Seminar on Mathematical Aspects of Theoretical Physics -Partial differential equations II-

Handout for the talk by David Klein - 11 June, 2012
References:
C. Bär, N. Ginoux, F. Pfaeffle - Wave Equations on Lorentzian Manifolds and Quantization [1]
C. Bär, K. Fredenhagen - Quantum Field Theory on Curved Spacetimes [2]

1 Introduction

We are interested in techniques which deal with linear wave equations on Lorentzian manifolds.

Recall 1. The wave equation in $\mathbb{R} \times \mathbb{R}^3$ is given by

$$\Box u = 0\,, \quad \Box := \partial_t^2 - \sum_{i=1}^{3} \partial_{x_i}^2 \tag{1}$$

with $\Box$ denoting the d'Alembert operator.

We observe the following properties of the solutions to 1: $\Box$ is linear, therefore the kernel of $\Box$ is a vector space. Possible solutions to 1 are: $(t,x) \mapsto \cos(nt)\cos(nx_1)$, $n \in \mathbb{Z}$, therefore the kernel of $\Box$ is of infinite dimensions.

However from our intuition we can see, that for each height and speed of a specific wave, there has to be a unique solution, corresponding to this solution.

The aim of this talk is, to prove the existance and uniqueness of solutions to the wave equation not on Minkowski space, but on the more generalized setting of a Lorentzian manifold. From the talk Partial differential equations I we know, that for the search for a solution to a given PDE, we should focus on the search for the fundamental solution to the given differential operator. For this, we need to generalize several mathematical concepts.

In the following, (M^n, g) will denote an n-dim. Lorentzian manifold.

Def. 1.1. A **generalized d'Alembertian** P on M is a linear differential operator of second order whose principal symbol is given by minus the metric. In the scalar setting (which we will keep throughout this talk), P can be written in local coordinates by

$$P = -\sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x)\frac{\partial}{\partial x^j} + B_1(x)$$

where A_j and B_1 are matrix-valued coefficients depending smoothly on x and g^{ij} is the inverse matrix of g_{ij} with $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$.

Example 1.2. The d'Alembertian acts on smooth functions on (M^n, g) by $\Box f := -\mathrm{tr}_g(\mathrm{Hess}(f))$, where $\mathrm{Hess}(X,Y) := \langle \nabla_X \mathrm{grad} f, Y \rangle$. In normal coordinates,

$$\Box f = -\mu_x^{-1} \sum_{j=0}^{n-1} \partial_{x_j}(\mu_x (\mathrm{grad} f)_j)$$

with $\mu_x := |\det((g_{ij})_{ij})|^{\frac{1}{2}}$. Therefore, the principal symbol is given by minus the metric.

Def. 1.3. Let P be a generalized d'Alembertian in the setting above. The **generalized wave equation** associated with P is

$$Pu = f$$

with $f \in C^\infty(M, \mathbb{K})$

2 Distributions and Fundamental Solutions

Def. 2.1. The space of $\mathbb{K}$-valued distributions on M is defined by

$$\mathcal{D}'(M,\mathbb{K}) := \{T\colon \mathcal{D}(M,\mathbb{K}) \to \mathbb{K}, \text{linear and continuous}\}$$

where $\mathcal{D}(M,\mathbb{K}) := \{\varphi \in C^\infty(M,\mathbb{K}), \operatorname{supp}(\varphi) \text{ is compact}\}$ denotes the space of $\mathbb{K}$-valued **test functions** on M.

How do differential operators act on distributions? Given $T \in \mathcal{D}'(M,\mathbb{K})$ and a linear differential operator P on M, one can define

$$PT[\varphi] := T[P^*\varphi]$$

with $\varphi \in \mathcal{D}(M,\mathbb{K})$ and P^* being the formal adjoint of P, i.e. on a Hilbert-space it fulfills $\langle Pu, v\rangle = \langle u, P^*v\rangle$.

How can functions be understood as distributions? For a fixed $f \in C^\infty(M,\mathbb{K})$ the map (which is called again f)

$$\varphi \mapsto \int_M f(x)\varphi(x)\mathrm{d}x$$

defines a $\mathbb{K}$-valued distribution on M.

Def. 2.2. Let P be a generalized d'Alembertian in the setting above and $x \in M$. A **fundamental solution** for P at x on M is a distribution $F \in \mathcal{D}'(M,\mathbb{K})$ with

$$PF = \delta_x \,, \tag{2}$$

where δ_x denotes the Dirac distribution in x, i.e. $\delta_x[\varphi] = \varphi(x)$, $\forall\varphi \in \mathcal{D}(M,\mathbb{K})$

After this short recall, we focus on finding fundamental solutions to generalized d'Alembertians on Lorentzian manifolds.

3 Riesz Distributions on Minkowski space

Our aim will be first to describe a fundamental solution to the generalizes d'Alembertian at the origin 0 on the Minkoswki space $(\mathbb{R}^n, \gamma)$, $\gamma := \langle\langle\cdot,\cdot\rangle\rangle_0$.

Def. 3.1. For any complex number α with $\mathfrak{Re}(\alpha) > n$ let $R_+(\alpha)$ and $R_-(\alpha)$ be the functions defined on $\mathbb{R}^n$ by

$$R_\pm(\alpha)(X) := \begin{cases} C(\alpha,n)\gamma(X)^{\frac{\alpha-n}{2}}, & \text{if } X \in J_\pm(0), \\ 0, & \text{otherwise,} \end{cases}$$

where $C(\alpha,n) := \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{(\frac{\alpha}{2}-1)!(\frac{\alpha-n}{2})!}$, $J_\pm(0)$ being the causal future or past of 0 and $z \mapsto (z-1)!$ is the Gamma function.

Lemma 3.2. For all $\alpha \in \mathbb{C}$ with $\mathfrak{Re}(\alpha) > n$ we have

$$\Box R_\pm(\alpha+2) = R_\pm(\alpha) \tag{3}$$

In particular, the map $\alpha \mapsto R_\pm(\alpha)$, $\{\mathfrak{Re}(\alpha) > n\} \to \mathcal{D}'(\mathbb{R}^n,\mathbb{C})$ can be holomorphically extended on $\mathbb{C}$.

Proof 3.3. We first proof the following identity: $\gamma \cdot R_\pm(\alpha) = \alpha(\alpha - n + 2)R_\pm(\alpha+2)$. It follows from

$$\frac{C(\alpha,n)}{C(\alpha+2,n)} = \frac{2^{(1-\alpha)}\ (\frac{\alpha+2}{2}-1)!\ (\frac{\alpha+2-n}{2})!}{2^{(1-\alpha-2)}\ (\frac{\alpha}{2}-1)!\ (\frac{\alpha-n}{2})!} = \alpha\,(\alpha-n+2).$$

Second, we proove the identity $(\operatorname{grad}\gamma) \cdot R_\pm(\alpha) = 2\alpha \operatorname{grad} R_\pm(\alpha+2)$: We choose a Lorentzian orthonormal basis $e_1, \dots, e_n$ of V and we denote differentiation in direction e_i by ∂_i. We fix a testfunction φ and integrate by parts:

$$\begin{aligned}
\partial_i \gamma \cdot R_\pm(\alpha)[\varphi] &= C(\alpha,n) \int_{J_\pm(0)} \gamma(X)^{\frac{\alpha-n}{2}} \partial_i \gamma(X) \varphi(X)\, dX \\
&= \frac{2C(\alpha,n)}{\alpha+2-n} \int_{J_\pm(0)} \partial_i (\gamma(X)^{\frac{\alpha-n+2}{2}}) \varphi(X)\, dX \\
&= -2\alpha C(\alpha+2,n) \int_{J_\pm(0)} \gamma(X)^{\frac{\alpha-n+2}{2}} \partial_i \varphi(X)\, dX \\
&= -2\alpha R_\pm(\alpha+2)[\partial_i \varphi] \\
&= 2\alpha \partial_i R_\pm(\alpha+2)[\varphi],
\end{aligned}$$

From the last identity it follows that

$$\begin{aligned}
\partial_i^2 R_\pm(\alpha+2) &= \partial_i \left(\frac{1}{2\alpha} \partial_i \gamma \cdot R_\pm(\alpha) \right) \\
&= \frac{1}{2\alpha} \left(\partial_i^2 \gamma \cdot R_\pm(\alpha) + \partial_i \gamma \cdot \left(\frac{1}{2(\alpha-2)} \partial_i \gamma \cdot R_\pm(\alpha-2) \right) \right) \\
&= \frac{1}{2\alpha} \partial_i^2 \gamma \cdot R_\pm(\alpha) + \frac{1}{4\alpha(\alpha-2)} (\partial_i \gamma)^2 \frac{(\alpha-2)(\alpha-n)}{\gamma} \cdot R_\pm(\alpha) \\
&= \left(\frac{1}{2\alpha} \partial_i^2 \gamma + \frac{\alpha-n}{4\alpha} \cdot \frac{(\partial_i \gamma)^2}{\gamma} \right) \cdot R_\pm(\alpha)\,.
\end{aligned}$$

Putting these pieces together, we conclude

$$\square\, R_\pm(\alpha+2) = \left(\frac{n}{\alpha} + \frac{\alpha-n}{4\alpha} \cdot \frac{4\gamma}{\gamma} \right) R_\pm(\alpha) = R_\pm(\alpha).$$

□

Def. 3.4. We call $R_\pm(\alpha)$ the **advanced (resp. retarded) Riesz distribution** on $\mathbb{R}^n$ for $\alpha \in \mathbb{C}$.

Lemma 3.5. The Riesz distribution satisfies

1. For any $\alpha \in \mathbb{C}$ one has $\operatorname{supp}(R_\pm(\alpha)) \subset J_\pm(0)$
2. $R_\pm(0) = \delta_0$

Corollary 3.6. $R_\pm(2)$ satisfies $\square R_\pm(2) = \delta_0$, and $\operatorname{supp}(R_\pm(2)) \subset J_\pm(0)$.

Therefore, $R_\pm(2)$ is an advanced (resp. retarded) fundamental solution to $\square$ in the origin.

4 Local fundamental solution

We took the first step on the way to a fundamental solution to a generalized d'Alembertian on a Lorentzian manifold, by finding the Riesz distribution and it's properties on Minkowski space. We will make use of this distribution to take the second step. But how exactly can we construct a solution out of it? Locally, we can try to pull the Riesz distribution back from the tangent space at a point onto a neighbourhood of that point.

4.1 First attempt

Def. 4.1. Let Ω be a geodesically starshaped neighbourhood of a point x in a Lorentzian manifold (M^n, g). Let $\exp_x \colon \exp^{-1}(\Omega) \to \Omega$ be the exponential map and $\mu_x := |\det((g_{ij})_{ij})|^{\frac{1}{2}}$. We define the Riesz distribution at x on Ω to the parameter $\alpha \in \mathbb{C}$ by

$$R_\pm^\Omega(\alpha, x) \colon \mathcal{D}(\Omega, \mathbb{C}) \to \mathbb{C}\,, \quad \varphi \mapsto R_\pm(\alpha)[(\mu_x \varphi) \circ \exp_x]$$

Where μ_x takes into account the difference between the volumeforms of M and $T_x M$.

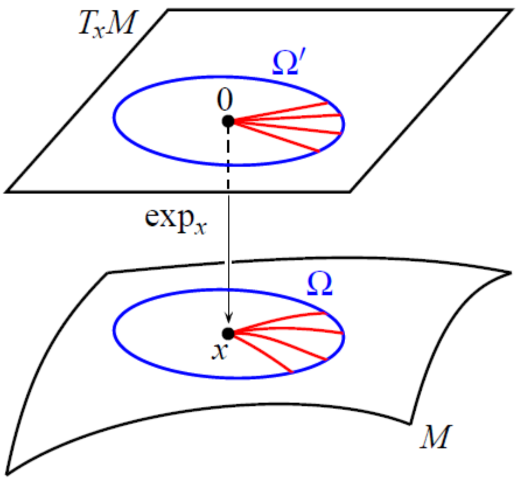

Figure 1: Ω is geodesically starshaped w.r.t. x. Picture source: [1]

Lemma 4.2. In the given setting, the Riesz distribution at x on Ω satisfies

1. $R^{\Omega}_{\pm}(0, x) = \delta_x$
2. $\mathrm{supp}(R^{\Omega}_{\pm}(\alpha, x)) \subset J^{\Omega}_{\pm}(x)$
3. $\Box R^{\Omega}_{\pm}(\alpha + 2, x) = \left(\frac{\Box\Gamma_x - 2n}{2\alpha} + 1\right) R^{\Omega}_{\pm}(\alpha, x)$

With $\Gamma_x := \gamma \circ \exp_x^{-1} : \Omega \to \mathbb{R}$.

We observe that (1) and (2) makes $R^{\Omega}_{\pm}(2, x)$ a promising candidate for a fundamental solution for $\Box$ at x, but (3) let this attempt fail because $\Box\Gamma_x - 2n$ does not vanish in general.

4.2 Formal Ansatz

In the given setting, we look for a fundamental solution of the form

$$T_{\pm}(x) := \sum_{k=0}^{\infty} V_x^k R^{\Omega}_{\pm}(2 + 2k, x)$$

where for each k, V_x^k is a smooth coefficient depending on x. This series is only formal but by plugging it into the equation $PT_{\pm} = \delta_x$ and using previously seen properties of the Riesz distribution, we deduce for the coefficients the following:

$$\nabla_{\mathrm{grad}\,\Gamma_x} V_x^k - \left(\tfrac{1}{2}\Box\Gamma_x - n + 2k\right) V_x^k = 2k\, P V_x^{k-1} \tag{4}$$

for every $k \geq 1$ as well as $V^0_{x,x} = 1$.

Def. 4.3. Let $\Omega \subset M$ be convex. A **sequence of Hadamard coefficients** for P on Ω is a sequence $(V^k)_{k \geq 0}$ of $C^{\infty}(\Omega \times \Omega, \mathbb{C})$ which fulfills 4 and $V^0_{x,x} = 1$, for all $x \in \Omega$ and $k \geq 1$, where we denote by $V_x^k := V^k_{x,\cdot} \in C^{\infty}(\Omega, \mathbb{C})$.

Γ_x denotes the parallel-transport of x which exists, because we have choosen our neighbourhood Ω to be geodesically starshaped. Therefore, 4 is also called *transport equation* and we benefit from it since it turns out to be a single differential equation which can be solved without any further assumptions.

For simplicity we consider from now on a generalized d'Alembertian P, which has no first order term. This is because such a term would involve the parallel transport of the connection which is canonically associated with P.

Proposition 4.4. Let Ω be a convex open subset in a Lorentzian manifold (M^n, g) and P be a generalized d'Alembertian on M of the form $P = \square + b$, $b \in C^\infty(M, \mathbb{K})$. Then there exists an unique sequence of Hadamard coeff. for P on Ω. It is given for all $x, y \in \Omega$ by $V^0_{x,y} = \mu_x^{\frac{1}{2}}(y)$ and, for all $k \geq 1$

$$V^k_{x,y} = -k\mu_x^{\frac{1}{2}}(y) \int_0^1 \mu_x^{\frac{1}{2}}(\Phi(y,s))s^{k-1} \cdot (P_{(2)}V_x^{k-1}(\Phi(y,s)))\mathrm{d}s \quad ,$$

where $\Phi(y,s) := \exp_x(s\exp_x^{-1}(y))$, $\Phi\colon \Omega \times [0,1] \to \Omega$. The index "(2)" in $P_{(2)}V^{k-1}$ stands for P acting on $z \mapsto V^{k-1}(x,z)$.

This leads us to the following Definition.

Def. 4.5. Let Ω and P be given as above. Let $(V^k)_{k\geq 1}$ be the sequence of Hadamard coefficients for P on Ω. The **advanced (resp. retarded) formal fundamental solution** for P at $x \in \Omega$ is the formal series

$$R^\Omega_\pm(x) = \sum_{k=0}^{\infty} V^k_x R^\Omega_\pm(2+2k, x) \tag{5}$$

4.3 Exact local fundamental solution

The existance of the Hadamard coefficients still does not provide any (local) fundamental solution, since the series 5 may diverge.

The idea is now to keep the first term of the formal fundamental solution unchanged, while multiplying the higher terms by a cutoff function.

More precisely, let Ω' be a convex open subset in M. Let $\sigma\colon \mathbb{R} \to [0,1]$ be a smooth function with $\mathrm{supp}(\sigma) \subset [-1,1]$ and $\sigma|_{[-\frac{1}{2},\frac{1}{2}]} = 1$. Fix an integer $N \geq \frac{n}{2}$ to insure that $R^{\Omega'}_\pm(2+2k, x)$ is continuous for any $k \geq N$, and a sequence $(\epsilon)_{j\geq N}$ of real positive numbers.

Set

$$\tilde{R}_\pm(x) := \sum_{j=0}^{N-1} V^j_x \cdot R^{\Omega'}_\pm(2+2j,x) + \sum_{j=N}^{\infty} \sigma(\frac{\Gamma_x}{\epsilon_j}) V^j_x \cdot R^{\Omega'}_\pm(2+2j,x)$$

for every $x \in \Omega$, with $\Gamma_x := \gamma \circ \exp_x^{-1}$, $\gamma := -\langle\langle \cdot, \cdot \rangle\rangle_0$.

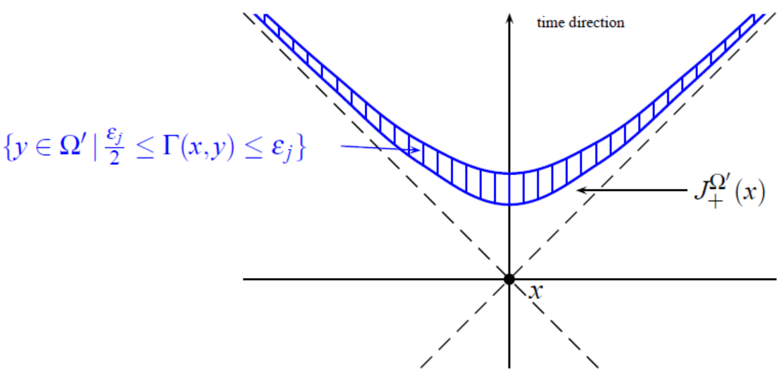

Figure 2: The cutoff function and Γ "squeeze" the support into the blue area. Picture source: [1].

Remark 4.6. This does not a priori define a fundamental solution, since it does not even define a distribution. The tactic here is that for ϵ_j small enough, both conditions are almost fulfilled.

Proposition 4.7. Given Ω' as above and $\Omega \subset\subset \Omega'$ relatively compact. Fix an integer $N \geq \frac{n}{2}$, then there exists a sequence $(\epsilon)_{j \geq N}$ of positive real numbers such that for all $x \in \bar{\Omega}$, $\tilde{R}_{\pm}(x)$ defines a distribution on Ω satisfying

1. $P_{(2)}\tilde{R}_{\pm}(x) - \delta_x = K_{\pm}(x, \cdot)$, where $K_{\pm} \in C^{\infty}(\bar{\Omega} \times \bar{\Omega}, \mathbb{C})$
2. $\operatorname{supp}(\tilde{R}_{\pm}(x)) \subset J_{\pm}^{\Omega'}(x)$
3. $y \mapsto \tilde{R}_{\pm}(y)[\varphi]$ is smooth on Ω for all $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$

In other words: choosing suitably ϵ leads to a distribution depending smoothly on the base point and which is nearly a fundamental solution in the sense that the difference $P_{(2)}\tilde{R}_{\pm}(x) - \delta_x$ is a smooth function. How can we obtain now a "true" solution out of this? The idea is to get rid of the error term by using methods of functional analysis. We set

$$\mathcal{K}_{\pm}(u) := \int_{\bar{\Omega}} K_{\pm}(\cdot, y)u(y)\mathrm{d}y$$

It follows, that (1) of the last proposition can be written as

$$P_{(2)}\tilde{R}_{\pm}(\cdot)[\varphi] = (\mathrm{Id} + \mathcal{K}_{\pm})\varphi$$

So we should look for an inverse to the operator $(\mathrm{Id} + \mathcal{K}\pm)$. For any given bounded endomorhpism A of a Banach space, $(\mathrm{Id} + A)$ is invertible as soon as $\|A\| < 1$.

Proposition 4.8. Let $\Omega \subset\subset \Omega'$ be a relatively compact causal domain in Ω' and assume that

$$Vol(\bar{\Omega}) \cdot \|K_{\pm}\|_{C^0(\bar{\Omega} \times \bar{\Omega})} < 1 \quad . \tag{6}$$

While $\|\cdot\|$ denotes the maximum norm. In particular, $K_{\pm}$ scales like the volume of the subset Ω. Since we are free to choose the neighbourhood, we can choose it small enough that $(\mathrm{Id} + \mathcal{K}_{\pm})$ becomes an isomorphism for all $k \in \mathbb{N}$ and is therefore invertible.

Setting

$$F_{\pm}^{\Omega}(\cdot)[\varphi] := (\mathrm{Id} + \mathcal{K}_{\pm})^{-1}(y \mapsto \tilde{R}_{\pm}(y)[\varphi])$$

for all $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$ we obtain the main result of this talk, namely the exact (local) fundamental solution which we will proof.

$$\begin{aligned}
(PF_{\pm}^{\Omega}(x))[\varphi] &= F_{\pm}^{\Omega}(x)[P^*\varphi] \\
&= \{(\mathrm{Id} + \mathcal{K}_{\pm})^{-1}(y \mapsto \tilde{R}_{\pm}(y)[P^*\varphi])\}(x) \\
&= \{(\mathrm{Id} + \mathcal{K}_{\pm})^{-1}\underbrace{(y \mapsto P_{(2)}\tilde{R}_{\pm}(y)[\varphi])\}}_{(\mathrm{Id}+\mathcal{K}_{\pm})\varphi}(x) \\
&= \varphi(x)
\end{aligned}$$

Therefore, we finally we arrived at

$$PF_{\pm}^{\Omega}(x) = \delta_x \tag{7}$$

which leaves us with the following conclusion:

Corollary 4.9. Let P be a generalized d'Alembertian on a Lorentzian manifold (M^n, g). Then every point on M posesses a relatively compact causal neighbourhood Ω such that for every $x \in \Omega$, there exist fundamental solutions $F_{\pm}^{\Omega}(x)$ on Ω for P at x satisfying

1. $\operatorname{supp}(F_{\pm}^{\Omega}(x)) \subset J_{\pm}^{\Omega}(x)$
2. $x \mapsto F_{\pm}^{\Omega}(x)[\varphi]$ is smooth for all $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$

5 Global fundamental solution

We want to construct now a global fundamental solution. A first idea could be to take the fundamental solutions constructed in the previous chapter and glue them together. There arises a problem namely, which equations should be solved in each coordinate patch not containing the point at which the fundamental solution is sought after? It becomes clear, that the global topology of the manifold could set up some serious problems. Therefore we restrict our search on a "nice" class of manifolds, i.e. on globally hyperbolic manifolds, even if uniqueness and existance of the results can be extended to a broader class of spacetimes.

The technique can be understood step by step: We first start by solving the Cauchy problem which provides us with a local-to-global construction. Afterwards we will use this to extend our local fundamental solution to globally hyperbolic manifolds.

5.1 Results from the Cauchy problem

Recall 2. Let P be a generalized d'Alembertian on a globally hyperbolic spacetime (M^n, g) and $S \subset M$ be a (smooth) spacelike hypersurface with unit tangent vector field ν. Let $f \in C^\infty(M, \mathbb{K})$ and $u_0, u_1 \in C^\infty(S, \mathbb{K})$. The **Cauchy problem** for P with Cauchy data (f, u_0, u_1) is the system of equations

$$\begin{aligned} Pu &= f \text{ on M} \\ u|_s &= u_0 \\ \partial_\nu u &= u_1 \text{ on S .} \end{aligned}$$

The main results of this analysis are the following

Proposition 5.1. Under the assumption of 4.8, there exists for every $v \in \mathcal{D}(\Omega, \mathbb{C})$ a function $u_\pm \in C^\infty(\Omega, \mathbb{C})$ such that

$$\begin{aligned} Pu_\pm &= v \\ \operatorname{supp}(u_\pm) &\subset J_\pm^\Omega(\operatorname{supp}(v)) \end{aligned}$$

Proof 5.2. *Sketch*: We use our construction of $F_\pm^\Omega$ to set

$$u_\pm[\varphi] := \int_\Omega v(x) F_\pm^\Omega(x)[\varphi] dx\,, \quad \text{for every } \varphi \in \mathcal{D}(\Omega, \mathbb{C}).$$

We need to show first the support condition, second, that $u_\pm$ is a solution of $Pu_\pm = v$, and last, that $u_\pm$ is in fact a smooth section. For this, see [2]. □

Corollary 5.3. Let P be given as above. Then there exists at most one advanced (resp. retarded) fundamental solution for P in x.

Theorem 5.4. Let (M^n, g), S and ν be given as above. Then for each open subset Ω of M satisfying the properties of 4.8 and such that $S \cap \Omega$ is a Cauchy hypersurface of Ω, the following holds: For all $u_0, u_1 \in \mathcal{D}(S \cap \Omega, \mathbb{C})$ and each $f \in \mathcal{D}(\Omega, \mathbb{C})$ there exists a unique $u \in C^\infty(\Omega, \mathbb{C})$ with

$$\begin{aligned} Pu &= f \\ u|_S &= u_0 \\ \partial_\nu u &= u_1 \quad . \end{aligned}$$

Furthermore, $\operatorname{supp}(u) \subset J_+^\Omega(K) \cup J_-^\Omega(K)$, where $K := \operatorname{supp}(u_0) \cup \operatorname{supp}(u_1) \cup \operatorname{supp}(f)$.

Proof 5.5. This is proven by using 5.1. $\square$

Theorem 5.6. Let P, M, S and ν be given as above.

1. For all $(f, u_0, u_1) \in \mathcal{D}(M, \mathbb{C}) \oplus \mathcal{D}(S, \mathbb{C}) \oplus \mathcal{D}(S, \mathbb{C})$ there exists a unique $u \in C^\infty(\Omega, \mathbb{C})$ such that

$$\begin{aligned} Pu &= f \\ u|_S &= u_0 \\ \partial_\nu u &= u_1 \quad . \end{aligned}$$

   Moreover, $\mathrm{supp}(u) \subset J_+^\Omega(K) \cup J_-^\Omega(K)$, with $K := \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

2. The map $\mathcal{D}(M, \mathbb{C}) \oplus \mathcal{D}(S, \mathbb{C}) \oplus \mathcal{D}(S, \mathbb{C}) \to C^\infty(M, \mathbb{C})$, $(f, u_0, u_1) \mapsto u$, where u is the solution of (1), is linear continuous.

Proof 5.7. This is proven by using 5.4. $\square$

5.2 Global existance of a fundamental solution

We are now able to put the pieces of the puzzle together and recieve

Theorem 5.8. Let P be a generalized d'Alembertian on a globally hyperbolic manifold spacetime M. Then there exists for each $x \in M$ a unique fundamental solution $F_+(x)$ with past compact support, and $F_-(x)$ with future compact support for P at x. They satisfy

1. $\mathrm{supp}(F_\pm(x)) \subset J_\pm^\mu(x)$
2. for every $\varphi \in \mathcal{D}(M, \mathbb{C})$ the map $M \to \mathbb{C}$, $x \mapsto F_\pm(x)[\varphi]$ is a smooth function with

$$P^*(x \mapsto F_\pm(x)[\varphi]) = \varphi$$

Proof 5.9. This Theorem is proven by adding 4.8, 5.1 and 5.4. $\square$

In conclusion, the wave equation $Pu = f$ with $f \in \mathcal{D}(M, \mathbb{C})$ posesses a unique solution $u_\pm \in C^\infty(M, \mathbb{C})$ with $\mathrm{supp}(u_\pm) \subset J_\pm^\mu(\mathrm{supp}(f))$ or equivalently with $\mathrm{supp}(u_\pm)$ being past (resp. future) compact on a globally hyperbolical spacetime M.