Local retarded off-shell intertwiners of covariant phase spaces – towards a nonperturbative construction

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Motivation and Setup

In this talk, we shall be interested in the following problem: consider (for concreteness; watch out for Final Considerations)...

- A scalar field $\phi \in \mathscr{C}^\infty(\mathscr{M})$ in a globally hyperbolic spacetime $(\mathscr{M}, g)$, and
- Two (1st-order) action functionals

$$S_i[\phi] = \int_{\mathscr{M}} \sqrt{|\det g(x)|}\mathrm{d}x \mathscr{L}_i(x, \phi(x), \partial^1\phi(x)),\ i = 1, 2$$

with (semilinear, strictly hyperbolic) Euler-Lagrange derivatives

$$S_{i(1)}[\phi] = \nabla_a \frac{\partial \mathscr{L}_i}{\partial \nabla_a \phi} - \frac{\partial \mathscr{L}_i}{\partial \phi},$$

such that $S_2 \doteq S$ is quadratic ("free") and $S_1 - S_2 = \lambda F(h) = \lambda \int_{\mathscr{M}} \sqrt{|\det g(x)|}\mathrm{d}x h(x)\mathscr{L}_{int}(x, \phi(x), \partial^1\phi(x))$ with $h \in \mathscr{C}^\infty_c(\mathscr{M})$ ("spacetime-cutoff" interaction term), $\lambda > 0$ and $F(h)_{(1)}[\phi]$ depends pointwise on ϕ and at most its first derivatives $\nabla\phi$, with $F(h)_{(1)}[0] = 0$.

We want to...

Main Goal & Definition

Prove the existence of a map $r_{S_1,S_2} : \mathscr{C}^\infty(\mathscr{M}) \to \mathscr{C}^\infty(\mathscr{M})$ such that

$$\begin{aligned} S_{1(1)} \circ r_{S_1,S_2} &= S_{2(1)}, & (1) \\ r_{S_1,S_2}(\phi)(x) &= \phi(x), x \notin J^+(\mathrm{supp} h). & (2) \end{aligned}$$

We call r_{S_1,S_2} the retarded Møller operator of S_1 w.r.t. S_2.

- When acting on solutions of $S_{2(1))}[\phi] = 0$ r_{S_1,S_2} can be seen as an intertwiner of (on-shell) covariant phase spaces or, equivalently, as the solution of a "covariant" Cauchy problem. Moreover, it formally satisfies $r_{S,S} = \mathbb{1}$ and $r_{S_1,S_3} = r_{S_1,S_2} \circ r_{S_2,S_3}$.
- (1)–(2) also mean that $r_{S_1,S_2}(\phi)$ solves an inhomogeneous (off-shell) nonlinear hyperbolic PDE with prescribed initial conditions in the past of supp$h$ $\Rightarrow$ very few rigorous well-posedness results exist!
- r_{S_1,S_2} appears naturally in the context of perturbative algebraic QFT (Dütsch–Fredenhagen CMP'03, Brunetti–Fredenhagen arXiv:0901.2063, Brunetti–Dütsch–Fredenhagen arXiv:0901.2038), where h plays both the role of an IR regulator and of a localization for the algebra of perturbative interacting fields.

Coupling as an off-shell flow parameter ⇒ Main Claim

- It's clear that r_{S_1,S_2} exist on shell whenever local well posedness for $S_{1(1)}[\psi] = (S + \lambda F(h))_{(1)}[\psi] = 0$ in a ngb. of supp$h$ holds. More in general, in the future of supp$h$ (1) tells us that $\psi = r_{S_1,S_2}(\phi) - \phi$ solves $S_{(1)}[\psi] = 0$ ⇒ finding r_{S_1,S_2} boils down to finding it locally!
- Differentiating (1) w.r.t. λ leads to

$$(S_{(1)}+\lambda F(h)_{(1)})^{(1)}[r_{S+\lambda F(h),S}(\phi)]\circ \frac{\mathrm{d}}{\mathrm{d}\lambda} r_{S+\lambda F(h),S}(\phi)+F_{(1)}[r_{S+\lambda F(h),S}(\phi)] = 0. \quad (3)$$

Now invoking (2) and applying the retarded fundamental solution $\Delta^R_{S+\lambda F(h)}[r_{S+\lambda F(h),S}(\phi)]$ of the linearised Euler-Lagrange operator $(S + \lambda F(h))^{(1)}_{(1)}[r_{S+\lambda F(h),S}(\phi)]$ around the background $r_{S+\lambda F(h),S}(\phi)$ to the left of both sides of (3) (notice that no other choice is possible!), we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} r_{S+\lambda F(h),S}(\phi) = -\Delta^R_{S+\lambda F(h)}[r_{S+\lambda F(h),S}(\phi)] \circ F_{(1)}(h)[r_{S+\lambda F(h),S}(\phi)], \quad (4)$$

which shows that $\psi(\lambda) \doteq r_{S+\lambda F(h),S}(\phi)$ is the unique solution of the flow equation (4) with initial condition $\psi(0) = \phi$.

- Formally integrating (4) w.r.t. λ on both sides and using the initial condition above, we arrive at

$$r_{S+\lambda F(h),S}(\phi) = \phi - \int_0^\lambda \mathrm{d}\lambda' \Delta^R_{S+\lambda' F(h)}[r_{S+\lambda' F(h),S}(\phi)] \circ F_{(1)}(h)[r_{S+\lambda' F(h),S}(\phi)]. \tag{5}$$

- We could keep proceeding formally by using (4) and write $r_{S+\lambda F(h),S}(\phi)$ as a formal power series of (n-fold) retarded products [Dütsch–Fredenhagen *ibid.*]; however, our nonperturbative aim is achieved by looking at the map

$$\psi(\lambda) \mapsto \phi(\lambda) = \psi(\lambda) + \int_0^\lambda \mathrm{d}\lambda' \Delta^R_{S+\lambda' F(h)}[\psi(\lambda')] \circ F_{(1)}(h)[\psi(\lambda')], \tag{6}$$

which just defines the inverse $r^{-1}_{S+\lambda F(h),S}$ of $r_{S+\lambda F(h),S}$.

Main Claim

The map (6) is invertible in a neighbourhood of zero in $\mathscr{C}^1([0,\Lambda],\mathscr{C}^\infty(\mathscr{M}))$; its inverse satisfies (1), (2).

Towards a proof of Main Claim

- It feels tempting to apply a fixed-point strategy to (1); however, our argument will show that this path is not tenable off shell!
- The central step in our proof is to obtain a priori estimates on $\Delta^R_{S+\lambda F(h)}[\psi]$ in terms of both the linear and the nonlinear (background) arguments. These are essentially refined energy estimates for $S^{(1)}_{(1)} + \lambda F(h)^{(1)}_{(1)}$ which state explicitly their dependence on the latter's coefficients, and were originally obtained by Klainerman [Klainerman '78–'80–'82].
- From now on, for the sake of pedagogy we shall set $(\mathscr{M}, g) = \mathbb{R}^{1,d-1} \ni (x^0 = t, x)$. Suppose that there exist $0 < T$ such that $\mathrm{supp} h$ is contained in the interior of the slab $\{(t,x) : 0 \leq t \leq T\}$, and define the energy norms

$$\|\psi\|_{E^k} \doteq \sup_{t' \in [0,T]} \|\psi(t',.)\|_{H^{(k+1)}_x} + \sup_{t' \in [0,T]} \|\partial_t \psi(t',.)\|_{H^{(k)}_x}.$$

Proposition

For $\phi, \delta\phi \in E^\infty \doteq \{\psi : \|\psi\|_{E^k} < +\infty, \forall k \geq 0\}$ we have

$$\|\Delta^R_{S+\lambda F(h)}[\phi]\delta\phi\|_{E^0} \leq C \sup_{t' \in [0,T]} \|\delta\phi\|_{L^2_x}, \tag{7}$$

$$\|\Delta^R_{S+\lambda F(h)}[\phi]\delta\phi\|_{E^k} \leq C \left(\|\delta\phi\|_{E^{k-1}} + \sup_{t' \in [0,T]} |(hF^{(1)}_{(1)})(t', .)|_{\mathscr{C}^k_x} \|\delta\phi\|_{E^0} \right), \; k \geq 1, \tag{8}$$

where C is a constant which depends only on k, d, T and $\|\phi\|_{\mathscr{C}^1(\mathrm{supp}\,h)}$.

- Applying Sobolev inequalities and Schauder estimates to the spatial $\mathscr{C}^k$ norms of $(hF^{(1)}_{(1)})(t', .)$ in (8), we arrive at

$$\|\Delta^R_{S+\lambda F(h)}[\phi]\delta\phi\|_{E^k} \leq C'(\|\delta\phi\|_{E^{k-1}} + \|\phi\|_{E^{k+1+[\frac{d+1}{2}]}} \|\delta\phi\|_{E^0}), \tag{9}$$

where $[s]$ gives the integer part of s.

Nash–Moser–Hörmander iteration scheme

- A variant of the argument above shows that one loses $1+[\frac{d+1}{2}]$ derivatives at each iteration when trying to solve (1)–(2) by a fixed-point method. This phenomenon has no on-shell counterpart.
- Alternative: use a Newton iteration scheme $\Rightarrow$ if it converges, it does it superexponentially; not the case here, again due to loss of derivatives. This can be fixed by applying suitable smoothing operators that make a "multiscale" decomposition of momentum space at each iteration step. The result is the celebrated

Theorem (Nash–Moser–Hörmander)

Let $\Phi : \mathscr{U} \subseteq E^\infty \cap \{\psi : \|\psi - \psi_0\|_{E^\mu} < R\} \to E^\infty$, $\mu \in \bar{\mathbb{Z}}+$, $R > 0$ be twice Gâteaux differentiable satisfying for all $k \geq 0$ the tame estimates

$$\|\Phi(\psi)\|_{E^k} \leq C(1 + \|\psi\|_{E^{k+r_0}}) \text{ for some } r_0 > 0, \tag{10}$$

$$\|\Phi'(\psi)(\delta\psi)\|_{E^k} \leq C[(1 + \|\psi\|_{E^{k+r_1}})\|\delta\psi\|_{E^{s_1}} + \|\delta\psi\|_{E^{k+s_1}}] \text{ for some } r_1, s_1 > 0, \tag{11}$$

$$\|\Phi''(\psi)(\delta_1\psi,\delta_2\psi)\|_{E^k} \leq C[(1+\|\psi\|_{E^{k+r_2}})\|\delta_1\psi\|_{E^{s_2}}\|\delta_2\psi\|_{E^{t_2}}+\|\delta_1\psi\|_{E^{s_2}}\|\delta_2\psi\|_{E^{k+t_2}}$$
$$+\|\delta_1\psi\|_{E^{k+t_2}}\|\delta_2\psi\|_{E^{s_2}}], \text{ for some } r_2, s_2, t_2 > 0, \tag{12}$$

and such that for all ψ in $\mathscr{V} \subset \{\psi : \|\psi-\psi_0\|_{E^{\mu'}} < R'\}$, $\mu' \in \bar{\mathbb{Z}}+$, $R' > 0$ there is a right inverse $\Psi(\psi)$ to $\Phi'(\psi)$ w.r.t. the linear factor satisfying for all $k \geq 0$ the tame estimates

$$\|\Psi'(\psi)(\delta\psi)\|_{E^k} \leq C[(1+\|\psi\|_{E^{k+a_1}})\|\delta\psi\|_{E^{b_1}} + \|\delta\psi\|_{E^{k+b_1}}] \text{ for some } a_1, b_1 > 0. \tag{13}$$

Then, for all k sufficiently large, there is a $R_k > 0$ such that for all $\phi \in E^\infty$ fulfilling $\|\phi\|_{E^{k+b_1}} < R_k$ the equation $\Phi(\psi) = \Phi(\psi_0) + \phi$ has a unique solution $\psi = \psi(\phi)$ such that $\|\psi(\phi) - \psi_0\|_{E^k} \leq R''\|\phi\|_{E^{k+b_1}}$. In particular, if ϕ also belongs to E^∞, so does $\psi(\phi)$.

In our problem, we take $\psi_0 \equiv 0$ and add a dependence in λ.

Tame (Gâteaux) differentiability of $\Delta^R_{S+\lambda F(h)}[\psi]$

To check that Φ_λ fulfills the hypotheses of the Theorem, first we collect some following formulae coming directly from the definition of a fundamental solution [Dütsch–Fredenhagen *ibid.*]:

$$\Delta^{R(1)}_{S+\lambda F(h)}[\psi](\delta\psi) = -\Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(2)}_{(1)}[\psi](\delta\psi, \Delta^R_{S+\lambda F(h)}[\psi]), \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta^R_{S+\lambda F(h)}[\psi] = -\Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(1)}_{(1)}[\psi] \circ \Delta^R_{S+\lambda F(h)}[\psi], \tag{15}$$

$$\begin{aligned} &\Delta^{R(2)}_{S+\lambda F(h)}[\psi](\delta_1\psi, \delta_2\psi) = \\ &= \Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(2)}_{(1)}(\delta_1\psi, \Delta^R_{S+\lambda F(h)}[\psi]) \circ F(h)^{(2)}_{(1)}(\delta_2\psi, \Delta^R_{S+\lambda F(h)}[\psi]) + \\ &+ \Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(2)}_{(1)}(\delta_2\psi, \Delta^R_{S+\lambda F(h)}[\psi]) \circ F(h)^{(2)}_{(1)}(\delta_1\psi, \Delta^R_{S+\lambda F(h)}[\psi]) + \\ &- \Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(3)}_{(1)}(\delta_1\psi, \delta_2\psi, \Delta^R_{S+\lambda F(h)}[\psi]). \end{aligned} \tag{16}$$

Equation (15) shows in particular that $\Delta^R_{S+\lambda F(h)}[\psi]$ is strongly differentiable (hence strongly continuous) in λ, thus allowing all the computations we need.

Tame estimates for iteration map, end of proof

From (14) and (16), one get the following formulae for the first two derivatives of the iteration map Φ_λ (6):

$$\Phi'_\lambda(\psi)(\delta\psi) = \delta\psi + \\ + \int_0^\lambda \mathrm{d}\lambda' \left(\Delta^R_{S+\lambda' F(h)}[\psi] \circ F(h)^{(1)}_{(1)}[\psi](\delta\psi) + \Delta^{R(1)}_{S+\lambda' F(h)}[\psi](\delta\psi) \circ F(h)_{(1)}[\psi] \right), \tag{17}$$

$$\Phi''_\lambda(\psi)(\delta_1\psi, \delta_2\psi) = \int_0^\lambda \mathrm{d}\lambda' \left(\Delta^R_{S+\lambda' F(h)}[\psi] \circ F(h)^{(2)}_{(1)}[\phi](\delta_1\phi, \delta_2\phi) + \right. \\ + \Delta^{R(1)}_{S+\lambda' F(h)}[\psi](\delta_1\psi) \circ F(h)^{(1)}_{(1)}[\phi](\delta_2\phi) + \Delta^{R(1)}_{S+\lambda' F(h)}[\psi](\delta_2\psi) \circ F(h)^{(1)}_{(1)}[\phi](\delta_1\phi) + \\ \left. + \Delta^{R(2)}_{S+\lambda' F(h)}[\psi](\delta_1\psi, \delta_2\psi) \circ F(h)_{(1)}[\phi] \right), \tag{18}$$

where $\Delta^{R(1)}_{S+\lambda' F(h)}[\psi]$ and $\Delta^{R(2)}_{S+\lambda' F(h)}[\psi]$ are respectively given by (14) and (16). Notice that $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi'_\lambda(\psi)$, seen as a linear map acting on $\delta\psi$ for fixed ψ, doesn't lose derivatives, due to the fact that the assumed loss in $F(h)_{(1)}$ is exactly compensated by the smoothing effect of $\Delta^R_{S+\lambda' F(h)}[\psi]$.

- The Proposition, together with Schauder estimates, show that Φ_λ satisfy the tame estimate (10) with $a_0 = [\frac{d+1}{2}] + 1$ for $\sup_{\lambda' \in [0,\lambda]} \|\psi(\lambda')\|_{E^{[\frac{d+1}{2}]+1}} < R$, that is, $\mu = [\frac{d+1}{2}] + 1$.
- Formulae (17)–(18) show that $\Phi'_\lambda(\psi)(\delta\psi)$ and $\Phi''_\lambda(\psi)(\delta_1\psi, \delta_2\psi)$ fulfill resp. the tame estimates (11) and (12) with $r_1 = r_2 = [\frac{d+1}{2}] + 1$ and $s_1 = s_2 = t_2 = 1$.
- Finally, due to (15) and the remark following (18), $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi'_\lambda(\psi)$ is a bounded and uniformly strongly continuous (in λ) linear map $\Rightarrow \Phi'_\lambda(\psi)$ be inverted by means of a Dyson series. Iterating the tame estimate for $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi'_\lambda(\psi])$, together with the argument for the convergence for the Dyson series, leads to the tame estimate (13) with $a_1 = [\frac{d+1}{2}] + 1$, $b_1 = 1$ and $\mu' = [\frac{d+1}{2}] + 2$ for the right inverse.
- Now... Just plug in the data above, run the "Nash–Moser–Hörmander machine", and we get local existence and uniqueness of $r_{S+\lambda F(h),S}$ in E^∞. The intertwining relation (1) shows that actually $r_{S+\lambda F(h),S}(\phi) \in \mathscr{C}^\infty$ for $\phi \in \mathscr{C}^\infty$. □

Coda: final considerations

- We've shown the existence of r_{S_1,S_2} for "sufficiently small" field configurations around a given one. This latter condition can be controlled in general by adjusting λ (coupling strength) or supp$h$ (lifespan).
- If the Cauchy problem for $S_{1(1)}[\psi] = 0$ is well-posed in the large, one can use the composition property of r_{S_1,S_2} to remove the cutoff (i.e. dependence on h) $\Rightarrow$ probably impossible off shell (?)
- We illustrated our strategy for the case of a scalar field in $\mathbb{R}^{1,d-1}$, but the argument carries through for arbitrary sections in any globally hyperbolic spacetime $\Rightarrow$ one has a local energy estimate of the same form as (7)–(8) by combining Klainerman's argument with the estimates in [Hawking–Ellis '73]; only the control of the extra error terms due to curvature and the absence of Killing fields is more cumbersome.
- The more general quasilinear case (e.g. general relativity) seems to pose, however, some new difficulties, and it'll be the aim of further scrutiny.