Exercises for quantum field theory
Wintersemester 2009/10

to be submitted on 28.10.2009 after the lecture

1. For every permutation $\sigma \in \mathrm{S}_n$ let
$$\big(U(\sigma)\Phi\big)(x_1,\dots,x_n) = \Phi(x_{\sigma(1)},\dots,x_{\sigma(n)})$$
be a linear operator in $\mathrm{L}^2(\mathbb{R}^n)$. We set
$$E_+ = \frac{1}{n!}\sum_{\sigma\in\mathrm{S}_n} U(\sigma)\ ,\ E_- = \frac{1}{n!}\sum_{\sigma\in\mathrm{S}_n} \mathrm{sign}(\sigma)U(\sigma)\ .$$
Show:
   (a) $\sigma \mapsto U(\sigma)$ is a unitary representation of the permutation group.
   (b) $[E_\pm, U(\sigma)] = 0$ for all $\sigma \in \mathrm{S}_n$.
   (c) E_+ and E_- are orthogonal projections, i.e. $E_\pm^2 = E_\pm = E_\pm^*$.
   (d) E_+ und E_- are mutually orthogonal.

2. Let $\mathfrak{H}_n$, $n = 0, 1, 2, \dots$ be a sequence of Hilbert spaces. Show that the set of sequences $(\Phi_0, \Phi_1, \Phi_2, \dots)$ with $\Phi_n \in \mathfrak{H}_n$ and $\sum_{n=0}^\infty \|\Phi_n\|^2 < \infty$ is a Hilbert space with respect to the scalar product
$$\big\langle(\Phi_0,\Phi_1,\Phi_2,\dots),(\Psi_0,\Psi_1,\Psi_2,\dots)\big\rangle = \sum_{n=0}^\infty \big\langle\Phi_n,\Psi_n\big\rangle\ .$$

3. (a) Determine the eigenvalues and eigenfunctions of the operator
$$a = x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}$$
on $\mathrm{L}^2(\mathbb{R})$.
   (b) Show that the operator
$$a^* = x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}$$
has no eigenvalues.

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4. Show for the creation and annihilation operators as defined in the lectures

   (a)
   $$\langle \Phi, a(f)\Psi \rangle = \langle a(f)^*\Phi, \Psi \rangle \ ,$$
   $f \in \mathfrak{H}_1$, $\Phi, \Psi \in \mathfrak{H}^+$, $\langle \Phi, N\Phi \rangle, \langle \Psi, N\Psi \rangle < \infty$, N particle number operator.

   (b) $[a(f), a(g)^*] = \langle f, g \rangle$

   (c) $[a(f), a(g)] = 0 = [a(f)^*, a(g)^*]$

   (d) Let $\Phi \in \mathfrak{H}^+$, $\langle \Phi, N\Phi \rangle < \infty$, with
   $$a(f)\Phi = 0$$
   for all $f \in \mathfrak{H}_1$. Then
   $$\Phi = \lambda\Omega \ ,$$
   $\lambda \in \mathbb{C}$, $\Omega = (1, 0, 0, \ldots)$ vacuum

5. Let A and B $m \times m$-matrices. The exponential function of a matrix is defined as the series
   $$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$
   (Convergence in the sense of convergence of all matrix entries). Show

   (a)
   $$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A, [A, \ldots [A}_{n}, B] \ldots]]$$

   (b) Let $[[A, B]A] = 0 = [[A, B], B]$. Then
   $$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

6. Calculate the mean particle number, and the expectation values of the particle number density and the energy density of the ground state of the (suitably renormalized) Hamiltonian in the Fock space of a nonrelativistic spinless boson,
   $$H = T + \mu N + \bar{c}a(\mathbf{y}) + ca^*(\mathbf{y})$$
   (T kinetic energy, N particle number, $\mu > 0$, $a(\mathbf{y})$ annihilation operator).

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7. Show that a system of identical nonrelativistic bosons with unbounded particle number and attractive 2-particle potential which vanishes at infinity, does not have a lower bound for the energy for any value of the chemical potential.
Hint: Look at the expectation value of the Hamiltonian in coherent states.

8. Show in the sense of distributions
$$\lim_{\varepsilon\downarrow 0}\frac{1}{x+i\varepsilon}=P\frac{1}{x}-i\pi\delta(x)\ .$$
Here P denotes Cauchy's principal value
$$\int dx(P\frac{1}{x})h(x):=\lim_{\varepsilon\downarrow 0}\int_{|x|>\varepsilon}dx\frac{h(x)}{x}$$

9. Let $\mathfrak{H}$ be a real Hilbert space, and let $b(f), f\in\mathfrak{H}$ generate an associative algebra CAR($\mathfrak{H}$) with unit by the relations
   - $f\mapsto b(f)$ is real linear
   - $b(f)^2=\|f\|^2$

   Let J be an isometric (i.e. $\|Jf\|=\|f\|$) for all $f\in\mathfrak{H}$) operator on $\mathfrak{H}$ with $J^2=-1$.

   (a) Show that $\mathfrak{H}$ gets the structure of a complex Hilbert space $\mathfrak{H}_{\mathbb{C}}$ by identifying J with the multiplication by i and by defining an appropriate complex valued scalar product.

   (b) Show that the elements
$$a(f)=\frac{1}{2}(b(f)+ib(Jf))\ ,\ a(f)^*=\frac{1}{2}(b(f)-ib(Jf))\ ,\ f\in\mathfrak{H}$$
   satisfy the canonical anti-commutation relations for fermionic annihilation and creation operators over the single particle space $\mathfrak{H}_{\mathbb{C}}$.

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10. Which SU(2)-matrices A satisfy the equation

$$A\,\mathbf{x}\cdot\vec{\sigma}\;A^* = (R\mathbf{x})\cdot\vec{\sigma}\;,\;\;\mathbf{x}\in\mathbb{R}^3\;,$$

    where R is the rotation with the Euler angles φ, θ and ψ? ($\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ Pauli matrices)

11. Show that every Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$ can be written as product of a Lorentz boost and a rotation.

12. Study the following representations of $\mathrm{SL}(2,\mathbb{C})$ with respect to equivalence.

$$\begin{aligned}\pi_1 &: A \mapsto A\\ \pi_2 &: A \mapsto (A^*)^{-1}\\ \pi_3 &: A \mapsto \overline{A}\end{aligned}$$

    (A^* adjoint matrix, $\overline{A}$ complex conjugated matrix)

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13. Calculate for $p^2 = m^2, p_0 > 0$ the positive definite square root of the matrix
$$\frac{1}{m}\begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}$$
and show that it describes a Lorentz boost.

14. Determine the Lie algebra of the Poincaré group.

15. Determine the little group G_x of a lightlike vector x in Minkowski space, $G_x = \{\Lambda \in \mathcal{L}_+^\uparrow | \Lambda x = x\}$.

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16. Calculate the eigenvalues of the operator $|L|^2$ in the representations

$$V_s(R)\xi \otimes \cdots \otimes \xi := R\xi \otimes \cdots \otimes R\xi$$

of SU(2) on the symmetrized $2s$-fold tensor power of $\mathbb{C}^2$.

17. Let $V_{s_1 s_2}$, $s_1, s_2 \in \mathbb{N}_0/2$ be the following representation of $\mathrm{SL}(2,\mathbb{C})$

$$V_{s_1 s_2}(A)\underbrace{\xi \otimes \cdots \otimes \xi}_{2s_1} \otimes \underbrace{\eta \otimes \cdots \otimes \eta}_{2s_2} := \underbrace{A\xi \otimes \cdots \otimes A\xi}_{2s_1} \otimes \underbrace{\overline{A}\eta \otimes \cdots \otimes \overline{A}\eta}_{2s_2}$$

on the $(2s_1 + 2s_2)$-fold tensor power of $\mathbb{C}^2$, symmetrized in the first $2s_1$ and in the last $2s_2$ factors. After restriction of the representation V_{s_1,s_2} to SU(2) it can be decomposed into irreducible subrepresentations. Which irreducible representations of SU(2)occur? Perform the decomposition in the case $s_1 = s_2 = 1/2$ explicitly.

18. Let φ be a complex valued solution of the Klein-Gordon equation.

    (a) Show that

    $$j_\mu(x) = \frac{i}{2}\big(\overline{\varphi(x)}\partial_\mu\varphi(x) - (\partial_\mu\overline{\varphi(x)})\varphi(x)\big)$$

    is a conserved current.

    (b) Let

    $$\Sigma = \{x \in \mathbb{M}, nx = c\}\ ,\ n \in V_+, n^2 = 1, c \in \mathbb{R}$$

    be a spacelike hyperplane in Minkowski space. Let

    $$Q(\Sigma) = \int \mathrm{d}^4x\,\delta(nx - c)n^\mu j_\mu(x)\ .$$

    Show that $Q(\Sigma)$ is independent of the choice of the hyperplane.

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19. Let ψ be a solution of the Dirac equation.

    (a) Show that
    $$j_\mu(x) = \big\langle \psi(x), \gamma_0\gamma_\mu\psi(x)\big\rangle$$
    is a conserved current.

    (b) Express j_μ by the corresponding 2-component solution of the Klein-Gordon equation.

20. Compute
$$\Delta_+(x) = (2\pi)^{-3}\int \frac{\mathrm{d}^3\mathbf{p}}{2\omega(\mathbf{p})} e^{-ipx}|_{p_0=\omega(\mathbf{p})} \ .$$

21. Let φ be the neutral scalar free field in the Fock space of a spinless massive particle. Show that the transformation
$$\varphi(x) \mapsto \varphi(x) + f(x)$$
with a real solution f of the Klein-Gordon equation preserves the commutation relations, and find a unitary operator U with
$$U\varphi(x)U^{-1} = \varphi(x) + f(x) \ .$$

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22. Show by using the momentum space representation that $\Delta_+^2(x)$ is a well defined distribution.

23. Find a state in the Fock space of the free scalar field for which the expected value of the energy density

$$h(x) = \frac{m^2}{2} :\varphi(x)^2: + \frac{1}{2} :\dot{\varphi}(x)^2: + \frac{1}{2} :\nabla\varphi(x)^2:$$

is negative within some region of Minkowski space.

24. The Feynman propagator Δ_F is defined by

$$i\Delta_F(x) = \begin{cases} \Delta_+(x) & x^0 > 0 \\ \Delta_+(-x) & x^0 < 0 \end{cases}$$

Show that Δ_F is invariant under the full Lorentz group and compute its Fourier transform.

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25. Let φ denote a free hermitean scalar quantum field. Compute the 3-point Wightman function of the Wick squares $:\varphi(x)^2:$ and show that it is a well defined distribution.

26. Show that the charge conjugation operator C,

$$(C\Phi)(p) = V_{s0}(\underset{\sim}{p}\zeta)\Phi(p)$$

with $\zeta = i\sigma_2$, is a unitary map from the single particle space of a particle with unit mass and spin s, with the scalar product

$$\langle \Phi, \Psi \rangle = \int \frac{d^3\mathbf{p}}{2\omega\mathbf{p}} \langle \Phi(p), V_{s0}(\tilde{p})\Psi(p) \rangle \ ,$$

to the same space where in the scalar product V_{s0} is replaced by V_{0s}. Here $V_{s_1 s_2}$ is the representation of $\mathrm{SL}(2,\mathbb{C})$ given by

$$V_{s_1 s_2}(A) \underbrace{\xi \otimes \cdots \otimes \xi}_{2s_1} \otimes \underbrace{\eta \otimes \cdots \otimes \eta}_{2s_2} = \underbrace{A\xi \otimes \cdots \otimes A\xi}_{2s_1} \otimes \underbrace{\overline{A}\eta \otimes \cdots \otimes \overline{A}\eta}_{2s_2} \ .$$

27. Find a local field A_μ in the Fock space of a massive particle with spin $s = 1$ which transforms under Poincaré transformations as

$$U(x,\Lambda)A_\mu(y)U(x,\Lambda)^{-1} = A_\nu(\Lambda y + x)\Lambda^\nu{}_\mu \ .$$

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28. Let ψ denote the quantized free Dirac field. Let

$$j_\mu(x) =: \overline{\psi(x)}\gamma_\mu\psi(x): \ .$$

Compute the expectation value of j_μ in a two particle state and show that it satisfies the continuity equation .

29. Let j_μ be as in exercise 28. Compute the 2-point Wightman function

$$\left\langle \Omega, j_\mu(x) j_\nu(y)\Omega \right\rangle$$

and its Fourier transform .

30. The free Dirac field satisfies the equal time anticommutation relations

$$\{\psi_\alpha(t,\mathbf{x}), \psi^*_\beta(t,\mathbf{y})\} = \delta_{\alpha,\beta}\delta(\mathbf{x}-\mathbf{y})$$

These relations are identical to fermionic anticommutation relations for annihilation and creation operators. Show that the corresponding vacuum state, characterized by the equation $\psi_\alpha(t,\mathbf{x})\Omega = 0$ is the groundstate for the charge $Q = \int d^3\mathbf{x} j_0(t,\mathbf{x})$.