Klausur Quantum Field Theory I
Wintersemester 2009/2010

Montag, 25.1.2010, Hörsaal 3, 12:00-13:30

1. Let χ be the mapping from $\mathbb{C}^4$ into the set of complex 2×2-matrices, defined by

$$\chi : x \equiv (x^0, x^1, x^2, x^3) \mapsto \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} . \tag{1}$$

   (a) Show that χ is bijective.

   (b) Let

$$\Lambda(A, B)x = \chi^{-1}(A\chi(x)B^*) \ , \ A, B \in \mathrm{SL}(2, \mathbb{C}) . \tag{2}$$

   Show that $\Lambda(A, B)$ is a complex Lorentz transformation, i.e.

$$(\Lambda(A, B)x)^\mu (\Lambda(A, B)x)_\mu = x^\mu x_\mu \ \forall \ x \in \mathbb{C}^4 .$$

   *H*int: Compute the determinant of $\chi(x)$.

2. Let

$$H = \frac{1}{i}\vec{\alpha} \cdot \nabla + \beta m \tag{3}$$

   be the Dirac Hamiltonian on the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ with hermitean 4×4-matrices $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \beta$ which satisfy the relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \ , \ \alpha_i \beta + \beta \alpha_i = 0 \ , \ \beta^2 = 1 \ \ , \ i, j = 1, 2, 3 . \tag{4}$$

   Determine the spectrum of H and the subspace of $\mathfrak{H}$ corresponding to negative energies.

3. Draw all graphs which occur in the calculation of the time ordered product of Wick cubes of a free hermitean scalar field

$$T \frac{:\varphi(x)^3:}{3!} \frac{:\varphi(y)^3:}{3!} \frac{:\varphi(z)^3:}{3!} \tag{5}$$

   and determine the contribution of the connected tree graphs.

4. Compute the commutator

$$[:\varphi(x)^n:, \varphi(y)] \tag{6}$$

   for a free hermitean scalar field φ.