\mathcal{O})) = \pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0 \circ \rho(\mathfrak{A}(\mathcal{O}'))' \quad .$$

Together with locality, Haag duality follows for $\mathcal{O}$. Now let $\mathcal{O}_1$ be arbitrary. Then there exists a unitary operator $U_1 \in \mathfrak{A}$ with $\rho = \text{Ad}U_1 \circ \rho \in \Delta(\mathcal{O}_1)$. We have $\rho_1(\mathfrak{A}) = U_1\rho(\mathfrak{A})U_1^{-1} = U_1\mathfrak{A}U_1^{-1} = \mathfrak{A}$ and therefore $\pi \circ \rho_1$ satisfies Haag duality for $\mathcal{O}_1$. Since $\pi_0 \circ \rho$ and $\pi_0 \circ \rho_1$ are unitarily equivalent, $\pi_0 \circ \rho$ satisfies Haag duality for $\mathcal{O}_1$, too.

We now conversely assume that $\pi_0 \circ \rho$ satisfies Haag duality. Then for $\mathcal{O}_1 \supset \mathcal{O}$ holds

$$\pi_0 \circ \rho(\mathfrak{A}(\mathcal{O}_1)) = \pi_0 \circ \rho(\mathfrak{A}(\mathcal{O}_1'))' = \pi_0(\mathfrak{A}(\mathcal{O}_1'))' = \pi_0(\mathfrak{A}(\mathcal{O}_1')) \quad ,$$

i.e. $\rho(\mathfrak{A}(\mathcal{O}_1)) = \mathfrak{A}(\mathcal{O}_1)$. Since $\bigcup_{\mathcal{O}_1 \supset \mathcal{O}} \mathfrak{A}(\mathcal{O}_1)$ is dense in $\mathfrak{A}$ and ρ is continuous, $\rho(\mathfrak{A}) = \mathfrak{A}$ follows. □

As a first example we consider a free Majorana field with gauge group $\mathbb{Z}_2$,

$$\alpha_\kappa(B(f)) = -B(f) \quad .$$

The vacuum state is invariant under α_κ. Hence, in Fock space $\mathcal{H}$ there is a unitary operator $U(\kappa)$ with

$$U(\kappa)B(f_1)\cdots B(f_n)\Omega = (-1)^n B(f_1)\cdots B(f_n)\Omega \quad .$$

$\mathcal{H}$ decomposes into a direct sum

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \quad ,$$

the summands corresponding to the irreducible representation of $\mathbb{Z}_2$.

As a field operator, which possesses partial intertwining properties between π_0 and π_1, one can take the Majorana field itself smeared with a test function f with compact support. We have

$$B(f)\pi_0(A) = \pi_1(A)B(f) \quad , \qquad \text{supp}f \subset \mathcal{O},\ A \in \mathfrak{A}(\mathcal{O}') \quad .$$

We choose f invariant under Γ_0 with $\|f\|^2 = 1$. Then $B(f)$ is unitary, and we obtain for the endomorphism ρ the formula

$$\rho(b(g,h)) = B(f)b(g,h)B(f) = b(f,g)b(h,f) \quad .$$

ρ is invertible, $\rho = \rho^{-1}$, it acts trivially on $\mathfrak{A}(\mathcal{O}')$,

$$\begin{aligned} \text{supp}g,\ \text{supp}h \subset \mathcal{O}' \quad \Rightarrow \quad \rho(b(g,h)) &= b(f,g)b(h,f) \\ &= b(g,f)b(f,h) \\ &= b(g,h) \quad , \end{aligned}$$

and it is transportable,

$$\begin{aligned} f_1 \in \mathcal{D}(\mathcal{O}_1),\ \Gamma_0 f_1 = f_1,\ \|f_1\| = 1 \quad \Rightarrow \quad \rho_1(b(g,h)) &= b(f_1,g)b(h,f_1) \\ &= b(f_1,f)b(f,g)b(h,f)b(f,f_1) \\ &= \text{Ad}U \circ \rho(b(g,h)) \end{aligned}$$

with $U = b(f_1, f)$.

As a second example we consider the free scalar charged Bose field

$$\varphi = \varphi_1 + i\varphi_2 \quad ,$$

where φ_1 and φ_2 are two commuting free hermitian scalar Bose fields. This model possesses the gauge symmetry

$$U(1) \ni e^{i\alpha} \mapsto (\varphi \mapsto e^{i\alpha}\varphi) \quad .$$

The vacuum is again invariant and the Fock space decomposes into the direct sum

$$\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n \quad ,$$

corresponding to the irreducible representations $e^{i\alpha} \mapsto e^{in\alpha}$, $n \in \mathbb{Z}$ of $U(1)$. An operator interpolating between the sectors is e.g.

$$\varphi(f) = \varphi_1(f) + i\varphi_2(f) \quad , \qquad \text{with supp} f \subset \mathcal{O},\ f \text{ real} \quad .$$

$\varphi(f)$ is a normal operator. Its polar decomposition

$$\varphi(f) = V_f|\varphi(f)|$$

yields a unitary operator V_f possessing the intertwining properties

$$V_f^n \pi_0(A) = \pi_n(A) V_f^n \quad .$$

The corresponding endomorphism is

$$\rho^{(n)}(A) = V_f^n A V_f^{-n} \quad .$$

The unitary operator $U = V_{f_1}^n V_f^{-n}$, supp$f_1 \subset \mathcal{O}_1$, f_1 real, transports the charge to $\mathcal{O}_1$, such that

$$\rho_1^{(n)} = \text{Ad}U \circ \rho^{(n)}$$

is localized in $\mathcal{O}_1$.

3. Field Multiplets and Cuntz Algebra

We now want to examine the sectors π_σ belonging to non-Abelian representations σ of G. Let ρ be an endomorphism of $\mathfrak{A}$ with $\pi_0 \circ \rho \simeq \pi_\sigma$, $\rho \in \Delta(\mathcal{O})$. Since π_σ is irreducible, $\omega = \omega_0 \circ \rho$ is a pure state. Let

$$\mathcal{H}_\omega = \left\{ \phi \in \mathcal{H} \,\middle|\, (\phi, A\phi) = \omega(A)\|\phi\|^2,\ A \in \mathfrak{A} \right\}$$

be the set of vectors inducing ω.

At first we convince ourselves of the fact that $\mathcal{H}_\omega$ is a subspace of $\mathcal{H}$. Let $\phi_1, \phi_2 \in \mathcal{H}_\omega$. If $\phi_1 + \phi_2 = 0$ then nothing has to be shown. Now let $\phi_1 + \phi_2 \neq 0$. Then,

$$\begin{aligned}(\phi_1, A\phi_1) + (\phi_2, A\phi_2) &= \omega(A)(\|\phi_1\|^2 + \|\phi_2\|^2) \\ &= \frac{1}{2}(\phi_1 + \phi_2, A(\phi_1 + \phi_2)) + \frac{1}{2}(\phi_1 - \phi_2, A(\phi_1 - \phi_2)) \quad .\end{aligned}$$

Since ω is pure, the state incuced by $\phi_1 + \phi_2$ has to coincide with ω, thus $\phi_1 + \phi_2 \in \mathcal{H}_\omega$. $\mathcal{H}_\omega$ is invariant under gauge transformations. Its dimension is d_σ.

THEOREM. *For every $\phi \in \mathcal{H}_\omega$ there exists a uniquely determined $\psi \in \mathcal{F}(\mathcal{O})$ with the properties*

(i) $\psi^*\Omega = \phi$,
(ii) $\psi A = \rho(A)\psi$, $A \in \mathfrak{A}$.

PROOF. Because of $\rho \in \Delta(\mathcal{O})$, for $A \in \mathfrak{A}(\mathcal{O}')$

$$(\phi, A\phi) = \omega(A)\|\phi\|^2 = \omega_0 \circ \rho(A)\|\phi\|^2 = \omega_0(A)\|\phi\|^2 = (\Omega, A\Omega)\|\phi\|^2 \quad .$$

Let $F' \in \mathcal{F}(\mathcal{O}')$. Then with E_ω as the projection onto $\mathcal{H}_\omega$ because of $m(F'^*F') \in \mathfrak{A}(\mathcal{O}')''$

$$E_\omega m(F'^*F')E_\omega = (\Omega, m(F'^*F')\Omega)E_\omega = \|F'\Omega\|^2 E_\omega \quad ,$$

thus

$$\|F'\Omega\|^2 \leq \|\phi\|^2 \mathrm{Tr} E_\omega F'^* F' E_\omega = \|\phi\|^2 \mathrm{Tr} E_\omega m(F'^*F')E_\omega = d_\sigma \|\phi\|^2 \|F'\Omega\|^2 \quad .$$

We now define an operator ψ by

$$\psi^* F'\Omega = F'\phi \quad , \qquad F' \in \mathcal{F}(\mathcal{O}') \quad .$$

ψ^* is defined densely, since Ω is cyclic for $\mathcal{F}(\mathcal{O})'$. According to the calculation before ψ^* (and therefore $\psi = (\psi^*)^*$) is bounded,

$$\|\psi^*\| \leq \sqrt{d_\sigma} \quad .$$

Furthermore, $\psi^* \in \mathcal{F}(\mathcal{O})'' = \mathcal{F}(\mathcal{O})$. We compare ψ^* to the unitary operator between the GNS Hilbert spaces to Ω and ϕ (for $\phi \neq 0$)

$$U\rho(A)\Omega = A\phi\|\phi\|^{-1} \quad , \qquad A \in \mathfrak{A} \quad .$$

For $A \in \mathfrak{A}(\mathcal{O}')$ holds $A \in \mathcal{F}(\mathcal{O})'$, thus (with $\rho(A) = A$)

$$\psi^*\rho(A)\Omega = \psi^* A\Omega = A\phi = \|\phi\| U\rho(A)\Omega \quad .$$

Since Ω is cyclic for $\mathfrak{A}(\mathcal{O}')$ in $\mathcal{H}_0$,

$$\psi^*\Big|_{\mathcal{H}_0} = \|\phi\| U \quad , \qquad \text{in particular } \psi^*\rho(A)\Omega = A\phi \quad .$$

Now let $\mathcal{O}_1 \supset \mathcal{O}$ and $A \in \mathfrak{A}(\mathcal{O}_1)$. Then for all $F_1' \in \mathcal{F}(\mathcal{O}_1)'$,

$$\begin{aligned} \psi^*\rho(A)F_1'\Omega = \psi^* F_1'\rho(A)\Omega &= F_1'\psi^*\rho(A)\Omega \quad \text{(because of } \psi^* \in \mathcal{F}(\mathcal{O}_1) \supset \mathcal{F}(\mathcal{O})) \\ &= F_1' A\phi = AF_1'\phi = A\psi^* F_1'\Omega \quad , \end{aligned}$$

hence

$$\psi A = \rho(A)\psi \quad , \qquad \forall A \in \Omega \quad .$$

The uniqueness follows from the fact that $\psi \in \mathcal{F}(\mathcal{O})$ is uniquely fixed by $\psi^*\Omega = \phi$. □

We now consider the set

$$H_\rho = \left\{\psi \in \mathcal{F},\ \psi A = \rho(A)\psi,\ A \in \mathfrak{A}\right\} \quad .$$

H_ρ is by definition a linear norm-closed subspace of $\mathcal{F}$. Let $\psi_1, \psi_2 \in H_\rho$. Then,

$$\psi_1^*\psi_2 \in \mathfrak{A}' \cap \mathcal{F} = U(G)'' \cap \mathcal{F} \quad .$$

The latter intersection is trivial. Namely, let $F_0 \in U(G)'' \cap \mathcal{F}$ with $F|_{\mathcal{H}_0} = c\mathbb{1}_{\mathcal{H}_0}$, $c \in \mathbb{C}$. Then for $F = F_0 - c\mathbb{1}$

$$m(F^*F)\Big|_{\mathcal{H}_0} = 0 = \pi_0(m(F^*F))$$

and thus $m(F^*F) = 0$, since π_0 is faithful. Now let $\phi \in \mathcal{H}$. Then

$$0 = (\phi, m(F^*F)\phi) = \int dg(\phi, \alpha_g(F^*F)\phi) \quad ,$$

but the integral of a continuous positive function can vanish only if the function vanishes identically, hence $(\phi, F^*F\phi) = 0$, thus $F_0 = c\mathbb{1}$. Hence, we can define a scalar product on H_ρ by

$$\langle \psi_1, \psi_2 \rangle \mathbb{1} = \psi_1^* \psi_2 \quad .$$

Actually, H_ρ is a Hilbert space, thus complete with respect to the topology defined by the scalar product, for

$$\langle \psi_1 - \psi_2, \psi_1 - \psi_2 \rangle = \|(\psi_1 - \psi_2)^*(\psi_1 - \psi_2)\| = \|\psi_1 - \psi_2\|^2 \quad ,$$

and the operator norm coincides on H_ρ with the Hilbert space norm. The elements of H_ρ are multiples of isometries. One therefore also calls H_ρ a Hilbert space of isometries.

Such Hilbert spaces of isometries always exist on infinitely dimensional Hilbert spaces. Let e.g. $\mathcal{H} = l^2(\mathbb{Z})$. We consider the operators of the form $(e_n(k) = \delta_{nk},\ k \in \mathbb{Z})$

$$\psi_{\lambda,\mu} e_n = \lambda e_{2n} + \mu e_{2n+1} \quad , \qquad \lambda, \mu \in \mathbb{C} \quad .$$

We have

$$(e_m, \psi^*_{\lambda,\mu} \psi_{\lambda',\mu'} e_n) = (\overline{\lambda}\lambda' + \overline{\mu}\mu')\delta_{mn} \quad ,$$

thus $\psi^*_{\lambda,\mu}\psi_{\lambda',\mu'} = (\overline{\lambda}\lambda' + \overline{\mu}\mu')\mathbb{1}$.

Furthermore, we have $H_\rho^*\Omega = \mathcal{H}_{\omega_0 \circ \rho}$, for if $\psi \in H_\rho$, then for $A \in \mathfrak{A}$ holds

$$(\psi^*\Omega, A\psi^*\Omega) = (\Omega, \psi A \psi^*\Omega) = (\Omega, \rho(A)\psi\psi^*\Omega) = (\Omega, \rho(A)m(\psi\psi^*)\Omega) \quad .$$

However,

$$m(\psi\psi^*)\rho(A) = m(\psi\psi^*\rho(A)) = m(\rho(A)\psi\psi^*) = \rho(A)m(\psi\psi^*) \quad ,$$

i.e. $m(\psi\psi^*) = \mathbb{C}\mathbb{1}$ because of the irreducibility of $\pi_0 \circ \rho$. We have $(\Omega, m(\psi\psi^*)\Omega) = \|\psi^*\omega\|^2$, thus $(\psi^*\omega, A\psi^*\omega) = \omega_0 \circ \rho(A)\|\psi^*\omega\|^2$, thus $\psi^*\omega \in \mathcal{H}_{\omega_0\circ\rho}$. The converse inclusion has already been shown.

The following proposition shows that the elements of H_ρ are the typical elements of the field algebra.

PROPOSITION. *Let* (F_i), $i = 1, \dots d_\sigma$ *be a family of operators in* $\mathcal{F}(\mathcal{O})$ *transforming like an irreducible tensor according to the unitary representation* U_σ *of* G. *Then there exists a* $\rho \in \Delta(\mathcal{O})$, *a* $B \in \mathfrak{A}(\mathcal{O})$ *and* $\psi_1, \dots, \psi_{d_\sigma} \in H_\rho$ *with* $F_i = B\psi_i$, $i = 1, \dots, d_\sigma$.

PROOF. $F_i^*\Omega \in \mathcal{H}'_i \subset \mathcal{H}_{\overline{\sigma}}$, $\mathcal{H}'_i$ irreducible representation space of $\mathfrak{A}$, $\overline{\sigma}$ conjugate representation to σ. Let $\rho \in \Delta(\mathcal{O})$ with $\pi_0 \circ \rho(\,\cdot\,) \simeq \,\cdot\,|_{\mathcal{H}'_i}\ \forall i$. Then there exists a $\phi_0 \in \mathcal{H}_0$ with

$$(\phi_0, \rho(A)\phi_0) = (F_i^*\Omega, AF_i^*\Omega) \quad ,$$

where ϕ_0 can be chosen independent of i. We now define a operator T on $\mathcal{H}_0$ by

$$T\pi_0(A)\Omega = \pi_0(A)\phi_0 \quad , \qquad A \in \mathfrak{A}(\mathcal{O}') \quad .$$

T possesses the property

$$T\pi_0(A) = \pi_0(A)T \quad , \qquad \text{on } \mathcal{H}_0,\ A \in \mathfrak{A}(\mathcal{O}') \quad ,$$

thus $T \in \pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))$, i.e. $T = \pi_0(B^*)$ for a $B \in \mathfrak{A}(\mathcal{O})$. $\mathcal{H}'_i$ contains a vector ϕ'_i inducing the state $\omega \circ \rho$. Hence, according to the previous proposition there exists a $\psi'_i \in H_\rho$ with $\psi'_i\Omega = \phi'_i$. With it holds

$$\begin{aligned}(\psi'^*_i B^*\Omega, A\psi'^*_i B^*\Omega) &= (B^*\Omega, \rho(A)\psi'_i\psi'^*_i B^*\Omega) \\ &= (B^*\Omega, \rho(A)m(\psi'_i\psi'^*_i)B^*\Omega) \\ &= (B^*\Omega, \rho(A)B^*\Omega)m(\psi'_i\psi'^*_i) \\ &= (\phi_0, \rho(A)\phi_0)m(\psi'_i\psi'^*_i) \\ &= (F^*_i\Omega, AF^*_i\Omega) \quad ,\end{aligned}$$

i.e. $\psi'^*_i B^*\Omega = \lambda F^*_i\Omega$, ($\mathcal{H}'_i$ irreducible). By redefinition of ψ'_i one obtains the proposition. □

H_ρ with $\pi_0 \circ \rho = \pi_\sigma$ is an irreducible representation space of G,

$$\alpha_g\Big|_{H_\rho} \simeq U_\sigma \quad .$$

Let $\{\psi_i,\ i = 1, \ldots, d_\sigma\}$ be an orthonormal basis of H_ρ. Then $\sum \psi_i\psi_i^* \in U(G)' \cap \mathcal{F}(\mathcal{O}) = \mathfrak{A}(\mathcal{O})$. However,

$$\sum \psi_i\psi_i^* = m(\sum \psi_i\psi_i^*) = \sum m(\psi_i\psi_i^*) \in \mathbb{C}\mathbb{1} \quad .$$

since $\psi_i\psi_i^*$ are mutually orthogonal projections,

$$\sum \psi_i\psi_i^* = 1 \quad .$$

The endomorphism ρ is implemented by $\{\psi_1, \ldots, \psi_n\}$ in the following way

$$\rho(F) = \sum_{i=1}^{d} \psi_i F \psi_i^* \quad ,$$

and a left inverse ϕ of ρ is defined by

$$\phi(F) = \frac{1}{d}\sum_{i=1}^{d} \psi_i F \psi_i^* \quad .$$

We now consider the $*$-algebra ${}^0\mathcal{O}_d$ generated by $\psi_1, \ldots, \psi_d$ (the **Cuntz algebra**).

THEOREM. *The algebra ${}^0\mathcal{O}_d$ is simple (for $d \geq 2$).*

PROOF. Let $\mathcal{I} \subset {}^0\mathcal{O}_d$ be a (two-sided) ideal, and let $A \in \mathcal{I},\ A \neq 0$. Every $A \in {}^0\mathcal{O}_d$ can be written in the form

$$A = \sum_{\substack{\alpha,\beta \\ \text{finite}}} \lambda_{\alpha\beta}\psi_\alpha\psi_\beta^*$$

with α, β finite sequences of numbers $\alpha_i, \beta_j \in \{1, \ldots, d\}$, where the length of the sequence α is denoted by $l(\alpha)$. ψ_α is defined as the product

$$\psi_\alpha = \psi_{\alpha_1} \cdots \psi_{\alpha_n} \quad , \qquad n = l(\alpha) \quad .$$

Because of the relation $\sum_{i=1}^{d} \psi_i \psi_i^* = 1$ the representation above is not unique. One can always choose it in a way that the multiindices β have the same length.

We use the following notation:

$$\alpha \leq \beta \qquad \Leftrightarrow \qquad l(\alpha) \leq l(\beta) \text{ and } \alpha_i = \beta_i,\ i = 1, \ldots, l(\alpha)$$

and for $\alpha \leq \beta$ let $\beta - \alpha$ be defined by

$$(\beta - \alpha)_i = \beta_{l(\alpha)+i} \quad , \qquad i = 1, \ldots, l(\alpha) - l(\beta) \quad .$$

Let n be the smallest length of an index α occurring in the representation of A, and let $\lambda_{\alpha_0 \beta_0} \neq 0$ for $\alpha_0, \beta_0,\ l(\alpha_0) = n$. We have

$$A' = \psi_{\alpha_0}^* A \psi_{\beta_0} = \sum_{\alpha > \alpha_0} \lambda_{\alpha \beta_0} \psi_{\alpha - \alpha_0} + \lambda_{\alpha \beta_0} \mathbb{1} \quad .$$

Let $r = \max\{l(\alpha - \alpha_0),\ \lambda_{\alpha \beta_0} \neq 0\}$. Then,

$$A'' = \psi_1^{*r} A \psi_1^r = \lambda_{\underbrace{(1, \ldots, 1)}_{r}, \beta_0} \psi_1^r + \lambda_{\alpha \beta_0} \mathbb{1} \quad .$$

With A, also A' and A'' are in $\mathcal{I}$. However, then

$$\mathbb{1} = \lambda_{\alpha_0 \beta_0}^{-1} \psi_2^* A'' \psi_2 \quad ,$$

too and hence $I = {}^0\mathcal{O}_d$. $\square$

Now we want to show that ${}^0\mathcal{O}_d$ possesses a unique C^*-norm. We already know that ${}^0\mathcal{O}_d$ possesses non-vanishing representations in Hilbert space, whose operator norm induce C^*-seminorms on ${}^0\mathcal{O}_d$; since ${}^0\mathcal{O}_d$ is simple, these are always C^*-norms. We now make use of the existence of a 1-parameter automorphism group in ${}^0\mathcal{O}_d$,

$$t \mapsto \alpha_t,\ \alpha_t \psi_j = e^{it} \psi_j \quad , \qquad j = 1, \ldots, d \quad .$$

${}^0\mathcal{O}_d$ decomposes into a direct sum of subspaces, which correspond to the different representations of $U(1)$,

$${}^0\mathcal{O}_d = \bigoplus_{k \in \mathbb{Z}} {}^0\mathcal{O}_d^k \quad ,$$

and thus becomes a graded algebra

$${}^0\mathcal{O}_d^k\, {}^0\mathcal{O}_d^l \subset {}^0\mathcal{O}_d^{k+l} \quad .$$

An element $A \in {}^0\mathcal{O}_d^k$ possesses a representation

$$A = \sum \lambda_{\alpha\beta} \psi_\alpha \psi_\beta^*$$

with $l(\beta) = r$ sufficiently large and $l(\alpha) = r + k$. A defines an operator from $H^{\otimes r} \to H^{\otimes r+k}$, for

$$\begin{aligned} H^{\otimes r} &\to H^r \\ u &\mapsto \psi(u) \quad , \qquad \psi(e_1 \otimes \cdots \otimes e_r) = \psi_1 \cdots \psi_r \end{aligned}$$

defines an isomorphism between $H^{\otimes r}$ and H^r, and with $u, w \in H^{\otimes r}$, $v \in H^{\otimes r+k}$ we have

$$\psi(v)\psi(u)^*\psi(w) = \langle u, w\rangle\psi(v) = \psi(\langle u, w\rangle v) \quad .$$

Let us at first consider ${}^0\mathcal{O}_d^0$. ${}^0\mathcal{O}_d^0$ is a subalgebra of ${}^0\mathcal{O}_d$. It has the structure of an inductive limit of the algebra $\mathcal{B}(H^{\otimes r})$ ($\cong M_{\alpha^r}(\mathbb{C})$) with the embedding

$$\begin{aligned} \mathcal{B}(H^{\otimes r}) &\hookrightarrow \mathcal{B}(H^{\otimes r+1}) \\ A &\mapsto \mathbb{1}_H \otimes A \quad . \end{aligned}$$

Hence it possesses a unique C^*-norm. A C^*-norm on ${}^0\mathcal{O}_d$ fixes a unique C^*-norm on ${}^0\mathcal{O}_d^k$. Namely, let $A \in {}^0\mathcal{O}_d^k$. Then $A^*A \in {}^0\mathcal{O}_d^0$ and $\|A\| = \|A^*A\|^{\frac{1}{2}}$.

Now let $\|\cdot\|$ be a C^*-norm on ${}^0\mathcal{O}_d$. Every $A \in {}^0\mathcal{O}_d$ possesses a unique decomposition $A = \sum A_k$ with $A_k \in {}^0\mathcal{O}_d^k$. Then

$$\|A\| \leq \sum_k \|A_k\| \quad .$$

Since the right hand side is independent of the choice of the C^*-norm, there is a maximal C^*-norm $\|\cdot\|_{\max}$.

LEMMA. *For every C^*-norm holds $\|A\| \geq \|A_k\|$.*

PROOF. Let A_k, taken as an operator of $H^{\otimes r} \to H^{\otimes r+k}$, possess the representation

$$A_k = \sum a_i |u_i\rangle\langle v_i|$$

with orthonormal systems (u_i) and (v_i), $a_i \in \mathbb{C}$ and $|a_1| = \|A_k\|$. As an element of ${}^0\mathcal{O}_d^k$, A_k can be written in the form

$$A_k = \sum a_i \psi(u_i)\psi(v_i)^* \quad .$$

We think of r chosen large enough, such that A possesses a representation, in which all terms have r factors ψ_i^*. Then we have

$$\psi(u_1)^*A\psi(v_1) = a_1\mathbb{1} + \sum_{k\neq 0} A'_k \quad , \qquad A'_k \in H^k,\ k > 0,\ A'_k \in (H^*)^{|k|},\ k < 0 \quad .$$

Now on $H^{|k|}$ (and also on $(H^*)^{|k|}$) for the left inverse holds

$$\|\phi(A)\| = \alpha^{-1}\|A\|$$

because of

$$\phi(\psi_{i_1}\cdots\psi_{i_k}) = d^{-1}\sum_i \psi_i^*\psi_{i_1}\cdots\psi_{i_k}\psi_i = d^{-1}\psi_{i_2}\cdots\psi_{i_k}\psi_{i_1} \quad .$$

Hence for every C^*-norm on ${}^0\mathcal{O}_d$ and all $l \in \mathbb{N}$ holds

$$\|A\| \geq \|\psi(u_1)^*A\psi(v_1)\| \geq \|a_1 + \sum_{k\neq 0}\phi^l(A'_k)\| \geq |a_1| - \sum_{k\neq 0}\|(A'_k)\|d^{-l} \quad ,$$

thus $\|A\| \geq |a_1| = \|A_k\|$. □

PROPOSITION. *${}^0\mathcal{O}_d$ possesses a unique C^*-norm.*

PROOF. Let $\mathfrak{A}_{\max}$ be the C^*-algebra formed with the help of the maximal C^*-norm of ${}^0\mathcal{O}_d$, and let $\mathcal{I} = \{A \in \mathfrak{A}_{\max},\, \|A\| = 0\}$ be the ideal in $\mathfrak{A}_{\max}$, on which a given C^*-norm $\|\cdot\|$ vanishes. We want to show that $\mathcal{I}$ is a null ideal.

Let $A = \mathcal{I}$. Then $\|A\| = 0 = \|A^*A\|$. Let $A_n \in {}^0\mathcal{O}_d$ be a sequence with $\|A_n - A\|_{\max} \to 0$. Then

$$\|(A_n^* A_n)_0\| \leq \|A_n^* A_n\| \to 0 \quad .$$

Since on ${}^0\mathcal{O}_d$, $\|\cdot\|$ coincides with the maximal C^*-norm, we have

$$\|(A_n^* A_n)_0\|_{\max} \to 0 \quad .$$

Now the maximal C^*-norm is invariant under α_t. We have

$$(A_n^* A_n)_0 = m(A_n^* A_n) \to m(A^* A) \quad .$$

It follows

$$\|m(A^*A)\|_{\max} = 0 = \|\frac{1}{2\pi}\int_0^{2\pi} dt\alpha_t(A^*A)\| \quad .$$

However, $\frac{1}{2\pi}\int_0^{2\pi} dt\alpha_t(A^*A) = 0$ is possible only for $A^*A = 0 \Rightarrow A = 0$, i.e. $\mathcal{I}$ contains only the null element. □

DEFINITION. $\mathcal{O}_d$ is the unique C^*-algebra generated by ψ_i, $i = 1, \ldots, d$.

We now take a look at the action of a group $G \subset U(d)$ on $\mathcal{O}_d$. We have already seen that the automorphisms

$$\alpha_g(\psi_i) = \psi_j g_{ji}$$

act as unitary operators on the Hilbert space H. Since

$$H^k \cong \underbrace{H \otimes \cdots \otimes H}_{k} \quad ,$$

all the tensor products of the defining unitary representation are also contained in the algebra as subrepresentations.

We now consider the invariant subalgebra

$$\mathcal{O}_G = \left\{A \in \mathcal{O}_d,\, \alpha_g(A) = A\,\forall g \in G\right\} \quad .$$

Let $A \in \mathcal{O}_G$ be taken as an operator from $H^{\otimes r}$ to $H^{\otimes s}$. Then

$$U^{\otimes s}(g)A = AU^{\otimes r}(g) \quad , \qquad g \in G \quad ,$$

i.e. A is an intertwiner between the representations $U^{\otimes r}$ and $U^{\otimes s}$. The elements of $\mathcal{O}_G$ with this property can also be characterized with the help of the endomorphisms ρ. For we have (for $l(\alpha) = s$, $l(\beta) = r$)

$$\psi_\alpha \psi_\beta^* \rho^r(A) = \psi_\alpha \psi_\beta^* \sum_{\substack{\gamma \\ e(\gamma) = r}} \psi_\gamma A \psi_\gamma^* = \psi_\alpha A \psi_\beta^*$$

$$\rho^s(A)\psi_\alpha \psi_\beta^* = \sum_{\substack{\gamma \\ e(\gamma) = s}} \psi_\gamma A \psi_\gamma^* \psi_\alpha \psi_\beta^* = \psi_\alpha A \psi_\beta^* \quad ,$$

thus the intertwiners T from $U^{\otimes r}$ to $U^{\otimes s}$ in $\mathcal{O}_G$ are characterized by the property that they satisfy the relation

$$T\rho^r(A) = \rho^s(A)T \quad , \qquad A \in \mathcal{O}_G \quad .$$

The idea of the Doplicher-Roberts reconstruction is to take the relation between $\mathcal{O}_d$ and $\mathcal{O}_G$ as a model for the relation between $\mathcal{F}$ and $\mathfrak{A}$. The group G acts as an automorphism group on $\mathcal{O}_d$. The properties of its representations can be described with the help of the intertwiners. These are elements of $\mathcal{O}_G$ and can be identified if the endomorphism ρ is known. The **Tanaka-Krein theorem** characterizes a compact group by the following data:

(i) the representation spaces $\mathcal{H}_\sigma$,
(ii) the intertwiners $T : \mathcal{H}_\sigma \to \mathcal{H}_\tau$, $TU_\sigma(g) = U_\tau(g)T$.

If these data are given, then the group is defined by the set of maps

$$g : \sigma \to g(\sigma) \in \mathcal{U}(\mathcal{H}_\sigma)$$

with $Tg(\sigma) = g(\tau)T \; \forall T \in (\sigma, \tau)$ and $g(\sigma \otimes \tau) = g(\sigma) \otimes g(\tau)$.

The problem with the application of this theorem is, that the Hilbert spaces are not given at first. We will see, however, that $\mathcal{O}_G$ contains sufficient information about the Hilbert spaces in $\mathcal{O}_d$.

A second problem is easy to solve. In our construction we have obtained all the tensor products of the defining representation and its subrepresentations as Hilbert space representations in the algebra $\mathcal{O}_d$ and hence all the corresponding intertwiners in $\mathcal{O}_G$. If $G = U(d)$, then the conjugate representation is not contained in the tensor products of the defining representation. In general the following proposition holds:

PROPOSITION. *Let G be a compact group, and let σ be a faithful self-conjugate finite dimensional representation of G. Then every irreducible representation of G is contained as a subrepresentation in the multiple tensor product of σ with itself.*

PROOF. Let χ be the character of σ and χ_τ the character of any irreducible representation τ of G. τ is equivalent to a subrepresentation of $\sigma^{\otimes n}$ if and only if

$$\int_G dg \chi(g)^n \chi_\tau(g) \neq 0 \quad , \qquad (\chi(g) = \overline{\chi(g)} \text{ because of } \sigma \simeq \overline{\sigma}) \quad .$$

i.e. we must show that

$$\int dg e^{\lambda\chi(g)} \chi_\tau(g) \neq 0$$

for some λ. We now make use of the fact that $\chi_\tau(e) = d_\tau$ and that for $g \neq e$ because of the faithfulness of σ

$$0 < \mathrm{Tr}(U_\sigma(g) - 1)^*(U_\sigma(g) - 1) = 2(\chi(e) - \chi(g)) \quad .$$

We show

$$\frac{\int dg e^{\lambda\chi(g)} \chi_\tau(g)}{\int dg e^{\lambda\chi(g)}} \underset{\lambda\to\infty}{\longrightarrow} d_\tau \quad .$$

For that purpose it is obviously sufficient to show

$$\frac{\int dg e^{\lambda\chi(g)} f(g)}{\int dg e^{\lambda\chi(g)}} \underset{\lambda\to\infty}{\longrightarrow} 0$$

if f is continuous with $f(e) = 0$.

We choose for $\epsilon > 0$ a neighborhood U of e with $|f(g)| < \epsilon$ for $g \in U$. On $G \setminus U$ then

$$0 < \chi(e) - \sup_{g\in G\setminus U} \chi(g) = \delta \quad .$$

Now let $V \subset U$ be a neighborhood of e with $\chi(g) \geq \chi(e) - \frac{\delta}{2}$. Then it follows

$$\left|\frac{\int dg e^{\lambda\chi(g)} f(g)}{\int dg e^{\lambda\chi(g)}}\right| \leq \Big(\int dg e^{\lambda\chi(g)}\Big)^{-1} \Big(\Big|\int_U dg e^{\lambda\chi(g)} f(g)\Big| + \Big|\int_{G\setminus U} dg e^{\lambda\chi(g)} f(g)\Big|\Big) \quad .$$

The first term on the right side is obviously bounded by ϵ. The second term is estimated in the following way

$$\frac{\Big|\int_{G\setminus U} dg e^{\lambda\chi(g)} f(g)\Big|}{\int dg e^{\lambda\chi(g)}} \leq \frac{\Big|\int_{G\setminus U} dg e^{\lambda\chi(g)} f(g)\Big|}{\int_V dg e^{\lambda\chi(g)} f(g)} \leq \sup|f| \frac{\text{vol}(G\setminus U)}{\text{vol}(V)} e^{-\lambda\chi(e)\frac{\delta}{2}} \quad ,$$

it thus converges to zero for $\lambda \to \infty$. Since $\epsilon > 0$ was arbitrary, the proposition follows. □

We confine ourselves in the following to the case $G \subset \text{SU}(d)$. Then the totally antisymmetric subspace of $H^{\alpha-1}$ carries the conjugate representation of G. For

$$\hat{\psi}_i = \frac{1}{\sqrt{(d-1)!}} \sum_{\substack{p \in S_d \\ p(1) = i}} \text{sign}(p)\psi_{p(2)} \cdots \psi_{p(d)}$$

forms a basis of this subspace and

$$S = \frac{1}{\sqrt{d!}} \sum_{p\in S_d} \text{sign}(p)\psi_{p(1)} \cdots \psi_{p(d)}$$

is invariant under $G \subset \text{SU}(d)$. With $\hat{\psi}_i = \sqrt{d}\psi_i^* S$ the proposition follows.

We thus know that in this case every representation of $G \in \mathcal{O}_d$ and correspondingly every intertwiner is contained in $\mathcal{O}_G$. However, we need a criterion, when a given algebra is isomorphic to $\mathcal{O}_G$. Here we use that the algebra $\mathcal{O}_{\text{SU}(d)}$ is easy to characterize.

All intertwiners between tensor products of the defining representation of $\text{SU}(d)$ are generated by permutations of the factors in the tensor product and by the determinant. In the Cuntz algebra the determinant corresponds to the element S. The permutations are represented by

$$\varepsilon(p) = \sum_\alpha \psi_\alpha \psi^*_{\alpha_p} \quad , \qquad p \in S_n$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\alpha_p = (\alpha_{p(1)}, \ldots, \alpha_{p(n)})$. This representation respects the natural embedding $S_n \hookrightarrow S_{n+1}$. Furthermore holds

$$\rho(\varepsilon(p)) = \varepsilon(p')$$

with $p' \in S_{n+1},\ p'(1) = 1,\ p'(i) = p(i-1),\ i = 1, \ldots, n$, as well as

$$\phi(\varepsilon(p)) = \begin{cases} \varepsilon(p) & p(1) = 1 \\ d^{-1}\varepsilon(p') & p(1) \neq 1 \end{cases}$$

with $p'(i) = ((1p(1))p)(i+1)$. The algebra generated by $\varepsilon(p),\ p \in S_\infty$ is isomorphic to the group algebra $\mathbb{C}S_\infty$ modulo the ideal generated by

$$E_{\alpha+1} = \frac{1}{(d+1)!} \sum_{p \in S_{d+1}} \text{sign}(p)\varepsilon(p) \quad .$$

Futhermore holds

$$SS^* = E_d$$
$$S^*\rho(S) = (-1)^{d-1}d^{-1}\,\mathbb{1} \quad ,$$

for

$$\begin{aligned} SS^* &= \frac{1}{d!} \sum_{p,p' \in S_d} \text{sign}(pp')\psi_p\psi_{p'}^* \quad , \qquad \psi_p = \psi_{p(1)} \cdots \psi_{p(d)} \\ &= \frac{1}{d!} \sum_{p,p' \in S_d} \text{sign}(p)\psi_{p'}\psi_{p'p}^* \\ &= \frac{1}{d!} \sum_{p \in S_d} \text{sign}(p) \sum_{\alpha} \psi_\alpha\psi_{\alpha p}^* \\ &= E_d \end{aligned}$$

and

$$\begin{aligned} S^*\rho(S) &= \frac{1}{d!} \sum_{p,p' \in S_d, i} \psi_p^*\psi_i\psi_{p'}\psi_i^* \text{sign}(pp') \\ &= \frac{1}{d!} \sum_{\substack{i,p \\ p(1)=i}} \psi_i\psi_i^*(-1)^{d-1} = (-1)^{d-1}d^{-1} \quad . \end{aligned}$$

The following proposition holds (DR Inv. Math. **98**, 157-218 (1989) Thm 4.17).

PROPOSITION. *Let $\hat{\mathcal{O}}$ be a simple C^*-algebra with an endomorphism $\hat{\rho}$ and a unitary representation $\hat{\varepsilon}$ of S_∞ with the following properties:*

(i) $\hat{\varepsilon}(p) \in (\hat{\rho}^n, \hat{\rho}^n),\ p \in S_n$,
(ii) $\hat{\varepsilon}((1\,2\,\cdots\,n+1))\hat{T} = \hat{\rho}(\hat{T})\hat{\varepsilon}((1\,2\,\cdots\,m+1)),\ \hat{T} \in (\hat{\rho}^n, \hat{\rho}^m)$,
(iii) $\exists \hat{S} \in (\text{id}, \hat{\rho}^d)$ *with* $\hat{S}^*\hat{S} = \mathbb{1},\ \hat{S}^*\hat{\rho}(\hat{S}) = (-1)^{d-1}d^{-1}\,\mathbb{1}$ *and* $\hat{S}\hat{S}^* = \hat{E}_d$,
(iv) $\hat{\mathcal{O}}$ *is generated by the intertwiners* $\hat{T} \in (\hat{\rho}^n, \hat{\rho}^m),\ n, m \in \mathbb{N}$.

Then there is an up to conjugation unique closed subgroup $G \subset$ SU(d) and an embedding of $\hat{\mathcal{O}}$ in $\mathcal{O}_d$ with $\hat{\mathcal{O}} = \mathcal{O}_G$, such that $\rho|_{\mathcal{O}_G},\ \varepsilon = \hat{\varepsilon}$ and $\hat{S} = S$ hold.

PROOF. We are using that $\mathcal{O}_{\mathrm{SU}(d)}$ can be taken as a subalgebra of either $\mathcal{O}_d$ or $\hat{\mathcal{O}}$. We consider the algebra B_0 generated by $\hat{\mathcal{O}}$ and ${}^0\mathcal{O}_d$ with the following relations:

(i) $\psi_i A = \hat{\rho}(A)\psi_i,\ A \in \hat{\mathcal{O}},\ i = 1, \dots, d,$
(ii) $\hat{\varepsilon}(p) = \varepsilon(p),$
(iii) $\hat{S} = S.$

On this algebra the SU(d) acts by

$$\alpha_g(\psi_i) = \sum_i \psi_j g_{ji} \quad , \qquad g \in \mathrm{SU}(d)$$
$$\alpha_g(A) = A \quad , \qquad A \in \hat{\mathcal{O}} \quad .$$

B_0 possesses a unique C^*-norm, which is invariant under α_g. Let the corresponding C^*-algebra be denoted by $\mathcal{B}$. Because of the uniqueness of the C^*-norm on ${}^0\mathcal{O}_d$, $\mathcal{O}_d$ is a subalgebra of $\mathcal{B}$.

If $\hat{\mathcal{O}}$ is generated by $\hat{\varepsilon}(p)$ and $\hat{S}$, then obviously $\hat{\mathcal{O}} = \mathcal{O}_{\mathrm{SU}(d)} \subset \mathcal{O}_d = \mathcal{B}$, and we obtain $G = \mathrm{SU}(d)$. In general, however, there exist intertwiners $\hat{T} \in (\hat{\rho}^n, \hat{\rho}^m)$ with $\hat{T} \notin \mathcal{O}_{\mathrm{SU}(d)}$. We consider the operators of the form

$$t_{\beta\alpha} = \psi_\beta^* \hat{T} \psi_\alpha \quad , \qquad l(\alpha) = n,\ l(\beta) = m \quad .$$

Obviously in $\mathcal{B}$ holds

$$t_{\beta\alpha} A = \psi_\beta^* \hat{T} \rho^n(A) \psi_\alpha = \psi_\beta^* \rho^m(A) \hat{T} \psi_\alpha = A t_{\beta\alpha} \quad ,$$

thus $t_{\beta\alpha} \in \hat{\mathcal{O}}' \cap \mathcal{B}$. The following lemma holds:

LEMMA. $\hat{\mathcal{O}}' \cap \mathcal{B} = Z(\mathcal{B})$.

PROOF. $\hat{\mathcal{O}}' \cap \mathcal{B}$ is invariant under SU(d). Hence, this algebra is generated by irreducible tensors under SU(d). Let $F_i,\ i = 1, \dots, n$ be such a tensor. Then there exists a $m \in \mathbb{N}$ and a subspace $\tilde{H} \subset H^m$ transforming according to the same representation, with corresponding orthonormal basis $\tilde{\psi}_i,\ i = 1, \dots, n$. Thus,

$$B = \sum_i F_i \tilde{\psi}_i^*$$

is invariant under SU(d) and hence in $\hat{\mathcal{O}}$. Because of the orthonormality of the $\tilde{\psi}_i$,

$$B\tilde{\psi}_i = F_i \quad .$$

Since $B \in (\hat{\rho}^n, \mathrm{id})$ and $\tilde{\psi}_i \in (\mathrm{id}, \rho^n)$, with condition (ii) from the theorem and the corresponding property in $\mathcal{O}_d$

$$\hat{\rho}(B) = B\hat{\varepsilon}(n+1 \cdots 1)$$
$$\rho(\tilde{\psi}_i) = \varepsilon(1 \cdots n+1)\hat{\psi}_i \quad ,$$

thus

$$\psi_j F_i = \psi_j B \tilde{\psi}_i = \hat{\rho}(B)\rho(\tilde{\psi}_i)\psi_j = B\hat{\varepsilon}(n+1 \cdots 1)\varepsilon(1 \cdots n+1)\psi_i\psi_j = F_i\psi_j$$

because of $\varepsilon = \hat{\varepsilon}$, i.e. $F_i \in \hat{\mathcal{O}}' \cap \mathcal{O}_d' \cap B = Z(\mathcal{B})$. □

We thus see, that the operators $t_{\alpha\beta}$ lie in the center of the algebra $\mathcal{B}$. We now diagonalize the center and choose a point in the spectrum. This is a one-dimensional representation φ of $Z(\mathcal{B})$. We define the group G we are looking for by

$$G = \left\{ g \in \mathrm{SU}(d),\ \varphi \circ \alpha_g = \varphi \right\} \quad .$$

The embedding of $\hat{\mathcal{O}}$ in $\mathcal{O}_d$ is now given by the following formula:

$$(\hat{\rho}^n, \hat{\rho}^m) \ni \hat{T} \mapsto \sum_{\substack{\alpha,\beta \\ l(\alpha)=m \\ l(\beta)=n}} \varphi(\psi_\alpha^* \hat{T} \psi_\beta) \psi_\alpha \psi_\beta^* \quad .$$

This map is obviously a homomorphism with $\hat{\varepsilon}(p) \mapsto \varepsilon(p)$ and $\hat{S} \mapsto S$. Since $\hat{\mathcal{O}}$ is by definition simple, the map is also injective.

In order to see that the group G is unique up to conjugation, we make use of the fact that $\mathcal{B}^{\mathrm{SU}(d)} = \hat{\mathcal{O}}$. Hence, $\mathcal{B}' \cap \mathcal{B}^{\mathrm{SU}(d)} = Z(\hat{\mathcal{O}}) = \mathbb{C}\mathbb{1}$. With $Z(\mathcal{B}) \simeq C(\mathrm{sp}Z(\mathcal{B}))$ follows that the spectrum of $Z(\mathcal{B})$ only consists of one orbit under SU(d). All stability subgroups defined by points of the spectrum are thus conjugate. □

For later application we want to see how the statistics operators $\varepsilon(p)$ can be expressed with the help of the observables. For that purpose we consider an irreducible tensor $\psi_i' = U\psi_i \in \mathcal{F}(\mathcal{O}_1)$ with $\mathcal{O}_1 \subset \mathcal{O}'$, $U = \sum \psi_j'^* \psi_j$ unitary. We have

$$\begin{aligned}
\varepsilon((1\,2)) &= \sum_{i,j} \psi_i \psi_j \psi_i^* \psi_j^* \\
&= \sum_{i,j} \psi_i U^* \psi_j' \psi_i^* \psi_j^* \\
&= \sum_{i,j} \rho(U^*) \psi_i \psi_j' \psi_i^* \psi_j^* \\
&\pm \sum_{i,j} \rho(U^*) \psi_j' \psi_i \psi_i^* \psi_j^* \\
&\pm \sum_{i,j} \rho(U^*) U \underbrace{\psi_j \psi_i \psi_i^* \psi_j^*}_{=1} \quad .
\end{aligned}$$

$U \in \mathfrak{A}$ is here characterized by Ad$U \circ \rho \in \Delta(\mathcal{O}_1)$. The operator $\varepsilon = \rho(U^*)U$ is called the statistics operator of ρ. ε coincides with $\varepsilon((1\,2))$ up to a sign originating from the graded locality in $\mathcal{F}$.

CHAPTER IV

DHR theory and the DR Reconstruction Theorem

1. Localized Sectors and Statistics

We start in this chapter from a net of von Neumann algebras $\mathfrak{A}(\mathcal{O}) \in \mathcal{B}(\mathcal{H}_\sigma)$ which is local, Haag dual, Poincaré covariant and possesses a unique vacuum state. We study the sectors satisfying the DHR criterion

$$\pi\Big|_{\mathfrak{A}(\mathcal{O}')} \simeq \pi_0\Big|_{\mathfrak{A}(\mathcal{O}')} \quad , \qquad \text{for every } \mathcal{O}$$

and are Poincaré covariant. We have already seen that these representations possess the property that

$$\pi \simeq \pi_0 \circ \rho$$

holds with a localized transportable endomorphism $\rho \in \Delta$. The representation π in general violates Haag duality; we make the following assumption (**finite statistics**): There exists in

$$\mathfrak{A}_\pi = \overline{\bigcup_{\mathcal{O}} \pi(\mathfrak{A}(\mathcal{O}'))'}$$

a positive conditional expectation $E_\pi : \mathfrak{A}_\pi \to \pi(\mathfrak{A})$ with

$$E_\pi(A) \geq d_\pi^{-2} A \quad , \qquad A \geq 0$$

with $d_\pi < \infty$.

We are now able to define the product of two representations:

$$[\pi_1 \times \pi_2] = [\pi_0 \circ \rho_1\rho_2] \quad , \qquad \text{with } \pi_i = \pi_0 \circ \rho_i \quad .$$

PROPOSITION. *Let* $\pi_0 \circ \rho_i \simeq \pi_0 \circ \rho_i'$, $i = 1, 2$. *Then* $\pi_0 \circ \rho_1\rho_2 \simeq \pi_0 \circ \rho_1'\rho_2'$.

PROOF. $\pi_0 \circ \rho_i \simeq \pi_0 \circ \rho_i' \Rightarrow \exists U_i \in \mathfrak{A}$ with $\rho_i' = \mathrm{Ad}U_i \circ \rho_i$. Thus we have

$$\begin{aligned} \pi_0 \circ \rho_1'\rho_2' &= \pi_0 \circ \mathrm{Ad}U_1 \circ \rho_1 \mathrm{Ad}U_2 \circ \rho_2 \\ &= \pi_0 \circ \mathrm{Ad}U_1\rho_1(U_2) \circ \rho_1\rho_2 \\ &= \mathrm{Ad}\pi_0(U_1\rho_1(U_2))\pi_0 \circ \rho_1\rho_2 \quad .\square \end{aligned}$$

This proposition shows that the DHR product of two representations is well-defined. It satisfies the DHR criterion, too.

We now show that $\pi_0 \circ \rho_1\rho_2$ is Poincaré covariant if $\pi_0 \circ \rho_1$ and $\pi_0 \circ \rho_2$ are. Let U_{ρ_i} be the unitary representation of the covering group $\mathcal{P}$ of the Poincaré group with the property

$$\mathrm{Ad}U_{\rho_i}(L) \circ \rho_i = \rho_i \circ \alpha_L \quad .$$

We are looking for a representation $U_{\rho_1\rho_2}$ with the property

$$\mathrm{Ad}U_{\rho_1\rho_2}(L) \circ \rho_1\rho_2 = \rho_1\rho_2 \circ \alpha_L \quad .$$

We have

$$\begin{aligned}\rho_1\rho_2 \circ \alpha_L &= \rho_1 \circ \mathrm{Ad}U_{\rho_2}(L) \circ \rho_2 \\ &= \mathrm{Ad}U_{\rho_1}(L) \circ \rho_1 \circ \alpha_L^{-1} \circ \mathrm{Ad}U_{\rho_2}(L) \circ \rho_2 \\ &= \mathrm{Ad}U_{\rho_1}(L) \circ \rho_1 \circ \mathrm{Ad}U_0(L)^{-1}U_{\rho_2}(L) \circ \rho_2 \quad .\end{aligned}$$

We consider the operators ($\rho \in \Delta(\mathcal{O})$)

$$V_\rho(L) = U_0(L)^{-1}U_\rho(L) \quad , \qquad L \in \mathcal{P} \quad .$$

Let $\hat{\mathcal{O}} \supset \mathcal{O} \cup L^{-1}\mathcal{O}$. Then for $A \in \mathfrak{A}(\hat{\mathcal{O}}')$ holds $\rho(A) = A$, $\rho\alpha_L(A) = \alpha_L(A)$ and hence

$$\begin{aligned}V_\rho(L)A = V_\rho(L)\rho(A) &= U_0(L)^{-1}\rho\alpha_L(A)U_\rho(L) \\ &= U_0(L)^{-1}\alpha_L(A)U_\rho(L) = AU_0(L)U_\rho(L) = AV_\rho(L) \quad ,\end{aligned}$$

thus we have $V_\rho(L) \in \mathfrak{A}(\hat{\mathcal{O}}')' = \mathfrak{A}(\hat{\mathcal{O}}) \subset \mathfrak{A}$. Hence follows

$$\rho_1\rho_2 \circ \alpha_L = \mathrm{Ad}U_{\rho_1\rho_2}(L) \circ \rho_1\rho_2$$

with

$$U_{\rho_1\rho_2}(L) = U_{\rho_1}(L)\rho_1(V_{\rho_2}(L)) \quad .$$

It remains to show that $U_{\rho_1\rho_2}$ is a representation. However, this follows from

$$\begin{aligned}V_\rho(L_1L_2) &= U_0(L_2)^{-1}U_0(L_1)^{-1}U_\rho(L_1)U_\rho(L_2) \\ &= \alpha_{L_2^{-1}}(V_\rho(L_1))V_\rho(L_2) \\ U_{\rho_1\rho_2}(L_1L_2) &= U_{\rho_1}(L_1)U_{\rho_1}(L_2)\rho_1(\alpha_{L_2^{-1}}(V_{\rho_2}(L_1))V_{\rho_2}(L_2)) \\ &= U_{\rho_1}(L_1)\rho_1(V_{\rho_2}(L_1))U_{\rho_1}(L_2)\rho_1(V_{\rho_2}(L_2)) \\ &= U_{\rho_1\rho_2}(L_1)U_{\rho_1\rho_2}(L_2) \quad .\end{aligned}$$

Not so easy to see is that the spectrum condition is satisfied.

The locality of the net $\mathfrak{A}(\mathcal{O})$ leads to the local commutativity of endomorphisms.

PROPOSITION. *Let $\mathcal{O}_1 \subset \mathcal{O}_2'$ and $\rho_i \in \Delta(O_i)$. Then $\rho_1\rho_2 = \rho_2\rho_1$.*

PROOF. Let $A \in \mathfrak{A}(\mathcal{O})$ for an arbitrary double cone $\mathcal{O}$. We choose double cones $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_2$, $\tilde{\mathcal{O}}_1$, $\tilde{\mathcal{O}}_2$ with the properties:

$$\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2 \subset \mathcal{O}' \quad ,$$

$$\hat{\mathcal{O}}_1 \cup \mathcal{O}_1 \subset \tilde{\mathcal{O}}_1 \quad , \qquad \hat{\mathcal{O}}_2 \cup \mathcal{O}_2 \subset \tilde{\mathcal{O}}_2 \quad ,$$

$$\tilde{\mathcal{O}}_1 \subset \tilde{\mathcal{O}}_2' \quad .$$

Because of the transportability of ρ_1 and ρ_2 there are unitary operators $U_1 \in \mathfrak{A}(\tilde{\mathcal{O}}_1)$,

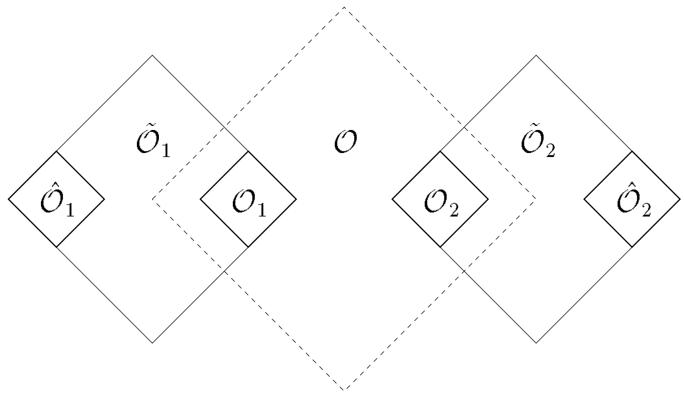

FIGURE IV.6. Double Cones

$U_2 \in \mathfrak{A}(\tilde{\mathcal{O}}_2)$ with

$$\mathrm{Ad}U_1 \circ \rho_1 \in \Delta(\hat{\mathcal{O}}_1) \quad , \qquad \mathrm{Ad}U_2 \circ \rho_2 \in \Delta(\hat{\mathcal{O}}_2) \quad .$$

Thus we have $\rho_1(A) = \mathrm{Ad}U_1^*(A)$, $\rho_2(A) = \mathrm{Ad}U_2^*(A)$ and therefore

$$\begin{aligned} \rho_1\rho_2(A) &= \rho_1 \circ \mathrm{Ad}U_2^*(A) = \mathrm{Ad}\rho_1(U_2^*)\mathrm{Ad}U_1^*(A) \\ &= \mathrm{Ad}U_2^*U_1^*(A) = \mathrm{Ad}U_1^*U_2^*(A) \\ &= \rho_2\rho_1(A) \quad , \end{aligned}$$

where have used $\rho_1(U_2^*) = U_2^*$ because of $\rho_1 \in \Delta(\mathcal{O}_1)$ and $U_2^* \in \mathfrak{A}(\tilde{\mathcal{O}}_2) \subset \mathfrak{A}(\mathcal{O}_1')$ and $U_1^*U_2^* = U_2^*U_1^*$. $\square$

We can now easily show that apart from unitary equivalence the product of representations does not depend on the order of the factors. Let $\rho_1, \rho_2 \in \Delta(\mathcal{O})$, let $\mathcal{O}_1, \mathcal{O}_2$ be spacelike separated double cones and U_1, U_2 unitary operators with $\hat{\rho}_i = \mathrm{Ad}U_i \circ \rho_i \in \Delta(\mathcal{O}_i)$, $i = 1, 2$. Then we have $\hat{\rho}_1\hat{\rho}_2 = \hat{\rho}_2\hat{\rho}_1$ and hence

$$\begin{aligned} \rho_2\rho_1 &= \rho_2 \circ \mathrm{Ad}U_1^* \circ \hat{\rho}_1 = \mathrm{Ad}\rho_2(U_1^*) \circ \rho_2\hat{\rho}_1 \\ &= \mathrm{Ad}\rho_2(U_1^*)U_2^* \circ \hat{\rho}_2\hat{\rho}_1 \\ &= \mathrm{Ad}\rho_2(U_1^*)U_2^* \circ \hat{\rho}_1\hat{\rho}_2 \\ &= \mathrm{Ad}\rho_2(U_1^*)U_2^*U_1\rho_1(U_2) \circ \rho_1\rho_2 \quad . \end{aligned}$$

The operator $\varepsilon(\rho_1, \rho_2) = \rho_2(U_1^*)U_2^*U_1\rho_1(U_2)$ is called **statistics operator**. It has remarkable properties:

(i) $\varepsilon(\rho_1, \rho_2)$ does not depend on the choice of U_1 and U_2 (for fixed $\mathcal{O}_1$ and $\mathcal{O}_2$), for if U_2' is unitary with $\mathrm{Ad}U_2' \in \Delta(\mathcal{O}_2)$, then $V_2 = U_2'U_2^* \in \mathfrak{A}(\mathcal{O}_2')' = \mathfrak{A}(\mathcal{O}_2)$, and we obtain (with $\hat{\rho}_1(V_2) = V_2$)

$$\begin{aligned} \rho_2(U_1^*)U_2'^*U_1\rho_1(U_2') &= \rho_2(U_1^*)U_2^*V_2^*U_1\rho_1(V_2)\rho_1(U_2) \\ &= \rho_2(U_1^*)U_2^*V_2^*\hat{\rho}_1(V_2)U_1\rho_1(U_2) \\ &= \rho_2(U_1^*)U_2^*V_2^*V_2U_1\rho_1(U_2) \\ &= \rho_2(U_1^*)U_2^*U_1\rho_1(U_2) \quad . \end{aligned}$$

A corresponding calculation holds for U_1.

(ii) $\varepsilon(\rho_1, \rho_2)$ does not change if $\mathcal{O}_1$ is replaced by $\hat{\mathcal{O}}_1 \supset \mathcal{O}_1$ and $\mathcal{O}_2$ by $\hat{\mathcal{O}}_2 \supset \mathcal{O}_2$ with $\hat{\mathcal{O}}_1 \subset \hat{\mathcal{O}}_2'$.

We now consider the set

$$\mathcal{K}_2 = \left\{(\mathcal{O}_1, \mathcal{O}_2) \middle| \mathcal{O}_1 \subset \mathcal{O}_2'\right\} \quad .$$

A path in $\mathcal{K}_2$ is a finite sequence $(\mathcal{O}_1^{(i)}, \mathcal{O}_2^{(i)}) \in \mathcal{K}_2$ with $\mathcal{O}_j^{(i)} \subset \mathcal{O}_j^{(i+1)}$ or $\mathcal{O}_j^{(i)} \supset \mathcal{O}_j^{(i+1)}$. $\varepsilon(\rho_1, \rho_2)$ then obviously depends on the connected component of $(\mathcal{O}_1, \mathcal{O}_2)$. Two pairs $(\mathcal{O}_1, \mathcal{O}_2)$ and $(\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2) \in \mathcal{K}_2$ are connected by a path if their centers (x_1, x_2) and $(\hat{x}_1, \hat{x}_2)$ are connected by a path in

$$\left\{(x, y) \middle| (x-y)^2 < 0\right\} \quad .$$

This set possesses in $d \geq 3$ dimensions exactly one connected component, in $d = 2$ dimensions exactly two connected components.

For further discussion it is useful to use the intertwiner calculus of DHR. The space

$$(\rho, \sigma) = \left\{T \in \mathfrak{A} \middle| T\rho(A) = \sigma(A)T,\ A \in \mathfrak{A}\right\}$$

has already been introduced for $\rho, \sigma \in \Delta$, also composition and adjunction

$$\begin{aligned} (\rho, \sigma) \times (\sigma, \tau) &\to (\rho, \tau) & \qquad (\rho, \sigma) &\to (\sigma, \rho) \\ (T, S) &\mapsto S \circ T = ST & T &\mapsto T^* \quad . \end{aligned}$$

This composition structure leads to another composition. For if $T_i \in (\rho_i, \sigma_i)$, $i = 1, 2$ then for $A \in \mathfrak{A}$ holds

$$\begin{aligned} T_1\rho_1(T_2)\rho_1\rho_2(A) = T_1\rho_1(T_2\rho_2(A)) &= T_1\rho_1(\rho_2(A)T_2) \\ &= T_1\rho_1\rho_2(A)\rho_1(T_2) = \rho_1\rho_2(A)T_1\rho_1(T_2) \end{aligned}$$

and $T_1\rho_1(T_2) = \sigma_1(T_2)T_1$. Thus by

$$\begin{aligned} (\rho_1, \sigma_1) \times (\rho_2, \sigma_2) &\to (\rho_1\rho_2, \sigma_1\sigma_2) \\ (T_1, T_2) &\mapsto T_1 \times T_2 = T_1\rho_1(T_2) = \sigma_1(T_2)T_1 \end{aligned}$$

a product between the intertwiner spaces is introduced. We have ($T_i \in (\rho_i, \sigma_i)$, $S_i \in (\sigma_i, \tau_i)$)

$$\begin{aligned} (S_1 \times S_2) \circ (T_1 \times T_2) &= S_1\sigma_1(S_2)T_1\rho_1(T_2) \\ &= S_1T_1\rho_1(S_2T_2) \\ &= (S_1 \circ T_1) \times (S_2 \circ T_2) \quad . \end{aligned}$$

If $\mathbb{1}$ is understood as an element of (ρ, ρ), then we write $\mathbb{1}_\rho$. There are embeddings of (σ, τ) into $(\rho\sigma, \rho\tau)$

$$T \to \mathbb{1}_\rho \times T = \rho(T)$$

and into $(\sigma\rho, \tau\rho)$

$$T \to T \times \mathbb{1}_\rho = T \quad .$$

One easily verifies that the $\times$-product is associative. We can write the statistics operator $\varepsilon(\rho_1, \rho_2)$ (as element of $(\rho_1\rho_2, \rho_2\rho_1)$)

$$\varepsilon(\rho_1, \rho_2) = (U_2^* \times U_1^*) \circ (U_1 \times U_2)$$

with $U_i \in (\rho_i, \hat{\rho}_i)$ unitary, $\hat{\rho}_i \in \Delta(\hat{\mathcal{O}}_i)$, $\hat{\mathcal{O}}_1 \subset \hat{\mathcal{O}}'_{2(r)}$ (right spacelike complement). Furthermore holds: If $S_i \in (\rho_i, \sigma_i)$, $i = 1, 2$ and $\rho_i, \sigma_i \in \Delta(\mathcal{O}_i)$, $\mathcal{O}_1 \subset \mathcal{O}'_2$ then $S_i \in \mathfrak{A}(\mathcal{O}_i)$ and $\rho_1(S_2) = S_2$, $\rho_2(S_1) = S_1$ and thus

$$S_1 \times S_2 = S_2 \times S_1 \quad .$$

From this one easily sees the invariance of the statistics operator for continuous deformations of the regions $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$.

We now show that the statistics operator also describes the commutation relations between intertwiners.

PROPOSITION. *Let* $\rho_1, \rho_2, \sigma_1, \sigma_2 \in \Delta$ *and* $T_i \in (\sigma_i, \rho_i)$, $i = 1, 2$. *Then,*

$$\varepsilon(\rho_1, \rho_2) \circ (T_1 \times T_2) = (T_2 \times T_1) \circ \varepsilon(\rho_1, \rho_2) \quad .$$

PROOF. Let $\hat{\mathcal{O}}_1 \in \hat{\mathcal{O}}'_{2(r)}$ and $U_i \in (\rho_i, \hat{\rho}_i)$, $V_i \in (\sigma_i, \hat{\sigma}_i)$ unitary with $\hat{\rho}_i, \hat{\sigma}_i \in \hat{\mathcal{O}}_i$. Then

$$\varepsilon(\rho_1, \rho_2) = (U_2^* \times U_1^*) \circ (U_1 \times U_2)$$
$$\varepsilon(\sigma_1, \sigma_2) = (V_2^* \times V_1^*) \circ (V_1 \times V_2) \quad .$$

We have $\hat{T}_i = U_i T_i V_i^* \in (\hat{\sigma}_i, \hat{\rho}_i)$ and hence (because of $\hat{\mathcal{O}}_1 \subset \hat{\mathcal{O}}'_{2(r)}$)

$$\hat{T}_2 \times \hat{T}_1 = \hat{T}_1 \times \hat{T}_2 \quad .$$

With this, it follows

$$\begin{aligned}
\varepsilon(\rho_1, \rho_2) T_1 \times T_2 &= (U_2^* \times U_1^*) \circ (U_1 \times U_2)(T_1 \times T_2) \\
&= (U_2^* \times U_1^*) \circ (\hat{T}_1 \times \hat{T}_2) \circ (V_1 \times V_2) \\
&= (U_2^* \times U_1^*) \circ (\hat{T}_2 \times \hat{T}_1) \circ (V_1 \times V_2) \\
&= (U_2^* \times U_1^*) \circ (U_2 \times U_1) \circ (T_2 \times T_1) \circ (V_1^* \times V_2^*) \circ (V_1 \times V_2) \\
&= T_1 \times T_2 \varepsilon(\sigma_1, \sigma_2) \quad . \square
\end{aligned} \tag{4}$$

Furthermore

$$\begin{aligned}
\varepsilon(\rho_1\rho_2, \rho_3) &= (U_3^* \times U_1^* \times U_2^*) \circ (U_1 \times U_2 \times U_3) \\
&= (U_3^* \times U_1^* \times U_2^*)(U_1 \times U_3 \times U_2)(U_1^* \times U_3^* \times U_2^*)(U_1 \times U_2 \times U_3) \\
&= \varepsilon(\rho_1, \rho_3) \circ (\mathbb{1}_{\rho_1} \times \varepsilon(\rho_2, \rho_3)) = \varepsilon(\rho_1, \rho_3)\rho_1(\varepsilon(\rho_2, \rho_3))
\end{aligned} \tag{5}$$

and correspondingly

$$\varepsilon(\rho_1, \rho_2\rho_3) = (\mathbb{1}_{\rho_2} \times \varepsilon(\rho_1, \rho_3)) \circ \varepsilon(\rho_1, \rho_2) = \rho_2(\varepsilon(\rho_1, \rho_3))\varepsilon(\rho_1, \rho_2) \quad .$$

Furthermore $\varepsilon(\rho, \mathrm{id}) = \varepsilon(\mathrm{id}, \rho) = \mathbb{1}$ and

$$\varepsilon(\rho, \sigma) = 1 \quad , \qquad \text{if } \sigma \in \Delta(\mathcal{O}_1),\ \rho \in \Delta(\mathcal{O}_2),\ \mathcal{O}_2 \in \hat{\mathcal{O}}'_{1(r)} \quad .$$

We call $\varepsilon(\rho,\rho)=\varepsilon_\rho$. ε_ρ possesses the properties

(i) $\varepsilon_\rho \in \rho^2(\mathfrak{A})'$,
(ii) $\varepsilon_\rho\rho(\varepsilon_\rho)\varepsilon_\rho = \rho(\varepsilon_\rho)\varepsilon_\rho\rho(\varepsilon_\rho)$.

(i) follows from the intertwiner property of ε_ρ. (ii) follows from (5) and (4):

$$\begin{aligned}\varepsilon_\rho\rho(\varepsilon_\rho) &= \varepsilon(\rho^2,\rho)\\ \varepsilon(\rho^2,\rho)\varepsilon_\rho &= \varepsilon(\rho^2,\rho)(\varepsilon_\rho\times\mathbb{1}_\rho) = (\mathbb{1}_\rho\times\varepsilon_\rho)\varepsilon(\rho^2,\rho) = \rho(\varepsilon_\rho)\varepsilon(\rho^2,\rho) \quad .\end{aligned}$$

In $d>2$ dimensions furthermore holds

$$\varepsilon_\rho^2 = 1 \quad .$$

The pair (ε_ρ,ρ) hence defines a representation of the braid group. The **braid group** B_n is defined as the group with the $n-1$ generators $\sigma_1,\ldots,\sigma_{n-1}$ and the following relations

$$\begin{aligned}\sigma_i\sigma_j &= \sigma_j\sigma_i \quad , \qquad \text{if } |i-j|>2\\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \quad .\end{aligned}$$

By

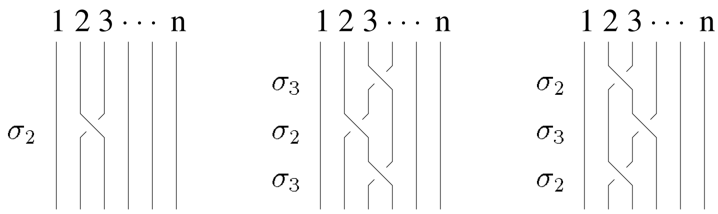

FIGURE IV.7. Braid Group

$$\varepsilon_\rho(\sigma_i) = \rho^{i-1}(\varepsilon_\rho)$$

a representation of the braid group is defined. If further $\varepsilon_\rho^2=1$ holds, then one obtains a representation of the permutation group S_n:

THEOREM. *If in addition to the braid relations $\sigma_i^2=1,\ i=1,\ldots,n-1$ holds, then the generators σ_i generate the permutation group with $\sigma_i=(i\,i+1)$.*

PROOF. In the permutation group S_n the following relations hold:

$$\begin{aligned}(i\,i+1)(j\,j+1) &= (j\,j+1)(i\,i+1) \quad \text{if } \{j,j+1\}\cap\{i,i+1\}=\emptyset\\ (i\,i+1)(i+1\,i+2)(i\,i+1) &= (i\,i+2)\\ (i+1\,i+2)(i\,i+1)(i+1\,i+2) &= (i\,i+2)\\ (i\,i+1)(i\,i+1) &= \mathbb{1} \quad .\end{aligned}$$

The group $\tilde{S}_n$ generated by $\sigma_i,\ i=1,\ldots,n-1$ is mapped homomorphically into S_n by $\sigma_i\mapsto(i\,i+1)$. Since S_n is generated by the transpositions $(i\,i+1)$, the map is onto. In order to show that it is also one to one we show that $\tilde{S}_n$ possesses at most $n!$ elements.

$\tilde{S}_1$ is generated by zero generators and thus contains only the $\mathbb{1}$. We want to show that every element of $\tilde{S}_{n+1}$ can be written in either of the following form:

$$g = \prod_{k=l}^{n} \sigma_k h \quad , \qquad h \in \tilde{S}_n,\ l = 1, \ldots, n+1 \tag{6}$$

Since l can take on $n+1$ values, it would follow

$$|\tilde{S}_{n+1}| \leq (n+1)|\tilde{S}_n|$$

and hence by induction $|\tilde{S}_{n+1}| \leq n!$.

In order to show (6) it suffices to prove that products of g with the generators $\sigma_1, \ldots, \sigma_n$ can again be written in the form (6). A product from the right with $\sigma_k,\ k < n$ obviously changes only h. A product from the left with $\sigma_k,\ k < l-1$, can be commuted past the factors $\sigma_k,\ k < l, \ldots, n$ and also changes only h. A product from the left with σ_{l-1} changes l to $l-1$. If we multiply from the left with σ_l, then l changes to $l+1$. If one multiplies from the left with $\sigma_k,\ k > l$, then

$$\begin{aligned} \sigma_k \sigma_l \sigma_{l+1} \cdots \sigma_n &= \sigma_l \cdots \sigma_{k-2} \sigma_k \sigma_{k-1} \sigma_k \sigma_{k+1} \cdots \sigma_n \\ &= \sigma_l \cdots \sigma_{k-2} \sigma_{k-1} \sigma_k \sigma_{k-1} \sigma_{k+1} \cdots \sigma_n \\ &= \sigma_l \cdots \sigma_{k-2} \sigma_{k-1} \sigma_k \sigma_{k+1} \cdots \sigma_n \sigma_{k-1} \quad . \end{aligned}$$

Thus, h is changed to $\sigma_{k-1}h$. There remains to consider the case that it is multiplied from the right with σ_n. For that purpose we assume that $h \in \tilde{S}_n$ possesses a representation analogous to (6),

$$h = \prod_{k'=l'}^{n-1} \sigma_{k'} h' \quad , \qquad h' \in \tilde{S}_{n-1} \quad .$$

Since σ_n commutes with $\tilde{S}_{n-1}$, one obtains

$$g\sigma_n = \prod_{k=l}^{n} \sigma_k \prod_{k'=l'}^{n} \sigma_{k'} h' \quad .$$

We have

$$\prod_{k=l}^{n} \sigma_k \prod_{k'=l'}^{n} \sigma_{k'} = \begin{cases} \displaystyle\prod_{k'=l'+1}^{n} \sigma_{k'} \prod_{k=l}^{n-1} \sigma_k & l \leq l' \\ \displaystyle\prod_{k'=l'}^{n} \sigma_{k'} \prod_{k=l-1}^{n-1} \sigma_k & l > l' \end{cases} \quad .$$

In both cases g has the representation we wanted. $\square$

We now want to see also geometrically why the braid and permutation groups occur here. We think of the double cones substituted by points in spacelike hyperplane $\mathbb{R}^{d-1}$. To a charge transporter U_k we associate a path γ_k from x_0 (the localization region of ρ) to another point x_k, where the points x_k are mutually distinct (corresponding to mutually relatively spacelike double cones, in which the endomorphisms $\mathrm{Ad}U_k \circ \rho$ are localized). The n factors in $U_1 \times \cdots \times U_n$ define n paths in $\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}$

$$\gamma_k \times \{k\} \times \{0\} \quad , \qquad k = 1, \ldots, n \quad .$$

We now connect the end points $(x_k, k, 0)$ of $\gamma_k \times \{k\} \times \{0\}$ in $\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}$ with $(x_k, p^{-1}(k), 1)$ and compose these paths with the paths

$$\gamma_k \times \{p^{-1}(k)\} \times \{0\} \quad , \qquad k = 1, \dots, n \quad .$$

In this way we obtain a family of paths connecting in $\mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}$ the points $(x_k, k, 0)$ with $(x_k, p^{-1}(k), 1)$.

The operator

$$\varepsilon(p) = (U^*_{p^{-1}(1)} \times \cdots \times U^*_{p^{-1}(n)}) \circ (U_1 \times \cdots \times U_n)$$

is constant under continuous changes of the points x_k. It hence depends only on the braid defined by this prescription, in $d > 2$ dimensions thus only on the permutation p.

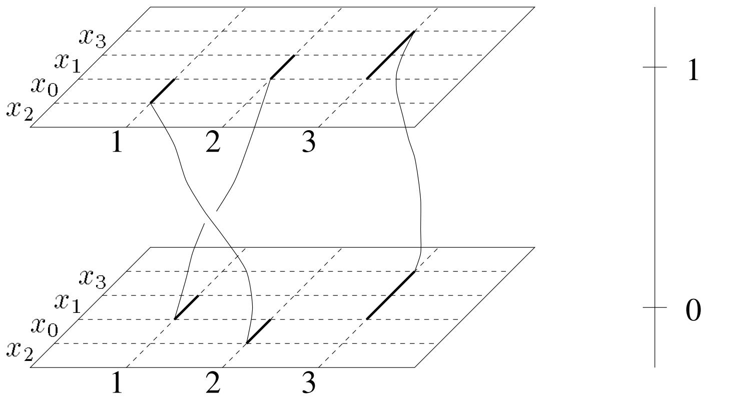

FIGURE IV.8. Geometric Understanding

2. Left Inverse and Conjugate Sector

For a further analysis one needs the existence of a left inverse. If ρ is an automorphism, then ρ^{-1} is the uniquely defined inverse. This case is characterized by the following equivalent conditions:

PROPOSITION. *The following conditions are equivalent for* $\rho \in \Delta$*:*

(i) ρ *is an automorphism.*
(ii) $\pi_0 \circ \rho$ *satisfies Haag duality.*
(iii) $\pi_0 \circ \rho^2$ *is irreducible.*
(iv) $\varepsilon(\rho, \rho)$ *is a multiple of* $\mathbb{1}$.

PROOF. The equivalence of (i) and (ii) has already been shown in section 2. (i) $\Rightarrow$ (ii) holds, since for an automorphism ρ, $\rho(\mathfrak{A})$ coincides with $\mathfrak{A}$, thus

$$\pi_0 \circ \rho^2(\mathfrak{A})' = \pi_0(\mathfrak{A})' = \mathbb{C}\mathbb{1}$$

because of the irreducibility of the vacuum representation. (iii) $\Rightarrow$ (iv), since $\varepsilon(\rho, \rho)$ lies in the commutant of $\rho^2(\mathfrak{A})$. However, this is trivial for $\pi_0 \circ \rho^2$ irreducible.

There remains to show (iv) $\Rightarrow$ (i). Let $\mathcal{O}$ be a double cone and $A \in \mathfrak{A}(\mathcal{O})$. We want to show that $A \in \rho(\mathfrak{A})$ holds if $\varepsilon(\rho, \rho) \in \mathbb{C}\mathbb{1}$. Let $\mathcal{O}_1 \subset \mathcal{O}'$ and U unitary with Ad$U \circ \rho \in \Delta(\mathcal{O}_1)$. Then $AU = U\rho(A)$, thus

$$A = AUU^* = U\rho(A)U^* = \rho(U)\rho(U^*)U\rho(A)U^*\rho(U)\rho(U^*) = \rho(U)\varepsilon_\rho\rho(A)\varepsilon^*_\rho\rho(U^*) \quad .$$

If $\varepsilon_\rho \in \mathbb{C}\mathbb{1}$ holds, then follows

$$A = \rho(UAU^*) \in \rho(\mathfrak{A}) \quad .\square$$

We now consider the general case. We are looking for a map $\phi : \mathfrak{A} \to \mathfrak{A}$ with the properties

$$\phi(\rho(A)B\rho(C)) = A\phi(B)C$$
$$\phi(AA^*) \geq 0$$
$$\phi(1) = 1 \quad .$$

ϕ is given on $\rho(\mathfrak{A})$ by ρ^{-1}. A continuation can be found in the following way. Let $\overline{\omega}$ be defined by

$$\overline{\omega} = \omega_0 \circ \rho^{-1}$$

as a state on $\rho(\mathfrak{A})$. According to the Hahn-Banach theorem every linear functional on $\rho(\mathfrak{A})$ can be continued to a linear functional on $\mathfrak{A}$, such that the norm is preserved. Let $\overline{\omega}_i$ be such a continuation. Then

$$\|\overline{\omega}_i\| = \|\overline{\omega}\| = 1 = \overline{\omega}_i(1) \quad ,$$

thus $\overline{\omega}_i$ is itself again a state. We suppress the index i but keep in mind that $\overline{\omega}$ is not uniquely determined.

We now consider the GNS construction to $\overline{\omega}$, $(\pi, \mathcal{H}, \Psi)$. The map

$$VA\Omega = \pi \circ \rho(A)\Psi \quad , \qquad A \in \mathfrak{A}$$

defines an isometry $V : \mathcal{H}_0 \to \mathcal{H}$,

$$\|VA\Omega\|^2 = \overline{\omega}(\rho(A^*A)) = \omega_0(A^*A) = \|A\Omega\|^2 \quad .$$

V satisfies the intertwiner relation

$$VA = \pi\rho(A)V \quad , \qquad \text{i.e. } V \in (\pi_0, \pi_\rho) \quad .$$

We now set

$$\phi(A) = V^*\pi(A)V \quad , \qquad A \in \mathfrak{A}$$

and show that ϕ is a left inverse of ρ. We have:

(i) $\phi(\rho(A)B\rho(C)) = V^*\pi(\rho(A)B\rho(C))V = AV^*\pi(B)VC = A\phi(B)C$,
(ii) $\phi(A^*A) = (\pi(A)V)^*(\pi(A)V) \geq 0$,
(iii) $\phi(1) = 1$.

Now we want to use the left inverse to examine the representation of the permutation group (we thus restrict ourselves to the case $\varepsilon_\rho^2 = 1$, the general case will be covered in the next semester). We obviously have

$$\phi(\varepsilon_\rho)\rho(A) = \phi(\varepsilon_\rho\rho^2(A)) = \phi(\rho^2(A)\varepsilon_\rho) = \rho(A)\phi(\varepsilon_\rho) \quad ,$$

thus $\phi(\varepsilon_\rho) \in \rho(\mathfrak{A})'$. For ρ irreducible this implies

$$\phi(\varepsilon_\rho) = \lambda_\rho \mathbb{1}$$

with the statistics parameter λ_ρ. In order to apply ϕ to a general permutation operator $\varepsilon(p)$, we use the same factorization as in section 3,

$$\varepsilon(p) = \begin{cases} \rho(\varepsilon(p')) & p(1) = 1 \\ \varepsilon_\rho \rho(\varepsilon(p')) & p(1) = 2 \\ \rho(\varepsilon(1\, p(1) - 1)\varepsilon_\rho \rho(\varepsilon(p'')) & p(1) > 2 \end{cases}$$

with p' as before and $p'' = (1\, p(1) - 1)(p')$. With it follows

$$\phi(\varepsilon(p)) = \begin{cases} \varepsilon(p') & p(1) = 1 \\ \lambda\varepsilon(p') & p(1) \neq 1 \end{cases} .$$

For the projections onto the totally symmetric, respectively antisymmetric part

$$E_n^{\rm a} = \frac{1}{n!} \sum_{p \in S_n} \varepsilon(p)\text{sign}(p)$$

$$E_n^{\rm s} = \frac{1}{n!} \sum_{p \in S_n} \varepsilon(p)$$

we can compute recursively

$$\phi(E_{n+1}^{\rm a}) = \frac{1}{(n+1)!} \Big\{ \sum_{\substack{p \,\in\, S_{n+1} \\ p(1) = 1}} \varepsilon(p')\text{sign}(p') - \sum_{i=2}^{n+1} \sum_{\substack{p \,\in\, S_{n+1} \\ p(1) = i}} \varepsilon(p')\lambda\text{sign}(p') \Big\}$$

$$= \frac{1 - \lambda n}{n+1} E_n^{\rm a}$$

$$\phi(E_{n+1}^{\rm s}) = \frac{1 + \lambda n}{n+1} E_n^{\rm s} \quad .$$

Since ϕ maps positive operators to positive operators, only the following values for λ are possible

$$\lambda = 0, \pm\frac{1}{d} \quad , \qquad d \in \mathbb{N} \quad .$$

In the case $\lambda = \frac{1}{d}$ the projections $E_n^{\rm a}$, $n > d$ have to vanish, for else $E_{n+1}^{\rm a}$ is mapped onto a negative operator. The corresponding holds in the case $\lambda = -\frac{1}{d}$ for the projections $E_n^{\rm s}$, $n > d$. One calls the case $\lambda = \frac{1}{d}$ para Bose statistics of the order d, the case $\lambda = -\frac{1}{d}$ para Fermi statistics of the order d and $\lambda = 0$ infinite statistics. These terms are motivated by the following observation: Let $\mathcal{K}$ be a d-dimensional vector space, and let U_n be the representation of S_n defined by

$$U_n(p)\Psi_1 \otimes \cdots \otimes \Psi_n = \Psi_{p(1)} \otimes \cdots \otimes \Psi_{p(n)} \quad .$$

Then in the case $\lambda = \frac{1}{d}$

$$\phi^n(\varepsilon(p)) = d^{-n}\text{Tr}U_n(p) \quad .$$

In the case of para Fermi statistics, $U_n(p)$ is substituted by $U_n(p)\text{sign}(p)$. In the case of para Bose statistics of the order d, $\phi^n \circ \varepsilon|_{S_n}$ coincides with the corresponding expression in the Cuntz algebra $\mathcal{O}_d$. In the case of para Fermi statistics, $\varepsilon(p)$ has to be substituted by its bosonized form $\hat{\varepsilon}(p) = \varepsilon(p)\text{sign}(p)$.

Now we make use of the assumption of finite statistics, i.e. there exists a conditional expectation $\mathcal{E}_\rho : \rho(\mathfrak{A}(\mathcal{O}'))' \to \mathfrak{A}(\mathcal{O})$ with $\mathcal{E}_\rho(A) \geq d_\rho^{-2}$, $A \geq 0$ for some $d_\rho < \infty$. From the formulae $\phi(E_d^{\mathrm{a}}) = \frac{1}{d^2} E_{d-1}^{\mathrm{a}}$ and the fact that $\rho_{-1}\mathcal{E}_\rho$ is a left inverse, it follows for irreducible ρ that $d \leq d_\rho$. We will see later that even $d_\rho = d$. We now want to construct the conjugate sector. There are different possibilities to do this. One is to show that every sector $[\pi]$ generated by $\overline{\omega}$ satisfies the DHR criterion. Then there exists a $\overline{\rho} \in \Delta(\mathcal{O})$ with $\pi \cong \pi_0 \circ \overline{\rho}$ and an isometry $R \in \mathfrak{A}(\mathcal{O})$ with

$$\overline{\rho}\rho(A)R = RA \quad .$$

This possibility has been used in the proof of the finiteness of the statistics for particle representations in massive theories. In this proof the spectral properties of the translations must be used explicitly.

A more modern possibility has been found by Longo. There one uses a PCT symmetry j and defines $\overline{\rho} = j\rho j$. In this construction (which keeps on making sense in the case of infinite statistics and then serves for the definition of the conjugate sector) one makes use of the modular theory. We will enter into these points in the next semester.

In this lecture I would like to introduce the construction developed by Doplicher, Haag, Roberts. It uses only the statistics operators and the Borchers property. The construction corresponds to the construction of a conjugate representation of a compact group. We have already seen that (in the case of para Bose statistics of the order d) the projections E_n^{a} for $n > d$ vanish. We now consider the subrepresentation of ρ^d determined by E_d^{a}. Let V be an isometry in $\mathfrak{A}(\hat{\mathcal{O}})$, $\hat{\mathcal{O}} \supset \overline{\mathcal{O}}$ with $VV^* = E_d^{\mathrm{a}}$, and let $\gamma(A) = V^*\rho^d(A)V$. Then $\gamma \in \Delta(\hat{\mathcal{O}})$.

PROPOSITION. *γ is an automorphism.*

PROOF. We show that ε_γ is a multiple of $\mathbb{1}$. This is equivalent to the fact that there exists a left inverse ϕ_γ of γ with $\phi_\gamma = \lambda\mathbb{1}$, $|\lambda| = 1$. Because of $V \in (\gamma, \rho^d)$ holds

$$(V \times V) \circ \varepsilon_\gamma = \varepsilon(\rho^d, \rho^d) \circ (V \times V) \quad ,$$

thus with $VV^* = 1$

$$\begin{aligned} \varepsilon_\gamma &= (V^* \times V^*) \circ \varepsilon(\rho^d, \rho^d) \circ (V \times V) \\ &= V^*\rho^d(V^*)\varepsilon(\rho^d, \rho^d)\rho^d(V)V \quad . \end{aligned}$$

A left inverse of γ is

$$\begin{aligned} \phi_\gamma(A) &= \phi^d(VAV^*)\phi^d(E_d^{\mathrm{a}})^{-1} \\ \Big(\phi^d(E_d^{\mathrm{a}}) &= \prod_{k=1}^{d} \frac{1 - \frac{d-k}{d}}{d-k+1} = d^{-d}\,\mathbb{1}\Big) \quad . \end{aligned}$$

Thus,

$$\phi_\gamma(\varepsilon_\gamma) = d^{-d}\phi^d(V\varepsilon_\gamma V^*) \quad .$$

We have

$$\begin{aligned} V\varepsilon_\gamma V^* &= (V \times \mathbb{1}_\gamma) \circ \varepsilon(\gamma,\gamma) \circ (V^* \times \mathbb{1}_\gamma) \\ &= \varepsilon(\gamma,\rho^d) \circ (\mathbb{1}_\gamma \times V) \circ (V^* \times \mathbb{1}_\gamma) \\ &= \varepsilon(\gamma,\rho^d) \circ (V \times V^*) \\ &= \varepsilon(\gamma,\rho^d) \circ (V^* \times \mathbb{1}_{\rho^d}) \circ (\mathbb{1}_{\rho^d} \times V) \\ &= (\mathbb{1}_{\rho^d} \times V^*) \circ \varepsilon(\rho^d,\rho^d) \circ (\mathbb{1}_{\rho^d} \times V) \\ &= \rho^d(V^*)\varepsilon(\rho^d,\rho^d)\rho^d(V) \end{aligned}$$

and hence

$$\phi_\gamma(\varepsilon_\gamma) = d^{-d} V^* \phi^d(\varepsilon(\rho^d,\rho^d))V \quad .$$

Now, $\varepsilon(\rho^d,\rho^d) = \varepsilon(p)$ with

$$p = \prod_{k=1}^{d} (k\, d + k) \quad .$$

Hence, $\phi^d(\varepsilon(p)) = d^{-d}\mathbb{1}$, thus $\phi_\gamma(\varepsilon_\gamma) = 1$. $\square$

Let now $W \in \mathfrak{A}(\hat{\mathcal{O}})$ be an isometry with $WW^* = E^{\mathrm{a}}_{d-1}$. Then we define a conjugate endomorphism by

$$\overline{\rho}(\,\cdot\,) = \gamma^{-1}(W^*\rho^{d-1}(\,\cdot\,)W) \quad .$$

One calculates with the left inverse

$$\overline{\phi}(\,\cdot\,) = \phi^{d-1}(W\gamma(\,\cdot\,)W^*)(\phi^{d-1}(E^{\mathrm{a}}_{d-1})^{-1} \quad , \qquad \phi^{d-1}(E^{\mathrm{a}}_{d-1}) = d^{2-d}\,\mathbb{1}$$

the statistical dimension of $\overline{\rho}$ to d (calculation as above). Now we have found an isometric intertwiner $R \in (\mathrm{id}, \overline{\rho}\rho)$:

$$R = \gamma^{-1}(W^*V) \quad .$$

For:

$$\begin{aligned} RA = \gamma^{-1}(W^*V)A &= \gamma^{-1}(W^*V\gamma(A)) = \gamma^{-1}(W^*\rho^d(A)V) \\ &= \gamma^{-1}(W^*\rho^d(A)W)\gamma^{-1}(W^*V) \\ &= \overline{\rho}\rho(A)R \quad . \end{aligned}$$

(Here we have used that $WW^* = E^{\mathrm{a}}_{d-1}$, $VV^* = E^{\mathrm{a}}_d$ and $E^{\mathrm{a}}_{d-1}E^{\mathrm{a}}_d = E^{\mathrm{a}}_d$ holds.) A left inverse is now defined by

$$\phi(A) = R^*\overline{\rho}(A)R \quad .$$

We have (also for reducible ρ)

$$\begin{aligned} \phi(\varepsilon_\rho) &= R^*\overline{\rho}(\varepsilon_\rho)R = \gamma^{-1}(V^*WW^*\rho^{d-1}(\varepsilon_\rho)E^{\mathrm{a}}_dV) \\ &= \gamma^{-1}(V^*\rho^{d-1}(\varepsilon_\rho)V) = \gamma^{-1}(V^*E^{\mathrm{a}}_d\rho^{d-1}(\varepsilon_\rho)E^{\mathrm{a}}_dV) \end{aligned}$$

Using $E^{\mathrm{a}}_{d+1} = 0$ one obtains

$$E^{\mathrm{a}}_d\rho^{d-1}(\varepsilon_\rho)E^{\mathrm{a}}_d = \frac{1}{d}E^{\mathrm{a}}_d$$

and hence

$$\phi(\varepsilon_\rho) = \frac{1}{d} \quad .$$

Furthermore $\overline{R} = \varepsilon(\overline{\rho}, \rho)R$ is an isometric intertwiner in $(\mathrm{id}, \rho\overline{\rho})$. We have

PROPOSITION. $\overline{R}^* \rho(R) = \overline{\rho}(\overline{R})^* R = \lambda_\rho \mathbb{1}$.

PROOF. We take $\overline{R}^* \rho(R)$ as an intertwiner in (ρ, ρ). Then

$$\begin{aligned}
\overline{R}^* \rho(R) &= (\overline{R}^* \times \mathbb{1}_\rho) \circ (\mathbb{1}_\rho \times R) \quad , \qquad \Big| \circ \varepsilon(\mathrm{id}, \rho) = 1 \\
&= (\overline{R}^* \times \mathbb{1}_\rho) \circ \varepsilon(\overline{\rho}\rho, \rho) \circ (R \times \mathbb{1}_\rho) \\
&= (\overline{R}^* \times \mathbb{1}_\rho) \circ (\varepsilon(\overline{\rho}, \rho) \times \mathbb{1}_\rho) \circ (\mathbb{1}_{\overline{\rho}} \times \varepsilon(\rho, \rho)) \circ (R \times \mathbb{1}_\rho) \\
&= (R^* \times \mathbb{1}_\rho) \circ (\mathbb{1}_{\overline{\rho}} \times \varepsilon(\rho, \rho)) \circ (R \times \mathbb{1}_\rho) \\
&= R^* \overline{\rho}(\varepsilon(\rho, \rho)) R = \phi(\varepsilon_\rho) = \lambda_\rho \mathbb{1} \quad .
\end{aligned}$$

Furthermore,

$$\lambda_\rho \mathbb{1} = \phi(\lambda_\rho \mathbb{1}) = R^* \overline{\rho}(\overline{R}^* \rho(R)) R = R^* (\overline{\rho}(\overline{R}^*) R) R = \lambda_{\overline{\rho}} \quad .$$

(In the case of braid group statistics we have $\overline{\overline{R}} \neq R$ and hence $R^* \overline{\rho}(\overline{R}) = \overline{\lambda_{\overline{\rho}}}$.) □

Now we can describe the inclusion of $\rho(\mathfrak{A})$ in $\mathfrak{A}$ explicitly, for we have

$$\begin{aligned}
A &= \lambda^{-1} \rho(R)^* \overline{R} A = \lambda^{-1} \rho(R)^* \rho\overline{\rho}(A) \overline{R} \\
&= |\lambda|^{-2} \rho(R)^* \rho\overline{\rho}(A) \overline{R}\overline{R}^* \rho(R) \\
&= d^2 \rho(R^* \overline{\rho}(A)) F \rho(R) \quad , \qquad F = \overline{R}\overline{R}^* \quad .
\end{aligned}$$

$\mathfrak{A}$ is thus generated by F and $\rho(A)$ (as a bimodule over $\rho(\mathfrak{A})$). Because of $\phi(\mathbb{1}) = 1$ holds (for $\overline{\rho}$ irreducible)

$$\phi(F) = d^{-2}$$

and $\phi(A) = R^* \overline{\rho}(A) R$ is the unique left inverse of ρ. We obtain for positive A the estimate for the conditional expectation $\mathcal{E}_\rho = \rho \circ \phi$

$$\begin{aligned}
\mathcal{E}_\rho(A) = \rho\phi(A) &= \rho(R^*) \rho\overline{\rho}(A^{\frac{1}{2}}) \mathbb{1} \rho\overline{\rho}(A^{\frac{1}{2}}) \rho(R) \\
&\geq \rho(R^*) \rho\overline{\rho}(A^{\frac{1}{2}}) F \rho\overline{\rho}(A^{\frac{1}{2}}) \rho(R) \\
&= d^{-2} A \quad ,
\end{aligned}$$

thus we have $d = d_\rho$. As a left module over $\rho(\mathfrak{A})$, $\mathfrak{A}$ possesses the Pimsner-Popo basis $\{d\overline{R}\}$. The formula above can be understood as an expansion in this basis:

$$A = d^2 \mathcal{E}_\rho(A \overline{R}^*) \overline{R} \quad .$$

We have presupposed that ρ is irreducible. Up to now we actually have only used that $\phi(\varepsilon_\rho) = \lambda_\rho \mathbb{1}$. For the DR reconstruction one needs endomorphisms ρ with $\phi(\varepsilon_\rho) = \pm\frac{1}{d}$ and the simple sector γ is the vacuum sector. This is achieved by substituting ρ with $\rho \oplus \overline{\rho}$ and d with $2d$. In the general case, where not all sectors are generated by an irreducible and its conjugate sector, one even needs arbitrarily large finite direct sums. We hence want to concern ourselves now with the case of reducible endomorphisms.

LEMMA. *Let $\rho_1, \rho \in \Delta,\ S, T \in (\rho_1, \rho)$ and ϕ, ϕ_1 be left inverses of ρ, ρ_1, respectively. Then*

(i) $\phi(SAS^*) = \phi_1(A)\phi(SS^*)$,
(ii) $\phi(S\varepsilon_{\rho_1}T^*) = T^*\phi(\varepsilon_\rho)S$.
(iii) *If ρ_1 is irreducible and $\phi(\varepsilon_{\rho_1}) = \lambda_1,\ S^*S = 1,\ SS^* = E$, then*

$$\phi(E\varepsilon_\rho E) = E\phi(\varepsilon_\rho)E = \lambda_1\phi(E)E \quad .$$

PROOF.

(i) ✓,
(ii) $(S \times \mathbb{1}_{\rho_1}) \circ \varepsilon_{\rho_1} \circ (T^* \times \mathbb{1}_{\rho_1}) = (\mathbb{1}_\rho \times T^*) \circ \varepsilon_{\rho_1} \circ (\mathbb{1}_{\rho_1} \times S)$,
(iii) $S^*\phi(\varepsilon_\rho)S = \phi(S\varepsilon_{\rho_1}S^*) = \phi_1(\varepsilon_{\rho_1})\phi(E) = \lambda_1\phi(E)$

$$\rho = \sum V_j\rho_j(\,\cdot\,)V_j^*$$
$$\phi = \sum c_j\phi_j(V_j^* \cdot V_j) \quad , \qquad \sum c_j = 1,\ c_j \geq 0$$
$$\phi(\varepsilon_\rho) = \sum_{j=1}^{m} c_jV_j\phi_j(\varepsilon_{\rho_j})V_j^* \quad .\square$$

3. Doplicher-Roberts Reconstruction

We now have provided all the ingredients to carry through the Doplicher-Roberts reconstruction. We confine ourselves here to the case that there exists an endomorphism $\rho \in \Delta(\mathcal{O})$ with statistical dimension d such that ρ^d contains the vacuum representation (this can always be achieved by adding to a sector the conjugate), and such that all sectors are contained in powers of ρ as subrepresentations.

We want to apply the theorem about the construction of the group from a subalgebra of the Cuntz algebra with a distinguished endomorphism. We consider the inductive limit of the intertwiner spaces

$${}^0\mathcal{O}_\rho^{(k)} = \bigcup_{\substack{n \geq 0 \\ n+k \geq 0}} (\rho^n, \rho^{n+k})$$

with the embedding

$$(\rho^n, \rho^{n+k}) \to (\rho^{n+1}, \rho^{n+1+k})$$
$$T \mapsto T \times \mathbb{1}_\rho \quad .$$

A product beween $T \in {}^0\mathcal{O}_\rho^{(k)}$ and $S \in {}^0\mathcal{O}_\rho^{(l)}$ is defined by choosing representers $T \in (\rho^n, \rho^{n+k}),\ S \in (\rho^{n+k}, \rho^{n+k+l})$, n sufficiently large, and setting

$$ST = S \circ T \quad .$$

This composition is obviously independent of n.

In this way one obtains a C^*-algebra ${}^0\mathcal{O}_\rho$. This contains the statistics operators (in bosonized form) $\varepsilon(p)$ as well as an isometry $S \in (\mathrm{id}, \rho^d)$ with $SS^* = E_d^{\mathrm{a}}$. The endomorphism $\rho \in \Delta(\mathcal{O})$ act in a natural way on the intertwiner spaces

$$(\rho^n, \rho^{n+k}) \to (\rho^{n+1}, \rho^{n+k+1})$$
$$T \mapsto \mathbb{1}_\rho \times T = \rho(T) \quad .$$

Furthermore, $S^*\rho^{d-1}(\,\cdot\,)S$ defines a left inverse of ρ. On has to show now that $\phi(\varepsilon_\rho) = \frac{1}{d}$. Then

$$S^*\rho(S) = \overline{S}^*\varepsilon(\rho,\rho^{d-1})\rho(S) = (-1)^{d-1}\overline{S}^*\rho(S) = d^{-1}(-1)^{d-1} \quad .$$

The C^*-algebra generated by $\varepsilon(p)$ and $\rho^n(S)$ is isomorphic to $\mathcal{O}_{\mathrm{SU}(d)}$. ${}^0\mathcal{O}_\rho$ has a unique C^*-norm and is simple. We are now able to use the theorem of Doplicher and Roberts and to identify $\mathcal{O}_\rho$, which is the normal closure of ${}^0\mathcal{O}_\rho$ with a subalgebra $\mathcal{O}_G \subset \mathcal{O}_d$, where G is an up to conjugation unique closed subgroup of SU(d).

One now considers the $*$-algebra generated by $\mathfrak{A}$ and $\mathcal{O}_d$ with the relations

$$\mathcal{O}_\rho \leftrightarrow \mathcal{O}_G$$
$$\psi_i A = \rho(A)\psi_i \quad .$$

It possesses a unique C^*-norm, and one obtains a unique net of field algebras $\mathcal{F}(\mathcal{O})$ with $\mathcal{F}(\mathcal{O})^G = \mathfrak{A}(\mathcal{O})$.