On the Algebraic Formulation of Classical General Relativity

(Über die algebraische Formulierung der klassischen Allgemeinen Relativitätstheorie)

Diploma thesis

submitted by

Jan-Christoph Weise

Erster Gutachter: Prof. Dr. Klaus Fredenhagen

Zweiter Gutachter: Dr. Mattias N. R. Wohlfarth

Hamburg, January 5, 2011

II. Institut für Theoretische Physik

Department Physik

Universität Hamburg

Abstract

In this work we discuss a possible algebraic formulation of classical General Relativity.

The algebraic formulation of classical field theories without gauge freedom is based on the definition of the Peierls bracket, a manifest covariant Poisson bracket which has first been introduced by PEIERLS in [19].

The definition of the Peierls bracket has been extended to classical gauge theories by DEWITT in [7a].

We examine the properties of the extended Peierls bracket and discuss to which extent the framework given in [7a] can be applied to classical General Relativity.

Zusammenfassung

In dieser Arbeit diskutieren wir eine mögliche algebraische Formulierung der klassischen Allgemeinen Relativitätstheorie.

Die algebraische Formulierung klassischer Feldtheorien ohne Eichfreiheit basiert auf der Definition der Peierlsklammer, einer manifest kovarianten Poissonklammer, welche zuerst von PEIERLS in [19] eingeführt wurde.

Die Definition der Peierlsklammer wurde von DEWITT in [7a] auf klasssische Eichfeldtheorien erweitert.

Wir untersuchen die Eigenschaften der erweiterten Peierlsklammer und diskutieren, inwieweit der Formalismus, welcher in [7a] vorgestellt wird, sich auf die klassische Allgemeine Relativitätstheorie anwenden lässt.

'Da steh ich nun [...]'
—Johann Wolgang von Goethe, Faust I

Contents

Introduction

A common approach to the construction of a Quantum Field Theory based on a classical field theory is the canonical quantization programme. On the classical level it relies on the construction of a classical Poisson algebra of observables based on the **canonical Poisson brackets**.

The canonical Poisson brackets are constructed using the the definition of the **canonical conjugate momenta** and are therefore closely related to the **Hamiltonian formulation** of the classical theory. The construction of the canonical Poisson brackets requires the definition of an explicit 'time' coordinate and a resulting foliation of spacetime into **Cauchy hypersurfaces** (see appendix A). We will refer to the resulting dependence of the definition of the canonical Poisson brackets as the loss of **manifest Lorentz covariance**.

When we are dealing with classical **gauge theories**, the unphysical degrees of freedom present in these theories result in a **singular Lagrangian**. This means that the relation between the time derivatives of the fields and the corresponding canonical conjugate momenta cannot be inverted and therefore the definition of the canonical Poisson brackets is not possible without further considerations.

Two common approaches to solve this problem are the following:

For some classical gauge theories (e.g. in case of the free Maxwell field) it is possible to modify the Lagrangian of the theory by adding a **gauge invariance breaking term**, resulting in a nonsingular Lagrangian. The resulting Poisson algebra based on this Lagrangian contains a subalgebra given by the gauge invariant functions on the phase space. It is, however, not possible to find such a gauge invariance breaking term for many important physical theories (see e.g. [14]).

Another approach makes use of Dirac's algorithm for singular Lagrangian systems (see [8]). In this formalism, the phase space of the theory is extended to allow the definition of all canonical conjugate momenta. The algebra of observables is obtained by dividing out the ideals generated by the arising **constraints**. The treatment of these constraints in a resulting quantized theory is often a severe difficulty (see e.g. [22]).

In [19], Peierls gave a definition of a Poisson bracket for classical field theories which is entirely based on the Lagrangian formulation of the classical theory and does not rely on a foliation of spacetime into Cauchy hypersurfaces and the definition of canonical conjugate variables. This poisson bracket is called the **Peierls bracket**. The Poisson algebra of observables based on the Peierls bracket has been shown to be isomorphic to the one based on the canonical Poisson bracket for any Lagrangian field theory which can be formulated as a Hamiltonian theory without constraints. The Peierls bracket is used in the **algebraic formulation of classical field theory** (see [4]).

In [19] the problems occurring when faced with singular Lagrangians in gauge theories are not adressed and the given method for the construction of the Peierls bracket is not defined for gauge theories.

There are few publications that pick up the idea of the Peierls bracket. First of all there are [7a, 7b] by DeWitt where an extension of the definition of the Peierls bracket for classical gauge theories is given. Marolf discusses further possible extensions in [17a, 17b].

In this work we are interested in the classical theory of General Relativity. The Lagrangian of General Relativity is singular due to the **diffeomorphism invariance** of the theory. A detailed discussion of the application of Dirac's algorithm for singular Lagrangian systems to General Relativity can be found in [22].

We want to keep the manifest Lorentz covariance of the theory. Therefore, in this work we examine to which extent the framework given by DeWitt in [7a] can be applied to give an algebraic formulation of classical General Relativity.

The outline of this work is the following:

In the first chapter, we give an introduction to the concepts of the algebraic formulation of classical field theory. This includes the definition of the phase space of the configuration space and phase space of a classical field theory, the linearized field equations and the original Peierls bracket.

In chapter 2 we discuss the structure of classical gauge theories.

Based on the structures defined in chapter 2, we define the extended Peierls bracket in the third chapter, where we also examine its properties. To get accustomed to the framework given in this chapter, we apply it to the classical theory theory of the free Maxwell field.

In chapter 4, we give a Lagrangian formulation of classical General Relativity in terms of a Lorentzian metric as dynamical field. We discuss the gauge symmetry of the theory and derive the linearized field equations. Finally, we apply the gauge fixing procedure introduced in chapter 3.

In the fifth chapter, a brief introduction to the formulation of classical General Relativity formulated in terms of a local connection form and tetrad fields is given. We present arguments that it is more complicated to apply the framework given in chapter 3 in this formulation of the theory.

In chapter 6, we give an overview on the occurring conceptual problems when applying the framework given in chapter 3 to General Relativity, including the problem of the existence of observables in General Relativity.

1. Algebraic Classical Field Theory

In this work we are exclusively concerned with classical field theory.

Usually classical fields are thought of as taking values in the **smooth sections** over a finite dimensional **vector bundle** $E \to M$ over a **spacetime** M (see appendix B).

For example, the classical real scalar field takes values in the smooth real valued functions on M and the vector potential of classical electrodynamics takes values in the smooth sections $\Gamma(T^*M)$ over the cotangent bundle T^*M.

In this chapter we always assume the base manifold M to be a **globally hyperbolic spacetime** with metric g (see appendix A).

In Quantum Field Theory the notion of a field is generalized. The fields are here thought of as operator valued distributions which, applied to test functions, yield linear operators on a Hilbert space. In this work we are working in the **algebraic formulation of classical field theory** which gives rise to a similar interpretation of the fields on the classical level.

In this chapter we introduce the algebraic formulation of classical field theory, following Brunetti and Fredenhagen in [4].

1.1. The space of smooth sections

We fix a finite dimensional vector bundle $E \to M$ over a spacetime M. Our interest lies in the vector space $\Gamma(E)$ of smooth sections over E, which provides the **configuration space** of the theory. We want to define a notion of smooth functions and a derivative on the configuration space.

To do so, we first need to endow $\Gamma(E)$ with a topology. The usual choice is to endow $\Gamma(E)$ with a vector topology generated by the following countable family of seminorms:

Definition 1.1.1
Let $\mathcal{K} := \{K_i\}_{i\in\mathcal{I}}$ be a countable covering of M by compact subsets. For each $K \in \mathcal{K}$ and multiindex k a seminorm on $\Gamma(E)$ is defined by

$$\|\Phi\|_{k,K} := \sup_{x\in K} |\partial^k \Phi| \quad \forall \Phi \in \Gamma(E) \tag{1.1.1}$$

Endowed with this topology, $\Gamma(E)$ is a **nuclear space**.

As a topological vector space we can identify $\Gamma(E)$ with its tangent spaces $T_\Phi\Gamma(E)$ at any point $\Phi \in \Gamma(E)$. We will do so throughout this work.

In this topology we have that $\Phi_i \to \Phi$ if and only if $\partial^k\Phi_i|_K \to \partial^k\Phi|_K$ uniformly for all multiindices k and compact sets $K \subset M$.

The subspace $\Gamma_0(E) \subset \Gamma(E)$ of sections with compact support over E will be of special interest. We endow it with the topology such that $\Phi_i \to \Phi$ if and only if $\partial^k\Phi_i \to \Phi$ uniformly for all multiindices k and there exists a compact set $K \subset M$ such that $\bigcup_i \text{supp } \Phi_i \subset K$. Endowed with this topology, $\Gamma_0(E)$ is a nuclear space.

Now that we have given topologies on the spaces $\Gamma(E)$ and $\Gamma_0(E)$, we can define **distributions** on these spaces and their tensor products:

Definition 1.1.2 (Distribution)
Let W be a topological vector space. A continuous linear map

$$D : \Gamma(E)^n \to W \tag{1.1.2}$$

is called a W-valued distribution on $\Gamma(E)^k$. The vector space of all W-valued distributions on $\Gamma(E)^k$, endowed with the weak topology, is denoted by $\mathcal{E}'(E^k, W)$.

Distributions on $\Gamma_0(E)^k$ are defined accordingly. The vector space of all W-valued distributions on $\Gamma_0(E)^k$ endowed with the weak topology is denoted by $\mathcal{D}'(E, W)$.

It is $\mathcal{E}'(E^k, W) \subset \mathcal{D}'(E^k, W)$.

The **weak topology** is defined such that $D_i \to D$ exactly if $D_i[\Phi] \to D[\Phi] \;\; \forall \Phi \in \Gamma(E)^k$.

Note: We use square brackets to indicate the arguments of a distribution.

Definition 1.1.3 (Support of a distribution)
Let $D \in \mathcal{E}'(E, W)$. We define

$$\text{supp } D := M \backslash \left\{ \bigcap U \subset M \text{ open} | \text{ supp } (\xi \in \Gamma_0(E)) \subset U \Rightarrow D[\xi] = 0 \right\} \tag{1.1.3}$$

By definition, the support of a distribution $D \in \mathcal{E}'(E, W)$ is compact.

We define **functionals** on the configuration space:

Definition 1.1.4 (Functional)
Let W be a topological vector space. A continuous map

$$F : \Gamma(E) \to W \tag{1.1.4}$$

is called a functional on $\Gamma(E)$.

We define the **functional derivative** on $\Gamma(E)$:

Definition 1.1.5 (Functional derivative)
Let F be a W-valued functional on $\Gamma(E)$.

The functional derivative of F at $\Phi \in \Gamma(E)$ in the direction of $\xi \in T_\Phi\Gamma(E) \simeq \Gamma(E)$ is defined as the linear map

$$\delta F(\Phi) : T_\Phi\Gamma(E) \simeq \Gamma(E) \to W \tag{1.1.5}$$

$$\delta F(\Phi)[\xi] := \frac{d}{d\lambda}|_{\lambda=0} F(\Phi + \lambda\xi) \tag{1.1.6}$$

if this limit exists.

If the limit exists $\forall \Phi \in \Gamma(E),\ \xi \in T_\Phi\Gamma(E)$ and

$$\delta F : \Gamma(E) \times \Gamma(E) \to W \tag{1.1.7}$$

*is continuous, F is called **continuously functionally differentiable**.*

The functional derivative can be iterated in the following sense:

Definition 1.1.6 (Iterated functional derivative)
Let F be a W-valued functional.

The k-fold iterated functional derivative of F is given by the linear map

$$\delta^k F(\Phi) : (T_\Phi\Gamma(E))^k \simeq \Gamma(E)^k \to W \tag{1.1.8}$$

$$\delta^k F(\Phi)[\xi_1, ..., \xi_k] := \frac{d}{d\lambda_1}|_{\lambda_1=0} \cdot ... \cdot \frac{d}{d\lambda_k}|_{\lambda_k=0} F(\phi + \sum_{i=1}^{k} \lambda_i \xi_i) \tag{1.1.9}$$

if this limit exists.

If this limit exists $\forall \Phi \in \Gamma(E),\ \xi \in T_\Phi\Gamma(E)^k \simeq \Gamma(E)^k$ and

$$\delta^k F : \Gamma(E) \times \Gamma(E)^n \to W \tag{1.1.10}$$

*is continuous, F is called **k-fold continuously functionally differentiable**.*

We are in particular interested in the class of **smooth functionals**:

Definition 1.1.7 (Smooth functional)
Let F be a W-valued functional on $\Gamma(E)$.

F is called smooth if it is continuously functionally differentiable in every order.

We will denote the vector space of smooth W-valued functionals on $\Gamma(E)$ by $\mathcal{F}(E, W)$.

By definition, the k-fold functional derivative of any smooth functional $F \in \mathcal{F}(E, W)$ yields a symmetric distribution $\delta^k F(\Phi) \in \mathcal{E}'(E, W)$.

Based on this property, we define the support of a smooth functional:

Definition 1.1.8 (Support of a functional)
Let $F \in \mathcal{F}(E, W)$. We define

$$\text{supp } F(\Phi) := \text{supp } (\delta F(\Phi) \in \mathcal{E}'(E, W)) \tag{1.1.11}$$

The vector space $\mathcal{F} := \mathcal{F}(E, \mathbb{R})$ of real valued smooth functionals on $\Gamma(E)$ forms a commutative, associative algebra via multiplication:

Definition 1.1.9 (Algebra of smooth functionals)
On the space $\mathcal{F}$ a multiplication is given by

$$(F_1 \cdot F_2)(\Phi) := F_1(\Phi) \cdot F_2(\Phi) \tag{1.1.12}$$

The resulting commutative, associative algebra is called the algebra of smooth functionals.

Our goal will be to endow $\mathcal{F}$ with an additional Poisson structure given by the Peierls bracket. The resulting Poisson algebra $\mathcal{F}, \{\cdot, \cdot\}$ is the essential object in the algebraic formulation of classical field theory.

1.2. The action principle

In the previous section we endowed the configuration space $\Gamma(E)$ with a topology and a differential structure. In this section we use this structure to determine the dynamics (i.e. the field equations) of a classical field theory based on the definition of an action functional.

A **local action functional** is usually defined to be a functional $S \in \mathcal{F}(E, \mathbb{R})$ which can be written in the form:

$$S(\Phi) = \int_M d^n x \, \mathcal{L}(\Phi)(x) \tag{1.2.1}$$

with $\mathcal{L}$ being a smooth functional valued in the smooth real scalar densities of rank 1, such that $\mathcal{L}(\Phi)(x)$ only depends on the value of Φ and its partial derivatives (up to finite order) at x. $\mathcal{L}$ is called the **Lagrangian density** of the theory.

Remark: A functional F will be called **Φ-jet-dependent** if $F(\Phi)$ depends only on the value of Φ and its partial derivatives up to finite order at a given point $x \in M$.

The field equations for Φ are usually given via the stationary point condition

$$\delta S(\Phi)[\xi] = 0 \;\; \forall \xi \in \Gamma_0(E) \tag{1.2.2}$$

However, we have to be more precise here. Due to the form of the action functional as an integral over a noncompact domain, it is in general not well defined all over $\Gamma(E)$.

This problem can be avoided by regarding the action as a distribution

$$\begin{aligned} &S(\Phi) : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathbb{R} \\ &S_\psi(\Phi) := S(\Phi)[\psi] := \int_M d^n x \; \mathcal{L}(\Phi)(x) \; \psi(x) \end{aligned} \tag{1.2.3}$$

We then define the functional derivative of S to be the distribution

$$\begin{aligned} &\delta S(\Phi) \in \mathcal{D}'(E, \mathbb{R}) \\ &\delta S(\Phi)[\xi] := \delta S_\psi(\Phi)[\xi] \quad \text{with } \psi \in \mathcal{C}_0^\infty(M, \mathbb{R}) : \psi|_{\text{ supp } \xi} = 1 \end{aligned} \tag{1.2.4}$$

The field equations are again obtained via the stationary point condition

$$\delta S(\Phi)[\xi] = 0 \quad \forall \xi \in \Gamma_0(E) \tag{1.2.5}$$

The set of stationary points will be referred to as the **shell** or **solution space** $\mathcal{S}$. The solution space provides the **phase space** of the theory.

The functional derivative of the action functional is iterated in the following sense:

$$\begin{aligned} &\delta^k S(\Phi) \in \mathcal{D}'(E^n, \mathbb{R}) \\ &\delta^k S(\Phi)[\xi_1, ..., \xi_k] := \delta^k S_\psi(\Phi)[\xi_1, ..., \xi_k] \text{ for any } \psi : \psi = 1 \text{ on } \bigcap_{i=1}^k \text{supp } \xi_i \end{aligned} \tag{1.2.6}$$

Due to the local form of the action functional, the iterated functional derivative of the action can be evaluated even on arguments with noncompact support, as long as the intersection of the support of all arguments is compact.

1.3. Linearized field theory

In this section we examine the differential structure on the solution space $\mathcal{S}$ by deriving the **linearized field equations**. We are interested in this structure since the construction of the Peierls bracket will be based on the definition of the linearized field equations.

We now derive the linearized field equations.

First of all, consider a smooth curve

$$\begin{aligned} &\Phi : \mathbb{R} \to \Gamma(E) \text{ smooth} \\ &\Phi(\lambda) \in \mathcal{S} \;\; \forall \lambda \in \mathbb{R} \end{aligned} \tag{1.3.1}$$

We expand $\Phi(\lambda)$ in terms of a formal power series:

$$\Phi(\lambda) \simeq \Phi(0) + \lambda \frac{d}{d\lambda} |_{\lambda=0} \Psi(\lambda) =: \Phi_0 + \lambda\chi, \quad \chi \in \Gamma(E) \tag{1.3.2}$$

By definition, $\Phi(\lambda)$ satisfies the field equations:

$$\delta S(\Phi(\lambda))[\xi] = 0 \;\; \forall \xi \in \Gamma_0(E) \tag{1.3.3}$$

The differentiation with respect to the parameter λ of this equation yields

$$\frac{d}{d\lambda} |_{\lambda=0} \delta S(\Phi(\lambda))[\xi] = \delta^2 S(\Phi_0)[\xi, \chi] = 0 \;\; \forall \xi \in \Gamma_0(E) \tag{1.3.4}$$

This result motivates the definition of the linearized field equations:

Definition 1.3.1 (Linearized field equation)
The linear partial differential equation for $\chi \in T_\Phi\Gamma(E) \simeq \Gamma(E)$, given by

$$\delta^2 S(\Phi)[\xi, \chi] = 0 \;\; \forall \xi \in \Gamma_0(E)\,, \tag{1.3.5}$$

is called the linearized field equation.

Note: The linearized field equations are defined all over $\Gamma(E)$, not only on the solution space $\mathcal{S}$. The existence of solutions to the linearized field equations is in general not ensured (see e.g. [9] on this problem in Yang Mills theory and General Relativity).

We introduce the notion of the formal adjoint of an operator:

Definition 1.3.2 (Formal adjoint)
Let $K : \Gamma(E^*) \to \Gamma(E)$ be a continuous linear operator.

If it exists a continuous linear operator

$$\tilde{K} : \Gamma(E^*) \to \Gamma(E) \tag{1.3.6}$$

satisfying

$$\int_M d^n x \; \xi \; (\tilde{K}\chi) = \int_M d^n x \; (K\xi) \; \chi \;\; \forall \xi, \chi \in \Gamma_0(E^*) \tag{1.3.7}$$

the operator $\tilde{K}$ is called the formal adjoint of K.

For the definition of the **dual bundle** E^* of a vector bundle E, see appendix B.

We define the differential operator induced by the second functional derivative of the action functional as the formally selfadjoint linear partial differential operator

$$\delta^2 S(\Phi) : \Gamma(E) \to \Gamma(E^*) \tag{1.3.8}$$

satisfying

$$\int_M d^n x \; \xi \; \delta^2 S(\Phi) \; \chi := \delta^2 S(\Phi)[\xi, \chi] \quad \forall \xi \in \Gamma_0(E), \; \chi \in \Gamma(E) \tag{1.3.9}$$

The operator $\delta^2 S(\Phi)$ is called the **Jacobian** of the theory.

The Jacobian is Φ-jet dependent, where we define an operator $P : \Gamma(E) \to \Gamma(E^*)$ to be Φ-jet dependent if all functionals of the form

$$A_\xi(\Phi)(x) := (\delta^2 S(\Phi) \; \xi)(x), \quad \xi \in \Gamma(E) \tag{1.3.10}$$

are Φ-jet dependent.

Using the definition of the Jacobian, we can express the linearized field equations as

$$\delta^2 S(\Phi) \; \chi = 0 \tag{1.3.11}$$

Solutions to the on-shell linearized field equations will be called **Jacobi fields** in this work, as it is done in [7a, 7b].

Remark: Any smooth curve on $\mathcal{S}$ yields a Jacobi field. However, the converse is in general not true. Any on-shell field history satisfying the converse is called **linearization stable** (see e.g. [11] on this subject in General Relativity).

1.4. Green's operators

In the previous section we derived the linearized field equations and defined the Jacobian. As we already mentioned, the construction of the Peierls bracket relies on the definition of the linearized field equations. To be more precise, it is based on the **Green's operators** of the Jacobian.

Definition 1.4.1 (Green's operator)
Let $P : \Gamma(M, E) \to \Gamma(M, E^*)$ be a linear partial differential operator.

A continuous linear map $G : \Gamma_0(M, E^*) \to \Gamma(M, E)$ is called a ***Green's operator*** *of P if it satisfies*

$$P \circ G = \mathrm{id}|_{\Gamma_0(M,E^*)} \tag{1.4.1}$$

$$G \circ P = \mathrm{id}|_{\Gamma_0(M,E)} \tag{1.4.2}$$

A special class of Green's operators is given by the **advanced** and **retarded Green's operators**:

Definition 1.4.2 (Advanced and retarded Green's operator)
A Green's operator G_+ *is called advanced Green's operator if*

$$\text{supp } G_+\xi \subset J^-(\text{supp } \xi) \; \forall \xi \in \Gamma_0(E) \tag{1.4.3}$$

A Green's operator G_- *is called retarded Green's operator if*

$$\text{supp } G_-\xi \subset J^+(\text{supp } \xi) \; \forall \xi \in \Gamma_0(E) \tag{1.4.4}$$

For the definition of the causal future J^+ and the causal past J_- of a domain, see appendix A.

Using the linearity and support properties, we can extend the action of the retarded Green's operator onto sections of past compact support, and the action of the advanced Green's operator onto sections of future compact support.

As we have already mentioned, we are interested in the Green's operators of the Jacobian. To be more precise, we are interested in the advanced and retarded Green's operators of the Jacobian. This rises the question under which conditions these operators exist. We discuss this now.

An important class of differential operators for which we know about the existence of unique advanced and retarded Green's operators are the so called **normally hyperbolic differential operators** (see appendix D):

Definition 1.4.3 (Normally hyperbolic differential operator)
Let $P : \Gamma(E) \to \Gamma(E^*)$ *be a linear partial differential operator.*

P is called ***normally hyperbolic*** *if it is of second order and its component in second order of the covariant derivative is of the form* $\gamma \circ \Box^\nabla$*, whereby* $\gamma : E \to E^*$ *is a vector bundle isomorphism.*

We introduce the **characteristic** of a linear partial differential operator $P : \Gamma(E) \to \Gamma(E^*)$ of finite order. It is defined in the following way:

We take into account the component of P in highest order in the covariant derivative ∇. For fixed $x \in M$, we replace all occurrences of the covariant derivative in this object by a covector $k \in T^*_x M$, which yields a map

$$\sigma_P : T^*M \to \text{Hom}(V, V) \tag{1.4.5}$$

whereby V denotes the fibre manifold of E and E^*. The map σ is called the **principal symbol** of P. The nontrivial kernel of this map is called the charakteristic of P and is denoted by char P.

In case of a normally hyperbolic operator P of the form given above, we have

$$\sigma_P(x, k \in T_x^*M) = \gamma(x) \circ g^{ab}(x)\ k_a k_b \tag{1.4.6}$$

We see that in this case it is char $P = \bigcup_{x \in M}(x, \bar{V}_x)$, where $\bar{V}_x$ denotes the lightcone in the cotangent space T_x^*M.

The most basic example of a normally hyperbolic differential operator is of course the well known d'Alembert operator:

$$\sqrt{-\det g}\ \Box^\nabla : \Gamma(M, E) \to \Gamma(M, E^*) \tag{1.4.7}$$

The Jacobian of the massless Klein-Gordon field is given by the d'Alembert operator.

For normally hyperbolic operators we have the following important theorem:

Theorem 1.4.1 (Corollary 3.4.3 in [1])
Let $P : \Gamma(M, E) \to \Gamma(M, E^*)$ be a normally hyperbolic differential operator and let M be globally hyperbolic.

There exist unique advanced and retarded Green's operators for P.

In case the differential operator P is formally selfadjoint and has unique advanced and retarded Green's operators, these are formally adjoint:

$$\begin{aligned} \tilde{G}_+ &= G_- \\ \tilde{G}_- &= G_+ \end{aligned} \tag{1.4.8}$$

The advanced and retarded Green's operators are used to define the **causal propagator** of the corresponding linear partial differential operator:

Definition 1.4.4 (Causal propagator)
Let $P : \Gamma(M, E) \to \Gamma(M, E^*)$ be a linear partial differential operator with unique advanced and retarded Green's operators G_+ and G_-.

The causal propagator Δ of P is defined as the linear operator

$$\Delta := (G_+ - G_-) : \Gamma_0(M, E^*) \to \Gamma(M, E) \tag{1.4.9}$$

The causal propagator obviously satisfies

$$\begin{aligned} P \circ \Delta &= 0 \,|_{\Gamma_0(M,E^*)} \\ \Delta \circ P &= 0 \,|_{\Gamma_0(M,E)} \end{aligned} \tag{1.4.10}$$

The use of identity (1.4.8) immediately yields

$$\tilde{\Delta} = -\Delta \tag{1.4.11}$$

We see that the causal propagator is antiselfadjoint.

Since it is defined as a linear combination of the retarded and advanced Green's operator, the action of the causal propagator can be extended onto sections of future as well as past compact support.

1.5. The Peierls bracket

In this chapter we introduce the **Peierls bracket** as it has originally been defined by PEIERLS in [19].

To understand the definition of the Peierls bracket, we will have to generalize the action of the linear operators defined in chapter 1.4 from the space of smooth sections $\Gamma_0(E^*)$ to the space of distributions $\mathcal{E}'(E, \mathbb{R})$, identified with their kernels (i.e. generalized sections over E^*). Since the definition of this extended action requires the introduction of the **wavefront set** of distributions in the framework of microlocal analysis, it is given separately in appendix D.

Peierls' aim has been the construction of a manifest covariant Poisson bracket on the algebra of smooth functionals $\mathcal{F}$.

Definition 1.5.1 (Poisson bracket)
Let F be a real associative algebra. A map

$$\{\cdot, \cdot\} : \mathcal{F} \times \mathcal{F} \to \mathcal{F} \tag{1.5.1}$$

*is called a **Poisson bracket** on $\mathcal{F}$, if it satisfies*

$$\begin{aligned}
\mathbb{R} - \textit{linearity} &: \ \{\lambda A, \mu B\} = \lambda\mu \ \{A, B\} \ \ \forall A, B \in \mathcal{F}, \ \lambda, \mu \in \mathbb{R} && (1.5.2)\\
\textit{antisymmetry} &: \ \{A, B\} = -\{B, A\} && (1.5.3)\\
\textit{derivative} &: \ \{A \cdot B, C\} = A \cdot \{B, C\} + \{A, C\} \cdot B && (1.5.4)\\
\textit{Jacobi identity} &: \ \{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 && (1.5.5)
\end{aligned}$$

The resulting algebra $(\mathcal{F}, \{\cdot, \cdot\})$ is called a **Poisson algebra**.

Definition 1.5.2 (Peierls bracket)
Let S be a smooth action functional such that the linearized field equations possess unique advanced and retarded Green's operators. Let Δ be the corresponding causal propagator.

*The **Peierls bracket** is defined as the map*

$$\begin{aligned}
&\{\cdot, \cdot\} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}\\
&\{A, B\} := \delta A \ [\Delta \ \delta B]
\end{aligned} \tag{1.5.6}$$

As stated in the definition, the Peierls bracket is only well defined for theories which yield linearized field equations with unique advanced and retarded Green's operators. This particularly includes theories which yield normally hyperbolic linearized field equations. Many important classical physical theories, such as gauge theories and General Relativity, do not satisfy this condition, which is the main problem we will adress in the following parts of this work.

As we see in appendix D, the map

$$\Delta : \mathcal{E}'(E, \mathbb{R}) \to \Gamma(E) \tag{1.5.7}$$

is not well defined on all distributions. This raises the question which restrictions we have to impose on the functionals in $\mathcal{F}$ to ensure the existence and closure of the resulting bracket algebra.

We will restrict the action of the bracket on the set of functionals in $\mathcal{F}$ that satisfy a condition $(*)$ such that

$$\begin{aligned} &A, B \in \mathcal{F} \text{ satisfy } (*) \Rightarrow A \cdot B \text{ satisfies } (*) \\ &A \in \mathcal{F} \text{ satisfies } (*) \Rightarrow (A \to \Delta\ \delta A \in \Gamma(E)) \text{ is well defined} \\ &A, B \in \mathcal{F} \text{ satisfy } (*) \Rightarrow \{A, B\} \text{ satisfies } (*) \end{aligned} \tag{1.5.8}$$

In case that the linearized field equations are normally hyperbolic, such a restriction is known to be given by the **wavefront set condition** (see e.g. [4]):

Definition 1.5.3 (Wavefront set condition)
Let $A \in \mathcal{F}$. A is said to satisfy the wavefront set condition if

$$WF(\delta^k A) \cap (\bar{V}_+^k \cup \bar{V}_-^k) = \emptyset \quad \forall k \in \mathbb{N} \tag{1.5.9}$$

We give a motivation for the wavefront set condition in appendix D.

From now on, by $\mathcal{F}$ we will denote the space of smooth real valued functionals satisfying the wavefront set condition.

We verify that the Peierls bracket provides a Poisson bracket on $\mathcal{F}$:

The linearity of the Peierls bracket results from the linearity of the action of the causal propagator and .

The antisymmetry of the Peierls bracket immediately follows from the fact that the causal propagator is antiselfadjoint.

The Peierls bracket is a derivative since the functional derivative satisfies the Leibniz rule.

it has been proven e.g. by Marolf in [17a] that the Peierls bracket satisfies the Jacobi identity . The proof relies on the symmetry of the functional derivative of the linearized field equations, i.e. on the symmetry of the third functional derivative of the action functional. We will give a more general proof when examining the properties of the extended Peierls bracket in section 3.2.

Endowed with the Poisson structure given by the Peierls bracket, $\mathcal{F}$ thus is indeed a Poisson algebra.

We are interested in **Poisson ideals** of this Poisson algebra.

Definition 1.5.4 (Poisson ideal)
Let $(\mathcal{F}, \{\cdot,\cdot\})$ be a Poisson algebra. A Poisson ideal of $(\mathcal{F}, \{\cdot,\cdot\})$ is a subspace $\mathcal{E} \subset \mathcal{F}$, such that

$$\begin{aligned} \{E, A\} \quad &\in \mathcal{E} \ \ \forall E \in \mathcal{E},\ A \in \mathcal{A} \qquad (1.5.10)\\ E \cdot A \quad &\in \mathcal{E} \ \ \forall E \in \mathcal{E},\ A \in \mathcal{A} \qquad (1.5.11) \end{aligned}$$

The field equations generate a Poisson ideal of $\mathcal{F}$. It is given by the subspace

$$\mathcal{E}_S = \{F \in \mathcal{F} : F(\Phi) = D(\Phi)[\delta S(\Phi)] \text{ for a } D : \Gamma(E) \to \mathcal{E}(E^*, \mathbb{R}) \text{ smooth}\} \quad (1.5.12)$$

We verify this. We have

$$\begin{aligned} &\{D[\delta S], A\} = \delta(D[\delta S])[\Delta\ \delta A] \\ &= D[\delta^2 S \circ \Delta\ \delta A] + \delta D[\delta S, \Delta\ \delta A] = \delta D[\delta S, \Delta\ \delta A] \subset \mathcal{E}_S \end{aligned} \quad (1.5.13)$$

where we used the fact that the causal propagator solves the linearized field equations.

We may divide the ideal $\mathcal{E}$ out of $\mathcal{F}$ to obtain the Poisson algebra

$$\begin{aligned} &\{\cdot,\cdot\} : \mathcal{F}/\mathcal{E} \times \mathcal{F}/\mathcal{E} \to \mathcal{F}/\mathcal{E} \\ &\{[A], [B]\} := [\{A, B\}] \end{aligned} \quad (1.5.14)$$

where the squared brackets denote the equivalence class of the argument. This algebra will be called the **on-shell** algebra generated by the Peierls bracket.

In [19], Peierls gave a proof that in case of a Hamiltonian theory without constraints, the on-shell algebra generated by the Peierls bracket is equivalent to the canonical Poisson algebra based on the canonical Poisson brackets.

2. Gauge Symmetries

In the previous chapter we mentioned that in gauge theories the linearized field equations possess no Green's operators and therefore the original Peierls bracket cannot be defined for these theories. In this chapter we examine the structure of gauge theories in order to be able to extend the definition of the Peierls bracket to these theories in the in next chapter.

First of all we will specify which class of theories we refer to by the indication classical gauge theories. We define a classical gauge theory by the following setting:

Let G be a (in general not finite dimensional) Lie group with Lie algebra $\mathrm{Lie}(G)$. Let G act smoothly and bijectively on $\Gamma(E)$, such that

$$(\Phi - g \cdot \Phi) \in \Gamma_0(E) \quad \forall g \in G \tag{2.0.1}$$

The action of G on $\Gamma(E)$ induces an action of the Lie algebra $\mathrm{Lie}(G)$ on $\Gamma(E)$. We will assume that we can identify $\mathrm{Lie}(G)$ (as a topological vectorspace) with the space $\Gamma_0(\mathfrak{g})$ of test sections over some finite dimensional vector bundle $\mathfrak{g} \to M$. We will furthermore assume that G acts **locally** on $\Gamma(E)$ in the sense that the induced action of $\mathrm{Lie}(G)$ on $\Gamma(E)$ is of the form

$$\begin{aligned} &f : \Gamma(E) \to T\Gamma(E) \simeq \Gamma(E) \\ &f \cdot \Phi := Q(\Phi) f \end{aligned} \tag{2.0.2}$$

where

$$Q(\Phi) : \Gamma(\mathfrak{g}) \to \Gamma(E) \tag{2.0.3}$$

is a Φ-jet dependent linear partial differential operator.

Remark: The classical Yang-Mills theories and General Relativity are classical gauge theories in this sense.

Via the pullback with respect to the action of G on $\Gamma(E)$, the group G also acts on the functionals $\mathcal{F}(E, W)$ on $\Gamma(E)$. The corresponding induced action of $\mathrm{Lie}(G)$ on a functional $F \in \mathcal{F}(E, W)$ is given by

$$\begin{aligned} &f : \mathcal{F}(E, W) \to \mathcal{F}(E, W) \\ &(f \cdot F)(\Phi) := \delta F(\Phi)[Q(\Phi) f] \end{aligned} \tag{2.0.4}$$

We now can define G**-invariants**:

Definition 2.0.5 (G-invariant)
A functional $F \in \mathcal{F}(E, W)$ is called a G-invariant if

$$(f \cdot F)(\Phi) = \delta F(\Phi)[Q(\Phi)f] = \tilde{Q}\ \delta F[f] = 0 \quad \forall f \in \Gamma_0(\mathfrak{g}) \tag{2.0.5}$$

where the action of a linear partial differential operator P on a distribution is defined in the following way:

$$\begin{aligned} &P : \Gamma(E^*) \to \Gamma(\mathfrak{g}),\ D \in \mathcal{D}'(E, W) \\ &(P\ D)[f] := D[\tilde{P}f] \quad \forall f \in \Gamma_0(\mathfrak{g}) \end{aligned} \tag{2.0.6}$$

Based on this notion, a theory based on an action functional S is called G**-invariant** with G being called the **gauge group** if the action functional is a G-invariant in the sense that

$$\delta S(\Phi)[Q(\Phi)f] = \tilde{Q}\ \delta S[f] = 0 \quad \forall f \in \Gamma_0(\mathfrak{g}) \tag{2.0.7}$$

In case the theory is G-invariant, $Q(\Phi)$ is called an **invariant flow** of the theory.

The functional derivative of identity (2.0.7) yields

$$\delta^2 S(\Phi)[Q(\Phi)f, (\cdot)] + \delta S(\Phi)[\delta Q(\Phi)[(\cdot)]f] = 0 \tag{2.0.8}$$

This implies that on-shell we have

$$\delta(\delta S(\Phi)[\xi])[Qf] = \delta^2 S(\Phi)[\xi, Q(\Phi)f] = 0 \quad \forall \xi \in \Gamma_0(E),\ f \in \Gamma_0(\mathfrak{g}) \tag{2.0.9}$$

We see that the field equations are G-invariant. This implies that the action of G on $\Gamma(E)$ maps the solution space $\mathcal{S}$ onto itself.

The second functional derivative of identity (2.0.7) yields

$$\begin{aligned} &\delta^3 S(\Phi)[Q(\Phi)f, (\cdot_1), (\cdot_2)] + \delta S(\Phi)[\delta^2 Q(\Phi)[(\cdot_1), (\cdot_2)]f] \\ &+ \delta^2 S(\Phi)[\delta Q[\cdot_1]f, (\cdot_2)] + \delta^2 S(\Phi)[\delta Q[(\cdot_2)]f, (\cdot_1)] = 0 \quad \forall f \in \Gamma_0(\mathfrak{g}) \end{aligned} \tag{2.0.10}$$

The identities (2.0.8) and (2.0.10) are of course valid for any invariant replacing the action functional.

Using identity (2.0.9), we see that on-shell all tangent vectors of the form

$$\xi = Q(\Phi)f \in T_\Phi \Gamma(E) \simeq \Gamma(E),\ f \in \Gamma(\mathfrak{g}) \tag{2.0.11}$$

provide Jacobi fields.

We will call Jacobi fields with past- or future compact support **unphysical Jacobi fields**. We will furthermore assume that all unphysical Jacobi fields of the theory are of the form (2.0.11).

The presence of unphysical Jacobi fields implies that no advanced and retarded Green's operators for the linearized field equations exist.

To verify this, let $\xi_\pm \in \Gamma(E)$ be an unphysical Jacobi field with future resp. past compact support. Now we assume the existence of advanced- and retarded Green's operators of the linearized field equations. Then on-shell we would have

$$\begin{aligned} &G_\pm \circ \delta^2 S(\Phi)\ \xi_\pm = G_\pm\ 0 = 0 \\ &G_\pm \circ \delta^2 S(\Phi)\ \xi_\pm = \mathrm{id}\ \xi = \xi \end{aligned} \tag{2.0.12}$$

which is clearly contradictory. We see that the existence of unphysical Jacobi fields implies the nonexistence of advanced and retarded Green's operators for the linearized field equations.

3. The extended Peierls bracket

In this chapter we extend the definition of the Peierls bracket to gauge theories, following DeWitt in [7a]. In the first section we give a method for the **gauge fixing** of the linearized field equations. In the second section we define the extended Peierls bracket and examine its properties. In the third section we investigate how the extended Peierls bracket is related to the symplectic formalism for classical field theories.

3.1. Gauge fixing

To extend the definition of the Peierls bracket to gauge theories, we are again concerned with the Jacobian of the theory:

$$\delta^2 S(\Phi) : \Gamma(M, E) \to \Gamma(M, E^*) \tag{3.1.1}$$

As we have seen in the previous chapter, no Green's operators G for this operator exist due to the existence of unphysical Jacobi fields generated by the invariant flow. In order to fix this problem, we have to modify the linearized field equations in a way such that the modified linearized field equations possess unique advanced and retarded Green's operators but preserve the physical structure of the theory. We will see what this means in detail later on in this section.

Following [7a], we use the following method for gauge fixing the linearized field equations:

We define a Φ-jet dependent linear partial differential operator

$$P(\Phi) : T_\Phi\Gamma(E) \simeq \Gamma(E) \to \Gamma(\mathfrak{g}^*) \tag{3.1.2}$$

such that no unphysical Jacobi fields $\xi \in \Gamma(E)$ satisfy the **gauge fixing condition**

$$P(\Phi)\ \xi = 0 \tag{3.1.3}$$

We furthermore define the Φ-jet dependent partial differential operator

$$\mathfrak{F}(\Phi) := P(\Phi) \circ Q(\Phi) : \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g}^*) \tag{3.1.4}$$

Since we assumed that all unphysical Jacobi fields are generated by the invariant flow, it is implied by the definition of P that $\mathfrak{F}$ possesses no nonvanishing solutions of future or past compact support.

As we have seen in section 1.4, the operator $\mathfrak{F}$ has this property if it is normally hyperbolic. In this case, $\mathfrak{F}$ possesses unique advanced and retarded Green's operators $\mathfrak{G}_+, \mathfrak{G}_-$. We will restrict our choice to operators P such that $\mathfrak{F}$ is normally hyperbolic at least in an open neighborhood $\mathcal{U}_{\mathfrak{F}}$ of the solution space $\mathcal{S}$, so that the causal propagator exists in this neighborhood and its functional derivative can be defined.

Having chosen an operator P possessing the required properties, we define the linear partial differential operator

$$\begin{aligned} &F(\Phi) : \Gamma(M, E^*) \to \Gamma(M, E) \\ &F(\Phi) := \delta^2 S(\Phi) + \tilde{P}(\Phi) \circ K(\Phi) \circ P(\Phi) \end{aligned} \tag{3.1.5}$$

with

$$K(\Phi) : \mathfrak{g}^* \to \mathfrak{g} \tag{3.1.6}$$

being a formally selfadjoint Φ-jet dependent vector bundle isomorphism.

The operator F is selfadjoint by construction.

We will restrict the choice of the operator P such that $F(\Phi)$ is normally hyperbolic at least in an open neighborhood $\mathcal{U}_F$ of the solution space. This again ensures that $F(\Phi)$ possesses unique advanced and retarded Green's operators in $\mathcal{U}_F$. The intersection of $\mathcal{U}_{\mathfrak{F}}$ and $\mathcal{U}_F$ is again an open neighborhood of $\mathcal{S}$ and will be denoted by $\mathcal{U} := \mathcal{U}_F \cap \mathcal{U}_{\mathfrak{F}}$.

For any choice of P and K satisfying the given conditions, the linear partial differential equation for $\xi \in T_\Phi \Gamma(E) \simeq \Gamma(E)$

$$F(\Phi)\ \xi = 0 \tag{3.1.7}$$

will be called the **gauge fixed linearized field equation**.

The physical structure of the original and gauge fixed linearized field equations is identical in the sense that on shell we have that

$$\delta^2 S(\Phi)\ \xi = 0 \quad \wedge \quad P(\Phi)\ \xi = 0 \Rightarrow F(\Phi)\ \xi = 0 \tag{3.1.8}$$

$$F(\Phi)\ \xi = 0 \quad \wedge \quad P(\Phi)\ \xi = 0 \Rightarrow \delta^2 S(\Phi)\ \xi = 0 \tag{3.1.9}$$

We see that the on-shell solutions of $F(\Phi)$ and $\delta^2 S(\Phi)$ satisfying the gauge fixing condition are identical.

The causal propagator of the gauge fixed linearized field equations is denoted by Δ_{gf}.

3.2. Definition and properties of the extended Peierls bracket

Since the gauge fixed linearized field equations possess unique advanced and retarded Green's operators at least in an open environment $\mathcal{U}$ of the solution space $\mathcal{S}$, we can define the corresponding causal propagator and, based on this definition, the **extended Peierls bracket**:

Definition 3.2.1 (Extended Peierls bracket)
The extended Peierls bracket is given by the map

$$\begin{aligned} &\{\cdot,\cdot\} : \mathcal{F}|_{\mathcal{U}} \times \mathcal{F}|_{\mathcal{U}} \to \mathcal{F}|_{\mathcal{U}} \\ &\{A, B\} := \delta A[\Delta_{\mathrm{gf}}\ \delta B] \end{aligned} \tag{3.2.1}$$

As we already did in section 1.5 for the original Peierls bracket, we restrict the action of the extended Peierls bracket to the functionals in $\mathcal{F}$ satisfying the wavefront set condition in order to ensure that the action of the bracket is well defined and the bracket algebra closes.

The form of the gauge fixed linearized field equations depends on the choice of the operators P and K. Therefore the corresponding causal propagator and the extended Peierls bracket also depend on this choice. We examine this dependency now.

To do so, we first need to calculate the dependence of the advanced and retarded Green's operators of F on the choice of the operators P and K.

A general variation of the operator F yields

$$\begin{aligned} &F \to F + dF,\ G_\pm \to G_\pm + dG_\pm \\ \Rightarrow\ &F \circ dG_\pm + dF \circ\ G_\pm = 0 \end{aligned} \tag{3.2.2}$$

This equation has the solution

$$dG_\pm = -G_\pm \circ dF \circ\ G_\pm \tag{3.2.3}$$

which can easily be checked by insertion in (3.2.2) and the definition of the Green's operators. This solution is unique due to the unique support properties of the advanced and retarded Green's operators.

The variation of F with respect to variations of P and K is given by

$$dF = d\tilde{P} \circ K \circ P + \tilde{P} \circ dK \circ P + \tilde{P} \circ K \circ dP \tag{3.2.4}$$

Using this in equation (3.2.3) yields

$$\begin{aligned} dG_\pm = &- G_\pm \circ d\tilde{P} \circ K \circ P \circ G_\pm \\ &- G_\pm \circ \tilde{P} \circ dK \circ P \circ G_\pm \\ &- G_\pm \circ \tilde{P} \circ K \circ dP \circ G_\pm \end{aligned} \tag{3.2.5}$$

In order to simplify this expression, we derive some additional relations in the following.

Using (2.0.8), on-shell or in case of a field independent invariant flow we have

$$F \circ Q = \tilde{P} \circ K \circ \mathfrak{F} \tag{3.2.6}$$

Acting with $G_\pm$ from the left and with $\mathfrak{G}_\pm$ from the right yields

$$Q \circ \mathfrak{G}_\pm = G_\pm \circ \tilde{P} \circ K \tag{3.2.7}$$

Acting with K^{-1} from the right results in

$$Q \circ \mathfrak{G}_\pm \circ K^{-1} = G_\pm \circ \tilde{P} \tag{3.2.8}$$

The formal adjoint of this equation is given by

$$K^{-1} \circ \mathfrak{G}_\pm \circ \tilde{Q} = P \circ G_\pm \tag{3.2.9}$$

Remark: The identities (3.2.8) and (3.2.9) are essential and will be used several times in the following calculations. Therefore, all subsequent calculations are only valid on-shell or assuming the field independence of the invariant flow.

Using identity (3.2.9), we see that for an invariant A we have

$$\begin{aligned} & F \circ G_\pm \; \delta A = \delta A \\ & P \circ G_\pm \; \delta A = K^{-1} \circ \mathfrak{G}_\pm \circ \tilde{Q} \; \delta A = 0 \\ \Rightarrow \; & \delta^2 S \circ G \; \delta A = \delta A \\ \Rightarrow \; & \delta^2 S \circ \Delta \; \delta A = 0 \end{aligned} \tag{3.2.10}$$

We see that the action of the causal propagator of $F(\Phi)$ on the functional derivative of an invariant yields a Jacobi field.

Using identities (3.2.8) and (3.2.9) in equation (3.2.5) yields

$$\begin{aligned} dG_\pm = & - G_\pm \circ d\tilde{P} \circ K \circ K^{-1} \circ \mathfrak{G}_\pm \circ \tilde{Q} \\ & - Q \circ \mathfrak{G}_\pm \circ K^{-1} \circ dK \circ K^{-1} \circ \mathfrak{G}_\pm \circ \tilde{Q} \\ & - Q \circ \mathfrak{G}_\pm \circ K^{-1} \circ K \circ dP \circ G_\pm \\ = & - G_\pm \circ d\tilde{P} \circ \mathfrak{G}_\pm \circ \tilde{Q} \\ & + Q \circ \mathfrak{G}_\pm \circ dK^{-1} \circ \mathfrak{G}_\pm \circ \tilde{Q} \\ & - Q \circ \mathfrak{G}_\pm \circ dP \circ G_\pm \end{aligned} \tag{3.2.11}$$

With this result we can calculate the variation of the extended Peierls bracket with respect to variations of the operators P and K:

$$\begin{aligned} d(\delta A[\Delta \; \delta B]) = & - \delta A[(G_+ \circ d\tilde{P} \circ \mathfrak{G}_+ - G_- \circ d\tilde{P} \circ \mathfrak{G}_-) \circ \tilde{Q} \; \delta B] \\ & - \delta A[Q \circ (\mathfrak{G}_+ \circ dK^{-1} \circ \mathfrak{G}_+ - \mathfrak{G}_- \circ dK^{-1} \circ \mathfrak{G}_-) \circ \tilde{Q} \; \delta B] \\ & - \delta A[Q \circ (\mathfrak{G}_+ \circ dP \circ G_+ - \mathfrak{G}_- \circ dP \circ G_-) \; \delta B] \end{aligned} \tag{3.2.12}$$

In case that A and B are invariants, this variation vanishes since in each term we have an invariant flow acting on a functional derivative of an invariant. We see that the Peierls bracket of two invariants does not depend on the choice of the operators P and K, at least along differentiable curves $P(\lambda), K(\lambda)$.

We will now examine the G-invariance of the extended Peierls bracket.

The functional derivative of the Peierls bracket yields

$$\begin{aligned} & \delta(\delta A[\Delta\ \delta B])[(\cdot)] \\ = \quad & \delta^2 A[\Delta\ \delta B, (\cdot)] - \delta^2 B[\Delta\ \delta A, (\cdot)] + \delta A[\delta\Delta[(\cdot)]\ \delta B] \\ = \quad & \delta^2 A[\Delta\ \delta B, (\cdot)] - \delta^2 B[\Delta\ \delta A, (\cdot)] \\ & - \delta A[G_+ \circ \delta F[(\cdot)] \circ G_+\ \delta B] + \delta A[G_- \circ \delta F[(\cdot)] \circ G_- \delta B] \end{aligned} \tag{3.2.13}$$

whereby we used expression (3.2.2) to calculate the functional derivative of the causal propagator.

The functional derivative of F is given by

$$\delta F[(\cdot)] = \delta(\delta^2 S)[(\cdot)] + \delta\tilde{P}[(\cdot)] \circ K \circ P + \tilde{P} \circ K \circ \delta P[(\cdot)] + \tilde{P} \circ \delta K[(\cdot)] \circ P \tag{3.2.14}$$

Acting with $G_\pm$ from the left and the right, we obtain

$$\begin{aligned} G_\pm \circ \delta F[(\cdot)] \circ G_\pm = \quad & G_\pm \circ \delta\tilde{P}[(\cdot)] \circ K \circ P \circ G_\pm \\ & + G_\pm \circ \tilde{P} \circ K \circ \delta P[(\cdot)] \circ G_\pm \\ & + G_\pm \circ \delta(\delta^2 S)[(\cdot)] \circ G_\pm \end{aligned} \tag{3.2.15}$$

We again apply the identities (3.2.8) and (3.2.9) to obtain

$$\begin{aligned} G_\pm \circ \delta F[(\cdot)] \circ G_\pm = \quad & G_\pm \circ \delta\tilde{P}[(\cdot)] \circ K \circ K^{-1} \circ \mathfrak{G}_\pm \circ \tilde{Q} \\ & + Q \circ \mathfrak{G}_\pm \circ K^{-1} \circ K \circ \delta P[(\cdot)] \circ G_\pm \\ & + G_\pm \circ \delta(\delta^2 S)[(\cdot)] \circ G_\pm \\ = \quad & G_\pm \circ \delta\tilde{P}[(\cdot)] \circ \mathfrak{G}_\pm \circ \tilde{Q} \\ & + Q \circ \mathfrak{G}_\pm \circ \delta P[\cdot] \circ G_\pm \\ & + G_\pm \circ \delta(\delta^2 S)[(\cdot)] \circ G_\pm \end{aligned} \tag{3.2.16}$$

Using this result in (3.2.13), we obtain for A, B being invariants:

$$\begin{aligned} \delta(\delta A[\Delta\ \delta B])[(\cdot)] = \quad & \delta^2 A[\Delta\ \delta B, (\cdot)] - \delta^2 B[\Delta\ \delta A, (\cdot)] \\ & - \delta A[G_+ \circ \delta(\delta^2 S)[(\cdot)] \circ G_+\ \delta B] \\ & + \delta A[G_- \circ \delta(\delta^2 S)[(\cdot)] \circ G_-\ \delta B] \\ = \quad & - \delta A[\delta Q[\Delta\ \delta B] f] + \delta B[\delta Q[\Delta\ \delta A] f] \\ & - \delta^3 S[G_-\ \delta A, Qf, G_+\ \delta B] \\ & + \delta^3 S[G_+\ \delta A, Qf, G_-\ \delta B] \end{aligned} \tag{3.2.17}$$

In the last step we applied identity (2.0.8) to the first two summands, whereby we assume to be on shell or the invariant flow to be field independent.

Now we can apply identity (2.0.10), again evaluated on-shell or assuming the field independence of the invariant flow, to the last two summands to obtain

$$\begin{aligned}
& -\delta A[\delta Q[\Delta\ \delta B]f] + \delta B[\delta Q[\Delta\ \delta A]f] \\
& - \delta^3 S[G_-\ \delta A, Qf, G_+\ \delta B] \\
& + \delta^3 S[G_+\ \delta A, Qf, G_-\ \delta B] \\
= & -\delta A[\delta Q[\Delta\ \delta B]f] + \delta B[\delta Q[\Delta\ \delta A]f] \\
& + \delta^2 S[G_-\ \delta A, \delta Q[G_+\ \delta B]f] + \delta^2 S[\delta Q[G_-\ \delta A]f, G_+\ \delta B] \\
& - \delta^2 S[\delta Q[G_+\ \delta A]f, G_-\ \delta B] - \delta^2 S[G_+\ \delta A, \delta Q[G_-\ \delta B]f]
\end{aligned} \tag{3.2.18}$$

We again use identity (2.0.8) on the last four summands, which yields

$$\begin{aligned}
& -\delta A[\delta Q[\Delta\ \delta B]f] + \delta B[\delta Q[\Delta\ \delta A]f] \\
& + \delta^2 S[G_-\ \delta A, \delta Q[G_+\ \delta B]f] + \delta^2 S[\delta Q[G_-\ \delta A]f, G_+\ \delta B] \\
& - \delta^2 S[\delta Q[G_+\ \delta A]f, G_-\ \delta B] - \delta^2 S[G_+\ \delta A, \delta Q[G_-\ \delta B]f] \\
= & -\delta A[\delta Q[\Delta\ \delta B]f] + \delta B[\delta Q[\Delta\ \delta A]f] \\
& + \delta A[\delta Q[G_+\ \delta B]f] + \delta B[\delta Q[G_-\ \delta A]f] \\
& - \delta B[\delta Q[G_+\ \delta A]] - \delta A[\delta Q[\delta G_-\ \delta B]f] \\
= & -\delta A[\delta Q[\Delta\ \delta B]f] + \delta B[\delta Q[\Delta\ \delta A]f] \\
& + \delta A[\delta Q[\Delta\ \delta B]f] - \delta B[\delta Q[\Delta\ \delta A]f] \\
= & \ 0 \quad \forall f \in \Gamma(\mathfrak{g})
\end{aligned} \tag{3.2.19}$$

We see that the Peierls bracket of two invariants again yields an invariant.

We still have to show that the extended Peierls bracket satisfies the Jacobi identity. As we already mentioned, the corresponding proof for the original Peierls bracket relies on the symmetry of the functional derivative of the linearized field equation. However, since it can in general not be derived from an action functional, the functional derivative of the gauge fixed linearized field equations can not be assumed to be symmetric.

We now examine under which conditions the Jacobi identity holds true for the extended Peierls bracket.

The iterated Peierls bracket is given by

$$\{\{A, B\}, C\} = \delta A[\delta\Delta[\Delta\ \delta C]\ \delta B] + \delta^2 A[\Delta\ \delta B, \Delta\ \delta C] - \delta^2 B[\Delta\ \delta C, \Delta\ \delta A] \tag{3.2.20}$$

When we add up the last two summands in this expression for all cyclic permutations of $\{A, B, C\}$, it is easily seen that the sum vanishes.

The first summand of (3.2.20) can be expressed as

$$\begin{aligned} \delta A[\delta\Delta[\Delta\ \delta C]\ \delta B] = & -\delta A[G_+ \circ \delta F[\Delta\ \delta C] \circ G_+\ \delta B] \\ & + \delta A[G_- \circ \delta F[\Delta\ \delta C] \circ G_-\ \delta B] \end{aligned} \tag{3.2.21}$$

If we assume that F can be derived from an action functional S_F such that

$$F(\Phi) = \delta^2 S_F(\Phi) \tag{3.2.22}$$

we have that expression (3.2.21) can be written as

$$\begin{aligned} \delta A[\delta\Delta[\Delta\ \delta C]\ \delta B] = & -\delta^3 S_F[G_-\ \delta A, G_+\ \delta B, \Delta\ \delta C] \\ & + \delta^3 S_F[G_+\ \delta A, G_-\ \delta B, \Delta\ \delta C] \\ = & -\delta^3 S_F[G_-\ \delta A, G_+\ \delta B, G_+\ \delta C] \\ & + \delta^3 S_F[G_-\ \delta A, G_+\ \delta B, G_-\ \delta C] \\ & + \delta^3 S_F[G_+\ \delta A, G_-\ \delta B, G_+\ \delta C] \\ & - \delta^3 S_F[G_+\ \delta A, G_-\ \delta B, G_-\ \delta C] \end{aligned} \tag{3.2.23}$$

If we add up this expression for all cyclic permutations of $\{A, B, C\}$, the sum vanishes due to the symmetry of the functional derivative of S_F.

In case that F cannot be derived from an action functional and that A, B, C are invariants, we can use identity (3.2.17) to obtain

$$\begin{aligned} \delta A[\delta\Delta[\Delta\ \delta C]\ \delta B] = & -\delta A[G_+ \circ \delta(\delta^2 S)[\Delta\ \delta C] \circ G_+\ \delta B] \\ & + \delta A[G_- \circ \delta(\delta^2 S)[\Delta\ \delta C] \circ G_-\ \delta B] \\ = & -\delta^3 S[G_-\ \delta A, G_+\ \delta B, \Delta\ \delta C] \\ & + \delta^3 S[G_+\ \delta A, G_-\ \delta B, \Delta\ \delta C] \\ = & -\delta^3 S[G_-\ \delta A, G_+\ \delta B, G_+\ \delta C] \\ & + \delta^3 S[G_-\ \delta A, G_+\ \delta B, G_-\ \delta C] \\ & + \delta^3 S[G_+\ \delta A, G_-\ \delta B, G_+\ \delta C] \\ & - \delta^3 S[G_+\ \delta A, G_-\ \delta B, G_-\ \delta C] \end{aligned} \tag{3.2.24}$$

If we add up this expression for all cyclic permutations of $\{A, B, C\}$, the sum vanishes due to the symmetry of the functional derivative of S.

we see that the Peierls bracket satisfies the Jacobi identity if restricted to act on invariants or in case the gauge fixed linearized field equations can be derived from an action functional.

For the original Peierls bracket, the field equations generate a Bracket ideal (see section 1.5). We will now see to which extent this holds true for the extended Peierls bracket.

We are again considering the set of functionals of the form

$$\mathcal{E}_S = \{F \in \mathcal{F} : F(\Phi) = D(\Phi)[dS(\Phi)] \text{ for a } D : \Gamma(M,E) \to \mathcal{E}(M,E^*,\mathbb{R}) \text{ smooth}\} \tag{3.2.25}$$

Let now $A = D[dS] \in \mathcal{E}$, $B \in \mathcal{F}$. Then it is

$$\{A,B\} = \delta(D[dS])[\Delta\ dB] = D[\delta^2 S \circ \Delta\ dB] - \delta D[dS, \Delta\ dB] \tag{3.2.26}$$

Using the definition of the operator F we obtain

$$\{A,B\} = D[(F - \tilde{P} \circ K \circ P) \circ \Delta\ \delta B] - \delta D[dS, \Delta\ \delta B] \tag{3.2.27}$$

If B is an invariant, the first term vanishes on-shell due to identity (3.2.10). The second term is contained in $\mathcal{E}_S$. Hence if we restrict the bracket algebra to the invariants, the field equations generate a bracket ideal $\mathcal{E}_S$ of $\mathcal{F}_{\text{inv}}$.

We **summarize our results** on the properties of the extended Peierls bracket obtained in this section:

In general, the extended Peierls bracket can only be defined on $\mathcal{F}|_{\mathcal{U}}$, where $\mathcal{U}$ is an open neighborhood of the solution space $\mathcal{S}$ in $\Gamma(E)$.

If the gauge fixed operator F can be derived from an action functional $S_F(\Phi)$, the restricted bracket algebra $(\mathcal{F}|_{\mathcal{U}}, \{\cdot,\cdot\})$ is a Poisson algebra.

On-shell, or assuming the field independence of the invariant flow, the invariants $\mathcal{F}_{\text{inv}}|_{\mathcal{U}}$ form a bracket subalgebra $(\mathcal{F}_{\text{inv}}|_{\mathcal{U}}, \{\cdot,\cdot\})$ which is independent from the choice of the operators P and K and is a Poisson algebra.

The field equations generate a bracket ideal of $(\mathcal{F}_{\text{inv}}, \{\cdot,\cdot\})$.

We will now have a closer look at the properties of the extended Peierls bracket in case that the invariant flow is field independent.

In this case, the operators P and K can be chosen to be field independent and the gauge fixed linearized field equations can be derived from the **gauge fixed action functional**

$$S_{\text{gf}}(\Phi) := S(\Phi) + \int_M d^n x\ \left\{\Phi\ (\tilde{P} \circ K \circ P\ \Phi)\right\} \tag{3.2.28}$$

Due to this property, the extended Peierls bracket coincides with the original Peierls bracket constructed with respect to this gauge fixed action functional, which is well defined at least in the open neighborhood $\mathcal{U}$ of the solution space since in this neighborhood the linearized field equations derived from this action functional are by definition normally hyperbolic. This implies that the extended Peierls bracket provides a Poisson bracket all over $\mathcal{U}$. In addition, we have already seen that the invariants generate a Poisson subalgebra $\mathcal{F}_{\text{inv}}$ of this Poisson algebra.

The **gauge fixed field equations** generate another Poisson ideal $\mathcal{E}_F$ given by

$$\mathcal{E}_F = \{A \in \mathcal{F} : A(\Phi) = D(\Phi)[\delta S_{\text{gf}}(\Phi)] \text{ for a } D : \Gamma(E) \to \mathcal{E}(E^*, \mathbb{R}) \text{ smooth}\} \tag{3.2.29}$$

Finally, we introduce the space $\mathcal{F}_P$ of smooth functionals generated by the gauge fixing operator P:

$$\mathcal{F}_P := \{F \in \mathcal{F} : \ F(\Phi) = D(\Phi)[P\ \Phi] \text{ for a } D : \Gamma(E) \to \mathcal{D}'(\mathfrak{g}^*, \mathbb{R}) \text{ smooth}\} \tag{3.2.30}$$

We examine the properties of $\mathcal{F}_P$.

First of all, we see that $(\mathcal{F}_P, \{\cdot,\cdot\})$ is a subalgebra of $(\mathcal{F}, \{\cdot,\cdot\})$. To verify this, let $A, B \in \mathcal{F}_P$. We then have

$$\begin{aligned} \{A, B\}(\Phi) = &\quad \{D_1(\Phi)[P\ \Phi], D_2(\Phi)[P\ \Phi]\} \\ = &\quad \delta D_1(\Phi)[P\ \Phi, \Delta\ \delta B] \\ &+ D_1(\Phi)[P \circ \Delta\ \delta D_2(\Phi)[P\ \Phi]] \\ &+ D_1(\Phi)[P \circ \Delta \circ \tilde{P}\ D_2(\Phi)] \end{aligned} \tag{3.2.31}$$

The first two summands are obviously contained in $\mathcal{F}_P$. The last term vanishes since

$$\begin{aligned} D_1(\Phi)[P \circ \Delta \circ \tilde{P}\ D_2(\Phi)] &= D_1[K^{-1} \circ \Delta_{\mathfrak{F}} \circ \tilde{Q} \circ \tilde{P}\ D_2(\Phi)] \\ &= D_1[K^{-1} \circ \Delta_{\mathfrak{F}} \circ\ \mathfrak{F}\ D_2(\Phi)] \\ &= 0 \end{aligned} \tag{3.2.32}$$

Furthermore, the subalgebra $\mathcal{F}_{\text{inv}}$ stabilizes $\mathcal{F}_P$ in $(\mathcal{F}, \{\cdot,\cdot\})$:

Let $A \in \mathcal{F}_{\text{inv}}$ and $B \in \mathcal{F}_P$. We then have

$$\begin{aligned} \{B, A\}(\Phi) &= \{D(\Phi)[P\ \Phi], A(\Phi)\} \\ &= D(\Phi)[P \circ \Delta\ \delta A(\Phi)] - \delta A(\Phi)[\Delta\ (\delta D(\Phi)[P\ \Phi])] \\ &= -\ \delta A(\Phi)[\Delta\ (\delta D(\Phi)[P\ \Phi])] \in \mathcal{F}_P \end{aligned} \tag{3.2.33}$$

Combining these results, we have that $\mathcal{F}_P + \mathcal{F}_{\text{inv}}$ forms a Poisson subalgebra of $\mathcal{F}$.

3.3. Relation to the symplectic formalism

In this section we examine the relation between the (extended) Peierls bracket and the symplectic formalism for classical field theories (see e.g. [6]). This formalism relies on the definition of a symplectic form (i.e. a closed nondegenerate 2-form) $\omega(\Phi)$ on the solution space $\mathcal{S}$. The formal inverse of the symplectic form then yields a Poisson bracket on the algebra of smooth functionals on the solution space. In this section we will see how the extended Peierls bracket can be inverted to construct such a symplectic form.

First of all we define the **Wronski operator** of a linear partial differential operator:

Definition 3.3.1 (Wronski operator)
Let $F : \Gamma(E) \to \Gamma(E^*)$ be a linear partial differential operator. A Wronski operator W of P is any antisymmetrical bidifferential operator

$$W : \Gamma(E) \times \Gamma(E) \to \Gamma(TM) \tag{3.3.1}$$

satisfying

$$\int_U d^n x \ \{\xi \ (P \ \chi) - \chi \ (P \ \xi))\} = \int_\Omega d\Omega_\mu \ W^\mu[\xi, \chi] \tag{3.3.2}$$

for all $\xi, \chi \in \Gamma(E)$ and open sets $U \subset M$ with smooth orientable boundary $\Omega = \partial U$.

We are interested in the Wronski operator $W_S(\Phi)$ for the linearized field equations. Given a Cauchy hypersurface Σ, we use $W_S(\Phi)$ to define the antisymmetric bilinear map

$$\begin{aligned} &\omega_\Sigma(\Phi) : \Gamma(E) \times \Gamma(E) \to \mathbb{R} \\ &\omega_\Sigma(\Phi)[\xi, \chi] := \int_\Sigma d\Sigma_\mu \ W_S^\mu(\Phi)[\xi, \chi] \end{aligned} \tag{3.3.3}$$

The action of $\omega_\Sigma(\Phi)$ is in general only well defined if (supp $\xi \cap$ supp $\chi \cap \Sigma$) is compact.

In the following, we restrict the action of $\omega_\Sigma(\Phi)$ to the solutions of the linearized field equations. We now examine the properties of ω_Σ.

First of all, we note that the action of $\omega_\Sigma(\Phi)$ is independent of the choice of the Cauchy hypersurface Σ. We verify this:

$$\begin{aligned} &\delta^2 S(\Phi) \ \xi = \delta^2 S(\Phi) \ \chi = 0, \ \{\Sigma, \Sigma'\} \text{ Cauchy hypersurfaces} \\ \Rightarrow \ &\omega_\Sigma(\Phi)[\xi, \chi] - \omega_{\Sigma'}(\Phi)[\xi, \chi] = \int_{\Sigma'}^{\Sigma} d^n x \ \left\{\xi \ (\delta^2 S \ \chi) - \chi \ (\delta^2 S \ \xi)\right\} = 0 \end{aligned} \tag{3.3.4}$$

We therefore denote ω_Σ simply by ω.

A direct consequence of this result is the following:

$$\begin{aligned} &\delta^2 S(\Phi) \ \xi = \delta^2 S(\Phi) \ \chi, \ (\text{supp } \chi) \text{ future or past compact} \\ \Rightarrow \ &\omega_\Sigma(\Phi)[\xi, \chi] = \omega_{\Sigma'}(\Phi)[\xi, \chi] = 0 \end{aligned} \tag{3.3.5}$$

since a Cauchy hypersurface Σ' exists such that $(\text{supp}\,\chi \cap \Sigma') = \emptyset$.

We see that $\omega(\Phi)$ vanishes when acting on at least one solution with past or future compact support of the linearized field equations.

It follows that in case that such solutions exist, the bilinear map $\omega(\Phi)$ is degenerate.

If we assume the existence of advanced and retarded Green's functions (and therefore the existence of the causal propagator Δ) for the linearized field equations, $\omega(\Phi)$ is non-degenerate.

This follows from the fact that it can then be inverted in the following sense:

Let $A \in \mathcal{F}$, Σ be an arbitrary Cauchy surface and Σ_+, Σ_- arbitrary Cauchy surfaces in the future resp. past of the support of A.

We then have that

$$\begin{aligned} \omega(\Phi)[\Delta\ \delta A, \xi] = &\quad \omega_\Sigma(\Phi)[G_+\ \delta A, \xi] - \omega_\Sigma(\Phi)[G_-\ \delta A, \xi] \\ = &\quad \omega_\Sigma(\Phi)[G_+\ \delta A, \xi] - \omega_{\Sigma_+}(\Phi)[G_+\ \delta A, \xi] \\ &- \omega_\Sigma(\Phi)[G_-\ \delta A, \xi] + \omega_{\Sigma_-}(\Phi)[G_-\ \delta A, \xi] \end{aligned} \tag{3.3.6}$$

where we used the support properties of the advanced and retarded Green's functions.

Using the definition of the Wronski operator, we can write this as

$$\begin{aligned} &- \int_\Sigma^{\Sigma_+} d^n x\ \left\{(G_+\ \delta A)\ (\delta^2 S\ \xi) - \xi\ (\delta^2 S \circ G_+\ \delta A)\right\} \\ &- \int_{\Sigma_-}^{\Sigma} d^n x\ \left\{(G_-\ \delta A)\ (\delta^2 S\ \xi) - \xi\ (\delta^2 S \circ G_-\ \delta A)\right\} \\ = &\int_\Sigma^{\Sigma_+} d^n x\ \left\{\xi\ (\delta^2 S \circ G_+\ \delta A)\right\} + \int_{\Sigma_-}^{\Sigma} d^n x\ \left\{\xi\ (\delta^2 S \circ G_-\ \delta A)\right\} \\ = &\int_\Sigma^{\Sigma_+} d^n x\ \{\xi\ \delta A\} + \int_{\Sigma_-}^{\Sigma} d^n x\ \{\xi\ \delta A\} \\ = &\int_{\Sigma_-}^{\Sigma_+} d^n x\ \{\xi\ \delta A\} = \delta A[\xi] \end{aligned} \tag{3.3.7}$$

We see that the inverse of ω is given by the causal propagator, and that the Wronski operator solves the initial value problem of the linearized field equations. The map $\omega(\Phi)$ for $\Phi \in \mathcal{S}$ is called the **symplectic form** of the theory.

Using this result, the original Peierls bracket can be expressed as

$$\{A, B\} = \omega[\Delta\ \delta A, \Delta\ \delta B] \tag{3.3.8}$$

In case that we are dealing with a gauge theory, we can derive a similar result:

Let A be an invariant and Φ be on-shell. Let furthermore $\xi \in \Gamma(E)$ be a Jacobi field.

We then have

$$\omega[\Delta_{\mathrm{gf}}\ \delta A, \xi] = \int_{\Sigma}^{\Sigma_+} d^n x\ \left\{ \xi\ (\delta^2 S \circ G_+\ \delta A) \right\} + \int_{\Sigma_-}^{\Sigma} d^n x\ \left\{ \xi\ (\delta^2 S \circ G_-\ \delta A) \right\} \quad (3.3.9)$$

where we just performed the same calculation as in (3.3.7).

As we calculated in (3.2.10), on-shell we have

$$P \circ G_{\mathrm{gf}}\ \delta A = 0 \quad (3.3.10)$$

for A being an invariant. Therefore it is

$$\delta^2 S \circ G_{\mathrm{gf}}\ \delta A = F \circ G_{\mathrm{gf}}\ \delta A = \delta A \quad (3.3.11)$$

and we have

$$\omega(\Phi)[\Delta_{\mathrm{gf}}\ \delta A, \xi] = \delta A[\xi] \quad (3.3.12)$$

For $\Phi \in \mathcal{S}$, the map $\omega(\Phi)$ is called the **presymplectic form** of the gauge theory, due to its degeneracy.

Using this result, the on-shell extended Peierls bracket of two invariants A, B can be expressed as

$$\{A, B\} = \omega[\Delta\ \delta A, \Delta\ \delta B] \quad (3.3.13)$$

since $\Delta\ \delta A$ provides a Jacobi field for A being an invariant (see identity 3.2.10).

3.4. Example: The Free Maxwell field

In this section we apply the construction scheme for the extended Peierls bracket on the theory of the free Maxwell field. We formulate the theory using the vector potential A as classical field.

The classical vector potential takes values in the smooth sections over the vector bundle T^*M. The covariant derivative on T^*M is given by the Levi-Civita connection.

The action functional for the vector potential is given by

$$S_\psi = \int_M d^n x\ \sqrt{-\det g}\ F_{ab}\ F^{ab}\ \psi, \quad F = dA \quad (3.4.1)$$

The functional derivative of the action functional yields the Maxwell equations:

$$\sqrt{-\det g}\ (\Box^\nabla A^a - \nabla^a \nabla_b A^b) = 0 \tag{3.4.2}$$

Due to the linearity of the theory, the linearized field equations are again given by the differential operator

$$\delta^2 S^a[\xi] := \sqrt{-\det g}\ (\Box^\nabla\ \xi^a - \nabla^a \nabla_b\ \xi^b) \tag{3.4.3}$$

This operator is not normally hyperbolic. It has the invariant flow

$$\begin{aligned} &Q : \mathcal{C}^\infty(M, \mathbb{R}) \to \Gamma(T^*M) \\ &Q_a\ f := \nabla_a\ f \end{aligned} \tag{3.4.4}$$

We follow the method of gauge fixing described in section (3.1) and make the following choices:

$$\begin{aligned} &P\ \xi := \sqrt{-\det g}\ \nabla^a \xi_a \\ \Rightarrow\ &\mathfrak{F}\ f = P \circ Q\ f = \sqrt{-\det g}\ \Box^\nabla\ f \\ &K := \frac{-1}{\sqrt{-\det g}} \\ \Rightarrow\ &(\tilde{P} \circ K \circ P)^a[\xi] = \sqrt{-\det g}\ \nabla^a \nabla_b \xi^b \\ \Rightarrow\ &F = \delta^2 S + \tilde{P} \circ K \circ P = \sqrt{-\det g}\ \Box^\nabla \end{aligned} \tag{3.4.5}$$

We see that $\mathfrak{F}, F$ are clearly normally hyperbolic and are therefore possess unique advanced and retarded Green's operators.

The chosen gauge fixing condition corresponds to the well known **Lorenz gauge**.

Since the invariant flow of the free Maxwell field is field independent, the extended Peierls bracket possesses the properties discussed at the end of the previous section.

The gauge fixed linearized field equations can be derived from the gauge fixed action functional

$$S_{gf}(A) := -\int_M d^n x\ \sqrt{-\det g}\ \{\nabla_a A^b\ \nabla^a A_b\} \tag{3.4.6}$$

which provides a basis for the definition of canonical conjugate variables.

The gauge fixed Maxwell equations generate a Poisson ideal $\mathcal{F}_F$ of the resulting Poisson algebra. The invariants generate a Poisson subalgebra $\mathcal{F}_{\text{inv}}$, which itself has an ideal $\mathcal{E}_S$ generated by the field equations. The gauge fixing condition generates a subalgebra $\mathcal{F}_P$ stabilized by $\mathcal{F}_{\text{inv}}$. In particular, the gauge fixing condition itself Poisson commutes with all invariants. We verify this:

Consider a functional of the form

$$B := D[P\ A], \quad D \in \mathcal{E}'(M \times \mathbb{R}, \mathbb{R}) \text{ field independent} \tag{3.4.7}$$

Let C be an invariant. Then it is

$$\{B, C\} = \delta B[\Delta\ \delta C] = D[P \circ \Delta\ \delta C] = 0 \tag{3.4.8}$$

We see that the gauge fixing condition Poisson commutes with all invariants.

4. Classical General Relativity

The main goal of this work is to which extent the formalism based on the extended Peirels bracket can be used to given an algebraic formulation of classical General Relativity. In this chapter, we therefore give a Lagrangian formulation of classical General Relativity (see e.g. [5, 23]) and derive the linearized Einstein equations to construct the gauge fixed linearized field equations.

4.1. Lagrangian formulation

A well known formulation of classical General Relativity is using the notion of a smooth Lorentzian metric g as dynamical field on a smooth (4-dimensional) manifold M. This approach is **background independent** in the sense that the causal and metric structure on M are not given in advance and determined only by the dynamical fields. However, this formulation still relies on a fixed smooth manifold structure on M.

General Relativity in terms of metric components can be formulated as a Lagrangian field theory . Using the framework given in chapter 1, we make the following choices:

Let $E := \Omega^2_{\mathrm{symm}}(M, \mathbb{R})$ be the vector bundle of symmetric $(0, 2)$-tensors on M. We are only interested in smooth sections $g \in \Gamma(E)$ which yield a globally hyperbolic Lorentzian spacetime (M, g). We will denote the subset of $\Gamma(E)$ satisfying these conditions by $\Gamma_{\mathrm{gh}}(E)$. This space provides the configuration space of the theory.

The configuration space $\Gamma_{\mathrm{gh}}(E)$ lies not open in $\Gamma(E)$. However, we still want to be able to use the differential structure on $\Gamma(E)$ given in section 1.1. Therefore we will only be considering functionals defined at least on an open neighborhood of $\Gamma_{gh}(E)$ in $\Gamma(E)$, so that the functional derivatives $\delta^k A(g)[\xi_1, ..., \xi_k]$ are well defined for any such functional A and $g \in \Gamma_{gh}(E),\ \xi_i \in \Gamma(E)$.

We will fix the covariant derivative ∇ to be the **Levi Civita connection**. The Levi Civita connection is the unique metric compatible and torsion free covariant derivative corresponding to the metric g. Its connection coefficients are called the **Christoffel symbols** and they are, in terms of the metric components, given by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \tag{4.1.1}$$

The **Riemann curvature** of the Levi Civita connection is given by

$$R^{\nabla}(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z := e_d\ R^{d\ \nabla}_{abc}\ X^b\ Y^c\ Z^a \tag{4.1.2}$$

which, expressed using the Christoffel symbols, yields

$$R^{d\ \nabla}_{\ abc} = \partial_b \Gamma^d_{ca} - \partial_c \Gamma^d_{ba} + \Gamma^d_{be}\Gamma^e_{ca} - \Gamma^d_{ce}\Gamma^e_{ba} \tag{4.1.3}$$

The Ricci curvature tensor is given by the contraction of the Riemann curvature:

$$R^{\nabla}_{ab} := R^{c\ \nabla}_{\ acb} \tag{4.1.4}$$

Finally, the **Ricci scalar** is given by the trace of the Ricci curvature:

$$R^{\nabla} := g^{ab}\ R^{\nabla}_{ab} \tag{4.1.5}$$

A possible choice for the action functional is the **Einstein Hilbert action**:

$$S(g) := \int_M d^n x\ \sqrt{-g}\ R^{\nabla} \tag{4.1.6}$$

To calculate the functional derivatives of the Einstein Hilbert action, we first derive some more elementary functional derivatives:

First of all, we will calculate the functional derivative of the inverse metric. We have

$$\begin{aligned} & g^{ab}\ g_{bc} = \delta^a_c \\ \Rightarrow\ & \delta(g^{ab}\ g_{bc})[\xi] = \delta g^{ab}[\xi]\ g_{bc} + g^{ab}\ \delta g_{bc}[\xi] = 0 \\ \Rightarrow\ & \delta g^{ab}[\xi] = -g^{ac} g^{bd}\ \delta g_{cd}[\xi] \end{aligned} \tag{4.1.7}$$

Using this result, we calculate the functional derivative of the Christoffel symbols:

$$\begin{aligned} \delta\Gamma^a_{bc}[\xi] =\ & -\frac{1}{2}\ g^{ae}\ g^{df}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})\ \xi_{ef} \\ & + \frac{1}{2} g^{ad}\ (\partial_b \xi_{cd} + \partial_c \xi_{bd} - \partial_d \xi_{bc}) \\ =\ & -g^{ad}\ \Gamma^f_{bc}\ \xi_{df} + \frac{1}{2}\ g^{ad}\ (\partial_b \xi_{cd} + \partial_c \xi_{bd} - \partial_d \xi_{bc}) \\ =\ & -\frac{1}{2}\ g^{ad}\ (\Gamma^e_{bc}\ \xi_{de} + \Gamma^e_{bc}\ \xi_{de} + \Gamma^e_{dc}\ \xi_{be} - \Gamma^e_{dc}\ \xi_{ce} + \Gamma^e_{bd}\ \xi_{ce} - \Gamma^e_{bd}\ \xi_{ce}) \\ & + \frac{1}{2}\ g^{ad}\ (\partial_b \xi_{cd} + \partial_c \xi_{bd} - \partial_d \xi_{bc}) \\ =\ & \frac{1}{2}\ g^{ad}\ (\nabla_b \xi_{cd} + \nabla_c \xi_{bd} - \nabla_d \xi_{bc}) \end{aligned} \tag{4.1.8}$$

The calculation of the functional derivative of the Riemann tensor then yields:

$$\begin{aligned} \delta R^{d\ \nabla}_{abc}[\xi] &= \partial_b \delta\Gamma^d_{ac}[\xi] - \partial_c \Gamma^d_{ab}[\xi] \\ &\quad + \Gamma^d_{be}\ \delta\Gamma^e_{ac}[\xi] + \Gamma^e_{ac}\ \delta\Gamma^d_{eb}[\xi] - \Gamma^d_{ec}\ \delta\Gamma^e_{ab}[\xi] - \Gamma^e_{ab}\ \delta\Gamma^d_{ce}[\xi] \\ &= \partial_b \delta\Gamma^d_{ac}[\xi] - \partial_c \Gamma^d_{ab}[\xi] + \Gamma^e_{bc}\ \delta\Gamma^d_{ae}[\xi] - \Gamma^e_{bc}\ \delta\Gamma^d_{ae}[\xi] \\ &\quad + \Gamma^d_{be}\ \delta\Gamma^e_{ac}[\xi] + \Gamma^e_{ac}\ \delta\Gamma^d_{eb}[\xi] - \Gamma^d_{ec}\ \delta\Gamma^e_{ab}[\xi] - \Gamma^e_{ab}\ \delta\Gamma^d_{ce}[\xi] \\ &= \nabla_b(\delta\Gamma^d_{ac}[\xi]) - \nabla_c(\delta\Gamma^d_{ab}[\xi]) \end{aligned} \tag{4.1.9}$$

which allows a straightforward calculation of the functional derivative of the Ricci tensor

$$\delta R^{\nabla}_{ac}[\xi] = \delta R^{d\ \nabla}_{adc}[\xi] = \nabla_d(\delta\Gamma^d_{ac}[\xi]) - \nabla_c(\delta\Gamma^d_{ad}[\xi]) \tag{4.1.10}$$

and the functional derivative of the Ricci scalar:

$$\begin{aligned}\delta R^{\nabla}[\xi] &= \delta(g^{ac}\ R^{\nabla}_{ac})[\xi] = \delta g^{ac}[\xi]\ R^{\nabla}_{ac} + g^{ac}\ \delta R^{\nabla}_{ac}[\xi] \\ &= -\ g^{ad}g^{ce}\ \xi_{de}\ R^{\nabla}_{ac} + g^{ac}\ (\nabla_d(\delta\Gamma^d_{ac}[\xi]) - \nabla_c(\delta\Gamma^d_{ad}[\xi]))\end{aligned} \tag{4.1.11}$$

Using the trace theorem (see e.g. [5], p. 162)

$$\delta(\det A) = \det A\ \mathrm{Tr}(A^{-1}\ \delta A) \tag{4.1.12}$$

we can calculate the functional derivative of $\sqrt{-\det g}$:

$$\delta\sqrt{-\det g}[\xi] = -\frac{1}{2\sqrt{-\det g}}\ \det g\ \mathrm{Tr}\ \xi = \frac{1}{2}\sqrt{-\det g}\ \mathrm{Tr}\ \xi \tag{4.1.13}$$

Now we can finally calculate the functional derivative of the Einstein Hilbert action:

$$\begin{aligned}\delta S(g)[\xi] &= \int_M d^n x\ \left\{\delta(\sqrt{-\det g})[\xi]\ R^{\nabla} + \sqrt{-\det g}\ \delta R^{\nabla}[\xi]\right\} \\ &= \int_M d^n x\ \sqrt{-\det g}\ \left\{\frac{1}{2}\mathrm{Tr}\ \xi\ R^{\nabla} - \xi_{de}\ R^{de} + \nabla_d(g^{ac}\ \delta\Gamma^d_{ac}[\xi]) - \nabla_c(g^{ac}\ \delta\Gamma^d_{ad}[\xi])\right\}\end{aligned} \tag{4.1.14}$$

Since we assume ξ to be of compact support, the last two summands in this expression yield integrations over the divergence of compactly supported vector fields and therefore vanish.

We are left with

$$\delta S(g)[\xi] = \int_M d^n x\ \sqrt{-\det g}\ (\frac{1}{2}\ R\ g^{ab} - R^{ab})\ \xi_{ab} =: -\int_M d^n x\ \sqrt{-\det g}\ G^{ab}\ \xi_{ab} \tag{4.1.15}$$

whereby we use this equation to define the **Einstein tensor** G^{ab}.

The stationary points of the Einstein Hilbert action are therefore given by the metric configurations satisfying the Einstein equations:

$$\begin{aligned}&\sqrt{-\det g}\ G^{ab} = 0 \\ \Leftrightarrow\ &G^{ab} = 0 \\ \Leftrightarrow\ &R^{ab} = 0\end{aligned} \tag{4.1.16}$$

4.2. Gauge symmetry

In this section, we examine the gauge invariance of General Relativity. Our goal is to find the invariant flows of the theory, so that we are able to use the framework given in chapter 3 to construct the extended Peierls bracket for General Relativity.

Since the formulation of General Relativity does only depend on the smooth manifold structure given on M, but not on any other tensorial objects on M, except for the dynamical field itself, it is called a **background independent** theory.

As a consequence, the theory is invariant under the action of diffeomorphisms $\phi : M \to M$ in the following sense:

Given a (compactly supported) diffeomorphism $\phi : M \to M$, we let ϕ act on g via the pullback:

$$g \to \phi^* g \tag{4.2.1}$$

Since the Einstein Hilbert action is defined in a coordinate independent way and does not depend on any tensorial objects on M other than g itself, we have that for compactly supported diffeomorphisms $\phi : M \to M$

$$S_\psi(g) = S_\psi(\phi^* g) \quad \forall \psi : \psi\mid_{\text{supp }\phi} = 1 \tag{4.2.2}$$

We se that in this sense the theory is $\mathrm{Diff}_0(M)$-invariant, whereby $\mathrm{Diff}_0(M)$ denotes the group of compactly supported diffeomorphisms $\phi : M \to M$.

The Lie algebra of $\mathrm{Diff}_0(M)$ is given by the Lie algebra of smooth compactly supported vector fields $\Gamma_0(TM)$ (see e.g. [18], p. 35ff).

The induced action of the Lie algebra on the metric field is given by

$$(X \cdot g)_{ab} := \mathcal{L}_X g_{ab} = \nabla_a X_b + \nabla_b X_a \quad \forall X \in \Gamma_0(TM) \tag{4.2.3}$$

Following the framework given in section (3.1), we see that the vector bundle $\mathfrak{g}$ is in the case of General Relativity given by the tangent bundle $\mathfrak{g} := TM$, and the invariant flow of the theory is given by the g-jet dependent differential operator

$$\begin{aligned} &Q(g) : \Gamma(TM) \to T_g\Gamma(E) \simeq \Gamma(E) \\ &Q(g)X := \mathcal{L}_X g \end{aligned} \tag{4.2.4}$$

We varify that $Q(g)$ indeed generates invariance transformations. We have

$$\begin{aligned}
\delta S(g)[Q(g)X] &= -\int_M d^n x \, \sqrt{-\det g} \, G^{ab} \, (Q(g)X)_{ab} \\
&= -2\int_M d^n x \, \sqrt{-g} \, G^{ab} \, (\nabla_a X_b + \nabla_b X_a) \\
&= 2\int_M d^n x \, \sqrt{-\det g} \, \left\{\nabla_a G^{ab} \, X_b + \nabla_b G^{ab} \, X_a\right\} \\
&= 0
\end{aligned} \tag{4.2.5}$$

due to the **Bianchi identity** (see. e.g. [5, 23]):

$$\nabla_a G^{ab} = 0 \tag{4.2.6}$$

4.3. The linearized theory

In this section we derive the linearized field equations for General Relativity.

The field equations are given by

$$\delta S(g) = -\sqrt{-\det g} \, G^{ab} = 0 \tag{4.3.1}$$

so that the linearized field equations are

$$(\delta^2 S(g) \, \xi)^{ab} =: -\sqrt{-\det g} \, \delta G^{ab}[\xi] - \delta\sqrt{-\det g}[\xi] \, G^{ab} = 0 \tag{4.3.2}$$

We calculate the functional derivative of the Einstein tensor:

$$\begin{aligned}
\delta G^{ab}[\xi] &= \frac{1}{2} \, \left\{\delta R[\xi] \, g^{ab} + R^{\nabla} \, \delta g^{ab}[\xi]\right\} - g^{ac} g^{bd} \, \delta R^{\nabla}_{cd}[\xi] \\
&= \frac{1}{2} \, g^{ab} \, (\delta g^{cd}[\xi] \, R_{cd} + g^{cd} \, (\nabla_e(\delta\Gamma^e_{cd}[\xi]) - \nabla_d(\delta\Gamma^e_{ce}[\xi]))) \\
&\quad \frac{1}{2} \, \delta g^{ab}[\xi] \, R^{\nabla} - g^{ac} g^{bd} \, (\nabla_e(\delta\Gamma^e_{cd}[\xi]) - \nabla_d(\delta\Gamma^e_{ce}[\xi])) \\
&= \frac{1}{2} \, g^{ab} g^{cd} \, (\nabla_e(\delta\Gamma^e_{cd}[\xi]) - \nabla_d(\delta\Gamma^e_{ce}[\xi])) \\
&\quad - g^{ac} g^{bd} \, (\nabla_e(\delta\Gamma^e_{cd}[\xi]) - \nabla_d(\delta\Gamma^e_{ce}[\xi])) \\
&\quad + \frac{1}{2} \left\{g^{ab} \, \delta g^{cd}[\xi] \, R_{cd} + \delta g^{ab}[\xi] \, R^{\nabla}\right\}
\end{aligned} \tag{4.3.3}$$

We already derived the functional derivative of the Christoffel symbols in equation (4.1.8). The use of this result yields

$$
\begin{aligned}
\delta G^{ab}[\xi] = & \frac{1}{2}\ g^{ab}\ \nabla^c \nabla^d \xi_{cd} - \frac{1}{4}\ g^{ab}\ \Box^\nabla \mathrm{Tr}\ \xi \\
& - \frac{1}{4}\ g^{ab}\ \Box^\nabla \mathrm{Tr}\ \xi - \frac{1}{4}\ g^{ab}\ \nabla^c \nabla^d \xi_{cd} + \frac{1}{4} g^{ab}\ \nabla^c \nabla^d \xi_{cd} \\
& - \frac{1}{2}\ g^{bc}\ \nabla^d \nabla^a \xi_{cd} - \frac{1}{2}\ g^{ac}\ \nabla^d \nabla^b \xi_{cd} + \frac{1}{2}\ (\Box^\nabla \xi)^{ab} \\
& + \frac{1}{2}\ \nabla^b \nabla^a \mathrm{Tr}\ \xi + \frac{1}{2}\ g^{ac}\ \nabla^b \nabla^d \xi_{cd} - \frac{1}{2}\ g^{ac}\ \nabla^b \nabla^d \xi_{cd} \\
& + \frac{1}{2} \left\{ g^{ab}\ \delta g^{cd}[\xi]\ R_{cd} + \delta g^{ab}[\xi]\ R^\nabla \right\} \\
= & \frac{1}{2}\ g^{ab}\ \nabla^c \nabla^d \xi_{cd} - \frac{1}{2}\ g^{ab}\ \Box^\nabla \mathrm{Tr}\ \xi - \frac{1}{2}\ \left(g^{bc}\ \nabla^d \nabla^a \xi_{cd} + g^{ac}\ \nabla^d \nabla^b \xi_{cd} \right) \\
& + \frac{1}{2}\ (\Box^\nabla \xi)^{ab} + \frac{1}{2}\ \nabla^b \nabla^a \mathrm{Tr}\ \xi \\
& + \frac{1}{2} \left\{ g^{ab}\ \delta g^{cd}[\xi]\ R_{cd} + \delta g^{ab}[\xi]\ R^\nabla \right\}
\end{aligned}
\tag{4.3.4}
$$

The presymplectic form for the Jacobian of the Einstein Hilbert action has been calculated by Crnkovic and Witten in [6]. It is given by

$$
\begin{aligned}
\omega(g)[\xi_1, \xi_2] &:= \int_\Sigma d\Sigma_a\ W^a[\xi_1, \xi_2] \\
W^a[\xi_1, \xi_2] &:= \delta\Gamma^a_{bc}[\xi_1]\ (\delta g^{bc}[\xi_2] + \frac{1}{2}\ g^{bc}\ \mathrm{Tr}\ \xi_2) \\
& \quad - \delta\Gamma^c_{bc}[\xi_1]\ (\delta g^{ab}[\xi_2] + \frac{1}{2}\ g^{ab}\ \mathrm{Tr}\ \xi_2) - \{\xi_1 \leftrightarrow \xi_2\}
\end{aligned}
\tag{4.3.5}
$$

We apply the method for gauge fixing given in section (3.1), and make the choices proposed in [7b], chapter 35.

We choose the gauge fixing operator $P(g)$ given by

$$
\begin{aligned}
P_a(g)\ \xi &:= \sqrt{-\det g}\ \left\{ \nabla^b \xi_{ba} + \nabla^b \xi_{ab} - \nabla_a \mathrm{Tr}\ \xi \right\} \\
\tilde{P}^{ab}(g)\ X &:= \sqrt{-\det g}\ \left\{ g^{ab}\ \nabla_c X^c - \nabla^a X^b - \nabla^b X^a \right\}
\end{aligned}
\tag{4.3.6}
$$

and the vector bundle isomorphism K given by

$$
K^{a\ b}(g) = \frac{1}{4\ \sqrt{-\det g}}\ g^{ab} \tag{4.3.7}
$$

This yields

$$
\begin{aligned}
\mathfrak{F}_a(g)\ X &:= P_a(g) \circ Q(g)\ X \\
&= \sqrt{-\det g}\ \left\{ 2\ \nabla^c (\nabla_c X_a + \nabla_a X_c) - \nabla_a (g^{bc} (\nabla_b X_c + \nabla_c X_b)) \right\} \\
&= 2\ \sqrt{-\det g}\ \left\{ \Box^\nabla X_a + \nabla_b \nabla_a X^b - \nabla_a \nabla_b X^b \right\} \\
&= 2\ \sqrt{-\det g} \left\{ \Box^\nabla + R^\nabla_{ab} \right\}\ X^b
\end{aligned}
\tag{4.3.8}
$$

which is clearly normally hyperbolic and therefore possesses unique advanced and retarded Green's operators when (M, g) is globally hyperbolic.

We construct the resulting gauge fixed linearized field equations. It is

$$\begin{aligned}
&\tilde{P}(g)^{ab} \circ K \circ P(g)\ \xi \\
&= \frac{1}{4}\ \sqrt{-\det g}\ \left\{(g^{ab}\ \nabla_c - 2\ \delta^b_c\ \nabla^a)\ (g^{cd}(2\ \nabla^e \xi_{de} - \nabla_d \mathrm{Tr}\ \xi))\right\} \\
&= \sqrt{-\det g}\ \left\{\frac{1}{2}\ 2\ g^{ab}\ \nabla^c \nabla^d \xi_{cd} - \frac{1}{4}\ g^{ab} \mathrm{Tr}\ \xi\right\} \\
&\quad \sqrt{-\det g}\ \left\{-\frac{1}{2}\ g^{bc}\ \nabla^a \nabla^d \xi_{cd} - \frac{1}{2}\ \ g^{ac} \nabla^b \nabla^d \xi_{cd} + \frac{1}{2}\ \nabla^a \nabla^b \mathrm{Tr}\ \xi\right\}
\end{aligned} \tag{4.3.9}$$

which yields

$$\begin{aligned}
F^{ab}(g)\ \xi :=&\ -\delta\sqrt{-\det g}[\xi]\ G^{ab} - \sqrt{-\det g}\ \delta G^{ab}[\xi] + \tilde{P}^{ab}(g) \circ K(g) \circ P(g)\ \xi \\
=&\ -\sqrt{-\det g}\ \left\{\frac{1}{2}\ (\Box^\nabla \xi)^{ab} - \frac{1}{4}\ g^{ab} \Box^\nabla \mathrm{Tr}\ \xi\right\} \\
&- \frac{1}{2}\ \sqrt{-\det g}\ \left\{(g^{bc}\ \nabla^a \nabla^d - g^{bc}\ \nabla^d \nabla^a + g^{ac} \nabla^b\ \nabla^d - g^{ac}\ \nabla^d \nabla^b)\ \xi_{cd}\right\} \\
&- \frac{1}{2}\ \sqrt{-\det g} \left\{g^{ab}\ \delta g^{cd}[\xi]\ R_{cd} + \delta g^{ab}[\xi]\ R^\nabla\right\} - \delta\sqrt{-\det g}[\xi]\ G^{ab}
\end{aligned} \tag{4.3.10}$$

The second line of the last expression can written in terms of the Riemann and Ricci curvature tensors:

$$\begin{aligned}
&(g^{bc}\ \nabla^a \nabla^d - g^{bc}\ \nabla^d \nabla^a + g^{ac} \nabla^b\ \nabla^d - g^{ac}\ \nabla^d \nabla^b)\ \xi_{cd} \\
&= g^{bc}\ [\nabla^a, \nabla^d]\ \xi_{cd} + g^{ac}\ [\nabla^b, \nabla^d]\ \xi_{cd} \\
&= g^{bc}\ (-R^{e\ \ ad}_{\ c}\ \xi_{ed} - R^{e\ \ ad}_{\ d}\ \xi_{ce}) + g^{ac}\ (-R^{e\ \ bd}_{\ c}\ \xi_{ed} - R^{e\ \ bd}_{\ d}\ \xi_{ce}) \\
&= -R^{cbad}\ \xi_{cd} - g^{bc}\ R^{da}\ \xi_{cd} - R^{cabd}\ \xi_{cd} - g^{ac}\ R^{db}\ \xi_{cd}
\end{aligned} \tag{4.3.11}$$

Using this result, the gauge fixed linearized field equations can be written as

$$\begin{aligned}
F^{ab}(g)\ \xi =&\ -\sqrt{-\det g}\ \left\{\frac{1}{2}\ (\Box^\nabla \xi)^{ab} - \frac{1}{4}\ g^{ab} \Box^\nabla \mathrm{Tr}\ \xi\right\} \\
&\frac{1}{2}\ \sqrt{-\det g}\ \left\{R^{cbad} + R^{cabd}\right\}\ \xi_{cd} \\
&- \frac{1}{2}\ \sqrt{-\det g} \left\{g^{ab}\ \delta g^{cd}[\xi]\ R_{cd} + \delta g^{ab}[\xi]\ R^\nabla - g^{bc}\ R^{da}\ \xi_{cd} - g^{ac}\ R^{db}\ \xi_{cd}\right\} \\
&- \delta\sqrt{-\det g}[\xi]\ G^{ab}
\end{aligned} \tag{4.3.12}$$

We examine whether this operator is normally hyperbolic:

The term in second order of the covariant derivative is given by

$$\begin{aligned}
& -\frac{1}{2}\,\sqrt{-\det g}\left\{(\Box^\nabla \xi)^{ab} - \frac{1}{2}\,\Box^\nabla \mathrm{Tr}\ \xi\right\} \\
= & -\frac{1}{4}\,\sqrt{-\det g}\left\{g^{ac}\ g^{bd}\ \Box^\nabla\ \xi_{cd} + g^{bc}\ g^{ad}\ \Box^\nabla \xi_{cd} - g^{ab}\ g^{cd}\ \Box^\nabla \xi_{cd}\right\} \\
=: & -\frac{1}{4}\sqrt{-\det g}\ \gamma^{ab}\circ \Box^\nabla\ \xi_{cd}
\end{aligned} \tag{4.3.13}$$

We see that the gauge fixed linearized field equations are normally hyperbolic if the vector bundle homomorphism given by

$$\gamma^{ab}(\xi) := (g^{ac}\ g^{bd} + g^{bc}\ g^{ad} - g^{ab}\ g^{cd})\ \xi_{cd} \tag{4.3.14}$$

is invertable and therefore a vector bundle isomorphism.

This is indeed the case, and the inverse of (4.3.14) is given by

$$\gamma^{-1}_{cd}(\xi) = \frac{1}{4}\ (g_{ac}\ g_{bd} + g_{bc}\ g_{ad} - g_{ab}\ g_{cd})\ \xi^{ab} \tag{4.3.15}$$

We verify this:

$$\begin{aligned}
(\gamma^{-1}\circ\gamma)_{cd}\ (\xi) &= \frac{1}{4}\ (g_{ac}\ g_{bd} + g_{bc}\ g_{ad} - \frac{1}{2}\ g_{ab}\ g_{cd})\ (g^{ae}\ g^{bf} + g^{be}\ g^{af} - g^{ab}\ g^{ef})\ \xi_{ef} \\
&= \frac{1}{4}\ (\delta^e_c\ \delta^f_d + \delta^f_c\ \delta^e_d - g_{cd}\ g^{ef} + \delta^e_d\ \delta^f_c + \delta^f_d\ \delta^e_c - g_{dc}\ g^{ef})\ \xi_{ef} \\
&\quad - \frac{1}{4}\ (g_{cd}\ g^{ef} + g_{cd}\ g^{fe} - g_{cd}\ g^{ef}\ \mathrm{Tr}\ g)\ \xi_{ef} \\
&= \delta^{ef}_{cd}\ \xi_{ef} + (-\frac{1}{2} - \frac{1}{2} + \frac{1}{4}\ \mathrm{Tr}\ g)\ g_{cd}\ \mathrm{Tr}\ \xi = \delta^{ef}_{cd}\ \xi_{ef} = \xi_{cd}
\end{aligned} \tag{4.3.16}$$

We see that γ provides a vector bundle isomorphism and therefore the gauge fixed linearized field equations are normally hyperbolic. This implies that for (M, g) being globally hyperbolic, F possesses unique advanced and retarded Green's operators.

We can, however, not assume that there exists an open neighborhood of the solution space on which (M, g) is globally hyperbolic. To be more precise, we can not even assume that g provides a Lorentzian metric in such a neighborhood. Therefore, there exist in general no Green's functions for the gauge fixed linearized field equations in an open neighborhood of $\Gamma_{gh}(E)$ in $\Gamma(E)$. This is a problem which we will address in more detail in chapter 6, where we will also discuss the problem of existence (resp. the nonexistence) of invariants in classical General Relativity.

5. The Palatini Action

In this section we give a brief introduction to the formulation of classical General Relativity using a local connection form and the tetrad fields as dynamical variables. This formulation relies on the notion of principal bundles and connections on associated vector bundles. A short overview on those concepts can be found in appendices B and C, which are based on [2] by BAUM.

In this chapter only we use greek indices to denote the components of tensors with respect to a coordinate base and latin indices to denote the components with respect to a noncoordinate base given by the tetrad fields.

5.1. The frame bundle and the tangent bundle

The formulation of classical General Relativity, based on a local connection form, relies on the definition of the **frame bundle**:

Definition 5.1.1 (Frame bundle)
Let M be an n-dimensional smooth parallelizable manifold. The set of bases of T_xM is given by

$$GL(M)_x : \{e_x = \{e_1, ..., e_n\} : e_x \text{ is a base of } T_xM\} \tag{5.1.1}$$

We define

$$GL(M) := \dot{\bigcup}_{x\in M} GL(M)_x \tag{5.1.2}$$

A right action of $GL(n,\mathbb{R})$ on $GL(M)_x$ is given by

$$\begin{aligned} &e_x \cdot f \to e'_x \in GL(M)_x \\ &e'_a := e_b\ f^b{}_a \end{aligned} \tag{5.1.3}$$

$GL(M)$ is a principal $GL(n,\mathbb{R})$-bundle endowed with the following structure:

The projection π is given by

$$\begin{aligned} &\pi : GL(M) \to M \\ &\pi(e_x \in GL(M)_x) = x \end{aligned} \tag{5.1.4}$$

Global trivialisations ϕ are given by the choice of a collection e of n smooth global base vector fields on M:

$$\phi(e'_x \in GL(M)_x) := f \in G : v_x \cdot f = e(x) \tag{5.1.5}$$

The smooth section over $GL(M)$ are then given by all collections of n smooth base vector fields on M.

Smooth manifolds M, which can be equipped with a metric g such that (M, g) yields a globally hyperbolic spacetime, are always parallelizable. This is implied by the fact that all globally hyperbolic spacetimes are isomorphic to $\sigma \times \mathbb{R}$ with σ being a 3-dimensional Riemannian manifold (see theorem (A.0.1) in appendix A). Since all 3-dimensional Riemannian manifolds are parallelizable, so is $\sigma \times \mathbb{R}$.

The tensor bundles over M are associated vector bundles of $GL(M)$:

Let $\rho_{(r,s)}$ be the induced representation of $GL(n, \mathbb{R})$ on $\mathbb{R}^{n^{(r+s)}}$:

$$(\rho_{(r,s)}(f) \cdot T)^{a_1,\ldots,a_r}{}_{b_1,\ldots,b_s} := g^{a_1}{}_{c_1} \cdot \ldots \cdot g^{a_r}{}_{c_r}\ T^{c_1,\ldots,c_r}{}_{d_1,\ldots,d_s}\ (g^{-1})^{d_1}{}_{b_1} \cdot \ldots \cdot (g^{-1})^{d_s}{}_{b_s} \tag{5.1.6}$$

The vector bundle $T^{(r,s)}$ of (r, s)-tensors on M is then isomorphic to

$$T^{(r,s)}(M) \simeq GL(M) \times_{(GL(n,\mathbb{R}),\rho_{(r,s)})} \mathbb{R}^{n^{(r+s)}} \tag{5.1.7}$$

In case of the tangent bundle TM, we have

$$TM \simeq GL(M) \times_{(GL(n,\mathbb{R}),\rho)} \mathbb{R}^n \tag{5.1.8}$$

with an isomorphism

$$\Phi : GL(M) \times_{(GL(n,\mathbb{R}),\rho)} \mathbb{R}^n \to TM \tag{5.1.9}$$

$$\Phi([e, V]) := e_a\ V^a \tag{5.1.10}$$

which is just the usual decomposition of tangent vectors in base vectors and components.

Given a connection form A on $GL(M)$, we can use theorem (C.0.6) in appendix (C) to construct a covariant derivative ∇^A on TM:

$$X = [e, V] \in \Omega^0(M, TM) \tag{5.1.11}$$

$$\nabla^A X := [e, dV + \rho_*(A \circ de)\ V] \tag{5.1.12}$$

In components, this can be written as

$$(\nabla^A X)_\mu = e_a\ (\partial_\mu V^a + A^a_{\mu\ b}\ V^b) \tag{5.1.13}$$

with $A^a_{\mu\ b}$ being the components of the local connection form wit respect to e in the given induced representation ρ_* of $\mathfrak{gl}(n, \mathbb{R})$.

We introduce local coordinates on M. Thich allows us to define the **tetrad fields** e^a_μ and their inverse e^μ_a, which map the components of vectors with respect to the base e to the respective components with respect to the coordinate base. Then it is

$$(\nabla^A e_a)(e_b) = e_c\ (\partial_\mu \delta^c_a + A^c_{\mu\ d}\ \delta^d_a)\ e^\mu_b = e_c\ A^c_{\mu\ a}\ e^\mu_b =: e_c\ \Gamma^c_{ba} \tag{5.1.14}$$

which defines the coefficients of the **spin connection**:

$$\Gamma^c_{ab} := A^c_{\mu\ b}\ e^\mu_a \tag{5.1.15}$$

This allows us to write equation (5.1.13) as

$$\begin{aligned} &(\nabla^A T)_\mu = e_\sigma\ e^\sigma_a\ (\partial_\mu(e^a_\nu\ T^\nu) + A^a_{\mu\ b}\ e^b_\nu\ T^\nu) \\ \Leftrightarrow\ &(\nabla^A T)^\sigma_\mu = e^\sigma_a(\partial_\mu(e^a_\nu\ T^\nu) + A^a_{\mu\ b}\ e^b_\nu\ T^\nu) \\ &= \partial_\mu T^\sigma + e^\sigma_a\ \partial_\mu e^a_\nu\ T^\nu + e^\sigma_a\ A^a_{\mu\ b}\ e^b_\nu\ T^\nu \\ &=: \partial_\mu T^\nu + \Gamma^\sigma_{\mu\nu}\ T^\nu \end{aligned} \tag{5.1.16}$$

which defines the usual connection coefficients

$$\Gamma^\sigma_{\mu\nu} := e^\sigma_a\ \partial_\mu e^a_\nu + e^\sigma_a\ A^a_{\mu\ b}\ e^b_\nu \tag{5.1.17}$$

A Lorentzian metric g on M is induced by

$$g_{\mu\nu} := e^a_\mu\ \eta_{ab}\ e^b_\nu \tag{5.1.18}$$

where η denotes the usual Minkowski metric $\eta = \text{diag}\ (-1, 1, 1, 1)$.

We require the connection to be a **metric connection**, i.e.

$$\partial(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y) \tag{5.1.19}$$

We calculate

$$\begin{aligned} &g(\nabla X, Y) + g(X, \nabla Y) \\ &= (\partial X^a + A^a{}_c\ X^c, Y^b)\ \eta_{ab}\ Y^b + X^a\ \eta_{ab}\ (\partial Y^a + A^a{}_c\ Y^c) \\ &= \partial(\eta(X, Y)) + A^a{}_c\ X^c\ \eta_{ab}\ Y^b + X^a \eta_{ab}\ A^b{}_c\ Y^c \end{aligned} \tag{5.1.20}$$

so that a metric connection has to satisfy

$$\begin{aligned} &\Gamma^a{}_c\ X^c\ \eta_{ab}\ Y^b + X^a \eta_{ab}\ A^b{}_c\ Y^c = 0 \\ \Rightarrow\ &\eta_{ab}\ A^b{}_c = -\eta_{cb}\ A^b{}_a \end{aligned} \tag{5.1.21}$$

which implies that the connection form is valued in $\mathfrak{so}(3, 1)$.

5.2. Lagrangian formulation

We want to give a Lagrangian formulation of classical General Relativity in terms of the tetrad field and the connection form. To do so, we have to express the curvature of g in these variables.

The Riemann tensor is given by

$$\begin{aligned} R(e_a, e_b)\ e_c &:= \nabla_{e_a}\ \nabla_{e_b} e_c - \nabla_{e_b}\ \nabla_{e_a}\ e_c - \nabla_{[e_a, e_b]} e_c \\ &= \nabla_{e_a}(e_d\ \Gamma^d_{bc}) - \nabla_{e_b}(e_d\ \Gamma^d_{ac}) - c^d_{ab}\ \nabla_{e_d} e_c \\ &= e_a(\Gamma^d_{bc})\ e_d - e_b(\Gamma^d_{ac})\ e_d + \Gamma^d_{bc}\ \Gamma^e_{ad}\ e_e + \Gamma^d_{ac}\ \Gamma^e_{bd}\ e_e - c^d_{ab}\ \Gamma^e_{dc}\ e_e \\ &= e^\mu_a\ \partial_\mu(A^d_{\nu\ c}\ e^\nu_b)\ e_d - e^\mu_b\ \partial_\mu(A^d_{\nu\ c}\ e^\nu_a)\ e_d \\ &\quad + A^e_{\mu\ c}\ A^d_{\nu\ e}\ e^\mu_b\ e^\nu_a\ e_d - A^e_{\mu\ c}\ A^d_{\nu\ e}\ e^\mu_a\ e^\nu_b\ e_d \\ &\quad - (e^\mu_a\ \partial_\mu e^\nu_b\ e^e_\nu - e^\mu_b\ \partial_\mu e^\nu_a\ e^e_\nu)\ A^d_{\rho\ c}\ e^\rho_e\ e_d \\ &= e_d\ (e^\mu_a\ \partial_\mu A^d_{\nu\ c}\ e^\nu_b + e^\mu_a\ A^d_{\nu\ c}\ \partial_\mu e^\nu_b - e^\mu_b\ \partial_\mu A^d_{\nu\ c}\ e^\nu_a - e^\mu_b A^d_{\nu\ c}\ \partial_\mu e^\nu_a) \\ &\quad + e_d\ (A^d{}_e \wedge A^e{}_c)_{\mu\nu}\ e^\mu_a\ e^\nu_b \\ &\quad - e_d\ (e^\mu_a\ \partial_\mu e^\nu_b - e^\mu_b\ \partial_\mu e^\nu_a)\ A^d_{\nu\ c} \\ &= e_d\ (dA^d{}_c + A^d{}_e \wedge A^e{}_c)_{\mu\nu}\ e^\mu_a\ e^\nu_b =: e_d\ R^d{}_{cab} \end{aligned} \tag{5.2.1}$$

So in components with respect to the basis e, the Riemann curvature tensor is given by

$$R^d{}_{cab} = (dA^d{}_c + A^d{}_e \wedge A^e{}_c)_{\mu\nu}\ e^\mu_a\ e^\nu_b \tag{5.2.2}$$

which allows us to define

$$R^d{}_{d\mu\nu} := (dA^d{}_c + A^d{}_e \wedge A^e{}_c)_{\mu\nu} \tag{5.2.3}$$

which is just the local curvature form of the connection form A.

The torsion of ∇ is given by

$$\begin{aligned} T(e_a, e_b) &= \nabla_{e_a} e_b - \nabla_{e_b} e_a - [e_a, e_b] \\ &= e_c\ \Gamma^c_{ab} - e_c\ \Gamma^c_{ba} - e_c\ e^c_\nu\ (e^\mu_a\ \partial_\mu e^\nu_b - e^\mu_b\ \partial_\mu e^\nu_a) \\ &= e_c\ (A^c_{\mu\ b}\ e^\mu_a - A^c_{\mu\ a}\ e^\mu_b - e^\mu_a\ \partial_\mu e^\nu_b\ e^c_\nu + e^\mu_b\ \partial_\mu e^\nu_a\ e^c_v) \\ &=: e_c\ T^c_{ab} \end{aligned} \tag{5.2.4}$$

In coordinate base the torsion is then given by

$$\begin{aligned} T^c_{\rho\sigma} &= T^c_{ab}\ e^a_\rho\ e^b_\sigma \\ &= A^c_{\rho\ b}\ e^b_\sigma - A^c_{\sigma_a}\ e^a_\rho - \partial_\rho e^\nu_b\ e^c_\nu\ e^b_\sigma + \partial_\sigma e^\nu_a\ e^c_\nu\ e^a_\rho \\ &= (de^c)_{\rho\sigma} + (A^c{}_b \wedge e^b)_{\rho\sigma} \end{aligned} \tag{5.2.5}$$

which is the expected result since following (5.1.17) we have

$$2e^c_\mu\ \Gamma^\mu_{[\rho\sigma]} = (de^c)_{\rho\sigma} + (A^c{}_b \wedge e^b)_{\rho\sigma} \tag{5.2.6}$$

We can now write the Einstein-Hilbert action in terms of the local connection form and the tetrad fields:

$$\begin{aligned} S_{EH}(A,e) &= \int_M d^n x \, \sqrt{-\det g} \; R^{\nabla}(A,e) = \int_M d^n x \, \sqrt{-\det g} \; R^a{}_{b\,\mu\nu} \, e^\mu_a \, e^\nu_c \, \eta^{bc} \\ &= \int_M d^n x \, \sqrt{-\det g} \; (dA^a{}_b + A^a{}_c \wedge A^c{}_b)_{\mu\nu} \, e^\mu_a \, e^\nu_d \, \eta^{bd} \end{aligned} \tag{5.2.7}$$

In this form the action is called the **Palatini action**.

The Palatini action is again $\mathrm{Diff}_0(M)$-invariant, where the diffeomorphism group acts via pullback on the tetrad fields and the connection form.

In addition, the Palatini action is by construction invariant under the action of the Lie group given by the smooth sections over $G := M \times O(3,1)$. $\Gamma(G)$ is a Lie group with

$$\begin{aligned} & g_1, g_2 \in \Gamma(G) \\ & (g_1 \cdot g_2)(x) := g_1(x) \cdot g_2(x) \end{aligned} \tag{5.2.8}$$

$\Gamma(G)$ acts on the tetrad fields via

$$(g \cdot e)_a(x) := e_b(x) \; g^b{}_a(x) \tag{5.2.9}$$

The action on the local connections form is, following theorem (C.0.3), given by

$$(g \cdot A)(x) := \mathrm{Ad}(g^{-1}(x)) \circ A + g^{-1}(x) \circ dg(x) \tag{5.2.10}$$

The functional derivative of the Palatini action vanishes iff the torsion and the Riemann tensor vanish. Therefore the resulting field equations are given by the usual Einstein equations and the demand that the covariant derivative ∇ is given by the Levi-Civita connection.

The derivation of this result can in detail be found for example in [13].

The additional gauge freedom present in this formulation results in the existence of an additional invariant flow and therefore the requirement to find an additional gauge fixing condition for the resulting linearized field equations. We conclude that for these reasons for the algebraic formulation of classical General Relativity the formulation in terms of metric components is preferable.

The presymplectic form for the Jacobian of the Palatini action has be calculated by Frauendiener and Sparling in [12].

6. Conceptual Problems

In this chapter we discuss some of the conceptual difficulties that occur when we attempt to construct the algebra of obvservables for classical General Relativity based on the extended Peierls bracket.

6.1. Existence of Green's operators

As we already mentioned in chapter 4, the space $\Gamma_{lor}(E)$ of smooth Lorentzian metrics on M is in general not an open subset of the space $\Gamma(E)$ of smooth symmetric $(0,2)$-tensor fields on M. This can be seen in the following example:

Let g_{ab} be a smooth Lorentz metric on a noncompact smooth manifold M. Furthermore, let $\phi : M \to \mathbb{R}^+$ be smooth and surjective. We define the smooth curve

$$\begin{aligned} &\gamma : \mathbb{R} \to \Gamma(E) \\ &\gamma(\lambda) := g - \lambda \cdot \phi \cdot g \end{aligned} \tag{6.1.1}$$

It is $\lim_{\lambda \to 0} \gamma(\lambda) = g$ with respect to the topology given on $\Gamma(E)$.

However, for any $\lambda \neq 0$ it exists a point $p_\lambda \in M$ with $\phi(p_\lambda) = \frac{1}{\lambda}$, so that $\gamma(\lambda)(p_\lambda) = 0$. We see that for this reason, $\gamma(\lambda) \notin \Gamma_{\mathrm{lor}}(E)$ for $\lambda \neq 0$ and therefore $\Gamma_{\mathrm{lor}}(E)$ lies not open in $\Gamma(E)$. This is by extension also true for the space $\Gamma_{\mathrm{gh}}(E)$ of smooth globally hyperbolic metrics on M.

As a consequence, the existence of the advanced and retarded Green's operators of the gauge fixed linearized field equations is not guaranteed on an open subset of $\Gamma(E)$, so that the functional derivative of the Green's operators is not defined even on $\Gamma_{gh}(E)$.

In chapter 3, we used the functional derivative of the causal propagator (and therefore of the Green's operators). To be more precise, we used the functional derivative of quantities of the form

$$F := \delta A[\Delta \; \delta B] \tag{6.1.2}$$

where A, B are compactly supported functionals defined over a neighborhood of $\Gamma_{\mathrm{gh}}(E)$. we will now see how we can define the functional derivative of a quantity of this form.

On $\Gamma_{\mathrm{gh}}(E)$, it is $\mathrm{supp}\ F \subset (J^+(\mathrm{supp}\ A) \cup J^-(\mathrm{supp}\ B)) \cap (J^+(\mathrm{supp}\ B) \cup J^-(\mathrm{supp}\ A))$, which is compact.

We define

$$\delta F(g \in \Gamma_{\text{gh}(E)})[\xi] := \delta F(g)[\xi'] \quad \forall \xi \in \Gamma(E),\ \xi' \in \Gamma_0(E),\ \xi'|_{\text{supp } F} = \xi|_{\text{supp } F} \quad (6.1.3)$$

The functional derivative of the causal propagator along ξ' can be defined, since for any $\chi \in \Gamma_0(E)$ and $g \in \Gamma_{\text{gh}}(E)$, the curve

$$\gamma(\lambda) := g + \lambda \cdot \chi \quad (6.1.4)$$

lies in $\Gamma_{\text{gh}}(E)$ at least for an open intervall $\lambda \in I,\ 0 \in I$, due to the boundedness of χ.

6.2. Observables in General Relativity

As we have seen in section 3.2, the classical algebra of observables of a gauge theory, based on the definition of the extended Peierls bracket, is an algebra of the invariants of the theory. This rises the question which are invariant quantities in General Relativity. We discuss this now.

First of all, let us consider local scalar quantities determined only by the metric field. A basic example is the contraction of the Riemann tensor with itself:

$$\psi_{p \in M}(g) := R_{abcd}(g)(p)\ R^{abcd}(g)(p),\ g \in \Gamma_{\text{gh}}(E) \quad (6.2.1)$$

This is not an invariant quantity with respect to the action of the diffeomorphism group, since the action of a diffeomorphism $\phi : M \to M$ on ψ yields

$$(\psi_p \circ \phi^*)(g) = R_{abcd}(\phi^* g)(p)\ R^{abcd}(\phi^* g)(p) \quad (6.2.2)$$

which is, for general $g \in \Gamma_{\text{gh}}(E)$ and $\phi \in \text{Diff}_0(M)$, not equal to ψ_p.

This is by extension true for any functional A with compact nonempty support $K \subset M$, since the action of any diffeomorphism $\phi : M \to M$ maps A onto the functional $A \circ \phi^*$ with support $\phi^{-1}(K)$. Constant functionals, however, are of course invariants.

Functionals with noncompact support provide no solution, since they are, as the action functional, not well defined over an open subset of $\Gamma(E)$.

One might consider a setting in which the base manifold itself is compact. In this case global quantities such as for example

$$\int_M d^n x\ \sqrt{-\det g}\ R_{abcd}\ R^{abcd} \quad (6.2.3)$$

are well defined and invariant with respect to the action of $\text{Diff}_0(M) = \text{Diff}(M)$.

However, following theorem (A.0.1) in appendix A, globally hyperbolic spacetimes can not be compact since they are always diffeomorphic to $\mathbb{R} \times \sigma$, σ being a three dimensional Riemannian manifold.

Numerous attempts to construct physical observables for General Relativity can be found in the literature (see e.g. [3] or [21]). In these attempts, the phase space of the theory is usually modified by the introduction of specific coordinate systems. For example, in [21], coordinates on the spacetime are induced by the coupling of General Relativity to a set of 4 test particles, which provide invariant reference points. However, this requires the specification of a fixed point and directions from and in which the particles emerge, which again destroys the manifest covariance of the theory in this formulation.

It is not clear how such modifications of the phase space can implemented in the formalism given in chapter 3, which specifically preserves the manifest covariance of the theory.

In conclusion, the construction of a manifest covariant algebra of observables for General Relativity formulated in terms of a metric field, based on the definition of the extended Peierls bracket, seems not possible due to the nonexistence of gauge invariant quantities.

7. Conclusions and Outlook

In this work, we discussed the properties of the extended Peierls bracket, defined by DeWitt in [7a]. We have see that it can be used to give a manifest covariant algebraic formulation of classical field theories, including gauge theories.

We have seen that the classical algebra of observables based on the notion of the extended Peierls bracket is an algebra of invariants, i.e. of gauge invariant quantities.

We examined to which extent this formalism can be applied to classical General Relativity.

To do so, we gave a Lagranigian formulation of classical General Relativity in terms of a metric field on a smooth manifold and derived the linearized field equations. Following [7b], we gave a choice for the gauge fixing and obtained the gauge fixed linearized field equations, which provide the basis for the construction of the extended Peierls bracket.

In chapter 6, we discussed the conceptual difficulties that result from the structure of the configuration space of the theory. Furthermore, we saw that the essential problem, which prevents the construction of an algebra of observables, is the nonexistence of gauge invariant quantities, at least in our manifest covariant formulation of the theory.

Apart from this crucial problem, there are other aspects which might be worth investigating.

First of all, although it can not be used to construct a Poisson algebra of observables for General Relativity, the extended Peierls bracket can be evaluated on functionals which are not gauge invariant. The resulting algebraic structure is discussed in [17b] and might be of interest.

Another question is how it it is possible to explicitely construct the Green's operators for the gauge fixed linearized field equations. A known method for the construction of Green's opeartors is the **Hadamard Parametrix construction** (see e.g. [1]). However, this method only allows the construction of the Green's operator in a geodesically starshaped neighborhood of any point $p \in M$, which is problematic in case of a dynamical background metric.

Finally, the next natural step is to consider the coupling of General Relativity to matter fields, which might also be a possible solution to the problem of the existence of observables (see [21]).

A. Lorentzian Geometry

In this chapter we summarize some basic concepts of Lorentzian geometry which we make use of in this work. It is based on section 1.3 of [1].

Throughout this chapter, M denotes a connected timeoriented Lorentzian manifold, i.e. a **Lorentzian spacetime**.

Definition A.0.1 (Inextendable curve)
*A piecewise C^1-curve in M is called **inextendable** if no piecewise C^1-reparametrisation of the curve can be continuously extended to any of the end points of the parameter interval.*

Definition A.0.2 (causal future and causal past)
*The **causal future** $J_+(p) \subset M$ of a point $p \in M$ is is the set of points that can be reached from p by future directed causal curves.*

*The **causal past** $J^-(p) \subset M$ of a point $p \in M$ is is the set of points that can be reached from p by past directed causal curves.*

Let $U \subset M$. The causal future resp. past of U is defined as $J_\pm(U) := \bigcup_{p \in U} J_\pm(p)$.

Definition A.0.3 (future and past compact)
*Let $K \subset M$ be closed. K is called **future compact** iff*

$$K \cup J_+(p) \text{ is compact } \forall p \in M \tag{A.0.1}$$

Accordingly, K is called past compact iff

$$K \cup J_-(p) \text{ is compact } \forall p \in M \tag{A.0.2}$$

Definition A.0.4 (Cauchy hypersurface)
*A subset $\Sigma \subset M$ of a Lorentzian spacetime M is a **Cauchy hypersurface** if each inextendable timelike curve on M intersects Σ exactly once.*

Definition A.0.5 (Globally hyperbolic spacetime)
*A Lorentzian spacetime is called **globally hyperbolic** iff it satisfies the **strong causality condition** and $\forall p, q \in M$ the intersection $J_+(p) \cap J_-(q)$ is compact.*

Theorem A.0.1 (Theorem 1.2.10 in [1])
Let M be a Lorentzian spacetime. Then the following are equivalent:

(1): M is globally hyperbolic.

(2): There exists a Cauchy hypersurface in M.

(3): M is isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta\, dt^2 + g_t$, where β is a smooth positive function, g_t is a Riemannian metric on Σ depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times \Sigma$ is a smooth spacelike Cauchy hypersurface in M.

B. Fibre bundles

In this chapter we give an introduction to the basic concepts of fibre bundles. This chapter is based on [2] by Baum.

B.1. Fibre bundles

Definition B.1.1 (Fibre bundle)
Let M be a smooth finite dimensional manifold.

*A smooth finite dimensional manifold E is called a **fibre bundle** over M if it has the following properties:*

(1) *It is given a smooth surjective map*

$$\pi : E \to M \tag{B.1.1}$$

*called the **projection** of the bundle.*

(2) *All preimages of points $x \in M$ with respect to the projections are isomorphic to a smooth manifold F, which is called the **fibre manifold** of the bundle:*

$$\pi^{-1}(x) \simeq F \quad \forall x \in M \tag{B.1.2}$$

(3) *It is given a countable covering of M by open subsets $U_{i \in \mathcal{I}} \subset M$ and a collection of smooth isomorphisms satisfying $\phi_{i \in \mathcal{I}}$*

$$\phi_i : \pi^{-1}(U_i) \to U_i \times F, \quad \phi(\pi^{-1}(x)) = \{x\} \times F \tag{B.1.3}$$

*and such that the **transition maps***

$$\phi_i^{-1} \circ \phi_j : \pi^{-1}(U_i \cap U_j) \to \pi^{-1}(U_i \cap U_j) \tag{B.1.4}$$

are smooth isomorphisms $\forall i, j \in \mathcal{I}$.

*The maps ϕ_i are called **local trivialisations** of E.*

We will denote the fibre bundle structure in short by (E, π, M, F).

A fibre bundle is called **trivial** if there exists a global trivialisation $\phi : E \to M \times F$.

Definition B.1.2 (Smooth section)
*Let E be a fibre bundle over M. A **smooth section** over E is a smooth map γ satisfying*

$$\gamma : M \to E, \quad \pi \circ \gamma = id \tag{B.1.5}$$

The space of smooth sections over E is denoted by $\Gamma(E)$.

Definition B.1.3 (Vertical tangent space)
*Let E be a fibre bundle over M. The **vertical tangent space** at $p \in E$ is defined as*

$$T_p^v E := Ker\ d\pi(p) : T_p E \to T_{\pi(p)} M \tag{B.1.6}$$

Definition B.1.4 (Horizontal tangent space)
Any vector space $T_p^h E \subset T_p M$ satisfying

$$T_p^h E \oplus T_p^v E = T_p E \tag{B.1.7}$$

*is called a **horizontal tangent space** of E at $p \in E$.*

B.2. Vector bundles

Definition B.2.1 (Vector bundle)
*A fibre bundle E over M is called a **vector bundle** if the fibre manifold is a topological vector space V and the transition maps generate vector space isomorphisms*

$$\phi_i \circ \phi_j^{-1} : \pi^{-1}(x) \simeq V \to \pi^{-1}(x) \simeq V \tag{B.2.1}$$

that depend smoothly on $x \in U_i \cap U_j$.

The smooth sections over a vector bundle form a vector space via the addition

$$(\gamma_1 + \gamma_2)(x) := \phi_i^{-1} \circ (\phi_i \circ \gamma_1(x) + \phi_i \circ \gamma_2(x)) \quad \forall x \in U_i \tag{B.2.2}$$

A smooth section over a vector bundle E is said to be of compact support if it $\phi \circ \gamma(x) \neq x \times 0$ only over a compact subset of M. The vector space of smooth sections of compact support is denoted by $\Gamma_0(E)$.

Definition B.2.2 (Dual bundle)
*Let (E, π, M, V) be a vector bundle. Given a vector space isomorphism $\psi : V \to V^*$, we define the the **dual vector bundle** (E, π, M, V^*) by replacing the local trivialisations ϕ_i of the original vector bundle by the local trivialisations*

$$\begin{aligned} &\phi_i' : \pi^{-1}(U_i) \to U_i \times V^* \\ &\phi_i'(p) := \psi \circ \phi_i(p) \quad \forall p \in \pi^{-1}(U_i) \end{aligned} \tag{B.2.3}$$

We will denote the dual vector bundle in short by E^*.

We will always assume the sections over E^* to be valued in the densities of rank 1, so that the action of $\Gamma(E)$ on $\Gamma(E)$ defined by

$$(\eta \cdot \gamma)(x) := (\psi \circ \eta)(x)(\phi \circ \gamma(x)) \quad \forall \eta \in \Gamma(E^*),\ \gamma \in \Gamma(E) \tag{B.2.4}$$

yields a smooth scalar density of rank 1 and can therefore by integrated over M.

B.3. Principal bundles

Definition B.3.1 (Principal bundle)
*A fibre bundle E over M is called a **principal** G**-bundle** if the fibre manifold is given by a Lie group G and the transistion maps generate smooth Lie group isomorphisms.*

Theorem B.3.1 (Theorem 2.5 in [2])
A principal G-bundle P is trivial exactly if it possesses at least one global section.

The Lie group G acts on E from the right via

$$R_g \cdot p = p \cdot g := \phi^{-1} \circ (\phi(p) \cdot g) \quad \forall g \in G,\ p \in E \tag{B.3.1}$$

and from the left via

$$L_g \cdot p = g \cdot p := \phi^{-1} \circ (g \cdot \phi(p)) \quad \forall g \in G,\ p \in E \tag{B.3.2}$$

Theses actions are by definition simply transitive on the fibres, i.e. we have that

$$\pi \circ L_g = \pi \circ R_g = \mathrm{id}\ \forall p \in E,\ g \in G \tag{B.3.3}$$

Definition B.3.2 (associated vector bundle)
Let (E, π, M, G) be a principal G-bundle. Let W be a vector space carrying a faithful left representation of G. On $P \times F$ we define a right action of G via

$$(p, w) \cdot g := (p \cdot g, g^{-1} \cdot w) \quad \forall (p, w) \in E \times W \tag{B.3.4}$$

We define the quotient space

$$\mathfrak{E} := (E \times W)/G =: E \times_G W \tag{B.3.5}$$

The equivalence class of (p, w) is denoted by $[p, w]$.

$\mathfrak{E}$ is a vector bundle with fibre manifold W and projection

$$\hat{\pi}([p, v]) := \pi(p) \tag{B.3.6}$$

Local trivialisations ψ_i of $\mathfrak{E}$ are given by fixing a family of smooth local sections $\gamma_i : U_i \to \pi^{-1}(U_i)$ over E and defining

$$\psi_i([p, w]) = v, \text{ whereby } [p, w] = [\gamma_i \circ \pi(p), v] \tag{B.3.7}$$

We see that $\mathfrak{E}$ is trivial if E is trivial.

C. Connections on principal bundles

In this chapter we introduce the notion of connections on principal bundles. We see how the local curvature form and covariant derivatives on asssociated vector bundles are defined using this definition. This chapter is again based on [2] by Baum.

Throughout this chapter, let (P, π, M, G) be a principal G-bundle over a smooth n-dimensional manifold, with $\mathfrak{g}$ being the Lie algebra of G. Let G be k-dimensional.

By the right action R_g on P, a corresponding action of $\mathfrak{g}$ on P is induced:

$$\begin{aligned} &X \in \mathfrak{g} : P \to TP, \ u \in P \to T_u P \\ &X \cdot u := \tilde{X}(u) \end{aligned} \tag{C.0.1}$$

The vector field $\tilde{X}$ is called the **fundamental vector field** of $X \in \mathfrak{g}$.

Since G acts simply transitive on the fibres of P, the fundamental vector fields span the vertical tangent space $T_u^v P$ at any point $u \in P$.

Definition C.0.3 (Connection)
A ***connection*** *assigns each point $u \in P$ a horizontal tangent space*

$$T_h : u \to T_u^h P \subset T_u P \tag{C.0.2}$$

such that the assignment satisfies the following properties:

$$\begin{aligned} &\textit{complementarity} \ : \ T_u^h \textit{ is a horizontal tangent space} && \text{(C.0.3)} \\ &\textit{right invariance} \ : \ dR_g(T_u^h P) = T_{(u \cdot g)}^h P \ \forall g \in G && \text{(C.0.4)} \\ &\textit{smoothness} \ : \ \textit{For all points } u \in P \textit{ it exists an open neighborhood} && \text{(C.0.5)} \\ &\qquad U \subset P \textit{ and a collection of } n \textit{ smooth tangent} && \text{(C.0.6)} \\ &\qquad \textit{vector fields } Y_i \in \Gamma(TP), \textit{ such that} && \text{(C.0.7)} \\ &\qquad \textit{span}(Y_1(p), ..Y_n(p)) = T_p^h P \ \forall p \in U && \text{(C.0.8)} \end{aligned}$$

The differential of the projection π

$$d\pi(u) : T_u^h P \to T_{\pi(u)} M \tag{C.0.9}$$

is then a linear isomorphism.

Definition C.0.4 (Connection form)
*A **connection form** on P is a one-form $A \in \Omega^1(P, \mathfrak{g})$ satisfying*

$$A(\tilde{X}) = X \ \forall X \in \mathfrak{g} \tag{C.0.10}$$
$$R_g^* A = A \circ dR_g = Ad(g^{-1}) \circ A \ \forall g \in G \tag{C.0.11}$$

The set of connection forms will be denoted by $\mathcal{C}(P)$.

Theorem C.0.2 (Theorem 3.1 in [2])
There exists a one-to-one correspondence between the set of connections and $\mathcal{C}(P)$.

Given a connection T^h, a connection form on P is given by

$$A(\tilde{X} \oplus Y) := X \ \forall \ X \in \mathfrak{g}, \ Y \in T^h P \tag{C.0.12}$$

Given a connection form A on P, a connection is given by

$$T^h : u \to T_u^h P := Ker \ A \tag{C.0.13}$$

Definition C.0.5 (Maurer Cartan form)
*The **Maurer Cartan form** $\theta_g \in \Omega^1(G, \mathfrak{g})$ is given by*

$$\theta_g := dL_{g^{-1}} \quad \forall g \in G \tag{C.0.14}$$

Given a covering of P by local sections (s_i, U_i) with transition functions g_{ij}, we will define

$$\theta_{ij} := dL_{g_{ij}^{-1}} \circ dg_{ij} \in \Omega^1(U_i \cap U_j, \mathfrak{g}) \tag{C.0.15}$$

Definition C.0.6 (Local connection form)
Let A be a connection form on P and (s, U) a local section over P. The one-form

$$A^s := A \circ ds \in \Omega^1(U, \mathfrak{g}) \tag{C.0.16}$$

*is called the **local connection form** determined by s.*

Theorem C.0.3 (Theorem 3.2 in [2])
Let A be a connection form on P, and let (s_i, U_i), (s_j, U_j) be local sections in P with transistion function $g_{ij} : U_i \cap U_j \to G$. Then it is

$$A^{s_i} = Ad(g_{ij}^{-1}) \circ A^{s_j} + \theta_{ij} \tag{C.0.17}$$

Given a covering of P by local sections (s_i, U_i) and a family of one-forms $\{A_i \in \Omega^1(U_i, \mathfrak{g})\}$ satisfying

$$A_i = Ad(g_{ij}^{-1}) \circ A_j + \theta_{ij} \tag{C.0.18}$$

there exists a connection form A on P satisfying $A^{s_i} = A_i$

We examine some special cases of Theorem (C.0.6):

In case G being a matrix group, the transformation behaviour of local connection forms is given by

$$A_i = g_{ij}^{-1} \circ A_j \circ g_{ij} + g_{ij}^{-1}\ dg_{ij} \tag{C.0.19}$$

In case P being a trivial principal G-bundle, a connection is defined by any global section s and one-form $A^s \in \Omega^1(M, \mathfrak{g})$.

Definition C.0.7 (Horizontal forms, forms of type ρ)
Let (P, π, M, G) be a principal G-bundle, $\rho : G \to GL(V)$ a representation of G and $E := P \times_{(G,\rho)} V$ the corresponding associated vector bundle.

A k-form $\omega \in \Omega^k(P, V)$ is called

$$\textbf{\textit{horizontal}} \quad \textit{if} \quad \omega(X_1, ..., X_k) = 0 \ \textit{if}\ X_i \in T_h P \ \textit{for any}\ i \in \underline{k} \tag{C.0.20}$$

$$\textbf{\textit{of type}}\ \rho \quad \textit{if} \quad \omega \circ dR_g = \rho(g^{-1}) \circ \omega \ \forall g \in G \tag{C.0.21}$$

The set of horizontal k-forms of type ρ will be denoted by $\Omega_h^k(P, V)^{(G,\rho)}$

Now consider the space $\mathcal{C}(P)$. For the difference of two connection forms $A_1, A_2 \in \mathcal{C}(P)$, we have

$$A_1 - A_2 \in \Omega_h^1(P, \mathfrak{g})^{(G,\mathrm{Ad})} \tag{C.0.22}$$

The reverse is also true. Assume $A \in \mathcal{C}(P)$, $\omega \in \Omega_h^1(P, \mathfrak{g})^{(G,\mathrm{Ad})}$. Then it is

$$\tilde{A} := A + \omega \in \mathcal{C}(P) \tag{C.0.23}$$

So $\mathcal{C}(P)$ is an affine space with respect to $\Omega_h^1(P, \mathfrak{g})^{(G,\mathrm{Ad})}$.

Theorem C.0.4 (Theorem 3.4 in [2])
The vector spaces $\Omega_h^k(P, V)^{G,\rho}$ and $\Omega^k(M, E)$ are isomorphic.

A vectorspace isomorphism is given by

$$\Phi : \Omega_h^k(P, V)^{(G,\rho)} \to \Omega^k(M, E) \tag{C.0.24}$$

$$\Phi(\bar{\omega}) = \omega \tag{C.0.25}$$

$$\omega_x(t_1, ..., t_k) := [p, \bar{\omega}_p(X_1, ..., X, k)] \tag{C.0.26}$$

whereby $t_i = d\pi_p(X_i)$ is arbitrary.

Definition C.0.8 (Covariant derivative)
A linear map

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E) \tag{C.0.27}$$

*is called **covariant derivative** on E, if*

$$\nabla(fe) = df \otimes e + f\nabla e \ \forall f \in \mathcal{C}^\infty(M),\ e \in \Omega^0(M, E) \tag{C.0.28}$$

Definition C.0.9 (total differential of a connection)
Let A be a connection form on P. The linear map

$$D_A : \Omega^k(P,V) \to \Omega^{k+1}(P,V) \tag{C.0.29}$$
$$(D_A\omega)_p(t_1,...,t_{k+1}) := d\omega(pr_h\ t_1,...,pr_h\ t_{k+1}) \tag{C.0.30}$$

is called the ***total differential*** *defined by A on P.*

Theorem C.0.5 (Theorem 3.9 in [2])
The total differential satisfies

$$D_A : \Omega_h^k(P,V)^{(G,\rho)} \to \Omega_h^{k+1}(P,V)^{(G,\rho)} \tag{C.0.31}$$
$$D_a\omega = d\omega + \rho_*(A) \wedge \omega\ \forall\omega \in \Omega_h^k(P,V)^{(G,\rho)} \tag{C.0.32}$$

The total differential D_A induces a map d_A on $\Omega^k(M,E)$:

$$d_A : \Omega^k(M,E) \to \Omega^{k+1}(M,E) \tag{C.0.33}$$
$$d_A\omega := \Phi(D_A\bar{\omega}) \tag{C.0.34}$$

Theorem C.0.6 (Theorem 3.10 in [2])
The induced differential d_A defines a covariant derivative ∇^A on E via

$$d_A\ |_{\Omega^0(M,E)} =: \nabla^A : \Omega^0(M,E) \to \Omega^1(M,E) \tag{C.0.35}$$

Furthermore, let $e = [s,v] \in \Gamma(U,E)$. Then it is

$$d_A e = [s, dv + \rho_*(A^s)v] \tag{C.0.36}$$

Definition C.0.10 (Curvature of a connection)
The two-form

$$F^A := D_A A \in \Omega_h^2(P,\mathfrak{g}) \tag{C.0.37}$$

is called ***curvature form*** *of A.*

The ***local curvature form*** *on M is given by*

$$F^s := F^A \circ ds \in \Omega^2(U,\mathfrak{g}) \tag{C.0.38}$$

Definition C.0.11 (Commutator of Lie algebra valued sections)
We fix a basis $\{a_i\}$ of $\mathfrak{g}$. Assume $\omega \in \Omega^k(M,\mathfrak{g})$, $\tau \in \Omega^l(M,\mathfrak{g})$. Then we can write

$$\omega = \omega^i\ a_i \tag{C.0.39}$$
$$\tau = \tau^i\ a_i \tag{C.0.40}$$

with $\omega^i \in \Omega^k(M,\mathbb{R})$, $\tau^i \in \Omega^l(M,\mathbb{R})$.

We can then define

$$[\omega,\tau]^\wedge := \omega^i \wedge \tau^j \otimes [a_i,a_j] \in \Omega^{k+l}(M,\mathfrak{g}) \tag{C.0.41}$$

Theorem C.0.7 (Theorem3.13 in [2]) *The curvature form satisfies*

$$\textit{Cartan structure equation} \ :\ F^A = dA + \frac{1}{2}[A,A]^\wedge \tag{C.0.42}$$
$$\textit{Bianchi identity} \ :\ D_A F^A = 0 \tag{C.0.43}$$

D. Microlocal Analysis

In this chapter, we introduce the **wavefront set** of distributions, following HÖRMANDER in [16]. The definition of the wavefront set will allow us to define the action of integral operators and distributions onto other distributions.

D.1. The wavefront set

Definition D.1.1 (Smooth distribution)
*A distribution $D \in \mathcal{D}'(E, \mathbb{R})$ is called **smooth** if there exists a smooth section*

$$\chi_D \in \Gamma(E^*) \tag{D.1.1}$$

such that

$$D[\xi] = \int_M d^n x \; \chi_D(x) \; \xi(x) \quad \forall \xi \in \Gamma_0(E) \tag{D.1.2}$$

Accordingly, a distribution $D \in \mathcal{E}'(E, \mathbb{R})$ is called smooth iff there exists a smooth section

$$\chi_D \in \Gamma_0(E^*) \tag{D.1.3}$$

such that

$$D[\xi] = \int_M d^n x \; \chi_D(x) \; \xi(x) \quad \forall \xi \in \Gamma(E) \tag{D.1.4}$$

*The smooth section χ_D is called the **kernel** of D.*

For now, let us consider distributions $D \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$.

Using Theorem 7.3.1 in [16], we know that the Fourier transform of a smooth function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$ decreases fast, i.e

$$\forall N \in \mathbb{N} \; \exists C_N \in \mathbb{R} : \; |\hat{\phi}(k)| \leq C_N (1 + |k|)^{-N} \quad \forall k \in \mathbb{R}^n \tag{D.1.5}$$

We use this property to study the singular behaviour of distributions.

We define the formal Fourier transform of such a distribution as

$$\hat{D}(k) := D[f], \; f(x) = e^{ikx} \tag{D.1.6}$$

We see that D is smooth iff its formal Fourier transform satisfies (D.1.5).

We define the **cone of singular directions**:

Definition D.1.2 (Cone of singular directions)
Let $D \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$. The cone of singular directions of D is defined as

$$\Sigma(D) := \{k \in \mathbb{R}^n \backslash \{0\} : \nexists \text{ conal neighborhood } V \text{ of } k, \text{ such that (D.1.5) holds on } V\} \tag{D.1.7}$$

We define the composition of distributions $D \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ with smooth compactly supported functions $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$:

$$\begin{aligned} &D \circ \psi \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}) \\ &(D \circ \psi)[\phi] := D[\psi \cdot \phi] \quad \forall \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R}), \ \phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \end{aligned} \tag{D.1.8}$$

In [16] is is shown that

$$\Sigma(D \circ \psi) \subset \Sigma(D) \quad \forall \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R}) \tag{D.1.9}$$

For any point $x \in \mathbb{R}^n$ we define

$$\Sigma_x(D) := \bigcap_{\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R}): \ x \in \text{supp } \psi} \Sigma(D \circ \psi) \tag{D.1.10}$$

Now we can define the **wavefront set** of a distribution:

Definition D.1.3 (Wavefront set)
Let $D \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$. The wavefront set of D is defined as

$$WF(D) := \{(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \simeq T^*\mathbb{R}^n \backslash \{0\} : k \in \Sigma_x(D)\} \tag{D.1.11}$$

In addition, we define

$$WF'(D) := \{(x, v) : \ (x, -v) \in WF(D)\} \tag{D.1.12}$$

We will now see how the notion of the wavefront set can be extended to distributions $D \in \mathcal{D}'(M \times \mathbb{R}, \mathbb{R})$ over a smooth n-dimensional manifold M.

Let $D \in \mathcal{D}'(M \times \mathbb{R}, \mathbb{R})$. We fix a covering of M by bijective coordinate charts

$$\kappa_i : U_i \subset M \to \mathbb{R}^n \tag{D.1.13}$$

and a corresponding decomposition of unity by smooth functions

$$\phi_i \in \mathcal{C}_0^\infty(M, \mathbb{R}), \ \text{supp } \phi_i \subset U_i, \ \sum_i \phi_i = 1|_M \tag{D.1.14}$$

D defines distributions $D_i \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ via

$$D_i[\psi] := (D \circ \phi_i)[\kappa_i^* \psi] \quad \forall \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R}) \tag{D.1.15}$$

The inverse construction also works in this setting. Any set of distributions $D_i \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ defines a distribution $D \in \mathcal{D}'(M \times \mathbb{R}, \mathbb{R})$ via

$$D[\psi] := \sum_i D_i[\kappa^i_*(\phi_i \cdot \psi)] \tag{D.1.16}$$

We define the wavefront set of D as

$$WF(D) := \bigcup_i \kappa_i^* WF(D_i) \tag{D.1.17}$$

D is again smooth iff its wavefront set is empty.

Finally we define the wavefront set of a distribution $D \in \mathcal{D}'(E, \mathbb{R})$, with E being a finite dimensional vector bundle over M. To do so, we use a similar method as above.

Let $D \in \mathcal{D}'(E, \mathbb{R})$. We fix a finite $\mathcal{C}^\infty$-base of $\Gamma_0(E)$, i.e. a finite collection $\{\chi_k\}$ of smooth sections in $\Gamma(E)$ such that

$$\forall \xi \in \Gamma_0(E) \; \exists! \; \{\phi_k\} \subset \mathcal{C}_0^\infty(M, \mathbb{R}) : \; \xi = \sum_k \phi_k \; \chi_k \tag{D.1.18}$$

D then defines distributions $D_k \in \mathcal{D}'(M \times \mathbb{R}, \mathbb{R})$ via

$$D_k[\phi] := D[\phi \cdot \chi_k] \;\; \forall \phi \in \mathcal{C}_0^\infty(M, \mathbb{R}) \tag{D.1.19}$$

Again, the inverse construction is also possible. Given a collection of distributions $D_k \in \mathcal{D}'(M \times \mathbb{R}, \mathbb{R})$, a distribution $D \in \mathcal{D}'(E, \mathbb{R})$ is defined via

$$D[\xi] = D[\sum_k \phi_k \; \chi_k] := \sum_k D_k[\phi_k] \tag{D.1.20}$$

The wavefront set of D is then defined as

$$WF(D) = \bigcup_k WF(D_k) \tag{D.1.21}$$

D is again smooth iff its wavefront set is empty.

D.2. Composition of distributions and integral operators

Throughout this section, E denotes a finite dimensional vector bundle over a smooth n-dimensional manifold M.

Given a continuous linear map

$$\Delta : \Gamma(E^*) \to \Gamma(E) \tag{D.2.1}$$

we can define the associated distributions

$$\begin{aligned} & D_\Delta \in \mathcal{D}'(E^{*2}, \mathbb{R}), \;\; D_{\Delta\chi} \in \mathcal{D}'(E^*, \mathbb{R}) \\ & D_\Delta[\xi, \chi] := D_{\Delta\chi}[\xi] := \int_M d^n x \; \xi \; (\Delta \; \chi) \;\; \forall \xi, \chi \in \Gamma_0(E^*) \end{aligned} \tag{D.2.2}$$

Now let $D'_i \in \mathcal{E}'(E, \mathbb{R})$ be a sequence of smooth distributions converging to a (not necessarily smooth) distribution $D' \in \mathcal{E}'(E, \mathbb{R})$. Let $\chi_{D'_i} \in \Gamma_0(E^*)$ be the kernel of D'_i.

We define the distribution

$$\begin{aligned} & D_{\Delta D'} \in \mathcal{D}'(E^*, \mathbb{R}) \\ & D_{\Delta D'}[\xi] := \lim \{ D_{\Delta \chi_{D'_i}}[\xi] \} \;\; \forall \xi \in \Gamma_0(E^*) \end{aligned} \tag{D.2.3}$$

if the limit exists $\forall \xi \in \Gamma_0(E^*)$.

Following theorem 8.2.13 in [16], this limit exists if

$$\nexists x \in M, \;\; v \in WF'(D') : (x, 0) \times v \in WF(D_\Delta) \tag{D.2.4}$$

and in this case it is

$$WF(D_{\Delta D'}) \subset \{ w \in T^*M | \; \exists v \in WF'(D') : w \times v \in WF(D_\Delta) \} \tag{D.2.5}$$

D.3. Properties of the causal propagator

We are in particular interested in the case in which Δ is the causal propagator of a normally hyperbolic partial differential operator F.

First of all, we want to estimate the wavefront set of D_Δ. Following theorem 8.3.1 in [16], for any linear partial differential operator P have that

$$WF(D) \subset WF(P \; D) \cup \; \text{char} \; P \tag{D.3.1}$$

Using this result, we can make the following estimation:

$$\begin{aligned} WF(D_\Delta) & \subset WF((1 \otimes F) \; D_\Delta) \cup \; \text{char} \; (1 \otimes F) \\ & = WF(0) \cup \bigcup_{x \in M} T^*M \times \bar{V}_x = \bigcup_{x \in M} T^*M \times \bar{V}_x \\ WF(D_\Delta) & \subset WF((F \otimes 1) \; D_\Delta) \cup \; \text{char} \; (F \otimes 1) \\ & = WF(0) \cup \bigcup_{x \in M} \bar{V}_x \times T^*M = \bigcup_{x \in M} \bar{V}_x \times T^*M \\ & \Rightarrow WF(D_\Delta) \subset \bigcup_{x \in M, \; y \in M} \bar{V}_x \times \bar{V}_y \end{aligned} \tag{D.3.2}$$

A even better estimation can be made using the **propagation of singularities** (see p.e. [20]).

With this result and using (D.2.4) we see that the distribution $D_{\Delta D'}$ is well defined for all distributions $D' \in \mathcal{E}'(E, \mathbb{R})$. In particular, if the wavefront set of D' does not intersect the lightcones $\bar{V}_x$, we can use (D.2.5) to see that in this case the wavefront set of $D_{\Delta D'}$ is empty and this distribution is smooth.

We denote the kernel of this smooth distribution by $\Delta\ D'$.

Bibliography

[1] C. Bär, N. Ginoux, F. Pfäffle: Wave Equations on Lorentzian Manifolds and Quantization, 2006

[2] H. Baum: Eichfeldtheorie. Lect. Notes Math., Humboldt-Universität Berlin, 2005

[3] P.G. Bergmann: Observables in General Relativity, Rev. Mod. Phys. 33 510-514 (1961)

[4] R. Brunetti, K. Fredenhagen: Quantum Field theory on Curved Backgrounds, Lect. Notes Phys. 786, 129-155, Springer, 2009

[5] S. M. Carroll: Spacetime and Geometry. Addison Wesley, 2004

[6] C.Crnkovic, E.Witten: Covariant description of canonical formalism in geometrical theories, in [15]

[7a] B. DeWitt: The Global Approach to Quantum Field Theory, Vol. 1. Oxford Sc. Pub., 2003

[7b] B. DeWitt: The Global Approach to Quantum Field Theory, Vol. 2. Oxford Sc. Pub., 2003

[8] P. A. M. Dirac: Lectures on Quantum Mechanics, Yeshiva University Press, New York, 1964

[9] M. Dubois-Violette: Remarks on the local structure of Yang-Mills and Einstein equations, Phys. Lett., Vol. 131B p. 323-326, 1983

[10] J. Ehlers, H. Friedrich (Eds.): Canonical Gravity: From Classical to Quantum, Springer 1994

[11] A. E. Fischer, J. E. Marsden: Linearization stablility of the Einstein equations, Bull. Am. Math. Soc., Vol. 79, Nr. 5, 1973

[12] J. Frauendiener, G. A. J. Sparling: On the Symplectic Formalism for General Relativity, Proc. R. Soc. Lond. A 1992 436, 141-153

[13] D. Giulini: Ashtekar Variables in Classical General Relativity, arXiv:gr-qc/9312032v2

[14] V. N. Gribov: Quantization of non-abelian gauge theories. Nuclear Physics B139 (1978), p.1-19

[15] S. W. Hawking, W. Israel (Eds.): Three hundred years of gravitation, Cambridge University Press, 1987

[16] L. Hörmander: The Analysis of Linear Partial Differential Operators I. Springer, 1983

[17a] D. Marolf: The Generalized Peierls Bracket, arXiv:hep-th/9308150

[17b] D. Marolf: Poisson Brackets on the Space of Histories, arXiv:hep-th/9308141v2

[18] K.-H. Neeb: Monastir Summer School: Infinite-Dimensional Lie Groups, 2006

[19] R. E. Peierls: The Commutation Laws of Relativistic Field Theory, Proc. Roy. Soc. London, Vol. 214(A), pp. 143-157, 1952

[20] M.J. Radzikowski, R. Verch: A local to global Singularity Theorem for Quantum Field Theory on Curved Space-Time, Commun. Math. Phys. 180 (1996) 1-22

[21] C. Rovelli: GPS observables in general relativity, arXiv:gr-qc/0110003v2

[22] T. Thiemann: Modern Canonical Quantum General Relativity. Cambridge Univ. Press, 2007

[23] R. M. Wald: General Relativity. The University of Chicago Press, 1984

Danksagung

Ich danke Herrn Prof. Dr. Fredenhagen für die interessante Aufgabenstellung und die intensive Betreuung über den gesamten Zeitraum der Arbeit.

Ich danke Katarzyna Rejzner, Dr. Thomas-Paul Hack, Dr. Kai J. Keller, Dr. Claudio Dappiaggi und Andreas Degner für ihre stete Bereitschaft, Fragen zu meiner Arbeit zu diskutieren.

Ich danke Andrey Saveliev und Benjamin Lang für ihre Geduld und Hilfsbereitschaft.

Schließlich danke ich meinen Eltern für die Ermöglichung meines Studiums.

Erklärung

Ich versichere, diese Arbeit selbständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel verfasst zu haben.

Ich gestatte die Veröffentlichung dieser Arbeit.

Hamburg, den ________________ ____________________________

Jan-Christoph Weise