Distinguished ground states in FRW spacetimes

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Outline of the Talk

- Motivations, *i.e.*, trivia about cosmological models,
- On the geometry of the background and on the cosmological horizon,
- On the underlying field theory: form the bulk to the horizon,
- Constructing distinguished states,
- On the Hadamard property of these distinguished states.

Based on

- C. D., Nicola Pinamonti, V. Moretti: Comm. Math. Phys. **285** (2009) 1129
- C. D., Nicola Pianmonti, V. Moretti: 0812.4033 [gr-qc] to appear on JMP

Playground

In the last 100 years and, hopefully, in the last talk we learned:

1) Interactions between matter consituents ⟶ QFT on **flat spacetime**...

2) "except" the gravitational one ⟶ **General Relativity**,

3) Algebraic Quantum Field Theory, ⟶ is the natural tool to formulate QFT on curved backgrounds ⟶ first step towards a true Quantum Gravity.

Natural playground ⟶ Cosmology

- unveils the structure and dynamics of the Universe,
- we can fully use QFT on curved background in the algebraic approach.

The Cosmological Principle and FRW - I

We wish to model the geometry of the Universe!

Occam's razor leads to the Cosmological principle, *i.e.*,

- **spacetime is homogeneous** $\longrightarrow$ at each instant of time, all space points look the same,

This means that $\exists\Sigma_t \in (M, g_{\mu\nu})$ with $t\in\mathbb{R}$ such that $M\sim\Sigma_t\times\mathbb{R}$ and $\forall p,q\in\Sigma_t$, one can found an isometry Ψ of $g_{\mu\nu}$ such that $\Psi(p)=q$.

- **spacetime is isotropic** $\longrightarrow$ there is at each point an observer who sees an isotropic Universe.

This means that $\exists$ a congruence of timelike curves (*a.k.a.*, observers) filling M and with tangent vectors ζ^μ such that, for every pair s_1^μ, s_2^μ at a point $p\in M$ such that $g^{\mu\nu}\zeta_\mu s_{i\nu}=0$, it exists an isometry $\widetilde{\Psi}$, such that $\widetilde{\Psi}(p)=p$ and $\widetilde{\Psi}_*\zeta_1^\mu=\zeta_2^\mu$

$$ds^2=-dt^2+a^2(t)\left[\frac{dr^2}{1+kr^2}+r^2(d\theta^2+\sin^2\theta d\varphi^2)\right].$$

The Cosmological Principle and FRW - II

A direct inspection of

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right],$$

shows that the metric is almost fully determined

- except the parameter $k = 0, \pm 1$ which fixes the topology spatial section: flat planes, spheres or hyperboloids,
- there is still no dynamical content. This determines $a(t)$ and, to this avail, one needs to choose $T_{\mu\nu}$, to solve

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.$$

The Cosmological Principle and FRW - III

Which $T_{\mu\nu}$? Let us start with *classical matter*

We know:

- part of mass-energy in the Universe is ordinary matter (stars, galaxies, clusters),
- their density is so low that they appear like "dust" with density ρ. Hence

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu, \quad \zeta^\mu\zeta_\mu = 1$$

- if we also include a contribution from pressure, then

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu + P\left(g_{\mu\nu} + \zeta_\mu\zeta_\nu\right),$$

which is the stress-energy tensor of a **perfect fluid**. This is the the most general choice for $T_{\mu\nu}$ if the matter is classical.

The Cosmological Principle and FRW - IV

- One needs a further assumption, namely an equation of state: $P = \gamma\rho$.

One can solve Einstein's equation (from now on $\Lambda = 0$ and $k = 0$), *i.e.*

$$G_{tt} = 8\pi T_{tt}, \quad G_{xx} = 8\pi T_{xx} \quad \text{and} \quad \nabla^\mu T_{\mu\nu} = 0.$$

This leads to

$$3\left(\frac{\dot{a}}{a}\right) = 8\pi\rho, \quad 3\frac{\ddot{a}}{a} = 4\pi(\rho + 3P),$$
$$\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a} = 0.$$

Notable choices are:

- $P = 0$ (pure dust) $\longrightarrow a(t) \sim t^{\frac{2}{3}}$ and $\rho a^3(t) = const$,
- $P = \frac{\rho}{3}$ (pure radiation) $\longrightarrow a(t) \sim \sqrt{t}$ and $\rho a^4(t) = const$.

The Cosmological Principle and FRW - V

The standard Cosmological model has several advantages:

1. allows for a description of cosmological redshift,
2. a natural playground to describe the evolution of (classical) matter,
3. above all, it is fairly simple:
   - homogeneity and isotropy,
   - the matter content is "classical",
   - there is an equation of state relating ρ and P.

It has also several problems

1. the homegeneity problem (fine tuning of inital condition),
2. the flatness problem,
3. the singularity problem $\longrightarrow$ ρ diverges whenever $a(t) \to 0$.

A possible way out: let us take seriously QFT!

Our goal

We shall thus consider a Universe filled with a scalar field:

- it provides a natural exit to many problems of classical cosmology,
- it is at the heart of many models such of inflation,
- it leaves many question unanswered: dark energy, dark matter...

In the previous talk we have seen

1. a massive scalar field on a FRW spacetime can be solved in a semiclassical regime:

$$R_{\mu\nu} - \frac{R}{2} = 8\pi \langle : T_{\mu\nu} : \rangle_\omega,$$

2. it leads to an effective cosmological constant,
3. it makes precise the role of quantum fluctuations!

Real problem: Does ω exist? Can a genuine ground state be constructed?

A distinguished class of "cosmological spacetimes" - I

Hyp. 1) Cosmological Principle $\Longrightarrow$

$$g_{FRW} = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 dS^2(\theta,\varphi)\right], \quad M \sim I \times X_3$$

and $a(t) \in C^\infty(I, R^+)$.

Immediate consequences:

1. Consider a co-moving observer as the integral line $\gamma(t)$ of ∂_t. If $M \setminus J^+(\gamma) \neq \emptyset$, then causal signals departing from each $x \in M \setminus J^+(\gamma)$ never reach $\gamma(t)$. Then we call $\partial J^+(\gamma)$ the (past) **cosmological horizon**,
2. if one introduces the conformal time $d\tau = \frac{dt}{a(t)}$ and rescales the metric as

$$g_{FRW} = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-\kappa r^2} + r^2 dS^2(\theta,\varphi)\right],$$

then τ ranges in $(\alpha,\beta) \subset \mathbb{R}$. Sufficient condition for the existence of an horizon is $\alpha > -\infty$ and/or $\beta < \infty$.

A distinguished class of "cosmological spacetimes" - I

Hyp. 2) We set $\kappa = 0$, *i.e.*, $M \equiv I \times \mathbb{R}^3$ hence the spacetime is conformally (a piece of) Minkowski.

Hyp. 3) We restrict the class of scale factors as:

$$a(\tau) = -\frac{1}{H\,\tau} + O\left(\tau^{-2}\right)\ ,$$

$$\frac{da(\tau)}{d\tau} = \frac{1}{H\,\tau^2} + O\left(\tau^{-3}\right), \quad \frac{d^2a(\tau)}{d\tau^2} = -\frac{2}{H\,\tau^3} + O\left(\tau^{-4}\right).$$

- Here H is chosen *positive* and the interval $I \doteq (-\infty, 0)$.

Consequences and Properties - I

1. If $a(\tau) = -\frac{1}{H\tau}$ then $\tau = -e^{-Ht}$, hence

$$ds^2 = -dt^2 + e^{-2Ht}(dr^2 + r^2 d\mathbb{S}^2(\theta, \varphi)), \quad t \in (-\infty, \infty).$$

   This is the cosmological de-Sitter spacetime.

2. For our choice of $a(\tau)$, as $\tau \to -\infty$, the background **"tends to"** de Sitter. Hence we are dealing with an exponential acceleration in the proper time t. This is the the prerequisite of all inflationary models.

Consequences and Properties - II

- There is always a **Cosmological horizon**. Under the coordinate change

$$U = \tan^{-1}(\tau - r)\ , \qquad V = \tan^{-1}(\tau + r),$$

the metric becomes:

$$g_{FRW} = \frac{a^2(U,V)}{\cos^2 U \cos^2 V}\left[-dUdV + \frac{\sin^2(U-V)}{4}dS^2(\theta,\varphi)\right].$$

Theorem:

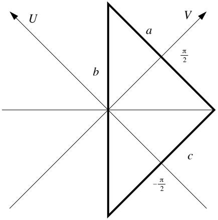
Under the previous assumptions the spacetime (M, g_{FRW}) can be extended to a larger spacetime $(\widehat{M}, \widehat{g})$ which is a conformal completion of the asymptotically flat spacetime at past (or future) null infinity $(M, a^{-2}g_{FRW})$, *i.e.*, "a" plays the role of the conformal factor. The cosmological horizon is

$$\Im^- \doteq \partial J^+(M;\widehat{M}) = \partial M \sim \mathbb{R}\times\mathbb{S}^2.$$

Consequences and Properties - III

- Conformall null infinity $\Im^-$ corresponds to the horizon *(region c in the figure)* and it is a null degenerate manifold with

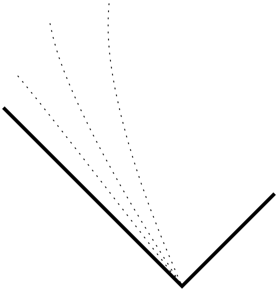
$$g|_{\Im^-} = 0 \cdot dl^2 + H^{-2} \left(d\mathbb{S}^2(\theta, \varphi) \right),$$



Consequences and Properties - III

Furthermore the manifold $M \cup \Im^-$ enjoys:

1. the vector field ∂_τ is a conformal Killing vector for $\widehat{g}$ in M,
2. the vector ∂_τ becomes tangent to $\Im^\pm$ approaching it and coincides with $-H^{-1}\widehat{\nabla}^b a$.



Aim of the analysis:

We want both to model a scalar QFT on a cosmological spacetime and to find a distinguished ground state

Hence from now on, we consider $\Phi : M \to \mathbb{R}$ such that

$$P\Phi = 0, \qquad P = -\Box_{g_{FRW}} + \xi R + m^2 \text{ and } \xi R + m^2 > 0$$

with smooth compactly supported initial data on a Cauchy surface,

N.B. FRW spacetimes are globally hyperbolic, hence Cauchy problems are meaningful.

- Each solution Φ is a smooth function on M, *i.e.*, $\Phi \in C^\infty(M)$.
- The set of solutions $S(M)$ of our equation is a symplectic space if endowed with the Cauchy-independent nondegenerate symplectic form:

$$\sigma(\Phi_1, \Phi_2) \doteq \int_\Sigma (\Phi_1 \nabla_N \Phi_2 - \Phi_2 \nabla_N \Phi_1)\, d\mu_g^{(\Sigma)}.$$

More on classical solutions

Next Problem :

We want to better characterise the space of solutions $S(M)$.

Any $\Phi \in \mathcal{S}(M)$ can be decomposed in modes ($\mathbf{k} \in \mathbb{R}^3$, $k = |\mathbf{k}|$,)

$$\Phi(\tau, \vec{x}) = \int_{\mathbb{R}^3} d^3\mathbf{k} \left[\phi_{\mathbf{k}}(\tau, \vec{x})\widetilde{\Phi}(\mathbf{k}) + \overline{\phi_{\mathbf{k}}(\tau, \vec{x})\widetilde{\Phi}(\mathbf{k})}\right],$$

with respect to the functions

$$\phi_{\mathbf{k}}(\tau, \vec{x}) = \frac{1}{a(\tau)} \frac{e^{i\mathbf{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}} \chi_{\mathbf{k}}(\tau),$$

$\chi_{\mathbf{k}}(\tau)$, is solution of the differential equation

$$\frac{d^2}{d\tau^2}\chi_{\mathbf{k}} + (V_0(\mathbf{k}, \tau) + V(\tau))\chi_{\mathbf{k}} = 0,$$

$$V_0(\mathbf{k}, \tau) := k^2 + \left(\frac{1}{H\tau}\right)^2 \left[m^2 + 2H^2\left(\xi - \frac{1}{6}\right)\right], \quad V(\tau) = O(1/\tau^3).$$

- Furthermore it holds the normalization

$$\frac{d\overline{\chi_{\mathbf{k}}(\tau)}}{d\tau}\chi_{\mathbf{k}}(\tau)-\overline{\chi_{\mathbf{k}}(\tau)}\frac{d\chi_{\mathbf{k}}(\tau)}{d\tau}=i.\quad \forall\tau\in(-\infty,0)$$

Idea: Construct a general solution treating $V(\tau)$ as a perturbation potential over solutions with $V=0$, *i.e.* those in purely de-Sitter background.

Thus for $V(\tau)=0$

$$\chi^0_{\mathbf{k}}(\tau)=\frac{\sqrt{-\pi\tau}}{2}e^{\frac{i\pi\nu}{2}}\overline{H^{(2)}_\nu(-k\tau)},$$

with

$$\nu=\sqrt{\frac{9}{4}-\left(\frac{m^2}{H^2}+12\xi\right)},$$

where $H^{(2)}_\nu$ is the Hankel function of second kind.

Perturbative solutions in the general case

- Let us consider the retarded fundamental solutions $S_{\mathbf{k}}$ of

$$\frac{d^2}{d\tau^2}\chi^0_k(\tau) + (V_0(k,\tau))\chi^0_k(\tau) = 0.$$

- Then the general solutions $\chi_{\mathbf{k}}$ can be constructed

$$\chi_{\mathbf{k}}(\tau) = \chi^0_{\mathbf{k}}(\tau) +$$
$$+\sum_{n=1}^{+\infty}(-1)^n \int_{-\infty}^{\tau} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n S_{\mathbf{k}}(\tau, t_1) S_{\mathbf{k}}(t_1, t_2)\cdots$$
$$S_{\mathbf{k}}(t_{n-1}, t_n) V(t_1) V(t_2) \cdots V(t_n) \chi_{\mathbf{k}}(t_n).$$

The series is convergent

if $|Re\nu| < 1/2$ and $V = O(\tau^{-3})$,
if $\frac{1}{2} \leq |Re\nu| < 3/2$ and $V = O(\tau^{-5})$.

From the bulk to the horizon ...

Bulk) A Weyl C^*-algebra $\mathcal{W}(M)$ can be associated to $(\mathcal{S}(M), \sigma)$. This is, up to $*$-isomorphisms, unique and its non vanishing generators $W_M(\phi)$ satisfy:

$$W_M(-\phi) = W_M(\phi)^*, \quad W_M(\phi)W_M(\phi') = e^{\frac{i}{2}\sigma(\phi,\phi')}W_M(\phi+\phi'),$$

Horizon) The symplectic space of real wavefunctions is:

$$\mathcal{S}(\Im^-) = \left\{\psi \in C^\infty(\mathbb{R}\times\mathbb{S}^2) \mid \psi \in L^\infty, \partial_\ell\psi \in L^1, \widehat{\psi} \in L^1, k\widehat{\psi} \in L^\infty\right\},$$

$$\sigma_{\Im^-}(\psi,\psi') = \int\limits_{\mathbb{R}\times\mathbb{S}^2} \left(\psi\frac{\partial\psi'}{\partial\ell} - \psi'\frac{\partial\psi}{\partial\ell}\right). \quad \forall\psi,\psi' \in \mathcal{S}(\Im^-)$$

Algebra) Since $\sigma_{\Im^-}$ is nondegenerate, we can construct a Weyl C^*-algebra $\mathcal{W}(\Im^-)$ as

$$W_{\Im^-}(\psi) = W^*_{\Im^-}(-\psi), \qquad W_{\Im^-}(\psi)W_{\Im^-}(\psi') = e^{\frac{i}{2}\sigma_{\Im^-}(\psi,\psi')}W_{\Im^-}(\psi+\psi').$$

Distinguished state on $\Im^-$

We can introduce a *distinguished state* $\lambda : \mathcal{W}(\Im^-) \to \mathbb{C}$ unambiguously defined

$$\lambda\left(W(\psi)\right) = e^{-\frac{\mu(\psi,\psi)}{2}}, \quad \forall W(\psi) \in \mathcal{W}(\Im^-)$$

where $\forall \psi, \psi' \in \mathcal{S}(\Im^-)$

$$\mu(\psi,\psi') = \int\limits_{\mathbb{R}\times\mathbb{S}^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\varphi)}\widehat{\psi}'(k,\theta,\varphi)dkdS^2(\theta,\varphi),$$

being $\widehat{\psi}(k), \widehat{\psi}'(k)$ the Fourier-Plancherel transform

$$\widehat{\psi}(k) = \int\limits_{\mathbb{R}} dl\, \frac{e^{ikl}}{\sqrt{2\pi}}\psi(l,\theta,\varphi).$$

The state λ enjoys the following (almost straightforward) properties:

- it is quasifree and pure,
- referring to its GNS triple $(\mathcal{H}, \Pi, \Upsilon)$, it is invariant under the left action α of the horizon symmetry group.

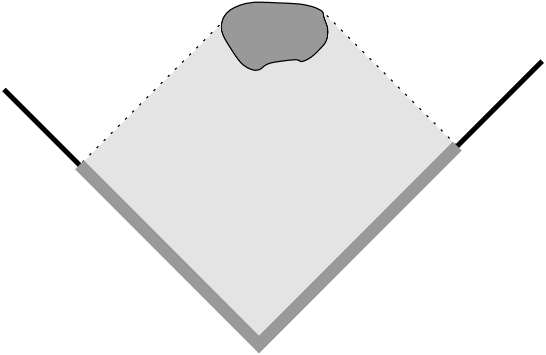
Properties of λ

Let us consider the timelike future directed vector field ∂_τ whose projection on the horizon is $\widetilde{Y} \propto \partial_l$ (also a generator of the algebra of horizon simmetries). Then

- then λ is the **unique** quasifree pure state on $\mathcal{W}(\Im^-)$ which is invariant under $\alpha_{\exp(s\partial_l)}$ $(s \in \mathbb{R})$ and the unitary group implementing such representation leaving fixed the cyclic GNS vector is strongly continuous with nonnegative self-adjoint generator,
- Each folium of states on $\mathcal{W}(\Im^-)$ contains at most one pure state which is invariant under $\alpha_{\exp(s\partial_l)}$,
- and much more...

Back to the bulk

Notice: $\phi \in S(M)$ can be extended to a unique smooth solution of the same equation on $M \cup \Im^- \longrightarrow \Gamma\phi \doteq \phi|_{\Im^-} \in C^\infty(\Im^-)$.



Back to the bulk - II

Theorem 1

If $\phi \in S(M)$ and $0 < \epsilon < \frac{3}{2} - \nu$, then

- $\Gamma\phi$ decays faster than $1/l^{\epsilon}$ whereas $\partial_l \Gamma\phi$ faster than $1/l^{1+\epsilon}$,
- $\Gamma\phi \in \mathcal{S}(\Im^-)$,
- $\sigma_{\Im^-}(\Gamma\phi, \Gamma\phi') = H^2 \sigma(\phi, \phi')$.

Particularly it exists an isometric $*$-homomorphism:

$$\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^-),$$
$$\imath(W(\phi)) \doteq W(\Gamma\phi).$$

Back to the bulk - III

- Any state $\widetilde{\lambda} : \mathcal{W}(\Im^-) \to \mathbb{C}$ can be pulled back to $\imath^*(\widetilde{\lambda}) : \mathcal{W}(M) \to \mathbb{C}$.
- Particularly the preferred state

$$\lambda_M(a) := \lambda(\imath(a)). \quad \forall a \in \mathcal{W}(M)$$

- In the de Sitter spacetime, λ_M is the Bunch-Davies state,
- it is invariant under the natural action of any bulk isometry Y on the algebra. The one-parameter U^Y_s group implementing such an action leaves fixed the cyclic vector in the GNS representation of λ_M,
- if Y is everywhere timelike and future-directed in M, then the 1-parameter group U^Y_s has positive self-adjoint operator.

Glimpses of Hadamard(ology)

Recall) A quasi-free state ω is fully characterized by its two-point function.

Local description: A two-point function $\omega(x,y)$ of a state ω is **Hadamard** if, for any normal neighbourhood $\mathcal{O}_p$,

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\ln\sigma_\epsilon(x,y) + W(x,y).$$

Global description: using **microlocal analysis**, a state ω of a real smooth K.-G. field is of Hadamard form if and only if the Schwartz kernel of the two-point function satisfies

$$WF(\omega) = \left\{((x,k_x),(y,-k_y)) \in (T^*M)^2 \setminus 0 \mid (x,k_x) \sim (y,k_y), k_x \triangleright 0\right\}.$$

Is λ_M Hadamard?

- Hadamard states $\longrightarrow$ ultraviolet behaviour of the ground state in Minkowski!
- Hadamard states $\longrightarrow$ quantum fluctuations of the (smeared) components of $T_{\mu\nu}$ are finite

To investigate λ_M, we first write its Schwartz kernel as the quadratic form

$$\lambda_M(f,g) = \int_{\mathbb{R}\times\mathbb{S}^2} 2k\Theta(k)\overline{\widehat{\psi}_f(k,\theta,\varphi)}\widehat{\psi}_g(k,\theta,\varphi)dkd\mathbb{S}^2(\theta,\varphi),$$

where $\psi_f = \Gamma E(f)$ and $\psi_g = \Gamma E(g)$.

Theorem

λ_M inviduates a distribution on $\mathcal{D}'(M\times M)$ such that

$$WF(\lambda_M) = \mathcal{V} =$$

$$= \left\{((x,k_x),(y,-k_y)) \in (T^*M)^2 \setminus 0 \mid (x,k_x) \sim (y,k_y), k_x \triangleright 0\right\},$$

thus it is Hadamard.

On the inclusion $\supset$

Since it holds

$$\lambda_M(f \otimes Pg) = \lambda_M(Pf \otimes g) = 0, \qquad \lambda_M(f \otimes g) - \lambda_M(g \otimes f) = E(f \otimes g),$$

then the inclusion $\supset$ descends from $\subset$ by means of the theorem of propagation of sigularities proved by Hörmander (see Radzikowski and many others).

Sketch of the proof. $\subset$

- Let us read λ_M as follows: introduce

$$K = (T \otimes I)(\Gamma E \otimes \Gamma E) \in \mathcal{D}'((\Im^- \times \Im^-) \times (M \times M)),$$
$$T = \frac{1}{H^2\pi^2(l - l' - i\epsilon)^2} \otimes \delta(\theta - \theta')\delta(\varphi - \varphi').$$

- introduce a sequence of cut-off functions $\chi_n \in C_0^\infty(\Im^-; \mathbb{C})$ and

$$\lambda_n \doteq \mathcal{K}(\chi_n \otimes \chi_n),$$

where $\mathcal{K} : C_0^\infty(\Im^- \times \Im^-) \to \mathcal{D}'(M \times M)$ is the map associated with the kernel ${}^tK$ in view of Schwarz kernel theorem.

Big Fat Final Theorem:

The sequence λ_n are such that:

1. $WF(\lambda_n) \subset \mathcal{V}$
2. $\lambda_n \to \lambda_M$ in the weak sense in $\mathcal{D}'(M \times M)$
3. $\sup_n \sup_{k \in V} |k|^N |\widehat{h\lambda_n}| < \infty$ for all $N \geq 1$ and for all $h \in C_0^\infty(M \times M; \mathbb{C})$ where V is any cone closed in $(T^*M)^2 \setminus 0$ lying in the complement of $\mathcal{V}$.

Hence λ_M satisfies $\subset$ and its of Hadamard form.

What lies in front of us?

Summary:

- A distinguished Hadamard state for a scalar field theory exists in a large class of FRW backgrounds,
- It has interesting properties of uniqueness and it is Hadamard,
- natural ground state in cosmology (to deal with interactions).

Open Questions:

- How can we connect this results to present observations?
- Is a free scalar field theory enough?[1]
- How can we describe interacting theories in our scenario?
- Is the road to mathematically precise inflationary models open?

[1] C.D., Klaus Fredenhagen and Nicola Pinamonti, Phys. Rev. D**77** (2008) 104015