Coherent States and the Semi-Classical Einstein Equation

Master thesis

by

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به یاری خدا و توفیق نیک وی، گویم که فن جبر و مقابله فنی است علمی ...

By the help of God and with His precious assistance,
I say that Algebra is a scientific art...
Omar Khayyam (1048-1131)

Contents

Zusammenfassung

In dieser Masterarbeit wird die semiklassische Einstein-Gleichung unter Zuhilfenahme von kohärenten Zuständen für das Klein-Gordon-Feld gelöst. Diese Zustände sind bezüglich eines (verallgemeinerten) Grundzustandes definiert und durch eine klassische Lösung f der Klein-Gordon-Gleichung charakterisiert. Auf einer gegebenen Raumzeit wird die Energiedichte eines allgemeinen kohärenten Zustandes berechnet und dann ein f gesucht so dass die semiklassische Einsteingleichung im entsprechenden kohärenten Zustand gelöst wird. Dieser allgemeine Ansatz wird in drei Fällen untersucht. Zunächst wird der Erwartungswert der Energiedichte des Grundzustandes in einer 3D Torusraumzeit – der Casimir-Effekt – berechnet und dazu verwendet um die semiklassische Einsteingleichung in dieser Raumzeit zu lösen. Dann werden Zustände niedriger Energie auf der de Sitter-Raumzeit untersucht und es wird versucht die semiklassische Einstein-Gleichung mit Hilfe von kohärenten Anregungen dieser Zustände zu lösen; dabei stellt sich heraus, dass eine Lösung nicht existiert. Schließlich wird die semiklassische Einstein-Gleichung auf allgemeinen Robertson-Walker Raumzeiten unter Zuhilfenahme von kohärenten Anregungen eines Zustandes niedriger Energie, unter der Annahme, dass die Energiedichte dieses Zustandes vernachlässigbar ist, gelöst.

Abstract

In this master thesis the semi-classical Einstein equation will be solved by using coherent states for the Klein-Gordon field. Coherent states are defined with respect to a (generalised) ground state and characterised by a classical solution f of the Klein-Gordon equation. On a given spacetime, we compute the energy density of a general coherent state and try to find an f such that the semi-classical Einstein equation is satisfied in the related coherent state. This general idea will be applied in three cases. First the expectation value of the ground state energy density in the 3D torus spacetime – the Casimir effect – will be calculated and will be used in order to solve the semi-classical Einstein equation in this spacetime. Then we consider states of low energy and their corresponding coherent states in de Sitter spacetime and try to solve the semi-classical Einstein equation by means of them; in this case it turns out that a solution does not exist. Finally, the semi-classical Einstein equation in general Robertson-Walker spacetimes will be solved by means of coherent states with respect to states of low energy under the assumption that the energy density of the latter is negligible.

Introduction

In the twentieth century two very significant theories have appeared, quantum field theory and general relativity. Each of them was profoundly successful in their own fields. Quantum field theory describes the behaviour of particles in Minkowski spacetime and general relativity describes the geometry of spacetime. At first sight these two theories are totally far from each other. But we should notice that indeed there exists no real Minkowski spacetime; every quantum field has a non-zero energy-momentum tensor and a non-zero energy-momentum tensor changes the geometry of spacetime. Therefore quantum field theory on curved spacetimes (QFT on CST), which is more general than quantum field theory on Minkowski spacetime, is necessary for a more fundamental description. QFT on CST can be applied to the cases where the spacetime curvature is large, but so low that quantum gravity effects are negligible. For obtaining a description of quantum field theory on curved spacetimes one should synthesize QFT with general relativity. One adapts the fundamental relations in QFT on Minkowski spacetimes to the curved setting and studies the resulting quantum fields. Although this sounds rather simple, a few complications arise.

Namely, a general curved spacetime has no symmetries. But in flat spacetime symmetries like the Poincaré group helped us to determine the vacuum state and to have a particle interpretation of states; hence, in a general curved spacetime we have no preferred state and related particle interpretation. For overcoming this problem one can formulate the quantum theory independent of a vacuum state and in terms of fields. This is done by using the algebraic approach, in which the basis of the theory is given by the algebra of observables [2]. States in algebraic quantum field theory are continuous linear functionals on the algebra of observables. In order to distinguish physical states from non-physical ones, the two-point functions of the states are considered. We compare the singularity structure of two-point function in curved spacetime with the one of the vacuum in flat spacetime and say that physical states in a curved spacetime are the ones that have the same singularity structure [8] [3]. To do this in a precise way, we introduce the notion of wave front set to define the singularity structure in a way which encompasses both curved and flat spacetime and we call each state whose two-point function has the correct wave front set a *Hadamard state*.

One problem is solved; now we have a definition for physical states in a curved spacetime. But still we have no idea about the physical interpretation of different Hadamard states. To solve this problem in Robertson-Walker spacetime one can smear the energy density along the timelike curve of an isotropic observer with a suitable test function and search for a state in which the smeared energy density is minimal. These states are called *states of low energy* and have been introduced by Olbermann [9].

In this master thesis we are searching for the solution of the semi-classical Einstein equation for the homogeneous and isotropic case:

$$G_{\mu\nu}(x) = 8\pi G \omega(: T_{\mu\nu}(x) :)$$

where $G_{\mu\nu}$ is the Einstein tensor, G is the gravitational constant and $\omega(: T_{\mu\nu} :)$ is the expectation value of the quantum stress-energy tensor. We consider the case that the spacetime metric, and hence the left hand side, is given and look for a state whose stress-tensor expectation value matches the right hand side. We try to find such a state among the homogeneous and isotropic coherent states with respect to states of low energy. In this case, the semi-classical Einstein equation is equivalent to the covariant conservation of $\omega(: T_{\mu\nu} :)$ and the semi-classical Friedman equation:

$$3H^2(t) = 8\pi G \omega(: \rho(t) :)$$

where $H(t)$ is the Hubble expansion rate and $\omega(: \rho(t) :)$ is the expectation value of the energy density in the state ω.

Coherent states are the quantum states which in a sense have the most classical behavior among all quantum states and related to this the lowest possible Heisenberg uncertainty. Usually coherent states are defined as the eigenstates of the annihilation operator whereas in the algebraic framework one can define them as follows. We define $\mathscr{A}$ as the algebra of the Klein-Gordon fields, and α_f as an automorphism $\alpha_f : \mathscr{A} \to \mathscr{A}$ by:

$$\alpha_f[\phi(g)] := \phi(g) + \int f(x)g(x)dx, \qquad \phi(g) \in \mathscr{A}, \qquad \phi(g) := \int \phi(x)g(x)dx$$

where $g(x)$ is a test function and $f(x)$ a solution of the Klein-Gordon equation. Then if ω_0 is a state on $\mathscr{A}$ which we consider as a generalised "ground state", $\omega_f := \omega_0 \circ \alpha_f$ is another state, which we call a coherent state with respect to ω_0. As a result, the one-point function of ω_f is given by f, i.e. $\omega_f(\phi(x)) = f(x)$.

We insert the expectation value of the stress energy tensor in a coherent state ω_f with respect to a state of low energy ω_0 in the semi-classical Einstein equation and try to find a homogeneous and isotropic, i.e. only time-dependent, solution of the Klein-Gordon equation f such the semi-classical Einstein equation holds.

We have investigated this general approach in three special cases: First, we used the ground state on the torus spacetime, which as a proper ground state is a state of low energy in particular, to construct the coherent state. After calculating the normal ordered energy density in the ground state on the three-dimensional torus spacetime following earlier results by Kay [15], i.e. the Casimir effect on this spacetime, we obtained a suitable solution of Klein-Gordon equation which solves the semi-classical Einstein equation.

Then we investigated the case of de Sitter spacetime, where we used results on the energy density in states of low energy computed in [16] for finding the solution of the semi-classical Einstein equation in this spacetime. In de Sitter spacetime, the left hand side of the semi-classical Friedman equation is a constant, while the energy density in a state of low energy grows towards the past. As it turned out that the contribution of a coherent excitation to the energy density can only be positive, a solution of the semi-classical Einstein equation was not possible by means of coherent states in this spacetime.

Finally we considered the case of a general spatially flat Robertson-Walker spacetime where we assumed that the energy density in the state of low energy is negligible in comparison with the square of the Hubble rate. This is motivated by recent results [6, 7]. Under this assumption the energy density of the coherent state ω_f is given by the classical energy density of the solution f and we solved the semi-classical Einstein equation both with and without a cosmological constant for the massless and conformally coupled case.

The thesis is organised as follows. In the first chapter we give a short abstract about quantum field theory in Minkowski spacetime. In chapter 2 we present some basic definitions about geometry and curved spacetimes. Then we study the scalar Klein-Gordon field on a general curved spacetime. We give a brief explanation about Robertson-Walker spacetime in chapter 3 and then introduce states of low energy in Robertson-Walker and de Sitter spacetime. In the last chapter we introduce coherent states and use them to solve the semi-classical Einstein equation.

Chapter 1

Quantum Field Theory on Minkowski Spacetime

1.1 The Klein-Gordon Equation

One of the equations that describes quantum fields is the Klein-Gordon Equation[29] [30]. Indeed it explains the free scalar fields' relativistic behaviour. Actually this equation is based on *special relativity*, which yields the Klein-Gordon equation in the following manner; If we use the quantum-theoretical correspondences:

$$\mathbf{p} \to \frac{\hbar}{i}\nabla \quad , \quad E \to i\hbar\frac{\partial}{\partial t} \tag{1.1}$$

it yields a relativistically invariant equation for a free scalar field:

$$E^2 - m^2c^4 + |\mathbf{p}|^2c^2 = 0 \quad \Rightarrow$$

$$[-\hbar^2\frac{\partial^2}{\partial t^2} + c^2h^2\Delta - m^2c^4]\phi = -c^2\hbar^2(\Box + \frac{m^2c^2}{\hbar^2})\phi = 0 \tag{1.2}$$

where in the first line m is the rest mass and $\mathbf{p}$ is the spatial momentum, and ϕ denotes a free scalar field and the d'Alembert operator is defined by, $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$. Then the equation is simplified and seems like,

$$(\Box + (\frac{mc}{\hbar})^2)\phi = 0 \tag{1.3}$$

We call this equation the **Klein-Gordon equation** and solutions of this equation are the Klein-Gordon fields. The corresponding action to the Klein-Gordon fields is:

$$S = \int \mathcal{L}(\phi(x), \partial_\mu\phi(x))\, d^4x = -\frac{1}{2}\int (\partial_\mu\phi\partial^\mu\phi + m^2\phi^2)\, d^4x \tag{1.4}$$

here $\mathcal{L}(\phi(x), \partial_\mu \phi(x))$ is the Lagrangian density for the Klein-Gordon fields, which gives again the Klein-Gordon equation by using Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

With the help of the Fourier transformation of the scalar field we obtain the general real solution by the following procedures. We represent a free scalar field by its Fourier transformation:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(p) e^{-ip_\mu x^\mu} \tag{1.5}$$

the tilde over ϕ means the Fourier transform of it. Then we impose the Klein-Gordon operator on it; From now on we use the natural units in which $\hbar = c = 1$:

$$(\Box + m^2)\phi(x) = \int \frac{d^4p}{(2\pi)^4} (m^2 - p^2)\tilde{\phi}(p) e^{-ip_\mu x^\mu} = 0 \tag{1.6}$$

this seems like the distributional product of delta, i.e. $x\delta(x) = 0$, therefore we have:

$$(m^2 - p^2)\tilde{\phi}(p) = 0 \quad \Rightarrow$$

$$\tilde{\phi}(p) = \delta(p^2 - m^2) f(p) \tag{1.7}$$

and also the delta function can be written as:

$$\delta(p^2 - m^2) = \delta(p_0^2 - \omega^2) = \left(\frac{\delta(p_0 - \omega) + \delta(p_0 + \omega)}{2\omega}\right) f(p) \tag{1.8}$$

$f(p)$ is a function of p. By the frequency here, ω, means that:

$$p^\mu = (\omega, \mathbf{p}) \tag{1.9}$$

and

$$\omega = \sqrt{\mathbf{p}^2 + m^2} \equiv \omega_{\mathbf{p}} \tag{1.10}$$

by using $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ as two coefficients, which is defined by, $a_{\mathbf{p}} = f(p_0, \mathbf{p})$ and $a_{\mathbf{p}}^\dagger = f(-p_0, -\mathbf{p})$, the general real solution of the Klein-Gordon equation seems as follows:

$$\begin{aligned} \phi(x) &= \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_{\mathbf{p}}} (a_{\mathbf{p}} \delta(p_0 - \omega) e^{-ip_\mu x^\mu} + a_{\mathbf{p}}^\dagger \delta(p_0 + \omega) e^{-ip_\mu x^\mu}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} (a_{\mathbf{p}} e^{-i\omega_{\mathbf{p}} t + i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{+i\omega_{\mathbf{p}} t - i\mathbf{p}\cdot\mathbf{x}}) \end{aligned} \tag{1.11}$$

where we replaced $\mathbf{p} \to -\mathbf{p}$ in the second term. We can see that the Klein-Gordon field $\phi(x)$ is a superposition of the plane wave solutions. Having the plane wave solutions in

Minkowski spacetime allows us to define the positive and negative frequency notion. If ψ_p is a plane wave solution, the positive frequency is defined as:

$$\partial_t \psi_p = -i\omega\psi_p, \qquad \omega > 0 \tag{1.12}$$

and the negative frequency solutions satisfy:

$$\partial_t \bar{\psi}_p = i\omega\bar{\psi}_p, \qquad \omega > 0 \tag{1.13}$$

where the positive frequency solutions correspond to a particle with positive energy.

1.2 Quantization of the free scalar Klein-Gordon Field

In classical mechanics, the canonical coordinates q_i and p_i, which construct the phase space, obey the Poisson brackets relations:

$$\begin{aligned} \{q_i, q_j\} = \{p_i, p_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned} \tag{1.14}$$

here, first we want to move to classical field theory and define the conjugate momentum for the free scalar Klein-Gordon fields which can be straightforwardly yielded from the Lagrange density:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{1.15}$$

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \tag{1.16}$$

in this case, means the Klein-Gordon fields, it is:

$$\pi = \partial_0\phi \tag{1.17}$$

for the second quantization, that means the canonical quantization of the scalar fields, first the canonical coordinates, $\phi(t, \mathbf{x})$ and $\pi(t, \mathbf{x})$ are promoted into the Hermitian operators and then in analogy to the classical mechanics we impose the canonical commutation relation (at the equal time) on them:

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}) \tag{1.18}$$

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] = 0 \tag{1.19}$$

consequently $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ which now have to be operators, should fulfil the following commutation relations:

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}] = 0 \tag{1.20}$$

$$[a_{\mathbf{p}}^{\dagger}, a_{\mathbf{p}'}^{\dagger}] = 0 \tag{1.21}$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger}] = (2\pi)^3 2\omega_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{p}') \tag{1.22}$$

which are obtained from 1.19 and 1.20. We should define a Hilbert space of many-particle state, also called *Fock* space. Then we define a unique and unitary equivalent representation for $a_{\mathbf{p}}$ with a vacuum state in the Fock space which is annihilated for all $a_{\mathbf{p}}$:

$$a_{\mathbf{p}}|0\rangle = 0 \qquad , \forall \mathbf{p} \tag{1.23}$$

where $|0\rangle$ is the vacuum state in the Fock space, i.e. a state without any particle. And we construct the N-particles state by acting $a_{\mathbf{p}}^{\dagger}$ on vacuum state, where $a_{\mathbf{p}}^{\dagger}$ is the adjoint of $a_{\mathbf{p}}$:

$$(a_{\mathbf{p_1}}^{\dagger})^{n_1}(a_{\mathbf{p_2}}^{\dagger})^{n_2}...(a_{\mathbf{p_f}}^{\dagger})^{n_f}|0\rangle = \sqrt{n_1!n_2!...n_f!}|\phi_N\rangle \tag{1.24}$$
$$N = n_1 + n_2 + ... + n_f$$

$|\phi_N\rangle$ is a state of N particles, which n_1 of them are with the momentum of p_1, n_2 of them with the momentum of p_2 and so on. We call $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^{\dagger}$ the annihilation and creation operators, respectively.

It would be useful to introduce the so called *number density operator*, which gives the number of particles with the momentum p:

$$N_p = a_p^{\dagger} a_p \tag{1.25}$$

also the classical Hamiltonian of the Klein-Gordon field is given by:

$$H = \int d^3x \, (\pi(\partial_0\phi) - \mathcal{L}) \tag{1.26}$$

after using (1.12), (1.16) and (1.18) it is:

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + (2\pi)^3 \omega_{\mathbf{p}} \delta^{(3)}(0)) \tag{1.27}$$

where we used the relation 1.23 for the last step. It is obvious that the ground state energy is infinity and makes the integral divergent, but since we can only measure energy differences, we put the ground state energy equal to zero, then we have:

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} \tag{1.28}$$

with the following commutation relation:

$$\begin{aligned} [H, a_{\mathbf{p}}^{\dagger}] &= \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} \\ [H, a_{\mathbf{p}}] &= -\omega_{\mathbf{p}} a_{\mathbf{p}} \end{aligned} \tag{1.29}$$

The energy of each Hamiltonian eigenstate can also be obtained by imposing Hamiltonian operator on it:

$$H|\phi_N\rangle = (n_1\omega_{\mathbf{p_1}} + n_2\omega_{\mathbf{p_2}} + ... + n_f\omega_{\mathbf{p_f}})|\phi_N\rangle \tag{1.30}$$

At the end of this section it is useful to define the classical **stress-energy tensor** or **energy-momentum tensor** in Minkowski spacetime as follows:

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \tag{1.31}$$

for the Klein-Gordon field in Minkowski spacetime it is:

$$T^\mu_\nu = -\frac{1}{2}\partial_\nu\phi\partial^\mu\phi - \frac{1}{2}\delta^\mu_\nu m^2\phi^2 \tag{1.32}$$

which obeys the following conservation law:

$$\partial_\mu T^\mu_\nu = 0 \tag{1.33}$$

and its relations with Hamiltonian and momentum are:

$$H = \int d^3x\, T^{00}, \qquad P^i = \int d^3x\, T^{0i} \qquad i = 1,2,3 \tag{1.34}$$

$T^{00} = \rho$ denotes the energy density and P^i is the spatial momentum carried by the field.

1.3 Green's Functions

One uses the *Green's function* or *fundamental solution* to solve the inhomogeneous differential equation with a specific initial condition or boundary condition. In our case for an inhomogeneous Klein-Gordon equation, $(\Box + m^2)\phi(x) = f(x)$, the Green's function should fulfil the following relation:

$$(\Box_x + m^2)G(x,y) = -i\delta^{(4)}(x-y) \tag{1.35}$$

then we have,

$$\phi(x) = i\int dy\, G(x,y)f(y) \tag{1.36}$$

one can obtain the Green's function from this equation with the help of Fourier ansatz and acting Klein-Gordon operator on it, but we do not present it here. We define also the causal propagator by the so-called retarded and advanced Green's function:

$$\Delta(x,y) = G_A(x,y) - G_R(x,y) \tag{1.37}$$

where $G_A(x,y)$ denotes the advanced Green's function (vanishes for $y^0 < x^0$) and $G_R(x,y)$ is the retarded one (vanishes for $y^0 > x^0$). It is important to mention that the causal propagator gives the vacuum expectation value of the commutator of two fields:

$$i\Delta(x,y) \equiv \langle 0|[\phi(x)\phi(y)]|0\rangle \tag{1.38}$$

As this propagator is named, it vanishes for $\eta_{\mu\nu}x^\mu y^\nu > 0$. ($\eta_{\mu\nu}$ denotes the Minkowski metric with signature $(-,+,+,+)$)

It would be also useful to introduce the Feynman propagator as the vacuum expectation value of the time ordered product of two fields:

$$i\Delta_F(x,y) \equiv \langle 0|T\phi(x)\phi(y)|0\rangle \tag{1.39}$$

where T is the time ordering operator,

$$T = \begin{cases} \phi(x)\phi(y) & for \quad x^0 > y^0 \\ \phi(y)\phi(x) & for \quad y^0 > x^0 \end{cases} \tag{1.40}$$

for the Feynman propagator and only in Minkowski spacetime we can have a particle interpretation; The Feynman propagator states the creation of a particle at the point x and its annihilation at the point y for $y^0 > x^0$ (and creation at y and annihilation at x for $x^0 > y^0$).

Chapter 2

Quantum Field Theory in Curved Spacetime

2.1 Spacetime and Geometry

For describing any physical phenomenon, one observes it from an inertial frame. In a flat spacetime an inertial frame is presented by an observer that travels in a straight line, i.e. moves without acceleration. The special relativity postulate declares that from all inertial frames one observes the same physics, i.e. the laws of physics are invariant under the change of inertial frames. With the existence of gravity, in a curved spacetime, the geometry of the spacetime changes, therefore there are no global inertial frames. The local light cone, equivalently the local inertial frame, depends on geometry of the spacetime, which is determined by a metric. Hence, first of all we should know precisely the spacetime and its geometry in which the phenomena occur [19].

Let us begin by introducing a topological differentiable manifold M together with a Lorentzian metric g. We also call T_pM the tangent space for each point $p \in M$ and T_p^*M denotes the cotangent space at p. We call a vector $X \in TM$, *i*) timelike if $g(X, X) < 0$, *ii*) spacelike if $g(X, X) > 0$ and, *iii*) lightlike or null if $g(X, X) = 0$. One defines ∇_X as the covariant derivative along a tangent vector X on a manifold M. Covariant derivatives are actually a covariant generalization of the partial derivative, which unlike the partial derivative, is independent of coordinates. One defines the covariant derivative by:

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\gamma} X^\gamma$$

where $X \in T_X M$ is vector field and $\Gamma^\nu_{\mu\gamma}$ is the **Christoffel symbol** and it is defined by the covariant derivative of the frame basis:

$$\nabla_{e_a} e_b := \Gamma^c_{ba} e_c$$

The next definition is the **Riemann curvature** tensor R^a_{bcd}:

$$R^a_{bcd} e_a = \nabla_c \nabla_d e_b - \nabla_d \nabla_c e_b - \nabla_{[e_c, e_d]} e_b \tag{2.1}$$

where $e_{a,b,c}$ are the frame basis and $\nabla_a = \nabla_{e_a}$. In a coordinate basis the Lie bracket of the basis vanishes, $[e_c, e_d] = 0$. The Ricci tensor is a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor field which is obtained by contraction of a covariant index and a contravariant index of the Riemann curvature tensor:

$$R_{ab} := R^d_{adb} \tag{2.2}$$

and the Ricci scalar is:

$$R := g^{ab} R_{ab} \tag{2.3}$$

A tensor T, for example the metric, has a symmetry, if it is constant along the integral curves of a vector field X; One means by the *integral curves* of a vector field, the curves which the vector field X is their tangent vector field. We present this symmetry by a specific derivative along the *flow* (integral curve) of X. It is called Lie derivative:

$$\mathscr{L}_X T = 0 \tag{2.4}$$

If c(t) is the integral curves of X with respect to a curve parameter t, there is a diffeomorphism ξ^X_t for small t and on an open neighborhood of $c(0)$. Then The Lie derivative reads:

$$\mathscr{L}_X Y := \lim_{t \to 0} \frac{(\xi^X_{-t})^* Y - Y}{t} \tag{2.5}$$

where Y is a vector on the manifold and $(\xi^X_{-t})^*$ is the pull back of diffeomorphism at parameter distance t along the flow of X.
we name X the **Killing vector**, if on a spacetime (M, g) the Lie derivative along it preserves the metric and it has also the following property:

$$(\mathscr{L}_X g)_{ab} = 0 \qquad \Rightarrow \qquad \nabla_{(a} X_{b)} = 0$$

the left hand side of the last relation means the anti-commutator of the indices.

A spacetime (M, g) is **time-orientable** if there exists a continuous vector field which is timelike everywhere. Then a **causal curve** is defined as a curve whose tangent vector field is timelike or null everywhere and it is future oriented when its tangent vectors are in the future light cone. We should notice that the causal structure is determined by the local light cone. By the **causal past** of $T \subset M$, which we present it by $J^-(T) \subset M$, one means the set of all points which can be connected to T with a future oriented causal curve. The causal future $J^+(T) \subset M$ is defined similarly. The **chronological future** $I^+(T)$ is a set of points that can be reached from T by a future directed timelike curve. The chronological past $I^-(T)$ is defined analogously. Additionally we have these two definitions:

$$
\begin{aligned}
I(T) :=& \quad I^+(T) \cup I^-(T) \\
J(T) :=& \quad J^+(T) \cup J^-(T)
\end{aligned}
\tag{2.6}
$$

If we could not connect two points in $T \subset M$ with a timelike curve, we call this subset **achronal**, i.e. $I^+(T) \cap T$ is empty. For instance, an edgeless spacelike hypersurface in the Minkowski spacetime is achronal. The **future domain of dependence** of T, $D^+(T)$, where T is a closed achronal set, is a set of all points p such that every past directed inextendible causal curve through p must intersect T. (We mean by inextendible that there is no point for which g as a metric is not invertible. $D^-(T)$ as the past domain of dependence is defined by replacing future with past). We define the domain of dependence $D(T)$ as follows:

$$
D(T) = D^+(T) \cup D^-(T) \tag{2.7}
$$

if $p \in D(T)$, we can predict what happens at this point by knowing what happens on T. By the **future Cauchy horizon** $H^+(T)$ we define the boundary of $D^+(T)$ and similarly the past Cauchy horizon $H^-(T)$ is the boundary of $D^-(T)$.
We call a closed achronal surface Σ, a **Cauchy surface** if its domain of dependence $D(\Sigma)$ is the entire manifold M. Now with these definitions, it is comprehensive that if we know what happens on a Cauchy surface, then we can predict what happens all over the corresponding spacetime. If there exists a Cauchy surface in a spacetime (M, g), we denote this spacetime, **globally hyperbolic**.

2.2 The Klein-Gordon Equation in Curved Spacetime

To write the quantum field theory in a curved spacetime we need some generalizations for this theory in the flat spacetime. In a curved spacetime there is no Poincarè symmetry, the symmetry which has helped us in the flat spacetime to distinguish the positive and negative frequencies and consequently, to determine the unique vacuum state. In this section we introduce some fundamental definitions of the Klein-Gordon field on the curved spacetime and in the next section we want to overcome this problem with the help of algebraic approach [1][11][18].

The first step to generalize the Klein-Gordon equation is to extend the scalar Klein-Gordon field's action on a curved spacetime. We start by defining an integral on a curved manifold. The integral of a function f on a metric manifold $(\mathbf{M}, g)$, which maps $\mathbf{M} \to \mathbb{R}$, looks like:

$$\int_{\mathbf{M}} f = \int_{\mathbb{R}^n} d^n x \sqrt{|g(x)|} F(x) \tag{2.8}$$

where $|g(x)|$ is the determinant of the metric and $F(x)$ is the function f on the inverse of the chart $\xi : \mathbf{M} \supset U \to \mathbb{R}^n$, i.e. $F(x) = f \circ \xi^{-1}(x)$. Similarly for the action on the curved spacetime we have:

$$S = \int d^4 x \sqrt{|g(x)|} \mathcal{L} \tag{2.9}$$

now we should determine the Lagrangian density on the curved spacetime. For this purpose, first we replace the flat-spacetime notions with the corresponding curved-spacetime expressions. Obviously the Minkowski metric $\eta_{\mu\nu}$ changes to the general metric on a curved spacetime $g_{\mu\nu}$ and the partial derivatives become covariant derivatives, $\partial_\mu \to \nabla_\mu$. It remains only a direct coupling to the curvature. Finally the Lagrangian density in the curved spacetime looks like:

$$\mathcal{L} = -\frac{1}{2}\sqrt{|g|}(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2 + \xi R\phi^2) \tag{2.10}$$

where R denotes the curvature scalar, it is called also Ricci scalar and $\xi \in \mathbb{R}$ is a dimensionless coupling constant. There are two common choices for ξ. One of them is the *minimal coupling* which vanishes the direct interaction with the curvature scalar, $\xi = 0$. The second choice, the conformal coupling, is:

$$\xi = \frac{(n-2)}{4(n-1)} \tag{2.11}$$

with n as the number of dimension. Then, for our case, i.e. for $n = 4$, the coupling constant is $\xi = \frac{1}{6}$.
Additionally, this Lagrangian is invariant under the following conformal transformation:

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{2.12}$$

where $\Omega(x)$ is a positive smooth function. Indeed, the causal structure of the curved spacetime is invariant under conformal transformation of the metric and the lightcones of $\tilde{g}$ agree with those of g:

$$0 = \tilde{g}(X, X) = \Omega^2(x) g(X, X) \qquad \Leftrightarrow \qquad g(X, X) = 0$$

The equation of motion which yields from the Lagrangian density, is given by:

$$(-\Box + m^2 + \xi R)\phi = 0 \tag{2.13}$$

the conjugate momentum in a general curved spacetime is defined by:

$$\pi = \frac{\partial \mathcal{L}}{\partial(\nabla_0 \phi)} \tag{2.14}$$

then for the Klein-Gordon fields,

$$\pi = \sqrt{|g|}\nabla_0 \phi \tag{2.15}$$

and by quantizing the theory, the canonical commutation relations are:

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] = 0 \tag{2.16}$$

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = \frac{i}{\sqrt{|g|}} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{2.17}$$

We define also the stress-energy tensor on the curved spacetime by the following relation:

$$T^{\mu\nu} = -2 \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} \tag{2.18}$$

the second fraction means the variation of the action with respect to the metric. For obtaining the variation of metric we follow the following procedure:

$$g^{ac} g_{cb} = \delta^a_b \overset{|\delta}{\Longrightarrow} \delta g^{ac} g_{cb} + g^{ac} \delta g_{cb} = 0$$
$$\delta g^{ab} = -g^{ac} g^{bd} \delta g_{cd}$$

and the variations of the determinant of the metric and the Ricci tensor are:

$$\delta(|g|) = (|g|) g^{ab} \delta g_{ab}$$
$$\delta R = -R^{\mu\nu} \delta g_{\mu\nu} + g^{\rho\sigma} g^{\mu\nu} (\delta g_{\rho\sigma;\mu\nu} - \delta g_{\rho\mu;\sigma\nu}) \tag{2.19}$$

where the indices after semicolon mean the covariant derivative with respect to them. After imposing variation on the action and using aforementioned variation of the metric and its determinant, the stress-energy tensor in the curved spacetime looks like:

$$T^{\mu\nu} = \nabla^\mu \phi \nabla^\nu \phi - \tfrac{1}{2} g^{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi + \tfrac{1}{2} g^{\mu\nu} m^2 \phi^2 - \xi (R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) \phi^2$$
$$+ \xi (g^{\mu\nu} \Box \phi^2 - \nabla^\mu \nabla^\nu \phi^2) \tag{2.20}$$

the same as stress-energy tensor in Minkowski spacetime, it vanishes by acting the covariant derivative:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2.21}$$

2.3 The Algebraic Approach

In the last chapter we tried to solve the Klein-Gordon equation and we described the corresponding quantum field by means of plane wave solutions of that equation.
In a general curved spacetime we have no plane wave basis to decompose our solutions to positive and negative frequencies. Therefore we should develop another construction to describe quantum field theory first in general and then on a specific curved spacetime, a construction that is independent of any plane wave basis. This is possible by means of the algebraic approach to quantum field theory.
In contrast to the presentation of quantum field theory in the last section, in which states are vectors in a Hilbert space and fields are operators on this Hilbert space, in the algebraic approach the fields are first elements of an abstract algebra and states are functionals on this algebra. The quantum fields in both representation are operator-valued distributions. But what is the advantage of using distributions in the quantum field theory? "Any constructive mathematical object must exist in the form of approximations expressible via finite symbolic sequence. This holds true for continuous functions and for generalized functions defined as linear functional, but not for the conventional arbitrary function. Furthermore if one obtains the value of a function's argument from some measurements involving say statistical errors, then all one can directly measure is an average of the function against a special test function which describe the distribution of errors."[5]
The quantum fields $\phi(x)$ don't have the proper value at every point in the spacetime and it results some problems such as UV divergences. By using operator-valued distributions on suitable test function can solve this problem [2][4].

Now with these definitions which we mentioned before, we can introduce the operator-valued distributions, the so-called smeared fields by:

$$\phi(f) = \langle \phi, f \rangle = \int dx\, \phi(x) f(x) \tag{2.22}$$

where $\phi(x)$ is the field at point x at time t and $f(x)$ is a test function. One interprets $\phi(f)$ as the spacetime average (weighted by f) of the field. These smeared fields are the elements of a defined algebra. This algebra with an operation constructs the commutation relations between fields. Then one can define states on this algebra which act on its elements and maps them to the complex number space.
We should also describe the causal propagator of the Klein-Gordon equation as the map $\Delta : \mathcal{D}(\Gamma) \to \mathcal{E}(\Gamma)$ and again $\Delta = G_A - G_R$, for G_A and G_R as the advanced and retarded Green's operators, respectively. If P would be the Klein-Gordon operator on a spacetime

M, then the advanced and retarded Green's operators satisfy:

i) $PG_{A,R} = G_{A,R}P\mid_{\mathcal{D}(M)} = id_{\mathcal{D}(M)}$

ii) $supp(G_A f) \subset J^+(supp\, f)$ for all $f \in \mathcal{D}(M)$

iii) $supp(G_R f) \subset J^-(supp\, f)$ for all $f \in \mathcal{D}(M)$

for the causal propagator of P we have, $\Delta := G_A - G_R$. For every solution of P, i.e. $Pw = 0$, with compactly supported initial conditions on the Cauchy surface, there is a test function $f \in \mathcal{D}(M)$ such that $w = \Delta f$ and also, for all of test functions $f \in \mathcal{D}(M)$ which satisfies $\Delta f = 0$ there is another test function $g \in \mathcal{D}(M)$ such that $f = \Delta g$.
As we mentioned before, lack of Poincarè symmetry in the curved spacetime and therefore lack of a unique vacuum state leads us to construct a quantum field theory with the help of algebraic relations between fields and then try to define corresponding states.

We begin with the *Borchers-Uhlmann* *-algebra $\mathscr{A}(M)$ of the free field on a topological manifold (M, g). The elements of this algebra are smeared fields ϕ, which are distributions on $\mathcal{D}(M)$, i.e. $\phi(f) = \langle \phi, f \rangle$ for $f \in \mathcal{D}(M)$. The * operation on $\phi(f)$ is defined as:

$$(\phi(f))^* = \phi(\overline{f}) \tag{2.23}$$

$$(\phi(f_1)...\phi(f_n))^* = \phi(\overline{f}_n)...\phi(\overline{f}_1) \tag{2.24}$$

and the canonical commutation relation is:

$$[\phi(f), \phi(g)] := i\Delta(f, g)\mathbb{I} \tag{2.25}$$

where by $\mathbb{I}$ we denote the identity operator and by Δ is meant causal propagator of the Klein-Gordon operator,

$$\Delta(f, g) = \langle f, \Delta g \rangle = \int f(x)\Delta(x, y)g(y)dx\, dy \tag{2.26}$$

and the causal condition implies that it vanishes if the supports of f and g are spacelike separated.
The commutation relation can be also represented with help of the symplectically smeared field. If we define $\mathscr{S}(M)$ as the solution space of the Klein-Gordon equation, for a given element $w \in \mathscr{S}(M)$ we have,

$$\{\exists f \mid w = \Delta f \quad for\, Pw = 0\} \tag{2.27}$$

P means here the Klein-Gordon operator. Then for the Klein-Gordon Lagrangian $\Omega(w_1, w_2)$ is a symplectic structure $\Omega : \mathscr{S}(M) \times \mathscr{S}(M) \to \mathbb{R}$ as follows:

$$\Omega(w_1, w_2) = \int_\Sigma d\Sigma[(\nabla_N w_1)w_2 - (\nabla_N w_2)w_1] \tag{2.28}$$

where Σ is a Cauchy surface and N is the normal vector field on it. Then if we define *symplectically smeared fields* $\Phi(w)$ as an equivalence of the smeared field $\phi(f)$, we have:

$$[\Phi(w_1), \Phi(w_2)] = i\Omega(w_1, w_2) \tag{2.29}$$

The states as we already mentioned, are functionals on a topological *-algebra $\mathscr{A}(M)$. These states are defined as continuous linear functionals on $\mathscr{A}(M)$ which is imposed on the algebra of the free scalar field, that maps $\mathscr{A}(M) \to \mathbb{C}$ and are normalised and positive:

$$\omega(I) = 1, \qquad \omega(A^*A) \geq 0, \qquad \forall A \in \mathscr{A}(M) \tag{2.30}$$

if one can describe a state ω as follows:

$$\omega = \mu\omega_1 + (1-\mu)\omega_2 \tag{2.31}$$

for $\omega_1 \neq \omega_2$ and $0 < \mu < 1$, then it is called a mixed state, otherwise it is a pure state.
By defining an even state, we introduce the quasifree or Gaussian state. We call a state on $\mathscr{A}(M)$ even, if it is invariant under $\phi(f) \to -\phi(f)$ and its n-point function vanishes for all odd n.
Then an even state on $\mathscr{A}(M)$ is **quasifree** or **Gaussian** if its n-point function fulfils the following relation:

$$\omega_n(f_1, ..., f_n) = \sum_{P_n \in S_n} \prod_{i=1}^{\frac{n}{2}} \omega_2(f_{P_n(2i-1)}, f_{P_n(2i)}) \tag{2.32}$$

where S_n is permutation group with n elements. And $P_n \in S_n$ should satisfy the following two conditions:

$$\begin{aligned} P_n(2i-1) &< P_n(2i) \quad for \quad 1 \leq i \leq \tfrac{n}{2} \\ P_n(2i) &< P_n(2i+1) \quad for \quad 1 \leq i < \tfrac{n}{2} \end{aligned} \tag{2.33}$$

the positivity condition should also hold for a quasifree state, $\omega_2(\overline{f}, f) \geq 0$ for every $f \in \mathcal{D}(M)$. Therefore we have the Cauchy-Schwartz inequality for it as follows:

$$\frac{1}{4}|\Delta(f,g) \leq \omega(f,f)\omega(g,g) \tag{2.34}$$

for two real-valued test functions f and g. Then a quasifree state is pure if and only if:

$$\omega(f,f) = \sup_g \frac{|\Delta(f,g)|^2}{4\omega(g,g)} \tag{2.35}$$

for $\omega(g,g) \neq 0$ and g is a real-valued test function in $\mathcal{D}(M,\mathbb{R})$.
Another notion that is worthy to mention is about the relation between solution space and its corresponding test function space. One can define an inner product μ on $\mathscr{S}(M)$ as symmetric part of a state such that if $w_1 = \Delta f_1$ and $w_2 = \Delta f_2$ for $w_1, w_2 \in \mathscr{S}(M)$ and $f_1, f_2 \in \mathcal{D}(M,\mathbb{R})$, then $\mu : \mathscr{S}(M) \times \mathscr{S}(M) \to \mathbb{R}$ is:

$$\mu(w_1, w_2) := \frac{1}{2}(\omega(f_1, f_2) + \omega(f_2, f_1)) \tag{2.36}$$

then the Cauchy-Schwartz inequality can be represented with the help of μ and knowing the relation between symplectic form Ω and the causal propagator, which is:

$$\Omega(w_1, w_2) = \Delta(f_1, f_2) \tag{2.37}$$

then we have:

$$\frac{1}{4}|\Omega(w_1, w_2)|^2 \leq \mu(w_1, w_1)\mu(w_2, w_2) \tag{2.38}$$

2.4 Hadamard States

In the quantum field theory in Minkowski spacetime, we defined some algebras which contain linear combination of products of free fields at separate points, like $\phi(x)\phi(y)$. We define the expectation value of these linear combinations of free fields by imposing a state on them:

$$\omega_2(f,g) = \omega(\phi(f)\phi(g)) = \lim_{\epsilon \downarrow 0} \frac{1}{4\pi^2} \frac{1}{(x-y)^2 + i\epsilon(x_0 - y_0) + \epsilon^2} \tag{2.39}$$

as it is obvious from this relation, the two-point function behaves smoothly when x and y are spacelike or timelike separated points, but it is singular for $(x-y)^2 = 0$, i.e. null related points. This singularity behaves so good as to be integrable with two proper test function. We can present also this singularity with the help of the wave front set (you can find the definition of wave front set in Appendix):

$$\begin{aligned} WF(\omega_2) = \{(x, y, k, -k) \in T^*M^2 \mid x \neq y, (x-y)^2 = 0, k \parallel (x-y), k_0 > 0\} \\ \cup\{(x, x, k, -k) \in T^*M^2 \mid k^2 = 0, k_0 > 0\} \end{aligned} \tag{2.40}$$

In the Minkowski spacetime for avoiding the singularity for expectation value of $\phi^2(x)$ and the other monomials like it, one introduces the notion of *normal ordering* of $\phi^2(x)$;

$:\phi^2(x):$, such that in the products of creation and annihilation operators, one arranges all creation operators on the left hand side of annihilation operators. By multiplying two normal ordered fields, like $:\phi^2(x)::\phi^2(y):$, the square of the two-point function appear. We already know that $\omega_2(x,y)$ has singularities but they are integrable with a proper test function. Now the problem is, if the singularities of $(\omega_2(x,y))^2$ are also integrable with the test function. The mode decomposition of fields to the positive and negative energy solved this problem and help us here to have a well-defined normal ordering in the Minkowski spacetime. But how about normal ordering in a general curved spacetime, in which there is no possibility to decompose the modes? In general, two distribution $\omega_1, \omega_2 \in \mathcal{D}'(M)$ can be multiplied give a well-defined distribution in $\mathcal{D}'(M)$ again, if their wave front sets satisfy a specific condition.
If $v_1, v_2 \in \mathcal{D}'(M)$, then one defines:

$$WF(v_1) \oplus WF(v_2) := \{(x, k_1 + k_2) \mid (x, k_1) \in WF(v_1), (x, k_2) \in WF(v_2)\} \tag{2.41}$$

one can defines the product of $\omega_1\omega_2$ such that it yields a well-defined distribution in $\mathcal{D}'(M)$, if $WF(\omega_1) \oplus WF(\omega_2)$ does not intersect the zero section,

$$WF(\omega_1) \oplus WF(\omega_2) = \{(x, k_1 + k_2) \neq (x, 0) \mid (x, k_1) \in WF(\omega_1), (x, k_2) \in WF(\omega_2)\} \tag{2.42}$$

in other words, it means that a distribution with a "two sides" wave front set can not be squared. For instance, let us check the wave front set of the δ-distribution. The δ-distribution is singular at $x = 0$ and its Fourier transform is a constant. Then its wave front set is:

$$WF(\delta) = \{(0, k) \mid k \in \mathbb{R}\backslash\{0\}\}$$

and it is obvious that δ-distribution does not have a "one-side" wave front set and ,consequently, it can not be squared.
As the conclusion, we should construct a state in general curved spacetime such that it can be squared, i.e. it should have a "one side" wave front set.

We call a state ω a **Hadamard state**, if its two-point function fulfils the **Hadamard condition**:

$$WF(\omega_2) = \{(x, y, k_x, -k_y) \in T^*M^2 \setminus \{0\} \mid (x, k_x) \sim (y, k_y), k_x \rhd 0\} \tag{2.43}$$

where $(x, k_x) \sim (y, k_y)$ means between x and y there is a null geodesic connection such that k_x is coparallel and cotangent of this null curve at the point x. $k_x \rhd 0$ also means that it is future directed. The last condition is the relation between k_x and k_y. It expresses that k_y is parallel transport of k_x at the point y along the null curve connecting x and y. In addition to the aforementioned Hadamard condition, we need some other notion to realize Hadamard states. we need to define the notion of distance between two points in curved spacetime. For a sufficiently small neighborhood U around x we introduce exponential map as follows: $\exp_x : T_xU \to U$, $X_x \to y$ for $x, y \in U \subset M$ and $X_x \in T_xU \subset T_xM$. Then the **half squared geodesic distance** $\sigma(x, y)$ for $x, y \in U$ is:

$$\sigma(x,y) := \frac{1}{2} g(\exp_x^{-1}(y), \exp_x^{-1}(y)) \tag{2.44}$$

since the geodesic distance vanishes also for null separated points in Lorentzian manifold and we want it vanishes only for two coincident points, we add an imaginary part to it and take the limit as it goes to zero:

$$\sigma_\epsilon(x,y) = \sigma(x,y) + 2i\epsilon(t_x - t_y) + \epsilon^2 \tag{2.45}$$

where t is a time function on $(\mathbf{M}, g)$. Then the two-point function of a Hadamard state must obey the following relation:

$$\omega_2(x,y) = \lim_{\epsilon \downarrow 0} \frac{1}{8\pi^2} \left(\frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y) \log\left(\frac{\sigma_\epsilon(x,y)}{\lambda^2}\right) + W(x,y) \right) \tag{2.46}$$

where λ is an arbitrary length scale and $U(x, y)$, $V(x, y)$ and $W(x, y)$ are smooth real-valued biscalar and are called **Hadamard coefficients**. They are regular at $x = y$ and $U(x, y) = 1$ for $x = y$. $W(x, y)$ is also the symmetric part of the two-point function which vanishes in causal propagator (since $i\Delta(x, y) = \omega_2(x, y) - \omega_2(y, x)$).
$V(x, y)$ is given by a series expansion in σ,

$$V = \sum_{n=0}^{\infty} v_n \sigma^n$$

here v_n are smooth biscalar coefficients.

Chapter 3

States of Low Energy

After defining Hadamard states the problem of finding a state with meaningful singular behavior on a general curved spacetime was solved. But still we can not define a ground state; indeed we do not know the interpretation of different Hadamard states. Parker wanted to solve this problem by considering states of minimal particle creation, the so-called adiabatic vacua, which are locally quasi-equivalent to the Hadamard class [13]. But it was not helpful because in a curved spacetime a particle interpretation is not meaningful.

After that, Fewster found out that the smeared energy density in a Hadamard state is bounded from below [21]. Actually, one smears the energy density with a test function which is compactly supported on a timelike worldline of an observer. As it is obvious that such an energy density is test function-dependent. Later Olbermann proved that on Robertson-Walker spacetime, there is a homogeneous and isotropic Hadamard state which is a state of low energy in this spacetime. In this way we bounded the renormalized expectation value of the energy density as a functional of the Hadamard states, which is smeared in time with the square of a fixed test function $g(t)$ along a timelike geodesic, from blow.

In this chapter we want to have an overview of state of low energy on the Robertson-Walker spacetime which is introduced by Olbermann [9].

3.1 The Energy-Momentum Tensor on general curved Spacetimes

As we mentioned before in the section 2.2, we defined the energy-momentum tensor for the Klein-Gordon fields by:

$$T_{ab} := \frac{2}{\sqrt{|g|}} \frac{\delta S_{KG}}{\delta g^{ab}} \tag{3.1}$$

where by $|g|$ one means determinant of the metric g on the spacetime M and S_{KG} is tha action of the Klein-Gordon field. For the case of minimal coupling, i.e. $\xi = 0$ it is:

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla_c \phi \nabla^c \phi - m^2 \phi^2) \tag{3.2}$$

this expression which consist of an ill defined pointwise product of distributions yield a divergent result. For solving this problem, we use the "point-splitting prescription" and restrict our energy-momentum to Hadamard states. Therefore we define the bidifferential operator as follows:

$$K_{ab}(x, x') := \nabla_a \otimes B_b^{b'}(x, x') \nabla_{b'} - \frac{1}{2} g_{ab}(x) (g_{cd}(x) \nabla^c B_e^d(x, x') \nabla^e - m^2) \tag{3.3}$$

where $B_b^{b'}(x, x')$ is a bitensor.
We define VM and WM as two vector bundles over M with vector spaces V and W, respectively. Then we call $VM \boxtimes WM$ the exterior tensor product of VM and WM; indeed it is a vector bindle over $M \times M$. A section of $VM \boxtimes WM$ is called a **bitensor**. Let us introduce point-coincidence limit for bitensors with following notation:

$$[L(x, x')] := \lim_{x \to x'} L(x, x') \tag{3.4}$$

Actually by this manner of representing stress-energy tensor by point-splitting prescription, we want to subtract the singularities from the whole expression.
Finally the stress-energy tensor in the Hadamard state ω looks like:

$$\omega(: \tilde{\hat{T}}_{ab} :) := [K_{ab}(\mathcal{W}_2^{\omega}(x, x') - \mathcal{G}(x, x'))] \tag{3.5}$$

where,

$$\mathcal{G}(x, x') = \lim_{\epsilon \downarrow 0} \frac{1}{8\pi^2} \left(\frac{U(x, x')}{\sigma_\epsilon(x, x')} + V(x, x') log\left(\frac{\sigma_\epsilon(x, x')}{\lambda^2} \right) \right) \tag{3.6}$$

is the singular part of the two-point function of a Hadamard state. We impose the covariant derivative on the energy momentum tensor:

$$\nabla^a \omega(: \tilde{\hat{T}}_{ab} :) = -\frac{1}{3} \nabla^a g_{ab} [P_x \mathcal{G}(x, x')] \tag{3.7}$$

where P_x is the Klein-Gordon operator with respect to x:

$$P_x := \nabla_a \nabla^a + m^2 + \xi R \equiv \Box_g + m^2 + \xi R \tag{3.8}$$

Since the energy-momentum tensor must obey the covariant conservation law, then we define again the stress-energy tensor which is now covariantly conserved:

$$\omega(:\hat{T}_{ab}:) := \omega(:\widetilde{\hat{T}}_{ab}:) + \frac{1}{3} g_{ab}[P_x \mathcal{G}(x,x')] + C_{ab}(\alpha,\beta,\nu,\rho) \tag{3.9}$$

where C_{ab} is a free tensor and given by:

$$C_{ab} = \alpha m^4 g_{ab} + \beta m^2 G_{ab} + \nu I_{ab} + \rho g_{ab} \tag{3.10}$$

G_{ab} is Einstein tensor and I_{ab} abd J_{ab} are:

$$I_{ab} := \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g^{ab}} \int_M R^2 d\mu_g = 2\nabla_a \nabla_b R + 2RR_{ab} - g_{ab}(\frac{1}{2}R^2 + 2\Box_g R) \tag{3.11}$$

$$J_{ab} := \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g^{ab}} \int_M R^{cd} R_{cd} d\mu_g = -\Box_g R_{ab} - \frac{1}{2} g_{ab}(R_{cd}R^{cd} + \Box_g R) + \nabla_a \nabla_b R + 2R^{cd}R_{cadb} \tag{3.12}$$

$d\mu_g = \sqrt{|det(g_{ab})|}$ is the volume element and α, β, ν and ρ are renormalisation parameters which must be determined by experiment.

3.2 Robertson-Walker Spacetime

The most straightforward spacetime for describing quantum field theory on a curved spacetime is a maximally symmetric spacetime, a spatially homogeneous and isotropic spacetime which is evolving in time, i.e. a universe consists of infinite spacelike slices such that each three-dimensional slice is maximally symmetric. Such a universe looks like $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ is time direction and Σ represents a maximally symmetric three-dimensional manifold. The metric on this spacetime is given by:

$$ds^2 = -dt^2 + a^2(t) d\sigma^2 \tag{3.13}$$

$a(t)$ denotes the scale factor and $d\sigma^2$ is the metric on Σ:

$$\kappa_{ij}(u) du^i du^j, \qquad i,j = 1,2,3 \tag{3.14}$$

where $u^{i,j}$ are coordinates on Σ and κ_{ij} is a three-dimensional maximally symmetric metric.We call such a homogeneous and isotropic spacetime **Robertson-Walker spacetime**.

The maximally symmetric metrics must obey the following relation with corresponding 3D metric's Riemann tensor:

$$\tilde{R}_{ijkl} = k(\kappa_{ik}\kappa_{jl} - \kappa_{il}\kappa_{jk}) \tag{3.15}$$

where k is normalised to {-1, 0, +1}. Again tilde over R insist the difference between this Riemann tensor and the Riemann tensor of a 4-dimensional metric.
A spacetime with $k = +1$ corresponds to the positive curvature on Σ and it is called also closed. $k = 0$ corresponds to the flat spacetime and $k = -1$ means Σ is open and has negative curvature.

3.3 States of Low Energy on Robertson-Walker Spacetime

On Robertson-Walker spacetime, we chose states which have the same symmetries as this spacetime has, i.e. they should be homogeneous and isotropic. We present a quasifree pure homogeneous and isotropic state ω on the FRW spacetime $(I \times \Sigma, g_{ab})$ of the Klein-Gordon field, with its two point function as follows:

$$\mathcal{W}_2^{\omega}(f, g) = \sum_{i,j=0}^{1} (\tilde{F}_i, \tilde{\omega}_{ij}\tilde{G}_j) \tag{3.16}$$

where by the parenthesis it means the scalar product in $L^2(\Sigma)$ and,

$$F_0 := \Delta f \mid_{\Sigma}, \quad F_1 := n^{\alpha}\nabla_{\alpha}(\Delta f) \mid_{\Sigma} \tag{3.17}$$

$\tilde{\Sigma}$ is Fourier transform on Σ. One means by $L(\Sigma)$ the symplectic space of initial values on the Cauchy surface:

$$L = \{(f_1, f_2), f_1, f_2 \in \mathcal{D}_{\mathbb{R}}(\Sigma)\} \tag{3.18}$$

n^{α} is also the future-directed normalised vector field normal to Cauchy surface Σ.
As always, the two point function must satisfy following conditions:

$$\begin{aligned} \mathcal{W}_2^{\omega}(f, g) \geq 0, \quad f, g \in \mathcal{D}(\Sigma) \\ \mathcal{W}_2^{\omega}(f, g) - \mathcal{W}_2^{\omega}(g, f) = i\Delta(f, g) \end{aligned} \tag{3.19}$$

then for quasifree pure state, these conditions make the *multiplication operator* $\hat{\omega}_{ij}$ to look like:

$$\begin{aligned} \hat{\omega}_{00} = a^6(t_0)|q(k)|^2 \\ \hat{\omega}_{11} = |p(k)|^2 \\ \hat{\omega}_{01} = \bar{\hat{\omega}}_{10} = a^3(t_0)\bar{q}(k)p(k) \end{aligned} \tag{3.20}$$

where $p(k)$ and $q(k)$ are bounded measured functions on $\tilde{\Sigma}$, which satisfy $\bar{q}p - \bar{p}q = i$. t_0 is also the Cauchy surface Σ time coordinate.

The Klein-Gordon operator on FRW spacetime is given by:

$$\Box_g + m^2 = \frac{\partial^2}{\partial t^2} + 3H(t)\frac{\partial}{\partial t} - a^{-2}(t)\Delta_\Sigma + m^2 \tag{3.21}$$

here $a(t)$ is the scale parameter and $H(t) = \frac{\dot{a}(t)}{a(t)}$ is the Hubble parameter. Δ_Σ is the Laplacian on the Cauchy surface Σ.

As it is obvious from 3.20 we can present solutions of the Klein-Gordon equation by a time dependent function times spatial one:

$$f_k(x,t) = S_k(t)J_k(x) \tag{3.22}$$

$J_k(x)$ is an eigenfunction of the Laplacian:

$$\Delta_\Sigma J_k = -E(k)J_k \tag{3.23}$$

and $S_k(t)$ is a solution of time part of the Klein-Gordon equation:

$$\ddot{S}_k + 3H\dot{S}_k + \omega_k^2 S_k = 0 \tag{3.24}$$

where,

$$\omega_k^2 = \frac{E(k)}{a^2} + m^2 \tag{3.25}$$

and S_k obeys the following condition:

$$\bar{S}_k\dot{S}_k - S_k\dot{\bar{S}}_k = ia^{-3} \tag{3.26}$$

which is concluded from 3.19. Let us denote S_k with the following initial condition at t_0 by T_k, where T_k is given by:

$$\begin{aligned} T_k(t_0) &= -p_k a^{-3}(t_0) \\ \dot{T}_k(t_0) &= q(k) \end{aligned} \tag{3.27}$$

we define a set of homogeneous and isotropic pure and quasifree states on $\mathscr{A}$, as the algebra of Klein-Gordon fields. Then there exists a unique state ω_g for which the smeared energy density functional:

$$E_g[\omega] := \int_\gamma g^2(t)\omega(:\hat{\rho}(t):)dt \tag{3.28}$$

is minimum. Here γ is an isotropic geodesic of observer and $f := g^2$ is a given test function and $g \in \mathcal{D}(I)$. We should again mention that here ω_g is a Hadamard state. The two-point function of the state ω in the mode function is given by:

$$\mathcal{W}_2^{\omega}(x,x') = \int d^3k J_{\mathbf{k}}(\mathbf{x}) \bar{J}_{\mathbf{k}}(x') T_k(t) \bar{T}_k(t') \tag{3.29}$$

$T_k(t)$ can be decomposed as follows:

$$T_k(t) = \lambda_k S_k(t) + \mu_k \bar{S}_k(t) \tag{3.30}$$

where $S_k(t)$ is again the solution of the time part of the Klein-Gordon equation which obeys 3.25. By putting $T_k(t)$ in 3.25 we have:

$$|\lambda_k|^2 - |\mu_k|^2 = 1 \tag{3.31}$$

λ_k and μ_k, which make 3.27 minimum, can be obtained by putting T_k in 3.27 and try to find the minimum for each k:

$$\begin{aligned} E_g &= \tfrac{1}{2} \int g^2(t)(|\dot{S}_k(t) + \mu \dot{\bar{S}}_k(t)|^2 + \omega^2(k)|\lambda S_k(t) + \mu \bar{S}_k(t)|^2)dt \\ &= \tfrac{1}{2} \int g^2(t)((|\lambda|^2 + |\mu|^2)(|\dot{S}_k(t)|^2 + \omega_k^2 |S_k(t)|^2) \\ &\quad + 2Re\{\mu\lambda(\dot{S}_k(t)^2 + \omega_k^2 S_k(t)^2)\}) \end{aligned} \tag{3.32}$$

by differentiating with respect to μ, one can obtain:

$$\mu_k = \sqrt{\frac{c_1(k)}{2\sqrt{c_1(k)^2 - |c_2(k)|^2}} - \frac{1}{2}} \tag{3.33}$$

and this yields:

$$\lambda_k = e^{i(\pi - \arg c_2(k))} \sqrt{\frac{c_1(k)}{2\sqrt{c_1(k)^2 - |c_2(k)|^2}} + \frac{1}{2}} \tag{3.34}$$

where c_1 and c_2 are:

$$c_1(k) = \int g^2(t)(|\dot{S}_k(t)|^2 + (E(k)a^{-2} + m^2)(k)|S_k(t)|^2)dt \tag{3.35}$$

$$c_2(k) = \int g^2(t)(\dot{S}_k(t)^2 + (E(k)a^{-2} + m^2)(k)S_k(t)^2)dt \tag{3.36}$$

3.4 States of Low Energy on de Sitter spacetime

The easiest example of Robertson-Walker spacetime is a spacetime with a constant Hubble parameter, i.e. $H = \frac{\dot{a}(t)}{a(t)} = constant$. Then the scale function is given by:

$$a(t) = e^{Ht} \tag{3.37}$$

here we want to investigate the state of low energy in this spacetime, which is precisely calculated in [16] and we give a brief review of it here.

By substituting $S_k(t)$, which is already introduced as a solution of time part of the Klein-Gordon equation in section 3 of this chapter, by $\eta_k(t) = S_k(t)a(t)$, then the time part of the Klein-Gordon equation, (3.24), is reformulated as follows:

$$\partial_\sigma^2 \eta_k(\sigma) + (k^2 + M_{m,\xi}(\sigma))\eta_k(\sigma) = 0 \tag{3.38}$$

where ∂_σ^2 means second derivative with respect to the conformal time σ, i.e. $\frac{\partial^2}{\partial\sigma^2}$ and $M_{m,\xi}$ is the time dependent mass and one defines it by:

$$M_{m,\xi}(\sigma(t)) = (m^2 + (\xi - \frac{1}{6})R(\sigma))a^2(t) \tag{3.39}$$

and,

$$\sigma(t) := \int_{t_0}^{t} a^{-1}(t')dt' \tag{3.40}$$

is the conformal time which makes the equations simpler in the Robertson-Walker spacetime.

The equation (3.26) is reformulated to:

$$\eta'_k\bar{\eta}_k - \eta_k\bar{\eta}'_k = i \tag{3.41}$$

the solutions of this equation are:

$$\eta_k(\sigma) = \frac{\sqrt{-\pi\sigma}}{2} e^{\frac{-i\pi\sigma}{2}} H_n^{(2)}(-k\sigma) \tag{3.42}$$

here $H_n^{(2)}$ is the Hankel function of the second kind:

$$H_n^{(2)}(z) = \frac{1}{2\pi}\int_{-\infty}^{0} \frac{e^{(\frac{z}{2})(t-\frac{1}{t})}}{t^{n+1}} dt \tag{3.43}$$

and its derivative is given by:

$$\frac{d}{dz}H_n^{(2)}(z) = \frac{1}{2}[H_{n-1}^{(2)}(z) - H_{n+1}^{(2)}(z)] \tag{3.44}$$

and n, the index of the Hankel function is given by:

$$n := \sqrt{\frac{9}{4} - 2(\frac{m^2}{2H^2} + 6\xi)} \tag{3.45}$$

If one presents the bidifferential operator in the FRW spacetime by $\mathcal{R}$, then it looks like:

$$\mathcal{R} := \frac{1}{2}(\partial_t\partial_{t'} + \frac{1}{a^2}\sum_{i=1}^{3}\partial_{x_i}\partial_{y_i} + m^2) \tag{3.46}$$

by imposing it on the [symmetry-reduced] two-point function, it yields:

$$[\mathcal{R}\tilde{\mathcal{W}}_2^{\omega}](\sigma) = \frac{H^4}{2(2\pi)^3} \int\limits_{\mathbb{R}^3} d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} [(1+2\mu_k^2)(|\dot{S}_k(t)|^2 + (k^2 e^{-2Ht} + m^2)|S_k(t)|^2)$$

$$+2\mu_k|\lambda_k| Re\{e^{i\, arg\lambda_k}(\dot{S}_k(t)^2 + (k^2 e^{-2Ht} + m^2)S_k(t)^2)\}] \tag{3.47}$$

where μ_k here is a real number.

Chapter 4

Solutions of the Semi-Classical Einstein Equation

The main motivation of this thesis is solving the semi-classical Einstein equation. We should find a proper state whose energy-momentum tensor is compatible with the semi-classical Einstein equation.
Coherent state can be the state we are looking for, since these states are states with minimum uncertainty and therefore the most classical state in the quantum field theory. Additionally, we can define the expectation value of the energy density easily. We will explain it precisely in the following sections.
In this chapter, first we will give a short introduction to the coherent state and then try it in different cases for the semi-classical Einstein equation.

4.1 Coherent States

The basic motivation of introducing coherent states was finding a state whose expectation value of the position operator behaves classically[14]:

$$\bar{x}(t) = \langle z|\hat{x}(t)|z\rangle \tag{4.1}$$

where $|z>$ is a coherent state and $\hat{x}(t)$ is the position operator. Then we want to $\bar{x}(t)$ obeys the classical equation of motion:

$$m\ddot{\bar{x}}(t) + \frac{\partial \bar{V}}{\partial x} = 0 \tag{4.2}$$

Actually coherent states are on the boundary of transition from classical mechanics to quantum mechanics. For coherent state the Heisenberg inequality changes to an equality:

$$< \Delta x >_z < \Delta p >_z = \frac{\hbar}{2} \tag{4.3}$$

where $< \Delta x >_z := [\langle z|x^2|z\rangle - (\langle z|x|z\rangle)^2]$. Furthermore coherent states are eigenstates of annihilation operator with eigenvalue of z:

$$a|z\rangle = z|z\rangle, \qquad z \in \mathbb{C} \tag{4.4}$$

one can obtain a coherent state from the ground state $|0\rangle$ of the harmonic oscillator as follows:

$$|z\rangle = e^{za^\dagger - \bar{z}a}|0\rangle \tag{4.5}$$

If we define $\mathscr{A}$ as the algebra on the Klein-Gordon fields, then α_f is an automorphism, which maps $\mathscr{A}$ to itself, $\alpha_f : \mathscr{A} \to \mathscr{A}$ and it is given as follows:

$$\alpha_f[\phi(g)] := \int \phi(x)g(x)dx + \int f(x)g(x)dx, \qquad \phi(x) \in \mathscr{A} \tag{4.6}$$

where $g(x)$ is a test function and $f(x)$ is a solution of Klein-Gordon equation. If ω is the ground state on $\mathscr{A}$:

$$\omega(\phi(g)\bar{\phi}(g)) \geq 0 \tag{4.7}$$

then,

$$\omega(\alpha_f(\phi(g))\alpha_f(\bar{\phi}(g))) = \omega \circ \alpha_f(\phi(g)\bar{\phi}(g)) \geq 0 \tag{4.8}$$

is another state, which is called a coherent state. It is the other definition of coherent state from algebraic point of view.

4.2 The Semi-Classical Einstein Equation

In the classical Einstein equation, the stress-energy density distribution on the spacetime determines the whole metric and curvature of the spacetime. In quantum field theory for studying the backreaction of quantum fields on the background spacetime we use the semi-classical Einstein equation:

$$G_{\mu\nu}(x) = 8\pi G\omega(: T_{\mu\nu} :) \tag{4.9}$$

where $G_{\mu\nu}$ denotes the Einstein tensor which is given by:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{4.10}$$

and G is Newton's gravitational constant. As it is obvious, in semi-classical Einstein equation we insert the expectation value of a proper Wick polynomial $: T_{\mu\nu} :$ with respect

to a suitable state ω instead of the stress-energy tensor of classical matter.
We said "a proper Wick polynomial" because one knows that for defining the notion of normal ordering in general curved spacetime one should find states whose two-point functions are singular but they are regular enough to allow multiplying them pointwise in normal ordering procedure. Therefore one has to define the Wick polynomial locally. The proper state for evaluating the energy-momentum tensor in semi-classical Einstein equation is Hadamard states. But the problem is that, we do not know which Hadamard state is the suitable one. Since we know that coherent states have the classical aspect in quantum theory, therefore use them in semi-classical Einstein equation.
In Minkowski spacetime we use the vacuum state to creating coherent states. But in general curved spacetime the notion of the vacuum state is meaningless. In last section we introduced the state of low energy which can supersede the notion of the vacuum state in general curved spacetime. Then we use them here to create coherent states.
Let us to represent the Einstein equation with a coherent state:

$$\omega_c(:T_{\mu\nu}:) = \frac{G_{\mu\nu}}{8\pi G} \tag{4.11}$$

where ω_c denotes the coherent state. Then we should find solutions for each spacetime. By specifying the spacetime, consequently, we know the right hand side of the equation. Then we should find a proper coherent state whose stress-energy density tensor's expectation value obeys the semi-classical Einstein equation on that spacetime.
We represent the expectation value of the stress-energy tensor of a coherent state as follows:

$$\omega_c(:T_{\mu\nu}:) = \omega_{SLE}(:T_{\mu\nu}:) + T_{\mu\nu}(f) \tag{4.12}$$

where ω_{SLE} denotes the state of low energy and f is a solution of Klein-Gordon equation. For the massless fields and the minimal coupling case we have:

$$\begin{aligned} \omega_c(:T_{\mu\nu}:) &= [\partial_\mu \omega_{SLE} \partial_\nu \omega_{SLE} - \tfrac{1}{2} g_{\mu\nu} (g^{\alpha\beta} \partial_\alpha \omega_{SLE} \partial^\alpha \omega_{SLE})] \\ &\quad + [\partial_\mu f \partial_\nu f - \tfrac{1}{2} g_{\mu\nu} (g^{\alpha\beta} \partial_\alpha f \partial^\alpha f)] \end{aligned} \tag{4.13}$$

we know the proper state of low energy and the expectation value of the stress-energy tensor evaluated by it before, then by determining the metric we should look for the proper function f. We want to find the solution on various spacetime. First we study the semi-classical Einstein equation in a cylinder spacetime. Then we try to find a solution in de Sitter spacetime. We will present them in the following sections.

4.3 Casimir Effect and solution of Semi-Classical Einstein equation in 3D torus Spacetime

In this section we want to calculate the expectation value of the energy density in 4-dimensional flat spacetime with the radius of R. Since in this spacetime the fields should obey the boundary conditions, i.e. $\phi(x) = \phi(x + 2\pi Rn)$, then it has the conditions for the Casimir effect.

Here $\rho(z) = T_{00}(z)$ denotes the energy density at a point z. We should know that $\rho(z)$ operates locally in a small neighborhood V of z. Then we should define the ground state k on the torus spacetime. But one should pay attention that the ground state on the flat torus spacetime is different from the vacuum state ω on the flat Minkowski spacetime.

Now we should represent the Klein-Gordon fields on the torus spacetime. The Klein-Gordon field on 3-dimensional torus spacetime is given by:

$$\phi(t,\mathbf{x}) = (2\pi R)^{-\frac{3}{2}} \sum_{\mathbf{n}\in\mathbb{Z}^3} (\frac{2|n|}{R})^{-1}[a_{\mathbf{n}} e^{-i[\frac{|n|}{R}t - \frac{\mathbf{n}.\mathbf{x}}{R}]} + a^{\dagger}_{\mathbf{n}} e^{i[\frac{|n|}{R}t - \frac{\mathbf{n}.\mathbf{x}}{R}]}] \tag{4.14}$$

where,

$$\mathbf{n} = \hat{i}n_{x_1} + \hat{j}n_{x_2} + \hat{k}n_{x_3}, \qquad |n| = (n^2_{x_1} + n^2_{x_2} + n^2_{x_3})^{\frac{1}{2}} \tag{4.15}$$

we consider $\frac{\mathbf{n}}{R}$ as the momentum in torus spacetime which can possess only discrete values because of the boundary conditions. The coefficient can be obtain by using the commutation relation:

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}) \tag{4.16}$$

We should take the expectation value of normal ordered energy density. We do the normal ordering with respect to vacuum state on Minkowski spacetime, then we have:

$$k(:\rho(z):) = k(\rho(z)) - \omega(\rho(z)) \tag{4.17}$$

the energy density at the point z is given by:

$$\rho(z) = \frac{1}{2}[(\frac{\partial\phi}{\partial t})^2 \mid_z + (\frac{\partial\phi}{\partial x_1})^2 \mid_z + (\frac{\partial\phi}{\partial x_2})^2 \mid_z + (\frac{\partial\phi}{\partial x_3})^2 \mid_z] \tag{4.18}$$

here we use the point-splitting prescription again and represent the expectation value of the normal ordered energy density by:

$$\begin{aligned} k(:\rho(z):) &= \lim_{x,y\to z}\{(k-\omega)[\tfrac{1}{2}(\partial_t\phi(x)\partial_{t'}\phi(y) + \partial_{x_1}\phi(x)\partial_{y_1}\phi(y) \\ &\quad + \partial_{x_2}\phi(x)\partial_{y_2}\phi(y) + \partial_{x_3}\phi(x)\partial_{y_3}\phi(y))]\} \end{aligned} \tag{4.19}$$

we can bring out the partial differential operators and rewrite it:

$$\lim_{x,y\to z}[\frac{1}{2}(\partial_t\partial_{t'}+\partial_{x_1}\partial_{y_1}+\partial_{x_2}\partial_{y_2}+\partial_{x_3}\partial_{y_3})[k(\phi(x)\phi(y))-\omega(\phi(x)\phi(y))]] \tag{4.20}$$

with the help of translation invariance and by putting the point z at zero point and $(t,\mathbf{y})$ equal to zero, we have to calculate the following limit:

$$\lim_{x\to 0}[-\frac{1}{2}(\partial_t^2+\Delta)[k(\phi(x)\phi(0))-\omega(\phi(x)\phi(0))]] \tag{4.21}$$

now we can use the Poisson resummation, which is given by:

$$\sum_{n\in\mathbb{Z}^3} f(n)=\sum_{m\in\mathbb{Z}^3}\int f(n)e^{-im2\pi n}dn \tag{4.22}$$

where f is a function of a integer number n. We reformulate the Klein-Gordon field with Poisson resummation,

$$k(\phi(x)\phi(0))=(2\pi R)^{-3}\sum_{\mathbf{n}\in\mathbb{Z}^3}(\frac{2|n|}{R})^{-1}e^{-i[\frac{|n|}{R}t-\frac{\mathbf{n.x}}{R}]}$$

$$=(2\pi R)^{-3}\sum_{\mathbf{m}\in\mathbb{Z}^3}\int(\frac{2|n|}{R})^{-1}e^{-i[\frac{|n|}{R}t-\frac{\mathbf{n.x}}{R}]}e^{-i2\pi\mathbf{n.m}}\,d\mathbf{n} \tag{4.23}$$

and then we can write the two-point function of k as the summation of two-point function of vacuum state:

$$k(\phi(x)\phi(0))=\sum_{\mathbf{n}\in\mathbb{Z}^3}\omega(\phi(x+2\pi Rn)\phi(0)) \tag{4.24}$$

then we have:

$$\begin{aligned}k(:\rho:) &= \lim_{x\to 0}\{-\tfrac{1}{2}(\partial_t^2+\Delta)[\sum_{\mathbf{n}\in\mathbb{Z}^3}\omega(\phi(x+2\pi Rn)\phi(0))-\omega(\phi(x)\phi(0))]\}\\ &= \lim_{x\to 0}\{-\tfrac{1}{2}(\partial_t^2+\Delta)[\sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{0\}}\omega(\phi(x+2\pi Rn)\phi(0))]\end{aligned}$$

$$=\lim_{x\to 0}\{-\frac{1}{2}\lim_{\epsilon\downarrow 0}(\partial_t^2+\Delta)[\sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{0\}}\frac{1}{(\mathbf{x}+2\pi R\mathbf{n})^2-t^2+i\epsilon}]\} \tag{4.25}$$

we drop ϵ and take the derivative on a spacelike Cauchy surface, then we have:

$$k(:\rho:)=\lim_{\mathbf{x}\to 0}[-\frac{1}{2}(\sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{0\}}\frac{16[(x_1+2\pi n_1R)^2+(x_2+2\pi n_2R)^2+(x_3+2\pi n_3R)^2]}{[(x_1+2\pi n_1R)^2+(x_22\pi n_2R)^2+(x_3+2\pi n_3R)^2]^3}$$

$$-\frac{12}{[(x_1+2\pi n_1R)^2+(x_22\pi n_2R)^2+(x_3+2\pi n_3R)^2]^2})] \tag{4.26}$$

we take $\mathbf{x}$ to zero:

$$k(:\rho:)=\sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{0\}}-\frac{2}{[(2\pi n_1R)^2+(2\pi n_2R)^2+(2\pi n_3R)^2]^2}] \tag{4.27}$$

the result of this summation is a number that we call it Γ_1. Now we should search for the solution of semi-classical Einstein equation.

We have for the coherent state the following relation:

$$\omega_c(:\rho(x):) = k(:\rho(x):) + \rho(f) \tag{4.28}$$

then we should find the proper function f.

On the flat torus spacetime we have no curvature, therefore:

$$R_{abcd} = 0, \qquad G_{\mu\nu} = 0 \tag{4.29}$$

then for a homogeneous and isotropic solution which only depends on t, we have:

$$\frac{1}{2}(\dot{f}(t))^2 + k(:\rho:) = 0 \tag{4.30}$$

the solution for function f is obtained by the following steps:

$$(\dot{f}(t))^2 = -2k(:\rho:)$$
$$(\dot{f}(t)) = (-2\Gamma_1)^{\frac{1}{2}}$$
$$f(t) = (-2\Gamma_1)^{\frac{1}{2}}t + C \tag{4.31}$$

where C is constant.

For the massive scalar Klein-Gordon field it is almost the same. The free scalar Klein-Gordon field in a 3D torus spacetime is given by:

$$\phi(t,\mathbf{x}) = (2\pi R)^{-\frac{3}{2}} \sum_{\mathbf{n}\in\mathbb{Z}^3} (2\omega_{\mathbf{n}})^{-\frac{1}{2}}[a_{\mathbf{n}} e^{-i[\omega_{\mathbf{n}} t - \frac{\mathbf{n}\cdot\mathbf{x}}{R}]} + a^\dagger_{\mathbf{n}} e^{i[\omega_{\mathbf{n}} t - \frac{\mathbf{n}\cdot\mathbf{x}}{R}]}] \tag{4.32}$$

$$\omega_{\mathbf{n}} = (\frac{n^2}{R^2} + m^2)^{\frac{1}{2}}$$

and the two-point function in massive case in 3D torus spacetime reads,

$$k(\phi(x)\phi(y)) = (2\pi R)^{-3} \sum_{\mathbf{n}\in\mathbb{Z}^3} (2\omega_{\mathbf{n}})^{-1} e^{-i[\omega_{\mathbf{n}}(t-t') - \frac{\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})}{R}]} \tag{4.33}$$

while the one in Minkowski spacetime, as we know, is:

$$\omega(\phi(x)\phi(y) = (2\pi)^{-3} \int (2\omega_{\mathbf{k}})^{-1} e^{-i[\omega_{\mathbf{k}}(t-t') - \mathbf{k}\cdot(\mathbf{x}-\mathbf{y})]}\, d\mathbf{k}$$

$$\omega_{\mathbf{k}} = (k^2 + m^2)^{\frac{1}{2}}$$

we compute the expectation value of normal ordered energy density at point z on 3D torus spacetime by:

$$k(:\rho(z:)) = \lim_{x,y\to z}[\frac{1}{2}(\partial_t\partial_{t'} + \partial_{x_1}\partial_{y_1} + \partial_{x_2}\partial_{y_2} + \partial_{x_3}\partial_{y_3} + m^2)[k(\phi(x)\phi(y)) - \omega(\phi(x)\phi(y))]] \tag{4.34}$$

in analogy to the massless case, we put z equal to zero and calculate the two-point function for $\phi(y) = \phi(0)$. Again we use the Poisson resummation to the two-point function in 3D torus in the two-point function of the vacuum state:

$$k(\phi(x)\phi(0)) = (2\pi R)^{-3} \sum_{\mathbf{n}\in\mathbb{Z}^3} (2\omega_{\mathbf{n}})^{-1} e^{-i[\omega_{\mathbf{n}} t - \frac{\mathbf{n}.\mathbf{x}}{R}]}$$

$$= (2\pi R)^{-3} \sum_{\mathbf{m}\in\mathbb{Z}^3} \int (2\omega_{\mathbf{n}})^{-1} e^{-i[\omega_{\mathbf{n}} t - \frac{\mathbf{n}.\mathbf{x}}{R}]} e^{-i2\pi \mathbf{n}.\mathbf{m}}\, d\mathbf{n} \tag{4.35}$$

then we have for the expectation value of the normal ordered energy density:

$$k(:\rho:) = \lim_{x\to 0}[-\frac{1}{2}(\partial_t^2 + \Delta + m^2)[\sum_{\mathbf{n}\in\mathbb{Z}^3} \omega(\phi(\mathbf{x} + 2\pi R\mathbf{n}, t)\phi(0)) - \omega(\phi(\mathbf{x}, t)\phi(0))]]$$

then,

$$k(:\rho:) = \lim_{\mathbf{x}\to 0}[-\frac{1}{2}(\partial_t^2 + \Delta + m^2) \sum_{\mathbf{n}\in\mathbb{Z}^3\backslash\{0\}} \omega(\phi(\mathbf{x} + 2\pi R\mathbf{n})\phi(0))] \tag{4.36}$$

the two-point function of the vacuum state in the massive case is given by:

$$\omega(\phi(x)\phi(0)) = \lim_{\epsilon\downarrow 0} \frac{4m}{(4\pi)^2(\sigma_\epsilon(x,0))^{\frac{1}{2}}} K_1(m(\sigma_\epsilon(x,0))^{\frac{1}{2}}) \tag{4.37}$$

where $\sigma_\epsilon(x, 0)$ is the half squared geodesic distance, in this case in the Minkowski spacetime and K_1 is a modified Bessel function of order one. For simplicity, we restrict to asymptotic case for which, $mR \gg 1$.
The modified Bessel function $K_1(x')$ behaves asymptomatically as follows:

$$K_1(x') \simeq e^{-x'}[\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{x'}} + O((\frac{1}{x'})^{3/2})]$$

then by using this approximation the expectation value of the normal ordered energy density looks like:

$$k(:\rho:) = \lim_{\mathbf{x}\to 0}\{-\frac{1}{2}(\partial_t^2+\Delta+m^2) \sum_{\mathbf{n}\in\mathbb{Z}^3\backslash\{0\}} \frac{4m}{(4\pi)^2((2\pi n_1 R + x_1)^2 + (2\pi n_2 R + x_2)^2 + (2\pi n_3 R + x_3)^2 - t^2)^{\frac{1}{2}}}$$

$$e^{-m((2\pi n_1 R+x_1)^2+(2\pi n_2 R+x_2)^2+(2\pi n_3 R+x_3)^2-t^2)^{\frac{1}{2}}}$$

$$[\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{m((2n_1\pi R + x_1)^2 + (2n_2\pi R + x_2)^2 + (2n_3\pi R + x_3)^2 - t^2)^{\frac{1}{2}}}}$$

$$+O((\frac{1}{m((2n_1\pi R + x_1)^2 + (2n_2\pi R + x_2)^2 + (2n_3\pi R + x_3)^2 - t^2)^{\frac{1}{2}}})^{3/2})]\}$$

we calculate the derivatives and impose the limit. Then we have:

$$k(:\rho:) = -\frac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^3\backslash\{0\}} [\frac{3\sqrt{m\pi/2}}{4\pi^2} \frac{e^{-m((2\pi n_1 R)^2+(2\pi n_2 R)^2+(2\pi n_3 R)^2)^{\frac{1}{2}}}}{((2\pi n_1 R)^2 + (2\pi n_2 R)^2 + (2\pi n_3 R)^2)^{\frac{7}{4}}}$$

$$+\frac{m\sqrt{m\pi/2}}{2\pi^2}\frac{e^{-m((2\pi n_1 R)^2+(2\pi n_2 R)^2+(2\pi n_3 R)^2)^{\frac{1}{2}}}}{((2\pi n_1 R)^2+(2\pi n_2 R)^2+(2\pi n_3 R)^2)^{\frac{5}{4}}}]$$

the result of the last summation is convergent and we call it Γ_2. Now we should find a solution of the massive Klein-Gordon equation whose expectation value of the energy density cancel the Casimir effect, because, as we know before, the Einstein tensor is zero:

$$\frac{1}{2}(\dot{f}^2+m^2f^2)+\Gamma_2=\frac{G_{\mu\nu}}{8\pi G}$$

$$\Rightarrow \dot{f}^2+m^2f^2+2\Gamma_2=0 \tag{4.38}$$

where the solution of this equation is as follows:

$$f=\frac{(2|\Gamma_2|)^{\frac{1}{2}}}{m}\sin(mt+c) \tag{4.39}$$

where c is a constant.

4.4 Finding a Solution for the Semi-Classical Einstein equation on de Sitter spacetime

For calculating the solution of the semi-classical Einstein equation we use the state of low energy on de Sitter spacetime for which we have given already a summary in section 3.4. Since we want to study cases, for which $\eta_k(\sigma) := S_k(t)a(t)$ can be expressed in terms of elementary functions, we restrict our calculations to minimal coupling, $\xi = 0$ and choose a mass for which the index of Hankel function of the second kind n, which is given by:

$$n := \sqrt{\frac{9}{4}-2(\frac{m^2}{2H^2}+6\xi)}$$

is equal to $\frac{1}{2}$ or $\frac{3}{2}$. Then the two cases corresponds to $m^2 = 2H^2$ and $m^2 = 0$, respectively. We consider the massive case.

For massive case, i.e. $m^2 = 2H^2$, we have $n = \frac{1}{2}$ and the corresponding Hankel function of the second kind is as follows:

$$H_n^{(2)}(x)=J_n(x)-iY_n(x) \quad \Rightarrow \quad H_{\frac{1}{2}}^{(2)}(x)=J_{\frac{1}{2}}(x)-iY_{\frac{1}{2}}(x)$$

where $J_n(x)$ is a Bessel function and $Y_n(x)$ is Bessel function of second kind. For $n = \frac{1}{2}$ these two functions are equal to:

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x\ , \qquad Y_{\frac{1}{2}}(x)=\frac{J_{\frac{1}{2}}(x)\cos\frac{1}{2}\pi-J_{-\frac{1}{2}}(x)}{\sin\frac{1}{2}\pi}=-J_{-\frac{1}{2}}(x)$$

then the Hankel function of the second kind reads:

$$H^{(2)}_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\sin x + i\sqrt{\frac{2}{\pi x}}\cos x$$

By inserting it in 3.42 we obtain the function $\eta_k(\sigma)$:

$$\eta_k(\sigma) = \frac{1}{2k}e^{ik\sigma}$$

Then the relation 3.47 looks like:

$$[\mathcal{R}\tilde{\mathcal{W}}_2^{\omega,s}](\sigma) = \frac{H^4\eta^2}{2(2\pi)^3}\int_{\mathbb{R}^3} d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}[(1+2\mu_k^2)(k\sigma^2+\frac{3}{2k}) \tag{4.40}$$

$$+2\mu_k|\lambda_k|Re\{e^{i(\arg\lambda_k+2k\sigma)}(\frac{3}{2k}+i\sigma)\}]$$

By choosing a Gaussian test function:

$$f(t) = e^{-\frac{(t-t_0)^2}{\epsilon^2}}$$

the only task to calculate is λ_k and μ_k.

For gaining λ_k and μ_k we should first calculate c_1 and c_2:

$$c_1(k) = \int_I dt f(t)(ke^{-4Ht}+\frac{3H^2}{2k}e^{-2Ht}) = \sqrt{\pi}\alpha e^{-3Ht_0}(ze^{4\alpha^2}+\frac{3}{2z}e^{\alpha^2})$$

where we changed the variables to $\alpha = \epsilon H$ and $z = kH^{-1}e^{-Ht_0}$. $c_2(k)$ Results from the following integral:

$$c_2 = \int_I dt f(t)(e^{2ik\sigma}(\frac{3H^2}{2k}e^{-2Ht} - iHe^{-3Ht}))$$

$$= \frac{3H^2}{2k}\int_{-\infty}^{\infty} dt \exp(-\frac{(t-t_0)^2}{\epsilon^2} - 2Ht + 2ik\sigma) - iH\int_{-\infty}^{\infty} dt \exp(-\frac{(t-t_0)^2}{\epsilon^2} - 2Ht + 2ik\sigma)$$

One uses an approximation to solve this integral. One assumes $\alpha \ll 1$ and then performs a Taylor expansion of $\sigma(t)$ to linear order around t_0. We write here only the results:

$$c_1 \approx \sqrt{\pi}\alpha e^{-3Ht_0}(z+\frac{3}{2z}) \tag{4.41}$$

$$c_2 \approx \sqrt{\pi}\alpha e^{-3Ht_0}e^{-\alpha^2z^2}\sqrt{1+\frac{9}{4z^2}}\exp(-i(\arctan\frac{2z}{3}+2z)) \tag{4.42}$$

Now for completing the calculation of expectation value of normal ordered energy density we should know $[\mathcal{R}\tilde{G}_1](\sigma)$. For $m^2 = 2H^2$ one has:

$$[\mathcal{R}\tilde{G}_1](\sigma) = \frac{H^4}{4\pi^2}(\frac{2\sigma^4}{r^4}+\frac{3\sigma^2}{2r^2}+\frac{23}{240})$$

$$= \{\lim_{\epsilon\to 0} \frac{H^4}{2(2\pi)^3} \int_{\mathbb{R}^3} d\mathbf{p}\, e^{-\epsilon p} e^{i\mathbf{p}\cdot\mathbf{r}} (\sigma^4 p + \frac{3\sigma^2}{2p})\} + \frac{23H^4}{960\pi^2} \tag{4.43}$$

At the end, we need only $[P_x\mathcal{G}_1]$ and $(\partial_t)^a(\partial_t)^b C_{ab}$ where $(\partial_t)^a$ is the worldline tangent of an isotropic observer γ.

For our case, $m^2 = 2H^2$ in the de Sitter spacetime, it yields:

$$[P_x\mathcal{G}_1] = \frac{1}{4\pi^2}\frac{H^4}{20} \tag{4.44}$$

and,

$$(\partial_t)^a(\partial_t)^b C_{ab} = 4\alpha H^4 - 6\beta H^4 \tag{4.45}$$

By imposing a renormalization condition for the energy density as follows:

$$\lim_{t\to\infty} \omega(:\rho(t):) = 0$$

the coefficients of $(\partial_t)^a(\partial_t)^b C_{ab}$ must obey:

$$4\alpha - 6\beta = \frac{19}{240}\frac{1}{4\pi^2}$$

For simplicity one introduces the auxiliary function by:

$$z \mapsto u(z) := \frac{c_1}{2\sqrt{c_1^2 - |c_2|^2}} = \frac{z^2 + \frac{3}{2}}{2\sqrt{(z^2+\frac{3}{2})^2 - e^{-2\alpha^2 z^2}(\frac{9}{4}+z^2)}}$$

Then the energy density for the state of low energy which is smeared by Gaussian test function is given by:

$$\omega_{SLE}(:\rho:) = \frac{H^4}{4\pi^2} e^{-2(t-t_0)} \int_{\mathbb{R}_+} dz\, z[2(u(z) - \frac{1}{2})(z^2 e^{-2(t-t_0)} + \frac{3}{2})$$
$$- \frac{\sqrt{u(z)^2 - \frac{1}{4}}}{\sqrt{1+\frac{4z^2}{9}}}(3\cos(2z(1-e^{-H(t-t_0)}))(\frac{4z^2 e^{-H(t-t_0)}}{9} + 1)$$
$$+2z\sin(2z(1-e^{-H(t-t_0)}))(e^{-H(t-t_0)} - 1))] \tag{4.46}$$

Now we can use this result for finding a solution for the semi-classical Einstein equation in the de Sitter spacetime. We have from (4.12) the following relation for the expectation value of the normal ordered energy density:

$$\omega_{cs}(:\rho:) = \omega_{SLE}(:\rho:) + \rho(f)$$

where $\omega_{cs}(:\rho:)$ denotes the expectation value of the normal ordered energy density of coherent state and $\omega_{SLE}(:\rho:)$ means the expectation value of the normal ordered energy density in the state of low energy. The energy density in the coherent state can be obtained by the semi-classical Einstein equation:

$$\omega_{cs}(:\rho:) = \frac{G_{00}}{8\pi G}$$

first we should calculate the zero-zero component of Einstein tensor in the sitter spacetime. The metric in the de Sitter spacetime is given by:

$$g_{\mu\nu} = diag(-1, e^{2Ht}, e^{2Ht}, e^{2Ht}) \tag{4.47}$$

and the Einstein tensor reads:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{4.48}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and the Ricci scalar respectively. The Ricci tensor is given by:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\alpha}_{\nu\mu} - \partial_\nu \Gamma^{\alpha}_{\alpha\mu} + \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\alpha\mu} \tag{4.49}$$

where $\Gamma^{\alpha}_{\nu\mu}$ is a Christoffel symbol and it is given by:

$$\Gamma^{\alpha}_{\nu\mu} = \frac{1}{2}g^{\alpha\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\nu\mu}) \tag{4.50}$$

then the zero-zero component of the Einstein tensor in the de Sitter spacetime is :

$$G_{00} = 3H^2 \tag{4.51}$$

where H, the Hubble parameter, in the de Sitter spacetime is a constant. Then the semi-classical Einstein equation becomes :

$$\omega_{SLE}(:\rho(t):) + \rho(f(t)) = \frac{3H^2}{8\pi G} \tag{4.52}$$

to be more precise:

$$\omega_{SLE}(:\rho:) + \frac{1}{2}(\dot{f}^2 + m^2 f^2) = \frac{3H^2}{8\pi G} \tag{4.53}$$

where f is a homogeneous isotropic solution of the Klein Gordon equation, which in the de Sitter spacetime looks like:

$$(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} + m^2)f = 0 \tag{4.54}$$

The right hand side of the relation (4.53) is a constant and this relation is correct for all $t \in I$. Since the function f is a real solution of the Klein-Gordon equation and from (4.54), we conclude that $\rho(f(t))$ takes only positive values for all t. By taking the limit of

$t \to -\infty$ the expectation value of the energy density in the state of low energy increases in the direction of infinity and it rises at a specific time above $\frac{3H^2}{8\pi G}$ and consequently, $\rho(f(t))$ should be negative. We can conclude that in the de Sitter spacetime there exists no solution for the semi-classical Einstein equation for all $t \in I$. But maybe we can solve this equation for when $\omega_{SLE}(:\rho(t):) < \frac{3H^2}{8\pi G}$.

For solving the semi-classical Einstein equation and finding the function f, we take a time derivative from the semi-classical Einstein equation:

$$\dot{\rho}_{SLE}(t) + \dot{f}\ddot{f} + m^2 f\dot{f} = 0$$

$$\dot{f}(\ddot{f} + m^2 f) = -\dot{\rho}_{SLE}(t) \tag{4.55}$$

where $\dot{\rho}_{SLE}(t)$ denotes the energy density in the state of low energy. By using the Klein-Gordon equation we have:

$$-3H\dot{f}^2 = -\dot{\rho}_{SLE}(t)$$

$$f = \int_I dt\,[\frac{\dot{\rho}_{SLE}(t)}{3H}]^{\frac{1}{2}} + C \tag{4.56}$$

where C is a constant. But unfortunately, this is not a solution of the Klein-Gordon equation. The real homogeneous solution of the Klein-Gordon equation in de Sitter spacetime is given by:

$$f(t) = \exp(\frac{-3H \pm \sqrt{9H^2 - 4m^2}}{2}) \tag{4.57}$$

and for the case of $m^2 = 2H^2$ they are:

$$f_1(t) = e^{-Ht} \qquad , \qquad f_2(t) = e^{-2Ht} \tag{4.58}$$

which are totally different from (4.56). If $F(t)$ denotes the general solution of the Klein-Gordon equation, we decompose it as follows:

$$F(t) := ae^{-Ht} + be^{-2Ht}$$

where a and b are proper constants. Now we try to find the solution of the semi-classical Einstein equation with this function. We use $F(t)$ in semi-classical Einstein equation and try to find proper values for constants a and b for which the equation holds approximately. For finding the constants a and b for which the equation is minimum we should solve the following relation:

$$\frac{\partial}{\partial a}[\int_I (\omega_{SLE}(:\rho(t):) + \rho(F(t,a,b)) - \frac{3H^2}{8\pi G})\,dt] = 0$$

$$\frac{\partial}{\partial b}[\int_I (\omega_{SLE}(:\rho(t):) + \rho(F(t,a,b)) - \frac{3H^2}{8\pi G})\,dt] = 0$$

where I is a suitable interval in which $\omega_{SLE}(:\rho(t):)$ has the lowest values in comparison with other t.
By solving and studying these relations numerically, we remarked that the minimum of the semi-classical Einstein equation is for a and b equal to zero. Therefore we can conclude that in the de Sitter spacetime which such an ansatz as the coherent state, there is no coherent state which can solve the semi-classical Einstein equation.

4.5 Solution of the Semi-Classical Einstein Equation on General Robertson-Walker Spacetimes

In Robertson-Walker spacetime the Hubble rate is not constant, i.e. it is time dependent. In general case, we do not know anything about scale factor a and consequently, Hubble rate. Therefore we replace the variable of the Klein-Gordon equation by scale factor; and for semi-classical Einstein equation analogously.
Let us to write the semi-classical Einstein equation in Robertson-Walker spacetime again:

$$\omega_{SLE}(:\rho:) + \rho(f) = \frac{3H^2}{8\pi G} \equiv 3m_P^2 H^2 \tag{4.59}$$

where m_P is the *Planck mass*. The *standard deviation* of the Gaussian test function ϵ, which is used in energy density of state of low energy is much bigger than Planck mass, $\epsilon \gg \frac{1}{m_p}$. It means that the expectation value of energy density evaluated by the state of low energy is much smaller than the right hand side of the last relation, $\omega_{SLE}(:\rho:) \ll 3m_P^2 H^2$. Therefore we neglect it and try to find a homogeneous and isotropic solution f of the Klein-Gordon equation which fulfils the semi-classical Einstein equation alone.
The homogeneous and isotropic Klein-Gordon equation for conformal coupling constant looks like:

$$(\partial_t^2 + 3H\partial_t + m^2 + \frac{R}{6})f(t) = 0 \tag{4.60}$$

where R is the Ricci scalar and in Robertson-Walker spacetime is, $R = \dot{H} + 2H^2$. If we define $h(t)$ by, $h(t) := a(t)f(t)$ and by knowing, $\partial_t = a^{-1}\partial_\sigma$, where σ is the conformal time, we can represent the Klein-Gordon equation much simpler by:

$$(\partial_\sigma^2 + a^2 m^2)h(t) = 0 \tag{4.61}$$

the energy density with conformal coupling constant, which is given by:

$$\rho(f(t)) := \frac{1}{2}\dot{f}^2(t) + \frac{1}{2}m^2 f^2(t) - \xi(R^{00} - \frac{1}{2}g^{00}R)f^2(t) \tag{4.62}$$

$$= \frac{1}{2}\dot{f}^2(t) + \frac{1}{2}(m^2 - H^2)f^2(t)$$

is reformulated to:

$$\rho(f(t)) = \frac{1}{2a^4}((\partial_\sigma h)^2 - 2aH(\partial_\sigma)h + a^2m^2h^2) \tag{4.63}$$

where for the last step of (4.60) we use (4.48) and (4.51). The left hand side of (4.61) is given by the semi-classical Einstein equation,

$$G_{00} = 8\pi G\rho \tag{4.64}$$

by putting the Planck mass equal to one, $m_P = 1$, we have:

$$\rho = 3H^2$$

then the semi-classical Einstein equation looks like:

$$\frac{1}{2a^4}((\partial_\sigma h)^2 - 2aH(\partial_\sigma h)h + a^2m^2h^2) = 3H^2 \tag{4.65}$$

now one should gain $h(\sigma(t))$ from (4.59) and (4.63). But the problem is that we do not know H explicitly in function of $\sigma(t)$ in general case. Actually the scale factor should be first determined then the Hubble rate can be formulated as a function of σ. Since we want to solve the semi-classical Einstein equation for all scale factors, we solve the Klein-Gordon and the semi-classical Einstein equation with respect to $a(t)$. We replace the partial derivatives with respect to σ with scale factor dependent one, i.e. $\partial_\sigma = a^2H\partial_a$. Consequently, the Klein-Gordon equation is represented by:

$$[(a^2H\partial_a)^2 + m^2a^2]h = 0 \tag{4.66}$$

and for semi-classical Einstein equation it yields:

$$\frac{1}{2}H^2(\partial_a h)^2 - \frac{H^2}{a}(\partial_a h)h + \frac{m^2}{2a^2}h^2 = 3H^2 \tag{4.67}$$

for solving these equations first we try to find a solution for massless case. Obviously, for massless case the Klein-Gordon equation is:

$$(a^2H\partial_a)^2h = 0 \tag{4.68}$$

and the semi-classical Einstein equation is:

$$\frac{1}{2}(\partial_a h)^2 - \frac{1}{a}(\partial_a h)h = 3 \tag{4.69}$$

the two solutions of the last equation is:

$$h(a(t)) = \sqrt{6}a \sinh\left(c_1 - \log(a)\right),$$
$$h(a(t)) = \sqrt{6}a \sinh\left(c_1 + \log(a)\right) \tag{4.70}$$

where c_1 is a constant. By inserting these functions in massless Klein-Gordon equation, one gains the Hubble parameter:

$$H(a(t)) = \frac{c_2}{a^3} \tag{4.71}$$

and c_2 is another constant. But we know from observations that the Hubble parameter is given by:

$$\frac{H^2}{H_0^2} = \frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \Omega_\Lambda \tag{4.72}$$

where H_0 is the Hubble parameter at the time of observation, Ω_r and Ω_m denote the radiation density and the matter density today, respectively. Ω_Λ is the cosmological constant or vacuum density today. By considering that Ω_r is negligible and by using Taylor expansion the Hubble parameter looks like:

$$H(t) \approx H_0\sqrt{\Omega_\Lambda}(1 + \frac{\Omega_m}{2\Omega_\Lambda a^3})$$

if we compare this Hubble parameter with the one which is yielded from the semi-classical Einstein equation in (4.67), we remark that the calculated one lacks a constant that should be added to it. For calculating the Hubble parameter which is more similar to the Hubble rate in (4.72), we add the cosmological constant to the semi-classical Einstein equation for the massless case:

$$\frac{1}{2}H^2(\partial_a h)^2 - \frac{H^2}{a}(\partial_a h)h + \Omega_\Lambda = 3H^2 \tag{4.73}$$

if one uses the Klein-Gordon equation, the first derivative of the function $h(a(t))$ is, $\partial_a h(a(t)) = \frac{c}{a^2 H}$. We insert it in the semi-classical Einstein equation and then we have:

$$\partial_a H = \frac{H(18a^4H^2 - 6\Omega_\Lambda a^4 + 3c^2)}{a(-6a^4H^2 - 2\Omega_\Lambda a^4 - c^2)}.$$

For simplifying this equation we define a new function by, $K(a(t)) = 3H^2 - \Omega_\Lambda$. Finally, the last differential equation is reformulated to:

$$\frac{a\,\partial \ln(K + \Omega_\Lambda)}{\partial a} = -6(\frac{2a^4K + c^2}{2a^4(K + \Omega_\Lambda) + c^2}). \tag{4.74}$$

For solving this equation, we assume the case for which $a \gg 1$ and consequently, $K \ll 1$. Then $K(a)$ is obtained from the following equation:

$$\frac{a\,\partial \ln(K + \Omega_\Lambda)}{\partial a} = -6(1 - \frac{1}{1 + c^2/4a^4\Omega_\Lambda}) \tag{4.75}$$

we solve this equation in the following steps:

$$\int_{\ln(K+\Omega_\Lambda)}^{\ln\Omega_\Lambda} d\ln(K'+\Omega_\Lambda) = \int_a^\infty (-6(1-\frac{1}{de^{-4\ln a'}}))\, d\ln a'$$

$$-\ln(\frac{K+\Omega_\Lambda}{\Omega_\Lambda}) = [-6(a' - \frac{1}{-4}(-4a' - \ln(1+de^{-4\ln a'})))]\ |_a^\infty$$

$$\frac{K+\Omega_\Lambda}{\Omega_\Lambda} = (1+\frac{d}{a^4})^{\frac{3}{2}}$$

where $d = c^2/4\Omega_\Lambda$. If one expands the last relation for large a then it yields:

$$K \approx \frac{3}{2}\frac{d\,\Omega_\Lambda}{a^4} + O(a^{-8}) = \frac{3}{4}\frac{c^2}{a^4} + O(a^{-8})$$

now in this relation we gained the cosmological constant part and also the radiation part of the Hubble parameter in (4.72). Maybe if one solves the semi-classical Einstein equation with the cosmological constant for the massive case, also the matter part of the Hubble parameter in (4.72) can be obtained.

Chapter 5

Conclusion

As a conclusion we should give a brief summary of what we have done in this master thesis. We have successfully calculated the Casimir effect in 3-dimensional torus spacetime and used the results to solve the semi-classical Einstein equation. The semi-classical Einstein equation has been solved properly with a suitable coherent state for both massless and massive scalar free fields in 3D torus spacetime.

We have tried to solve the semi-classical Einstein equation in de Sitter spacetime with the help of a state of low energy and its corresponding coherent state in this spacetime. As we know, the Hubble rate is constant in de Sitter spacetime and consequently the right hand side of semi-classical Einstein equation is constant. On the other hand, the energy density of state of low energy rises in the past and its value for a specific time becomes larger than the right hand side of the semi-classical Einstein equation and since both, the energy density of the state of low energy and also the right hand side of the semi-classical Einstein equation are positive-valued, then we should find a homogeneous and isotropic solution whose energy density is negative but such a solution does not exist. Therefore we can claim that there exists no coherent state with respect to a state of low energy in de Sitter spacetime which can solve the semi-classical Einstein equation.

We have also solved the semi-classical Einstein equation in a general spatially flat Friedman-Robertson-Walker spacetime. We have assumed that the energy density of state of low energy is negligible in comparison to the total energy density in Robertson-Walker spacetime. Therefore we solved the semi-classical Einstein equation by the classical energy density of the Klein-Gordon solution. After that we have solved it for the massless case and it yielded a suitable homogeneous and isotropic state in that spacetime, but the resulting Hubble rate was different from the Hubble rate which can be inferred from observations. Therefore, we reformulated the semi-classical Einstein equation by adding the cosmological constant to it. It yielded an ordinary differential equation and we solved it approximately for the case of large scale factors.

Chapter 6

Appendix

6.1 Distributions

Let us start by defining test function. If $\Gamma \subseteq \mathbb{R}^n$ would be an arbitrary domain, $\mathcal{C}_0^\infty(\Gamma)$ is the set of all infinitely differentiable continuous functions $f \in \Gamma$ with compact support. We call the closure of a set of point $x \in \Gamma$ the support of a function if we have $f(x) \neq 0$ for all $x \in supp\, f$. A sequence $\{f_k\}_{k=1}^\infty$ in the vector space $\mathcal{C}_0^\infty(\Gamma)$ is called to be convergent to $f \in \mathcal{C}_0^\infty(\Gamma)$ if and only if:
a)There is a compact set $L \subset \Gamma$ that $supp(f_k) \subset L$
b)$D^\alpha f_k$, the α-th derivative of f_k, converge uniformly to $D^\alpha f$ on L for each multi-index α
The space $\mathcal{C}_0^\infty(\Gamma)$, equipped with this convergence of sequence and a compact support, is called the fundamental space $\mathcal{D}(\Gamma)$ and we name its elements *test functions*. We denote also $\mathcal{E}(\Gamma)$ the fundamental space without any compact support.
A linear functional ϕ on $\mathcal{D}(\Gamma)$, which is continuous, i.e. if there is a convergent sequence $\{f_k\} \to f$ for $f_k, f \in \mathcal{D}(\Gamma)$ then $\phi(f_k) \to \phi(f)$, is called a generalized function or distribution. By $\mathcal{D}'(\Gamma)$ we denote the set of all distributions, which are linear maps $\mathcal{D}(\Gamma) \to \mathbb{R}$. Dirac delta δ_a for $a \in \Gamma$ as an arbitrary fixed point is the simplest example of linear continuous distribution:

$$\delta_a(f) = \langle \delta, f \rangle = f(a) \tag{6.1}$$

We define the dual pairing of $\phi(x) \in \mathcal{D}'(\Gamma)$ and $f(x) \in \Gamma$ as another example of distribution on $\mathcal{D}(\Gamma)$ by:

$$\phi(f) = \langle \phi, f \rangle := \int_\Gamma d^n x\, \phi(x) f(x) \tag{6.2}$$

two distribution are equal, i.e. $\phi(f) = \rho(f)$, $\forall f \in \Gamma$, if and only if $\phi = \rho$. We also denote $\mathcal{E}'(\Gamma)$ the space of distribution with compact support.

6.2 Schwartz Distributions

We start with the definition of a smooth function in $\mathcal{S}(\mathbb{R}^m)$. We call a function $f \in \mathcal{S}(\mathbb{R}^m)$ smooth, if:

$$\sup_x |x^\alpha \partial^\beta f(x)| < \infty \tag{6.3}$$

for all multi-indices α and β. Then we denote ψ a Schwartz distribution or tempered distribution, as an distribution in the topological dual of $\mathcal{S}(\mathbb{R}^m)$. A linear functional on $\mathcal{S}(\mathbb{R}^m)$ is a Schwartz distribution, i.e. $\psi \in \mathcal{S}'(\mathbb{R}^m)$, if and only if there exist an $N \in \mathbb{N}$ and a constant $C \in \mathbb{R}_+$, such that,

$$|\psi(f)| \leq C \sum_{|\alpha|+|\beta|\leq N} \sup_x |x^\alpha \partial^\beta f(x)| \tag{6.4}$$

6.3 Wave Front set

Let us start by studying more the singularity structure of distributions. The best way to resolve a singular distribution is with the help of its Fourier transformation.
We call a distribution $\phi \in \mathcal{E}^{'}(\mathbb{R}^m)$ smooth if and only if for every $n \in \mathbb{N}$ there is a constant $C_n \in \mathbb{R}$ we have:

$$|\hat{\phi}(p)| \leq C_n(1 + |p|)^{-n} \tag{6.5}$$

If we multiply a distribution $\psi \in \mathcal{D}'(\mathbb{R}^m)$ by a test function $f \in \mathcal{D}(\mathbb{R}^m)$, then ψf would be a distribution with compact support, i.e. $\psi f \in \mathcal{E}'(\mathbb{R}^m)$. From these explanations. we can now come to a conclusion that the singular support of a distribution $\phi \in \mathcal{D}'(\mathbb{R}^m)$ is the complement of the set of points $x \in \mathbb{R}^m$ such that there is a function $f \in \mathcal{D}(\mathbb{R}^m)$ with $f(x) = 1$ which $\widehat{\phi f}$ is rapidly decreasing.

We can have a detailed analysis of the singularity structure of ϕ by the set of directions in which the singularity occurs. We define an **open conic neighborhood** Γ as a neighborhood of $k_0 \in (\mathbb{R}^m)$ which is invariant under the action of $\mathbb{R}_+$ by multiplication, i.e. if $k \in \Gamma$ is a point in conic neighborhood of k_0 then implies $\mu k \in \Gamma$ for $\mu \in (0, \infty)$. So, we call a point $(x_0, k_0) \in \mathbb{R}^m \times (\mathbb{R}^m \ \{0\})$ a regular directed point of ϕ if there a test function $f \in \mathcal{D}(\mathbb{R}^m)$ which is non-zero at x_0 and also if there is a constant $C_n \in \mathbb{R}$ for every $n \in \mathbf{N}$, such that the Fourier transformation of ϕf obeys the following inequality:

$$|\widehat{\phi f}(k)| \leq C_n(1 + |k|)^{-n} \tag{6.6}$$

for all k in a conic neighborhood of k_0. We call the complement of the set of regular directed points of ϕ in $\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$, the **wave front set** of ϕ, $WF(\phi)$. Obviously if ϕ is smooth, then $WF(\phi)$ is empty.

In general curved spacetime the wave front set is subset of cotangent vector bundle on the manifold M, i.e. $WF(\phi) \subset T^*M$.

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Erklärung gemäß Prüfungsordnung

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Hamburg, 31. Oktober 2012 Reza Safari