On the Algebraic Theory of Kink Sectors: Application to Quantum Field Theory Models and Collision Theory

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To my parents and my brother

Abstract:

Several two dimensional quantum field theory models have more than one vacuum state. An investigation of super selection sectors in two dimensions from an axiomatic point of view suggests that there should be also states, called soliton or kink states, which interpolate different vacua.

Familiar quantum field theory models, for which the existence of kink states have been proven, are the Sine-Gordon and the ϕ_2^4-model. In order to establish the existence of kink states for a larger class of models, we investigate the following question:

Which are sufficient conditions a pair of vacuum states has to fulfill, such that an interpolating kink state can be constructed?

We discuss the problem in the framework of algebraic quantum field theory which includes, for example, the $P(\phi)_2$-models. We identify a large class of vacuum states, including the vacua of the $P(\phi)_2$-models, the Yukawa$_2$-like models and special types of Wess-Zumino models, for which there is a natural way to construct an interpolating kink state.

In two space-time dimensions, massive particle states are kink states. We apply the Haag-Ruelle collision theory to kink sectors in order to analyze the asymptotic scattering states. We show that for special configurations of n kinks the scattering states describe n freely moving non interacting particles.

Zusammenfassung:

Einige Modelle für Quantenfeldtheorien in zwei Raumzeitdimensionen besitzen mehr als ein Vakuum. Untersucht man im Rahmen der algebraischen Quantenfeldtheorie Superauswahlsektoren in $1+1$ Dimensionen, dann zeigt sich, daß Kink-Zustände, die unterschiedliche Vakua interpolieren, in natürlicher Weise auftreten.

Bekannte Beispiele für zweidimensionale Modelle, in denen die Existenz von Kink-Zuständen bewiesen wurde, sind das Sinus-Gordon- und das ϕ_2^4-Modell. Um die Existenz von Kink Zuständen für eine möglichst große Klasse von Modellen nachzuweisen, wird die folgende Fragestellung vom Standpunkt der algebraischen Quantenfeldtheorie aus diskutiert:

Welche Bedingungen sind hinreichend, um aus den Vakua des zu untersuchenden Modelles interpolierende Kink-Zustände konstruieren zu können?

Wir entwickeln ein allgemeines Konstruktionsverfahren für Kink-Zustände, welches auf operatoralgebraischen Methoden beruht. Ferner zeigen wir, daß sich unser Verfahren auf eine große Klasse von Modellen anwenden läßt, die unter anderem alle $P(\phi)_2$-Modelle, Yukawa$_2$- und spezielle Typen von Wess-Zumino-Modellen enthält.

In zweidimensionalen Quantenfeldtheorien sind massive Einteilchen-Zustände auch Kink-Zustände. Wir wenden die Haag-Ruelle-Streutheorie auf Kink-Sektoren an. Dabei zeigt sich, daß spezielle Konfigurationen von n Kinks sich asymptotisch wie n freie massive Teilchen verhalten.

Contents

1 Introduction and Overview

1.1 Introduction

In the early sixties, Rudolf Haag and Daniel Kastler proposed an algebraic approach to quantum field theory [45]. The purpose of this program was to understand the basic principles of quantum field theory in a mathematically rigorous way. The starting point of this approach, which can be seen in contrast to approaches based on classical Lagrangians, is an axiomatic formulation of a general quantum field theory by employing only the fundamental concepts of special relativity and quantum mechanics. This axiomatic approach is a successful tool to investigate general properties shared by different quantum field theories and it is a useful guideline for investigating and analyzing quantum field theories which have already been constructed.

On the other hand, the algebraic approach to quantum field theory seems to be less practicable for constructing concrete models. For the purpose to construct examples of quantum field theories explicitly, the Lagrangian approach (canonical quantization, functional integral) is more adequate, but it is always accompanied by serious mathematical problems. One is faced with many technical difficulties in order to prove that the investigated model exists and is compatible with the principles of special relativity and quantum mechanics. Nevertheless, approaches based on classical Lagrangians play an important role in high energy physics and they have successfully been applied for phenomenological purposes.

Besides the free field, many examples for interacting quantum field theories, fitting into the framework of algebraic quantum field theory, are known in two space-time dimensions, let us mention the $P(\phi)_2$ and Yukawa$_2$-like models. There are also some non-trivial examples in three space-time dimensions, for instance the

ϕ_3^4 and the Yukawa$_3$ model. It is still an open problem whether there are interacting quantum field theory models in four space-time dimensions.

There are different purposes for investigating low dimensional quantum field theories. Among them are, for instance, the following:

(i) Since there is a large set of different quantum field theories in two space time dimensions, these models can be viewed as "theoretical laboratories" in which the concepts of general quantum field theory can be tested.

(ii) In some cases, low dimensional quantum field theory models play a role as effective theories in solid state physics.

The concept of *kinks* appears naturally in the investigation of two-dimensional quantum field theories. We shall see later that the occurrence of kinks can be verified for a large class of quantum field theory models.

In Section 1.1.1, we make some preliminary remarks on algebraic quantum field theory. We do not give a complete survey in this field, we only give the definitions and notions which will be used later to carry through our analysis.

For a detailed treatment of local quantum physics and its mathematical description from an algebraic point of view, we refer to Haag's book [43] and references, given there. Introductory articles with the emphasis on the theory of super selection sectors can also be found in [54].

We briefly motivate and illustrate the concept of kinks in Section 1.1.2 and give a review of some well-known results in Section 1.1.3 and Section 1.1.4. During the last twenty years, many different approaches have been developed in order to analyze kinks in quantum field theory. We shall give a brief description of those examples which are nearest related to our analysis.

1.1.1 Preliminary Remarks on Local Quantum Physics

Haag-Kastler nets and superselection sectors: The framework of algebraic quantum field theory has turned out to be a successful formalism to describe physical concepts like observables, states , superselection sectors (charges) and statistics. These notions can appropriately be described by mathematical concepts like C*-algebras, positive linear functionals and equivalence classes of representations. For the convenience of the reader, we shall state the relevant definitions and assumptions here.

Let $\mathcal{O} \subset \mathbb{R}^{1,s}$ be a region in space-time. We denote by $\mathfrak{A}(\mathcal{O})$ the algebra generated by all observables which can be measured within $\mathcal{O}$. For technical reasons we always suppose that $\mathfrak{A}(\mathcal{O})$ is a C*-algebra and $\mathcal{O}$ is a double cone, i.e. a bounded and causally complete region. Motivated by physical principles, we make the following assumptions:

(1) The assignment

$$\mathfrak{A} : \mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$$

is an isotonous net of C*-algebras, i.e. if $\mathcal{O}_1$ is contained in $\mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1)$ is a C*-sub-algebra of $\mathfrak{A}(\mathcal{O}_2)$. The isotony encodes the fact that each observable which can be measured within $\mathcal{O}$ can also be measured in every larger region. Furthermore, the C*-inductive limit

$$C^*(\mathfrak{A})$$

of the net $\mathfrak{A}$ can be constructed since the set of double cones is directed. We refer to [69] for this notion.

(2) Two local operations which take place in space-like separated regions should not influence each other. The *principle of locality* is formulated as follows: If the regions $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated, then the elements of $\mathfrak{A}(\mathcal{O}_1)$ commute with those of $\mathfrak{A}(\mathcal{O}_2)$.

(3) Each operator a which is localized in a region $\mathcal{O}$ should have an equivalent counterpart which is localized in the translated region $\mathcal{O}+x$. The *principle of translation symmetry* is encoded by the existence of an automorphism group $\{\alpha_x; x \in \mathbb{R}^{1,s}\}$ which acts on the C*-algebra $C^*(\mathfrak{A})$ such that α_x maps $\mathfrak{A}(\mathcal{O})$ onto $\mathfrak{A}(\mathcal{O} + x)$.

A net of C*-algebras which fulfills the conditions *(1)* to *(3)* is called a *translationally covariant Haag-Kastler net*.

In order to discuss particle-like concepts, we select an appropriate class $\mathfrak{S}$ of normalized positive linear functionals, called states, of $C^*(\mathfrak{A})$. We require that the states $\omega \in \mathfrak{S}$ fulfill the conditions:

(1) There exists a strongly continuous unitary representation of the translation group $U : x \mapsto U(x)$ on the GNS[1]-Hilbert space $\mathcal{H}$ which implements the translations in the GNS-representation π, i.e.

$$\pi(\alpha_x a) = U(x)\pi(a)U(-x)$$

[1]Given a state $\omega \in \mathfrak{S}$, we obtain via GNS-construction a Hilbert space $\mathcal{H}$, a *-representation π of $C^*(\mathfrak{A})$ on $\mathcal{H}$ and a vector $\Omega \in \mathcal{H}$ such that $\langle \Omega, \pi(a)\Omega \rangle = \omega(a)$ for each $a \in C^*(\mathfrak{A})$. The triple $(\mathcal{H}, \pi, \Omega)$ is called the GNS-triple of ω.

for each $a \in C^*(\mathfrak{A})$.

(2) The stability of a physical system is encoded in the spectrum condition (positivity of the energy), i.e. the spectrum (of the generator) of $U(x)$ is contained in the closed forward light cone.

These conditions are also known as the *Borchers criterion*. States which satisfy the Borchers criterion and which are, in addition, translationally invariant are called *vacuum states*.

We denote by $\mathfrak{R}$ the set of *-representations π such that π is the GNS-representation of some state $\omega \in \mathfrak{S}$. Unitary equivalence classes $[\pi]$ of *-representations $\pi \in \mathfrak{R}$, which we shall call *sectors*[2], are used to investigate *quantum numbers* or *charges*. More precisely, the quantum numbers or charges of the physical system under consideration are labeled by the set of *irreducible* sectors.

In order to illustrate these notions we shall summarize some aspects of the theory of DHR super selection sectors here. For the analysis of quantum field theories with short range forces, the following selection criterion for physical representations, which is known as the *DHR criterion*, has been proposed by S. Doplicher, R. Haag and J. Roberts [16, 17, 18, 19]: The representations are unitarily equivalent to a given vacuum representation π_0 when being restricted to the C*-algebra $C^*(\mathfrak{A}, \mathcal{O}')$. [3]

If *Haag duality*, a maximality condition for local observables, holds in the vacuum representation π_0

$$\pi_0(\mathfrak{A}(\mathcal{O}))'' = \pi_0(C^*(\mathfrak{A}, \mathcal{O}'))'$$

then it follows that for each representation π, which fulfills the DHR criterion, there exists an endomorphism ρ of $C^*(\mathfrak{A})$ such that π is unitarily equivalent to $\pi_0 \circ \rho$. The endomorphism ρ can be chosen in such a way that it acts trivially on observables which are localized in the space-like complement of a given double cone $\mathcal{O}$, and ρ is interpreted as a charge $[\pi_0 \circ \rho]$ which is localized within the region $\mathcal{O}$.

The main feature of these endomorphisms is that they can be used to define the composition of sectors. Given two representations $\pi_1 = \pi_0 \circ \rho_1$ and $\pi_2 = \pi_0 \circ \rho_2$ the product of the sectors $\theta_1 = [\pi_1]$ and $\theta_2 = [\pi_2]$ is given by

$$\theta_1 \theta_2 = [\pi_0 \circ \rho_1 \rho_2] \ .$$

[2] For our purpose it is more convenient to use the notion *sector* also for non-primary representations. Furthermore, if π is an irreducible representation, then we shall call $[\pi]$ an irreducible sector.

[3] For an unbounded region G we denote by $C^*(\mathfrak{A}, G)$ the C*-sub-algebra of $C^*(\mathfrak{A})$ which is generated by all local algebras $\mathfrak{A}(\mathcal{O})$ with $\mathcal{O} \subset G$.

The sector $\theta_1\theta_2$ is independent on the particular choice of endomorphisms, it only depends on the sectors θ_1 and θ_2. The composition of irreducible sectors is interpreted as the fusion of quantum numbers. Furthermore, for each pair of DHR endomorphisms ρ_1, ρ_2 there exists a canonical unitary operator $\epsilon(\rho_1, \rho_2)$, called statistics operator, such that

$$\rho_1(\cdot)\epsilon(\rho_1, \rho_2) = \epsilon(\rho_1, \rho_2)\rho_2(\cdot) \ .$$

As a consequence, the fusion of DHR sectors is commutative, i.e.:

$$\theta_1\theta_2 = \theta_2\theta_1 \ .$$

Assuming that θ has finite statistics (see [18, 19, 57, 58]), it could be shown that for each irreducible DHR sector θ there exists a conjugate $\bar{\theta}$, representing the anti-charge of θ. It is determined uniquely by the property that the sector

$$\theta\bar{\theta} = \bar{\theta}\theta$$

contains the vacuum sector $[\pi_0]$ precisely once. Moreover, the product of two irreducible sectors θ_1 and θ_2 can be decomposed into a finite direct sum of irreducible sectors:

$$\theta_1\theta_2 = \bigoplus_{\theta} n^{\theta}_{\theta_1\theta_2}\ \theta$$

where the natural numbers $n^{\theta}_{\theta_1\theta_2}$ count the multiplicity of the sector θ in the product $\theta_1\theta_2$; they are called *fusion coefficients*.

An investigation which concerns charges, localized in space-like cones, has been carried out by D. Buchholz and K. Fredenhagen [10]. The sectors which have been studied by them are called *BF sectors* and they generalize the concept of DHR sectors.

The time-slice formulation in two dimensions: In two space-time dimensions, there are, besides the free field, many interacting quantum field theory models. In order to place their analysis into the framework of algebraic quantum field theory, it is convenient to work with the *time-slice formulation* which we briefly describe here. The time slice formulation has two main aspects:

Aspect 1: The *Cauchy data* with respect to a *space-like plane* Σ (the initial conditions at time $t = 0$) are given by an isotonous net

$$\mathfrak{M} : \mathcal{I} \longmapsto \mathfrak{M}(\mathcal{I})$$

which assigns to each bounded subset $\mathcal{I} \subset \Sigma$ a C*-algebra $\mathfrak{M}(\mathcal{I})$. It is assumed that the space-like translations act as an automorphism group $\{\alpha_{\mathbf{x}} : \mathrm{x} \in \mathbb{R}\}$ on the C*-inductive limit $C^*(\mathfrak{M})$ in such a way that $\alpha_{\mathbf{x}}$ maps $\mathfrak{M}(\mathcal{I})$ onto $\mathfrak{M}(\mathcal{I} + \mathrm{x})$. Moreover, locality holds, i.e. if the intersection of two intervals $\mathcal{I}_1, \mathcal{I}_2$ is empty, then the elements in $\mathfrak{M}(\mathcal{I}_1)$ commute with those in $\mathfrak{M}(\mathcal{I}_2)$.

Aspect 2: The *dynamics*, which describes the time evolution of a physical system, is given by a one-parameter automorphism group

$$\alpha = \{\alpha_t : t \in \mathbb{R}\} \quad \text{of } C^*(\mathfrak{M}).$$

Motivated by physical principles, α should satisfy the following list of axioms:

(i) The automorphisms α_t commute with the spatial translations $\alpha_{\mathbf{x}}$.

(ii) The propagation speed, which is induced by the automorphism group α, is not faster than the speed of light, i.e. if an operator a is localized in the region $\mathcal{I}$, then the operator $\alpha_t(a)$ localized in $\mathcal{I}_{|t|} := \mathcal{I} + (-|t|, |t|)$.

In Chapter 2, we shall describe how a translationally covariant Haag-Kastler net $\mathfrak{A}_\alpha$ can be constructed from a given net of Cauchy data and a given dynamics α. Furthermore, we shall give a brief introduction into $P(\phi)_2$- and Yukawa$_2$ models there.

1.1.2 The Concept of Kinks

Kinks already appear in classical field theories and the typical systems in which they occur are 1+1-dimensional. Familiar examples are the Sine-Gordon and the ϕ_2^4-model. We briefly describe the latter:

The Lagrangian density of the model is given by

$$\mathfrak{L}(\phi, x) = \frac{1}{2}\,\partial_\mu\phi(x)\partial^\mu\phi(x) - U(\phi(x))$$

where the potential U is given by

$$U(z) := \lambda/2\,(z^2 - a)^2 \ .$$

The energy of a classical field configuration ϕ is

$$E(\phi) = \int \mathrm{dx}\, \Big(\frac{1}{2}(\partial_0\phi(0,\mathrm{x}))^2 + \frac{1}{2}(\partial_1\phi(0,\mathrm{x}))^2 + U(\phi(0,\mathrm{x}))\Big) \ .$$

With the choice of U, given above, the absolute minimum value of U is zero and thus the energy functional $E : \phi \mapsto E(\phi)$ is positive.

There are two configurations $\phi_\pm$ with zero energy $E(\phi_\pm) = 0$:

$$\phi_\pm : (t,\mathrm{x}) \longmapsto \ \pm\ a \ .$$

These configurations are invariant under space-time translations and represent the vacua of the classical system.

There are two further configurations $\phi_s, \phi_{\bar{s}}$ which are stationary points of the energy functional E. They are given by

$$\phi_s : (t,\mathrm{x}) \longmapsto a\ \tanh(\ \sqrt{\lambda} a\mathrm{x}\ ) \quad \text{and} \quad \phi_{\bar{s}} : (t,\mathrm{x}) \longmapsto -\, a\ \tanh(\ \sqrt{\lambda} a\mathrm{x}\ ) \ .$$

These configurations represent the kinks of the classical system which *interpolate* the vacua $\phi_\pm$. Indeed, we have for the kink ϕ_s

$$\lim_{\mathrm{x}\to\pm\infty} \phi_s(t,\mathrm{x}) = \phi_\pm(t,\mathrm{x}) = \pm a \ . \tag{1.1}$$

The configuration $\phi_{\bar{s}}$, which interpolates the vacua $\phi_\pm$ in the opposite direction, represents the anti-kink of ϕ_s. Both of them have the same energy, namely

$$E(\phi_s) = E(\phi_{\bar{s}}) = \frac{4}{3}\sqrt{\lambda} a^3 \ .$$

From the classical example above, we see that the crucial properties of a kink are to interpolate vacuum configurations as well as to be a configuration of finite energy. Motivated by these properties, in quantum field theory a *kink state* ω is defined as follows:

The interpolation property: For each observable a, the limits

$$\lim_{\mathrm{x}\to\pm\infty} \omega(\alpha_{(t,\mathrm{x})}(a)) = \omega_\pm(a) \tag{1.2}$$

exist and $\omega_\pm$ are vacuum states. Note that equation (1.2) is the quantum version of the interpolation property (1.1).

Positivity of the energy: ω fulfills the Borchers criterion.

In the literature the concept of *kink* as described above is often called *soliton* (see [31, 32]) or more seldom *lump* (see [13]). In the subsequent, we shall use the word kink.

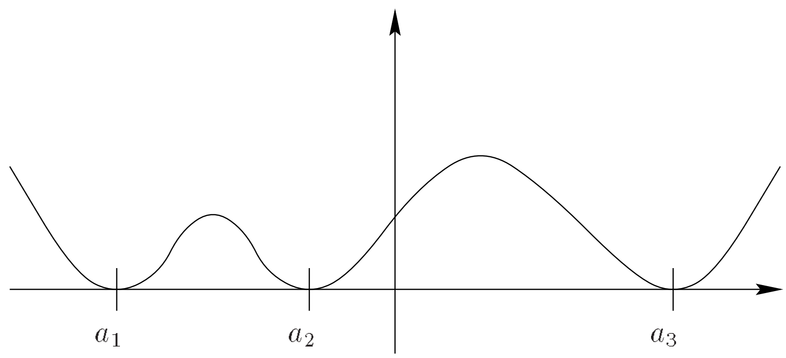

Figure 1.1: The figure shows an interaction potential U whose minima $a_1, \cdots, a_3$ are not related by a symmetry of the Lagrangian.

We close this section by mentioning the following: The above Lagrangian $\mathfrak{L}$ is invariant under the $\mathbb{Z}_2$-symmetry $\phi \mapsto -\phi$ and the vacuum configurations are related by this spontaneously broken symmetry. We can also imagine a situation where more than one vacuum is present but where there are no symmetries which connect different vacua. To illustrate this, we replace $U(z) = \lambda/2(z^2 - a)^2$ by a polynomial with local minima at $\{a_1, \cdots, a_n\}$ all of the hight zero, i.e.

$$U(a_1) = \cdots = U(a_n) = 0 \quad .$$

Hence we obtain n-distinct vacuum configurations

$$\phi_j : (t, \mathbf{x}) \longmapsto \phi_j(t, \mathbf{x}) = a_j \;\; ; \;\; j = 1, \cdots, n \quad .$$

The minima can be placed in such a way, see Figure 1.1, such that there is no symmetry of the Lagrangian $\mathfrak{L}$ which maps $\{a_1, \cdots, a_n\}$ onto $\{a_1, \cdots, a_n\}$ and we conclude that the vacuum configurations ϕ_j are not related by a symmetry of the Lagrangian.

1.1.3 Kinks in Quantum Field Theory Models: Models with ϕ^4_2-like Interactions

During the 70s, examples for interacting quantum field theory models were constructed. It was proven by J. Glimm, A. Jaffe and T. Spencer that two-dimensional models with $P(\phi)_2$- interaction exist, and their vacuum states satisfy the Wightman axioms [37, 41]. Interactions between fermions and bosons have also been studied, in particular the Yukawa$_2$ interactions [37, 38, 74, 75]. Furthermore, an investigation of the Sine-Gordon model has been carried out by J. Fröhlich an E. Seiler [35].

A few years later, a great deal of attention has been paid to the construction of new superselection sectors which are different from vacuum sectors. In 1976, the existence of kink sectors for the $(\Phi\cdot\Phi)^2_2$- and the Sine-Gordon model was established by J. Fröhlich [31]. To illustrate the ideas and techniques which have been used in [31], we give a short review of the construction of the kink sectors of the $(\Phi\cdot\Phi)^2_2$-model.

The Cauchy data on a space-like plane Σ are represented by the W*-algebras

$$\mathfrak{M} : \mathcal{I} \longmapsto \mathfrak{M}(\mathcal{I}) := \left\{ e^{i\Phi(f_1)+i\Pi(f_2)} \;\middle|\; f_j \in S(\Sigma,\mathbb{R}^2), \mathrm{supp}(f_j) \subset \mathcal{I} \right\}''$$

where $\Phi = (\Phi_1, \Phi_2)$ is a massive free two-component Bose field and Π its canonically conjugate, acting as operator valued distributions on the Fock space $\mathcal{H}_0$. The infrared regularized Hamiltonian of the model is

$$H(g) = H_0 + \int_\Sigma \mathrm{d}\mathbf{x}\; g(\mathbf{x}) \left(\lambda \; :(\Phi\cdot\Phi)(\mathbf{x})^2: \; + \sigma_1 : \Phi_1(x)^2 : \; + \sigma_2 : \Phi_2(x)^2 : \right) - E(g)\mathbf{1}$$

where H_0 is the free Hamiltonian and g is an appropriate test function with compact support. The constant $E(g)$ is chosen in such a way that $H(g)$ is a positive operator. The $O(2)$ invariance of $H(g)$ is explicitly broken unless $\sigma_1 = \sigma_2$. It remains a $\mathbb{Z}_2$ symmetry which is given by $\Phi \mapsto -\Phi$.

The basic ingredients for the construction of kink sectors of the $(\Phi\cdot\Phi)^2_2$-model have been taken from the work of J. Glimm, A. Jaffe and T. Spencer [37] which contains the following results:

(i) The uniform limit

$$\alpha_t(a) := \lim_{g\to 1} e^{itH(g)}\, a\, e^{-itH(g)}$$

exists for each local operator a and defines a dynamics α with propagation speed less or equal than one.

(ii) Since $H(g)$ is invariant under the $\mathbb{Z}_2$ action $\Phi \mapsto -\Phi$, the automorphism

$$\chi : \exp(i\Phi(f_1) + i\Pi(f_2)) \longmapsto \exp(-i\Phi(f_1) - i\Pi(f_2))$$

commutes with the dynamics α, i.e.:

$$\alpha_t \circ \chi = \chi \circ \alpha_t \ \ .$$

For appropriate values $\sigma_1, \sigma_2 < 0$, $\sigma_1 \neq \sigma_2$ and sufficiently small coupling constant λ, there are precisely two inequivalent vacuum states $\omega_\pm$ which are related by the symmetry χ:

$$\omega_+ \circ \chi = \omega_- \ \ .$$

The existence of the vacuum sectors and the properties of the dynamics of the $(\Phi \cdot \Phi)^2_2$-model can be used to construct interpolating kink sectors. It has been proven by J. Fröhlich [31] that besides the two vacuum sectors there are two kink sectors, a kink and its corresponding anti-kink. We summarize the main steps of his construction here.

Step 1: Let s be a smooth test function with the property: There exists a bounded interval $\mathcal{I} \subset \Sigma$ such that

$$s(\mathbf{x}) = \begin{cases} \pi & \text{if } \mathbf{x} \in \mathcal{I}_{RR} \\ 0 & \text{if } \mathbf{x} \in \mathcal{I}_{LL} \end{cases}$$

where $\mathcal{I}_{RR}$ is the right and $\mathcal{I}_{LL}$ is the left complement of $\mathcal{I}$. The graph of s is shown by Figure 1.2. The $O(2)$-valued function

$$g_s : \mathbf{x} \longmapsto g_s(\mathbf{x}) = \begin{pmatrix} \cos(s(\mathbf{x})) & \sin(s(\mathbf{x})) \\ -\sin(s(\mathbf{x})) & \cos(s(\mathbf{x})) \end{pmatrix} \in O(2)$$

induces a *Bogoliubov automorphism* ρ_s which is defined on the Weyl operators by

$$\rho_s : \exp(i\Phi(f_1) + i\Pi(f_2)) \longmapsto \exp(i\Phi(g_s f_1) + i\Pi(g_s f_2)) \ \ .$$

Since

$$g_s(\mathbf{x}) = \begin{cases} -\mathbf{1}_2 & \text{if } \mathbf{x} \in \mathcal{I}_{RR} \\ \mathbf{1}_2 & \text{if } \mathbf{x} \in \mathcal{I}_{LL} \end{cases} \ \ ,$$

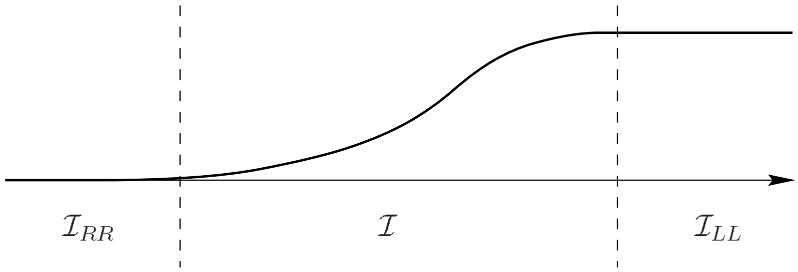

Figure 1.2: The figure shows the graph of the kink function s.

the automorphism ρ_s acts trivially on operators which are localized in $\mathcal{I}_{LL}$ and as the symmetry χ on those which are localized in $\mathcal{I}_{RR}$.

Obviously, the states $\omega_s := \omega_- \circ \rho_s$ and $\bar{\omega}_s := \omega_+ \circ \rho_s^{-1}$ fulfill the interpolation condition for kink states.

Step 2: The explicit knowledge of the dynamics α can be used to prove the existence of a strongly continuous function

$$\gamma : (t, \mathrm{x}) \longmapsto \gamma(t, \mathrm{x})$$

where $\gamma(t, \mathrm{x})$ is a unitary operator, localized in a sufficiently large interval $\mathcal{I}_{(t,\mathrm{x})}$. It implements the automorphism

$$\alpha_{(t,\mathrm{x})} \circ \rho_s \circ \alpha_{(-t,-\mathrm{x})} \circ \rho_s^{-1} = \mathrm{Ad}(\gamma(t, \mathrm{x}))$$

and satisfies the *cocycle* condition:

$$\gamma(t_1 + t, \mathrm{x}_1 + \mathrm{x}) = \alpha_{(t,\mathrm{x})}(\gamma(t_1, \mathrm{x}_1))\gamma(t, \mathrm{x}) \ . \tag{1.3}$$

The operators $\gamma(t, \mathrm{x})$ describe the translation by (t, x) of the kink charge $[\omega_- \circ \rho_s]$. It follows from the properties of γ that ω_s is translationally covariant and satisfies the spectrum condition. The same holds for the state $\bar{\omega}_s := \omega_+ \circ \rho_s^{-1}$. This implies that ω_s and $\bar{\omega}_s$ are kink states.

It is well known that the Bogoliubov automorphism ρ_r is implemented by a unitary operator u_r which is localized in $\mathrm{supp}(r)$ if the function r is smooth with compact support. This fact can be used to show that the set of sectors, consisting of the vacua $e_\pm := [\omega_\pm]$ and the kink sectors $\theta := [\omega_s], \bar{\theta} := [\bar{\omega}_s]$ is closed under fusion. In the same manner as DHR endomorphisms [18, 19, 43], the automorphisms ρ_s can be used to describe the *fusion* of kink sectors. We consider functions s_1, s_2 such that the state ω_{s_1} belongs to the sector θ and the state $\bar{\omega}_{s_2}$ belongs to the sector $\bar{\theta}$. The fusion of the sectors θ and $\bar{\theta}$ is

$$\theta\bar{\theta} := [\omega_- \circ \rho_{s_1}\rho_{s_2}^{-1}] = [\omega_- \circ \rho_{s_1 - s_2}] = e_- \quad .$$

Here the fact that $r = s_1 - s_2$ has compact support is used. Analogously, we obtain

$$\bar{\theta}\theta = e_+$$

which justifies the interpretation that $\bar{\theta}$ is the anti-kink sector with respect to θ. Furthermore, we have the remaining combinations:

$$\theta\theta := [\omega_- \circ \rho_{s_1}\rho_{s_2}] = e_- \quad \text{and} \quad \bar{\theta}\bar{\theta} := [\omega_+ \circ \rho_{s_1}^{-1}\rho_{s_2}^{-1}] = e_+ \quad . \tag{1.4}$$

As we shall see later, the combinations (1.4) do not correspond to the proper fusion of kink sectors.

In 1977 J. Fröhlich proved the existence of the kink states of the one- component ϕ_2^4-model [32] by using, in comparison to [31], an alternative method. The technical difficulties which arise here are due to the fact that one has to deal with a one-component Bose field. Therefore, there is no a priori choice for a Bogoliubov transformation ρ_s. We shall give a brief summary of the results of [32] to illustrate the main differences to the construction of the $(\Phi \cdot \Phi)_2^2$-kinks.

The construction of the vacuum sectors of the ϕ_2^4-model, which is presented in [32], uses the methods of Euclidean field theory. The vacuum states of the ϕ_2^4-model can be obtained from two measures $\mu_\pm$ on $S'(\mathbb{R}^2)$ which satisfy the *Osterwalder-Schrader axioms*. We briefly explain how the measures $\mu_\pm$ are constructed as limits of perturbations of the Gaussian measure μ_0.

Step I: Let μ_0 be the Gaussian measure on the space of tempered distributions $S'(\mathbb{R}^2)$ with mean zero and covariance C where the integral kernel of C is

$$C(x-y) = \int \mathrm{d}^2p\, (p^2 - m^2)^{-1}\, e^{ip(x-y)} \quad .$$

The regularized interaction part of the Euclidean action is

$$S_1(g, \phi) = \int \mathrm{d}^2x\, g(x)\, (\, \lambda \ :\phi(x)^4 :_{\mu_0} \, - \sigma \ :\phi(x)^2 :_{\mu_0} \,)$$

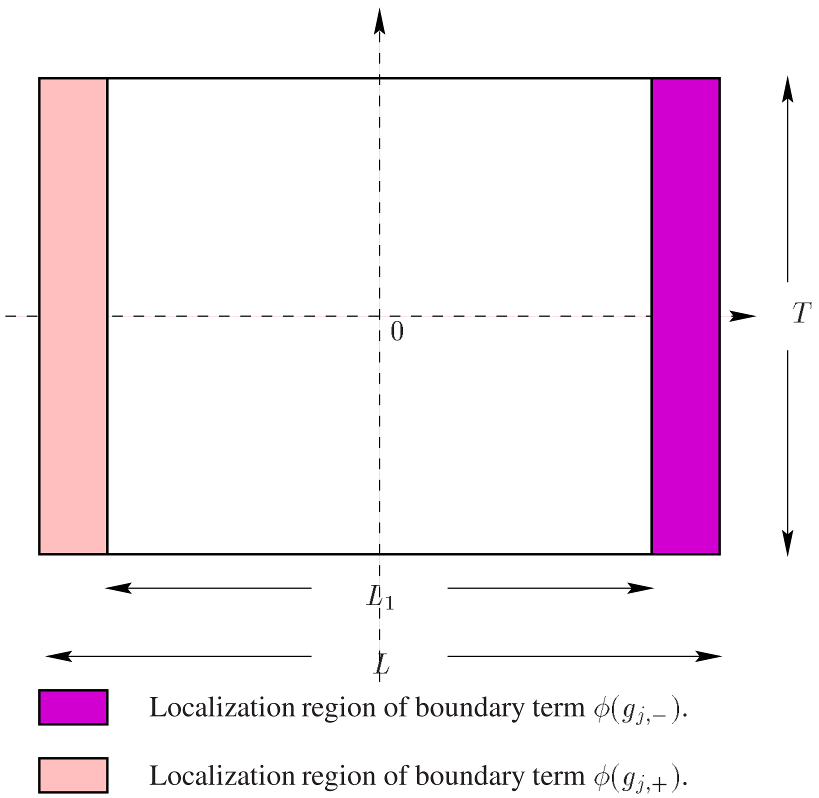

Figure 1.3: The finite volume region.

where $: \cdot :_{\mu_0}$ is the normal ordering with respect to the Gaussian measure μ_0 and g is a smooth test function. The action $S_1(g, \phi)$ is invariant under the substitution $\phi \mapsto -\phi$. To approximate one of the measures $\mu_\pm$ the $\mathbb{Z}_2$ symmetry has to be broken explicitly by introducing appropriate boundary terms.

The test function g can be chosen in such a way that it is one in the region $I_T \times I_L$ and zero outside a slightly larger region. Here the interval I_s is defined by $I_s := (-s/2, s/2)$. For $L_1 < L$ the region $I_L \backslash I_{L_1}$ has two connected components $I_\pm$ and there are two possibilities (corresponding to μ_+ or μ_-) to choose boundary conditions with respect to each of the regions $I_T \times I_\pm$ (See Figure 1.3 for an illustration). This gives four different boundary terms

$$\{\delta S_{j,\pm}(\phi) = \phi(g_{j,\pm}) + c_{j,\pm}; j = \pm\}$$

where $g_{j,\pm}$ are suitable test functions which have support in a neighborhood of $I_T \times I_\pm$ and $c_{j,\pm}$ are appropriate constants. The regularized interaction part of the Euclidean action with boundary terms is

$$S_{ij}(g,\phi) = S_1(g,\phi) + \delta S_{i,+}(\phi) + \delta S_{j,-}(\phi) \ .$$

Step II: To approximate the measure $\mu_\pm$, we perturb μ_0 by a positive L_1-function

$$\mathrm{d}\mu_{T,L,\pm}(\phi) \ := \ Z(T,L,\pm) \ \mathrm{d}\mu_0(\phi) \ \exp(\ -S_{\pm\pm}(g,\phi) \ )$$

where the constant $Z(T,L,\pm)$ is for normalization. According to J. Glimm, A. Jaffe and T. Spencer [41], the limits

$$\int \mathrm{d}\mu_\pm(\phi) \ \exp(\phi(f)) = \lim_{L\to\infty} \lim_{T\to\infty} \int \mathrm{d}\mu_{T,L,\pm}(\phi) \ \exp(\phi(f)) \ ,$$

which determine the measures $\mu_\pm$, exist for each test function f. Since the different choices for the boundary terms are related by the the map $\phi \mapsto -\phi$, i.e.

$$\phi(g_{+,\pm}) = -\phi(g_{-,\pm}) \ ,$$

the measures μ_+ and μ_- satisfy the relation

$$\mathrm{d}\mu_+(-\phi) = \mathrm{d}\mu_-(\phi) \ .$$

Step III: There are four Hamilton operators $\{H_{ij}(L); i,j = \pm\}$ acting on the Fock space $\mathcal{H}_0$ of the massive free scalar field. They are related to the unnormalized measures

$$\mathrm{d}\mu_{T,L,ij}(\phi) := \mathrm{d}\mu_0(\phi) \ \exp(\ -S_{ij}(g,\phi) \ )$$

by *Nelson's Feynman-Kac formula*:

$$\int \mathrm{d}\mu_{T,L,ij}(\phi) = \langle \Omega_0, \exp(\ -TH_{ij}(L) \ )\Omega_0 \rangle \ .$$

Here Ω_0 is the bare vacuum vector in $\mathcal{H}_0$. Let $\mathfrak{M} : \mathcal{I} \mapsto \mathfrak{M}(\mathcal{I})$ be the net of Cauchy data for the massive free scalar field. The dynamics of the ϕ_2^4-model can be obtained by the prescription

$$\alpha_t(a) := \lim_{L\to\infty} e^{itH_{ij}(L)} \ a \ e^{-itH_{ij}(L)}$$

where the limit is independent of the choice of the boundary conditions. Finally, by using the *Osterwalder-Schrader reconstruction theorem*, two vacuum states $\omega_\pm$ with respect to the dynamics α can be constructed from the measures $\mu_\pm$.

As for the $(\Phi \cdot \Phi)_2^2$-model, the construction of the kink states for the ϕ_2^4-model is placed into the framework of algebraic quantum field theory. In comparison to [31], the main difference is that the cocycle, $\gamma : (t, \mathrm{x}) \mapsto \gamma(t, \mathrm{x})$ (equation (1.3)) is constructed first. According to the results of J. Roberts [64], the total information of a kink sector is contained in its corresponding charge transporters. In other words, the kink sector can be reconstructed if the cocycle γ is given.

The crucial property which allows us to carry through the analysis of [32] is the following: Let $\mathcal{I}$ be a bounded interval, then the observables which are localized in the left complement $\mathcal{I}_{LL}$ of $\mathcal{I}$ are *statistically independent* of those which are localized in the right complement $\mathcal{I}_{RR}$. This means, formulated in the language of operator algebras, that the W*-tensor product

$$\mathfrak{M}(\mathcal{I}_{LL}) \,\overline{\otimes}\, \mathfrak{M}(\mathcal{I}_{RR})$$

is unitarily isomorphic to the W*-algebra

$$\mathfrak{M}(\mathcal{I}_{LL}) \vee \mathfrak{M}(\mathcal{I}_{RR}) \ .$$

Here the W*-algebras $\mathfrak{M}(\mathcal{J})$, which belong to half lines $\mathcal{J}$, are defined by

$$\mathfrak{M}(\mathcal{J}) = C^*(\mathfrak{M}, \mathcal{J})'' \ .$$

The statistical independence for half-line algebras can be proven by using the analysis which has been carried out in [8]. For the convenience of the reader, we give a complete proof in Appendix A. We now describe the main steps of the construction of the kink sectors of the ϕ_2^4-model.

Step 1: According to [15, 20], the statistical independence of $\mathfrak{M}(\mathcal{I}_{LL})$ and $\mathfrak{M}(\mathcal{I}_{RR})$ implies the existence of a unitary operator $u_\mathcal{I}$ which has the following properties: Let a and b be operators which are localized in $\mathcal{I}_{LL}$ and $\mathcal{I}_{RR}$ respectively. Then the relations

$$u_\mathcal{I}\, a\, u_\mathcal{I}^* = a \quad \text{and} \quad u_\mathcal{I}\, b\, u_\mathcal{I}^* = \chi(b) \quad \text{hold.}$$

Here χ is the Bogoliubov automorphism which is induced by the map $\phi \mapsto -\phi$.

Step 2: According to the results of [31], it can be shown that for each t the limit

$$\gamma^0(t) := \lim_{L\to\infty} \exp(\ itH_{++}(L)\ )u_{\mathcal{I}} \exp(\ -itH_{-+}(L)\ )u_{\mathcal{I}}^*$$

exists and that the operator $\gamma^0(t)$ is localized in a sufficiently large interval $\mathcal{I}_t$. Note that the Hamiltonian $H_{-+}(L)$ belongs to the following *interpolating* boundary conditions: The left boundary term is chosen with respect to the boundary conditions for the vacuum ω_- and the right boundary term is chosen with respect to the boundary conditions for the vacuum ω_+. Finally, the charge transporters are given by

$$\gamma(t,\mathbf{x}) := \alpha_{\mathbf{x}}(\gamma^0(t)u_{\mathcal{I}})u_{\mathcal{I}}^*$$

and the corresponding interpolating automorphism ρ can be obtained from γ by the uniform limit

$$\rho(a) = \lim_{\mathbf{x}\to-\infty} \gamma(t,\mathbf{x})\ a\ \gamma(t,\mathbf{x})^* \ .$$

It follows from its construction that ρ acts trivially on the observables which are localized in $\mathcal{I}_{LL}$ and acts as the symmetry χ on those which are localized in $\mathcal{I}_{RR}$. The kink sector and its anti-kink sector are

$$\theta = [\omega_+ \circ \rho] \quad \text{and} \quad \bar{\theta} = [\omega_+ \circ \rho^{-1}] \quad \text{respectively.}$$

This result is in complete analogy to the result for the $(\Phi\cdot\Phi)^2_2$-model, i.e. in both models the same four irreducible sectors appear.

We finally mention some further treatments of kink sectors:

(i) In [31, Chapter 5], the existence of kink states in general $P(\phi)_2$-models is discussed. However, this construction leads only to kink states which interpolate vacua which are connected by the internal symmetry transformation $\phi \mapsto -\phi$. We shall see later that we achieve a generalization of this result.

(ii) In the late 80s, J. Fröhlich and P.A Marchetti developed a quantization of kinks in terms of Euclidean functional integrals which has been applied to several lattice field theories [33, 61, 34].

(iii) Recently, a construction of kink sectors for a lattice version of the XY-model has been carried out by H. Araki [1]. We also refer to [52].

1.1.4 On the Axiomatic Characterization of Kink Sectors

An axiomatic characterization of kink sectors, which is placed into the framework of algebraic quantum field theory, has firstly been given in [31, Chapter 6]. Kink sectors are described there by equivalence classes of interpolating automorphisms. It is required for them to be translationally covariant and to satisfy appropriate interpolation conditions, more precisely, they act trivially on observables which are localized in a left space-like complement of a double cone $\mathcal{O}$ and as an internal symmetry on those which are localized in the right space-like complement of $\mathcal{O}$. We postpone a detailed description of interpolating automorphisms until Chapter 4, we only make a few remarks on this issue here.

(i) To determine a kink sector one has to choose a vacuum sector in addition to a given equivalence class of interpolating automorphisms.

(ii) The kink sectors which can be obtained from interpolating automorphisms have a special property. They interpolate vacua which are related by an internal symmetry transformation. However, there are candidates for models, possessing more than one vacuum sector, in which different vacuum sectors can not be related by an internal symmetry (see Section 1.1.2).

The investigation of massive one-particle states motivates to look for a more general analysis of kink sectors. Massive one-particle states in general $d > 1+1$-dimensional quantum field theories have been studied by D. Buchholz and K. Fredenhagen [10]. One of their main results states that each massive one-particle state is an excitation of a unique vacuum within a space-like cone. In other words, the GNS-representation of a massive one particle state is unitarily equivalent to a vacuum representation when being restricted to an algebra which belongs to the causal complement of a space-like cone. In $1+1$-dimensional quantum field theories, massive one-particle states have kink properties, i.e. *two* vacuum states correspond to each massive one-particle state. This is due to the fact that the space-like complement of a bounded region in two-dimensional Minkowski space has two connected components [22, 23, 71].

In 1994, an adequate representation of kink sectors was developed by K. Fredenhagen [23]. It is given in terms of algebra homomorphisms, called *kink homomorphisms*, which are generalizations of DHR and BF endomorphisms. To point out the relations between kink homomorphisms, interpolating automorphisms and DHR (BF) endomorphisms, we summarize some of the main properties of kink homomorphisms here.

(i) Kink homomorphisms are C*-algebra homomorphisms ρ which map a C*-algebra $\mathfrak{B}_{e_2}$, depending on the right (left) vacuum e_2, into a C*-algebra $\mathfrak{B}_{e_1}$, depending on the left (right) vacuum and e_1. The composition of two kink homomorphisms corresponds to the composition of kink sectors. In the particular case $e = e_1 = e_2$ the algebra $\mathfrak{B}_e$ is mapped into itself and ρ is nothing else but a BF endomorphism [10].

(ii) Conversely, DHR and BF endomorphisms are particular kink homomorphisms. They correspond to trivial kink sectors which interpolate a vacuum sector e with itself.

(iii) To each kink sector, a kink homomorphism can be related, no matter whether the vacua under consideration are related by a symmetry or not. In particular, for each pair which consists of an interpolating automorphism ρ and a vacuum sector e, there exists a unique kink homomorphism ρ_e, which is, however, an extension of ρ.

A general analysis of kink sectors has been carried out in [71]. Here kink homomorphisms are used as a tool to discuss direct sums subobjects and conjugation of kink sectors. A review of these results is given in Chapter 3.

1.2 Overview

The purpose of this section is to give an overview of new results. The reader can find here a summary of Chapter 4 and Chapter 5 in which we establish sufficient conditions for the existence of kink sectors. Moreover, we briefly summarize Chapter 6 where we apply the Haag-Ruelle collision theory to kinks.

1.2.1 A Review of Recent Results

Studying $1+1$-dimensional quantum field theories from an axiomatic point of view shows that kink sectors naturally appear in the theory of superselection sectors. This motivates the following question:

Question: If we consider any quantum field theory model in $1 + 1$ dimensions which possesses more than one vacuum state, which conditions for a pair of vacuum states will be sufficient such that an interpolating kink state can be constructed?

In Chapter 4, we give an answer to this question by developing a construction scheme for kink states which is based on general principles. In order to make the comprehension of the subsequent chapters easier we shall state the main ideas here.

The construction of an interpolating kink state is based on a simple physical idea: Let $\mathfrak{A}$ be a Haag-Kastler net of W*-algebras. Each double cone $\mathcal{O}$ splits our system into two infinitely extended laboratories, namely the laboratory which belongs to the left space-like complement $\mathcal{O}_{LL}$, and the laboratory $\mathcal{O}_{RR}$ which belongs the right space-like complement $\mathcal{O}_{RR}$. In order to prepare an interpolating kink state, we wish to prepare one vacuum state ω_1 in the left laboratory $\mathcal{O}_{LL}$, and another vacuum state ω_2 in the right laboratory $\mathcal{O}_{RR}$. This can only be done if the preparation of ω_1 does not disturb the preparation procedure of ω_2. In other words, the physical operations which take place in the laboratory on the left side $\mathcal{O}_{LL}$ should be statistically independent of those which take place in $\mathcal{O}_{RR}$. Note that in [32] the corresponding condition is needed for the Cauchy data.

Therefore, we require that there exists a vacuum representation π_0 such that the W*-tensor product

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{LL}) \,\overline{\otimes}\, \mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$$

is unitarily isomorphic to the von Neumann algebra

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{LL}) \vee \mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$$

where $\mathfrak{A}_{\pi_0}$ is the net in the vacuum representation π_0. [4] This condition is equivalent to the existence of a type I factor $\mathcal{N}$ which sits between $\mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$ and $\mathfrak{A}_{\pi_0}(\mathcal{O}_{R})$:

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{RR}) \subset \mathcal{N} \subset \mathfrak{A}_{\pi_0}(\mathcal{O}_{R}) \ .$$

Here $\mathcal{O}_R$ is the space-like complement of $\mathcal{O}_{LL}$. In other words, the inclusion

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{RR}) \subset \mathfrak{A}_{\pi_0}(\mathcal{O}_{R}) \tag{1.5}$$

is *split*.

A detailed investigation of standard split inclusions of W*-algebras has been carried out by S. Doplicher and R. Longo [20]. We also refer to the results of D. Buchholz [8], C. D'Antoni and R. Longo [15] and C. D'Antoni and K. Fredenhagen [14].

[4] for an unbounded region $\mathcal{U}$, $\mathfrak{A}_{\pi_0}(\mathcal{U})$ denotes the von Neumann algebra which is generated by all local algebras $\mathfrak{A}_{\pi_0}(\mathcal{O})$ with $\mathcal{O} \subset \mathcal{U}$.

Let ω_1 and ω_2 be two inequivalent vacuum states whose restrictions to each local algebra $\mathfrak{A}(\mathcal{O})$ are normal. Using the isomorphy

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{LL})\,\overline{\otimes}\,\mathfrak{A}_{\pi_0}(\mathcal{O}_{RR}) \cong \mathfrak{A}_{\pi_0}(\mathcal{O}_{LL}) \vee \mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$$

we conclude that the map

$$ab \longmapsto \omega_1(a)\omega_2(b) \;\; , a \text{ is localized in } \mathcal{O}_{LL} \text{ and } b \text{ is localized in } \mathcal{O}_{RR},$$

defines a state of the algebra $C^*(\mathfrak{A}, \mathcal{O}_{LL} \cup \mathcal{O}_{RR})$ which, by the Hahn-Banach theorem, can be extended to a state ω of the C*-algebra of all observables. The state ω interpolates the vacua ω_1 and ω_2 correctly, but for an explicit construction of an interpolating state which satisfies the *Borchers criterion*, some technical difficulties have to be overcome.

The condition that the inclusion (1.5) is split is sufficient to develop a general construction scheme for interpolating kink states. We shall give a brief description of it here.

Step 1: We consider the W*-tensor product of the net $\mathfrak{A}$ with itself:

$$\mathfrak{A}\,\overline{\otimes}\,\mathfrak{A} : \mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})\,\overline{\otimes}\,\mathfrak{A}(\mathcal{O})$$

The map α_F which is given by interchanging the tensor factors,

$$\alpha_F : a_1 \otimes a_2 \longmapsto a_2 \otimes a_1$$

is called the *flip automorphism*. Since the inclusion (1.5) is split, the flip automorphism is unitarily implemented on $\mathfrak{A}_{\pi_0}\,\overline{\otimes}\,\mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$ by a unitary operator θ which is contained in $\mathfrak{A}_{\pi_0}\,\overline{\otimes}\,\mathfrak{A}_{\pi_0}(\mathcal{O}_R)$ [15]. The adjoint action of θ induces an automorphism

$$\beta := (\pi_0 \otimes \pi_0)^{-1} \circ \mathrm{Ad}(\theta) \circ \pi_0 \otimes \pi_0$$

which maps local algebras into local algebras. For each observable a which is localized in the left space-like complement of $\mathcal{O}$ we have $\beta(a) = a$, and for each observable b which is localized in the right space-like complement of $\mathcal{O}$ we have $\beta(b) = \alpha_F(b)$. Note that β may depend on the choice of the vacuum representation π_0.

Step 2: It is obvious that the state

$$\omega := \omega_1 \otimes \omega_2 \circ \beta|_{C^*(\mathfrak{A})\otimes \mathbf{1}}$$

interpolates ω_1 and ω_2. Let π_1 and π_2 be the GNS-representations of ω_1 and ω_2 respectively. Then the GNS-representation

$$\pi = \pi_1 \otimes \pi_2 \circ \beta|_{C^*(\mathfrak{A}\otimes \mathbf{1})}$$

of ω is translationally covariant because the automorphism

$$\alpha_x \circ \beta \circ \alpha_{-x} \circ \beta$$

is implemented by a cocycle $\gamma(x)$ of *local* operators in $C^*(\mathfrak{A})$. The positivity of the energy can be proven by showing the additivity of the energy-momentum spectrum for automorphisms like β. This together implies that ω is an interpolating kink state.

In comparison to [31, 32], our construction scheme has the following advantages:

- $\oplus$ It is independent of specific details of the considered model because the split property (1.5), which is the crucial condition for applying the construction scheme, can be motivated by general principles.

- $\oplus$ It can be applied to pairs of vacuum sectors which are not related by a symmetry transformation.

Unfortunately, there is one disadvantage which is the price we have to pay for using a model independent analysis.

- $\ominus$ The split property for wedge algebras (1.5) has to be proven for the vacuum states of the model under consideration if we want to apply our construction scheme to it. It is believed that the vacuum states of the $P(\phi)_2$- and Yukawa$_2$ models fulfill this condition, but a rigorous proof is only known for the massive free Bose and Fermi field [14, 8, 78].

In Chapter 5, we investigate an alternative construction of kink states which can directly be applied to models. It is convenient to formulate our setup in the time slice formulation of a quantum field theory. We fix a *space-like plane* $\Sigma \subset \mathbb{R}^2$ and consider a net of Cauchy data $\mathfrak{M}$ which is faithfully represented on a Hilbert space $\mathcal{H}_0$. The Cauchy data of the $P(\phi)_2$- and the Yukawa$_2$ model are given by the nets of the corresponding free fields at time $t = 0$. For these Cauchy data it can be proven that the inclusion

$$\mathfrak{M}(\mathcal{I}_{RR}) \subset \mathfrak{M}(\mathcal{I}_R)$$

is split [8, 78, 72].

Let us briefly explain how kink states can be constructed if the following conditions are assumed:

(i) The dynamics of the model satisfies an appropriate *extendibility condition* which we shall explain later.

(ii) The vacuum states are local Fock states which is automatically satisfied for $P(\phi)_2$ and Yukawa$_2$ models [37, 75].

Step 1': We consider the twofold net

$$\mathfrak{M} \mathbin{\overline{\otimes}} \mathfrak{M} : \mathcal{I} \longmapsto \mathfrak{M}(\mathcal{I}) \mathbin{\overline{\otimes}} \mathfrak{M}(\mathcal{I}) \ .$$

Like in *Step 1* of our previous construction scheme, the split property implies that on $\mathfrak{M}(\mathcal{I}_{RR}) \mathbin{\overline{\otimes}} \mathfrak{M}(\mathcal{I}_{RR})$, the flip automorphism is implemented by a unitary operator $\theta_{\mathcal{I}}$ [15]. The adjoint action of $\theta_{\mathcal{I}}$ is an automorphism $\beta^{\mathcal{I}}$ which has the following properties:

(i) The automorphism $\beta^{\mathcal{I}}$ acts trivially on observables which are localized in the left complement of $\mathcal{I}$ and it acts like the flip on observables which are localized in the right complement of $\mathcal{I}$.

(ii) The automorphism $\beta^{\mathcal{I}}$ maps local algebras into local algebras.

Note that the automorphism $\beta^{\mathcal{I}}$ does not depend on the dynamics α.

Step 2': Let ω_1, ω_2 be two vacuum states with respect to a given dynamics α. The state

$$\omega := \omega_1 \otimes \omega_2 \circ \beta^{\mathcal{I}}|_{C^*(\mathfrak{M}) \otimes \mathbb{C}\mathbf{1}}$$

interpolates the vacua ω_1 and ω_2. Moreover, it is covariant under spatial translations since for each x the operator

$$\gamma(0, \mathrm{x}) = (\alpha_{\mathbf{x}} \otimes \alpha_{\mathbf{x}})(\theta_{\mathcal{I}})\theta_{\mathcal{I}}$$

is localized in a sufficiently large bounded interval. Indeed, the unitary operators

$$U(0,\mathrm{x}) := (U_1(0,\mathrm{x}) \otimes U_2(0,\mathrm{x}))\,(\pi_1 \otimes \pi_2)(\gamma(0,-\mathrm{x}))$$

implement the spatial translations in the GNS-representation of ω where U_1 and U_2 implement the translations in the GNS-representations π_1, π_2 of ω_1 and ω_2 respectively.

Step 3': It remains to be proven that ω is translationally covariant with respect to the dynamics α. For this purpose, we wish to construct a cocycle $\gamma(0,t)$ such that the operators

$$U(t,0) := (U_1(t,0) \otimes U_2(t,0))\,(\pi_1 \otimes \pi_2)(\gamma(-t,0))$$

implement the dynamics α in the GNS-representation of ω. The operator

$$\gamma(t,0) := (\alpha_t \otimes \alpha_t)(\theta_{\mathcal{I}})\theta_{\mathcal{I}}$$

is a formal solution. Unfortunately, the flip implementer $\theta_{\mathcal{I}}$ is *not* contained in any local algebra and the term $(\alpha_t \otimes \alpha_t)(\theta_{\mathcal{I}})$ has no mathematical meaning unless α is the free dynamics. However, it can be given a meaning in some cases. We shall see that for an interacting dynamics there exists a suitable cocycle of the operators $\gamma(t,0)$ such that $\gamma(t,0)$ is localized in a bounded interval whose size depends linearly on $|t|$.

In order to formulate a sufficient condition for the existence of $\gamma(t,0)$, we construct an extension of the net $\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}$. We define $\hat{\mathfrak{M}}(\mathcal{I})$ to be the von Neumann algebra which is generated by $\mathfrak{M}(\mathcal{I}) \,\overline{\otimes}\, \mathfrak{M}(\mathcal{I})$ and the operator $\theta_{\mathcal{I}}$. The net

$$\hat{\mathfrak{M}} : \mathcal{I} \longmapsto \hat{\mathfrak{M}}(\mathcal{I})$$

is an extension of $\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}$ which does not fulfill locality. This is due to the non-trivial implementation properties of $\theta_{\mathcal{I}}$. We shall call a dynamics α *extendible* if there exists a dynamics $\hat{\alpha}$ of $\hat{\mathfrak{M}}$ which is an extension of $\alpha \otimes \alpha$. Indeed,

$$t \longmapsto \gamma(t,0) := \hat{\alpha}_t(\theta_{\mathcal{I}})\theta_{\mathcal{I}}$$

is a cocycle which has the desired properties. Finally, we conclude like in Step 3 of our previous construction scheme that the state

$$\omega := \omega_1 \otimes \omega_2 \circ \beta^{\mathcal{I}}|_{C^*(\mathfrak{M})\otimes\mathbb{C}1}$$

is a kink state where ω_1, ω_2 are vacuum states with respect to the dynamics α.

Since the extendibility condition is very technical one might bother that it is only fulfilled for few exceptional cases. Fortunately, this is not true. There is a large class of quantum field theory models whose dynamics are extendible. In Chapter 5, we shall prove that the extendibility holds for the following models:

(i) $P(\phi)_2$-models.

(ii) Yukawa$_2$ models.

(iii) Special types of Wess-Zumino models.

Note that a Dirac spinor field contributes to the field content of the Yukawa$_2$ and Wess-Zumino models, and the nets of Cauchy data fulfill *twisted duality* instead of Haag duality [78]. According to recent results which have been established by M. Müger [62], our results remain true for these cases also.

Wess-Zumino models have been studied in several papers. We refer to the work of A. Jaffe, A. Lesniewski, J. Weitsman and S. Janowsky [47, 50, 51, 48, 49]. It has been proven in [48] that some Wess-Zumino models possess more than one vacuum sector. An application of our construction scheme proves the existence of kink states for these models.

1.2.2 Some Consequences and Miscellaneous Results

As one can see from the previous section, our construction scheme leads to kink states of the form:

$$\omega = \omega_1 \otimes \omega_2 \circ \beta|_{C^*(\mathfrak{A})\otimes \mathbf{1}} \tag{1.6}$$

where β is a suitable interpolating automorphism and ω_1, ω_2 are vacuum states. The homomorphism

$$\Delta := \beta|_{C^*(\mathfrak{A})\otimes \mathbf{1}} : C^*(\mathfrak{A}) \longrightarrow C^*(\mathfrak{A} \mathbin{\overline{\otimes}} \mathfrak{A})$$

may be interpreted as a co-product which induces a product on the set of locally normal linear functionals of $C^*(\mathfrak{A})$. We shall call it *interpolating product*. It can be used to compute the composition of kink sectors in a very convenient way which we illustrate by the following example: Let ω_1, ω_2 and ω_3 be vacuum states. The composition of the kink sectors $\theta_1 = [\omega_1 \otimes \omega_2 \circ \Delta]$ and $\theta_2 = [\omega_2 \otimes \omega_3 \circ \Delta]$ can be defined since the right vacuum of θ_1 coincides with the left vacuum of θ_2. We shall prove in Section 4.2.2 that the sector

$$[(\omega_1 \otimes \omega_2 \otimes \omega_3) \circ (\mathrm{id} \otimes \Delta) \circ \Delta]$$

is nothing else but the product sector $\theta_1\theta_2$. We refer to Chapter 3 and [22, 23, 70] where a precise definition of the composition of kink sectors is given.

It is not hard to show that the kink state, which is given by equation (1.6), is not a pure state. Its GNS-representation is

$$\pi = \pi_1 \otimes \pi_2 \circ \Delta$$

where π_1 and π_2 are the GNS-representations of ω_1 and ω_2 respectively. We are interested in irreducible sub-representations of π. It is not obvious whether π possesses any irreducible subrepresentation which will not be the case if the von Neumann algebra $\pi(C^*(\mathfrak{A}))''$ is, for example, type III.

To show reducibility we are faced with the problem to prove that the von Neumann algebra $\pi(C^*(\mathfrak{A}))''$ is type I. Unfortunately, the investigation of the type of the von Neumann algebra $\pi(C^*(\mathfrak{A}))''$ turns out to be difficult.

On the other hand, there are kink sectors induced by interpolating automorphisms for which the algebra $\pi(C^*(\mathfrak{A}))''$ is, in fact, type I. Such examples arise naturally in the investigation of the question how *large* the class of kink sectors is which can be obtained by applying our construction scheme.

We shall prove in Section 4.4.1 that this class contains *simple* kink sectors which are characterized by fulfilling wedge duality. Let π be a kink representation which belongs to a simple kink sector and let us assume that the split property holds for wedge algebras in the representation π. Then there exists an appropriate interpolating automorphism β such that the representation

$$\hat{\pi} := \pi_1 \otimes \pi_2 \circ \beta|_{C^*(\mathfrak{A})\otimes \mathbf{1}}$$

is unitarily equivalent to the countably infinite multiple of π; as a consequence, the algebra $\hat{\pi}(C^*(\mathfrak{A}))''$ is type I.

In Section 4.4.2, we shall establish a lower bound for the mass of those kinks which can be obtained via our construction scheme.

1.2.3 Kink Fields and Collision Theory

In Chapter 6, the Haag-Ruelle collision theory [44, 68] is applied to kink sectors. In comparison to [70], our analysis leads to an improved treatment of kink scattering. We refer to the work of S. Doplicher, R. Haag and J. Roberts [18, 19] and

D. Buchholz and K. Fredenhagen [10] where the Haag-Ruelle collision theory has been applied to DHR and BF superselection sectors in four space-time dimensions. Scattering of plektons (particles with braid group statistics in three space-time dimensions) has been investigated by K. Fredenhagen, M.R. Garberdiel und S.M. Rüger [28].

For the construction of kink collision states, it is useful to develop an adequate *field bundle formalism* [18, 19, 10] which can be applied to kink sectors. We shall see that such a formalism needs rather technical methods since, in contrast to DHR and BF sectors, kink sectors can not be composed arbitrarily.

To illustrate the concept of kink fields, we consider the following situation: The Hilbert space of all physical states splits into a direct sum of Hilbert spaces such that each of them carries an irreducible representation of the observable algebra:

$$\mathcal{H} = \Big(\bigoplus_e \mathcal{H}_e\Big) \oplus \Big(\bigoplus_\theta \mathcal{H}_\theta\Big) \ .$$

The first sum runs over all vacuum sectors e whereas the second sum is taken over all irreducible kink sectors θ. Given two irreducible kink sectors θ_1 and θ_2 such that the left vacuum of θ_2 coincides with the right vacuum of θ_1, then their composition $\theta_1\theta_2$ is well defined. We assume that the product sector $\theta_1\theta_2$ can be decomposed into a finite direct sum of irreducible sectors:

$$\theta_1\theta_2 = \bigoplus_\theta n^\theta_{\theta_1\theta_2}\ \theta \ .$$

Moreover, to be not too complicated, we let the fusion coefficients $n^\theta_{\theta_1\theta_2}$ be either one or zero. The Hilbert space which belongs to the sector $\theta_1\theta_2$ is

$$\mathcal{H}_{\theta_1\theta_2} := \bigoplus_\theta n^\theta_{\theta_1\theta_2}\ \mathcal{H}_\theta \ \subset\ \mathcal{H} \ .$$

Within this situation, there are *two* different types of charge carrying *field operators*. This kind of *doubling* appears here since a kink can be interpreted either as an excitation of its left or as an excitation of its right vacuum. We now describe the two different types of kink fields, carrying a kink charge θ, more precisely.

(1) Kink fields with *orientation* $+$ are linear operators $(\mathbf{a}^+(\theta), D^+(\theta))$ with domains

$$D^+(\theta) := \bigoplus_{\theta_1:\theta\theta_1 \text{ well defined}} \mathcal{H}_{\theta_1}$$

where $\mathbf{a}^+(\theta)$ maps $\mathcal{H}_{\theta_1}$ into $\mathcal{H}_{\theta\theta_1}$.

(2) Kink fields with *orientation* – are linear operators $(\mathbf{a}^-(\theta), D^-(\theta))$ with domains

$$D^-(\theta) := \bigoplus_{\theta_2 : \theta_2\theta \text{ well defined}} \mathcal{H}_{\theta_2}$$

where $\mathbf{a}^-(\theta)$ maps $\mathcal{H}_{\theta_2}$ into $\mathcal{H}_{\theta_2\theta}$.

As we shall see later, the orientation is related to the *localization properties* of a kink field: The localization region of a kink field with positive orientation is a right wedge region and vice versa. Such a structure, which is the main reason for technical difficulties, does not arise in the DHR and BF case where any two sectors can be composed in any order.

In order to give an idea how kink fields can be multiplied, we discuss a simple example. Let us consider five kink sectors $\theta_2, \cdots, \theta_0, \cdots, \theta_{-2}$ such that the product sector $\theta = \theta_2\theta_1\theta_0\theta_{-1}\theta_{-2}$ is well defined. There are six ways in which the field operators $\mathbf{a}^\pm(\theta_{\pm 2}), \mathbf{a}^\pm(\theta_{\pm 1})$ can be composed on $\mathcal{H}_{\theta_0}$:

$$\mathbf{a}_{(++--)} = \mathbf{a}^+(\theta_2)\mathbf{a}^+(\theta_1)\mathbf{a}^-(\theta_{-2})\mathbf{a}^-(\theta_{-1})$$

$$\mathbf{a}_{(+-+-)} = \mathbf{a}^+(\theta_2)\mathbf{a}^-(\theta_{-2})\mathbf{a}^+(\theta_1)\mathbf{a}^-(\theta_{-1})$$

$$\mathbf{a}_{(+--+)} = \mathbf{a}^+(\theta_2)\mathbf{a}^-(\theta_{-2})\mathbf{a}^-(\theta_{-1})\mathbf{a}^+(\theta_1)$$

$$\mathbf{a}_{(-++-)} = \mathbf{a}^-(\theta_{-2})\mathbf{a}^+(\theta_2)\mathbf{a}^+(\theta_1)\mathbf{a}^-(\theta_{-1})$$

$$\mathbf{a}_{(-+-+)} = \mathbf{a}^-(\theta_{-2})\mathbf{a}^+(\theta_2)\mathbf{a}^-(\theta_{-1})\mathbf{a}^+(\theta_1)$$

$$\mathbf{a}_{(--++)} = \mathbf{a}^-(\theta_{-2})\mathbf{a}^-(\theta_{-1})\mathbf{a}^+(\theta_2)\mathbf{a}^+(\theta_1)$$

Each of the composed operators $\mathbf{a}_{(\cdots)}$ is a well defined linear map from $\mathcal{H}_{\theta_0}$ into $\mathcal{H}_\theta$. We see from this example that factors of a product of kink fields, supposed to be defined, can be exchanged if their orientations are different. If in addition the localizing regions of two exchangeable field operators are space-like separated, then the exchange of them is induced by a unitary operator, called *quasi-statistics operator*. For example, assuming that $\mathbf{a}^+(\theta_2)$ and $\mathbf{a}^-(\theta_{-2})$ are localized in space-like separated regions, then there exists a unitary operator $\epsilon_{(-+-+|--++)}$ such that

$$\mathbf{a}_{(-+-+)}\psi = \epsilon_{(-+-+|--++)}\, \mathbf{a}_{(--++)}\psi$$

for each $\psi \in \mathcal{H}_{\theta_0}$. These operators are the analogues of the statistics operators within the DHR and BF situation. The crucial difference is that the DHR and BF statistics operators are related to arbitrary permutations of factors in a product of field operators, whereas the quasi-statistics operators are related to pairs of finite $\mathbb{Z}_2 = \{+,-\}$-valued sequences, in our example it is the pair $(-+-+|--++)$. To point out these differences, we use the word *quasi*. A detailed analysis of quasi-statistics relations between kink field operators is carried out in Section 6.2. We also refer to [70].

The Hilbert space $\mathcal{H}$ carries a unitary and strongly continuous representation of the translation group U which implements the translation of observables and which fulfills the spectrum condition. We assume that the kink superselection sectors under consideration are massive one-particle states which means that the spectrum of the translation group being restricted to an irreducible kink sector

$$U_\theta(x) = U(x)|_{\mathcal{H}_\theta}$$

consists of an isolated mass shell $H_m = \{p|p^2 = m^2; p_0 \geq 0\}$ and a subset of the set $C_{\mu+m} = \{p|p^2 \geq (m+\mu)^2; p_0 \geq 0\}$. The one-particle subspace, the space of all vectors in $\mathcal{H}_\theta$ whose spectral support is contained in H_m, will be denoted by $\mathcal{H}^1_\theta$.

Following the DHR and BF analysis, it is possible to construct one-kink operators, i.e. kink field operators which map a particular vacuum vector $\Omega_e \in \mathcal{H}_e$ into a one-particle space $\mathcal{H}^1_\theta$. If the kink field operator has orientation $+$, then the vacuum e is the left vacuum of θ, otherwise, if the orientation is $-$, then it is the right vacuum of θ.

The one-kink creation operators $\mathbf{a}(f,t)$ depend on a *energy-momentum distribution* f in which the *velocity* of the created kink is encoded, and it depends in addition on a time parameter t which compares, in a certain sense, the *free motion* of a kink with its motion when *interaction* is present. Let us suppose there is one kink alone in the world, then there is nothing else which can interact with it. Therefore, for a one-kink state there is no difference between the free and the interacting case. This means that applying $\mathbf{a}(f,t)$ to the corresponding vacuum vector Ω_e, the vector

$$\psi(f) = \mathbf{a}(f,t)\Omega_e$$

is independent of t.

A multi-kink state can be obtained by an application of multiple one-kink creation operators to a vacuum Ω_e

$$\psi(t) = \mathbf{a}(f_1,t)\cdots\mathbf{a}(f_n,t)\Omega_e$$

as long as we respect the composition rules for kink fields. Note that the expression $\psi(t)$ above may contain kink fields with different orientation. The vector $\psi(t)$ describes a configuration of n kinks at time t after scattering.

For large t we expect that the kinks decouple and behave like n non-interacting freely moving particles. This is indeed the case for a special class of kink configurations. It seems to be necessary to assume that

(i) the interpolated vacua are related by an internal symmetry which gives us the possibility to assign to each one-kink creation operator $\mathbf{a}(f_j, t)$ a bounded interpolation region $\mathcal{O}_j(t)$, which is, roughly speaking, the region in which the interpolation of the vacua takes place.

(ii) Another crucial property is the *velocity ordering* in order to ensure that the space-like distances between the interpolation regions increase to infinity when t tends also to infinity.

In Chapter 6, we shall present a detailed discussion on the assumptions which have to be made such that the strong limit

$$\psi^{\text{out}} = s - \lim_{t\to\infty} \mathbf{a}(f_1, t) \cdots \mathbf{a}(f_n, t)\Omega_e$$

exists and depends only on the one-particle vectors $\psi_j = \mathbf{a}(f_j, t)\Omega_{e_j}$. Here Ω_{e_j} are suitable vacuum vectors. Furthermore, we shall prove that the norm of the scattering state ψ^{out} is

$$||\psi^{\text{out}}|| = \prod_{j=1}^{n} ||\psi_j|| \ .$$

This implies that the assignment

$$\bigotimes_{j=1}^{n} \mathcal{H}^1_{\theta_j} \longrightarrow \mathcal{H} \ : \ \bigotimes_{j=1}^{n} \psi_j \longmapsto \psi^{\text{out}}$$

is an isometry where $\mathcal{H}^1_{\theta_j}$ are suitable one-particle spaces. As expected, the vector ψ^{out} represents a configuration of n non-interacting freely moving kinks. The asymptotic in-states can be obtained analogously.

2 Axiomatic Quantum Field Theory and Models in $1+1$-Dimensions

2.1 The Time-Slice Formulation of a Quantum Field Theory

As already mentioned in the introduction, for the analysis of quantum field theory models it is convenient to work with the time-slice formulation. In order to keep the present chapter self-contained, we repeat the main aspects of it. In addition to that, we describe how a Haag-Kastler net can be constructed from the Cauchy data and a given dynamics.

2.1.1 Cauchy Data and Dynamics of a Quantum Field Theory

Let us consider *Cauchy data* of a quantum field theory which are faithfully represented on a Hilbert space $\mathcal{H}_0$, i.e. the Cauchy data are given by a net of von Neumann algebras

$$\mathfrak{M} := \{\mathfrak{M}(\mathcal{I}) \subset \mathfrak{B}(\mathcal{H}_0);\ \mathcal{I} \text{ is a bounded interval in } \Sigma\}$$

This net satisfy the conditions:

(1) Isotony: If $\mathcal{I}_1 \subset \mathcal{I}_2$, then $\mathfrak{M}(\mathcal{I}_1) \subset \mathfrak{M}(\mathcal{I}_2)$.

(2) Locality: If $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, then $\mathfrak{M}(\mathcal{I}_1) \subset \mathfrak{M}(\mathcal{I}_2)'$.

(3) The group of spatial translations in $\Sigma \cong \mathbb{R}$ is represented unitarily and strongly continuous on $\mathcal{H}_0$

$$U : \mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H}_0) \ ; \ \mathbf{x} \longmapsto U(\mathbf{x})$$

such that the automorphism $\alpha_{\mathbf{x}} := \mathrm{Ad}(U(\mathbf{x}))$ maps $\mathfrak{M}(\mathcal{I})$ onto $\mathfrak{M}(\mathcal{I}+\mathbf{x})$.

The time evolution is described by the notion of dynamics. For convenience, we repeat its definition here.

Definiton 2.1.1 : A one-parameter group of automorphisms $\alpha = \{\alpha_t \in \mathrm{Aut}(\mathfrak{M}); t \in \mathbb{R}\}$ is called a *dynamics* of the net $\mathfrak{M}$ if the following conditions are fulfilled:

(1) The automorphism group α has propagation speed $\mathrm{ps}(\alpha) \leq 1$, where $\mathrm{ps}(\alpha)$ is defined by:

$$\mathrm{ps}(\alpha) := \inf\{\beta' | \alpha_t \mathfrak{M}(\mathcal{I}) \subset \mathfrak{M}(\mathcal{I}_{\beta'|t|}); \forall t, \mathcal{I}\} \ .$$

Here $\mathcal{I}_s := \mathcal{I} + (-s, s)$ denotes the interval, enlarged by $s > 0$.

(2) The automorphisms $\{\alpha_t \in \mathrm{Aut}(\mathfrak{M}); t \in \mathbb{R}\}$ commute with the automorphism group of spatial translations $\{\alpha_{\mathbf{x}} \in \mathrm{Aut}(\mathfrak{M}); \mathrm{x} \in \mathbb{R}\}$, i.e.:

$$\alpha_t \circ \alpha_{\mathbf{x}} = \alpha_{\mathbf{x}} \circ \alpha_t \quad ; \quad \forall \mathbf{x}, t \ .$$

The set of all dynamics of $\mathfrak{M}$ is denoted by $\mathrm{dyn}(\mathfrak{M})$.

Note that in many cases it is possible to choose the same net of Cauchy data for different models. For instance, the Cauchy data of the $P(\Phi)_2$-models can be chosen by the time-zero algebras of the massive free Bose field [37].

2.1.2 Haag-Kastler Nets for Cauchy Data

To distinguish different theories, we have to compare different dynamics. For this purpose, we shall construct a *universal Haag-Kastler net* with respect to a given net $\mathfrak{M}$ of Cauchy data.

Let $U(\mathfrak{M})$ be the group of unitary operators in $C^*(\mathfrak{M})$ and we define $\mathfrak{G}(\mathbb{R}, \mathfrak{M})$ to be the group which is generated by the set

$$\{(t,u) |\ t \in \mathbb{R} \text{ and } u \in U(\mathfrak{M})\ \}$$

modulo the following relations:

(1) We require for each $u_1, u_2 \in U(\mathfrak{M})$ and for each $t \in \mathbb{R}$:

$$(t, u_1)(t, u_2) = (t, u_1 u_2) \ \text{ and } \ (t, \mathbf{1}) = \mathbf{1} \ .$$

(2) If $u_1 \in \mathfrak{M}(\mathcal{I}_1)$ and $u_2 \in \mathfrak{M}(\mathcal{I}_2)$ such that $\mathcal{I}_1 \subset (\mathcal{I}_2)^c_{|t|}$, then we require for each $t_1 \in \mathbb{R}$:

$$(t_1 + t, u_1)(t_1, u_2) = (t_1, u_2)(t_1 + t, u_1) \ .$$

We conclude from relation *(1)* that (t, u) is the inverse of (t, u^*).

We see from the construction above that a localization region in $\mathbb{R} \times \Sigma$ can be assigned to each element in $\mathfrak{G}(\mathbb{R}, \mathfrak{M})$: A generator (t, u), $u \in \mathfrak{M}(\mathcal{I})$ is localized in $\mathcal{O} \subset \mathbb{R} \times \Sigma$ if $\{t\} \times \mathcal{I} \subset \mathcal{O}$. We denote by $\mathfrak{G}(\mathcal{O})$ the subgroup of $\mathfrak{G}(\mathbb{R}, \mathfrak{M})$ which is generated by the elements which are localized in the double cone $\mathcal{O}$. We easily observe that relation *(2)* implies that group elements commute if they are localized in space-like separated regions.

The translation group in $\mathbb{R}^2$ is naturally represented by group automorphisms of $\mathfrak{G}(\mathbb{R}, \mathfrak{M})$. They are defined by

$$(t, \mathbf{x}) \longmapsto \beta_{(t,\mathbf{x})}(t_1, u) := (t + t_1, \alpha_{\mathbf{x}} u)$$

and the subgroup $\mathfrak{G}(\mathcal{O})$ is mapped onto $\mathfrak{G}(\mathcal{O} + (t, \mathbf{x}))$ by $\beta_{(t,\mathbf{x})}$.

In order to construct the universal Haag-Kastler net, we build the group C*-algebra $\mathfrak{B}(\mathcal{O})$ with respect to $\mathfrak{G}(\mathcal{O})$ (compare also [56]). For convenience, we shall describe the construction of $\mathfrak{B}(\mathcal{O})$ briefly.

In the first step we build the *-algebra $\mathfrak{B}_0(\mathcal{O})$ which is generated by all complex valued functions a on $\mathfrak{G}(\mathcal{O})$, such that

$$a(u) = 0 \quad \text{for almost each } u \in \mathfrak{G}(\mathcal{O}) \ .$$

We write such a function symbolically as a formal sum

$$a = \sum_u a(u)\, u \ .$$

The product and the *-relation are given as follows:

$$\begin{aligned} ab = \sum_u a(u)\, u \cdot \sum_{u'} b(u')\, u' &= \sum_{u'} \Big(\sum_u a(u) b(u^{-1} u') \Big) u' \\ a^* &= \sum_u \bar{a}(u^{-1})\, u \end{aligned}$$

It is well known, that the algebra $\mathfrak{B}_0(\mathcal{O})$ has a C*-norm which is given by

$$||a|| := \sup_\pi ||\pi(a)||_\pi$$

where the supremum is taken over all *-representations π of $\mathfrak{B}_0(\mathcal{O})$. Finally, we define $\mathfrak{B}(\mathcal{O})$ as the closure of $\mathfrak{B}_0(\mathcal{O})$ with respect to this norm.

By construction, the group isomorphisms $\beta_{(t,\mathrm{x})}$ induce a representation of the translation group by automorphisms of the C*-inductive limit $C^*(\mathfrak{B})$ of the net

$$\mathfrak{B} : \mathcal{O} \longmapsto \mathfrak{B}(\mathcal{O})$$

and we conclude that $\mathfrak{B}$ is a translationally covariant Haag-Kastler net.

Proposition 2.1.2 : *Each dynamics $\alpha \in \mathrm{dyn}(\mathfrak{M})$ induces a C*-homomorphism*

$$\iota_\alpha : C^*(\mathfrak{B}) \longrightarrow C^*(\mathfrak{M})$$

such that for each $(t, \mathrm{x}) \in \mathbb{R}^2$,

$$\iota_\alpha \circ \beta_{(t,\mathrm{x})} = \alpha_{(t,\mathrm{x})} \circ \iota_\alpha \ .$$

In particular,

$$\mathfrak{A}_\alpha : \mathcal{O} \longmapsto \mathfrak{A}_\alpha(\mathcal{O}) := \iota_\alpha(\mathfrak{B}(\mathcal{O}))''$$

is a translationally covariant Haag-Kastler net.

Proof. Let α be a dynamics of $\mathfrak{M}$, then we conclude from $\mathrm{ps}(\alpha) \leq 1$ that the prescription

$$(t, u) \longmapsto \alpha_t u$$

defines a C*-homomorphism

$$\iota_\alpha : C^*(\mathfrak{B}) \longrightarrow C^*(\mathfrak{M}) \ .$$

In particular, ι_α is a representation of $\mathfrak{B}$ on the Hilbert space $\mathcal{H}_0$. These statements can be verified by using the following relations:

(a)
$$\iota_\alpha((t, u_1)(t, u_2)) = \alpha_t u_1 \alpha_t u_2 = \alpha_t(u_1 u_2) = \iota_\alpha(t, u_1 u_2)$$

(b) If (t_1, u_1) and $(t_1 + t, u_2)$ are localized in space-like separated regions, then we conclude from $\mathrm{ps}(\alpha) \leq 1$:
$$[\iota_\alpha(t_1, u_1), \iota_\alpha(t_1 + t, u_2)] = \alpha_{t_1}[u_1, \alpha_t u_2] = 0$$

(c)

$$\iota_\alpha(\beta_{(t,\mathrm{x})}(t_1,u)) = \iota_\alpha(t+t_1,\alpha_{\mathrm{x}}u) = \alpha_{(t,\mathrm{x})}\alpha_{t_1}u$$

□

In general, we expect that for a given dynamics α the representation ι_α is not faithful. Hence each dynamics defines a two-sided ideal

$$J(\alpha) := \iota_\alpha^{-1}(0) \in C^*(\mathfrak{B})$$

in $C^*(\mathfrak{B})$ which we call the *dynamical ideal* with respect to α and the algebras

$$\mathfrak{B}(\mathcal{O})/J(\alpha) \cong \mathfrak{A}_\alpha(\mathcal{O})$$

may depend on the dynamics α. Indeed, if $\mathcal{O}$ is a double cone whose base is *not contained* in Σ, then, in general, for different dynamics α_1, α_2 the algebras $\mathfrak{A}_{\alpha_1}(\mathcal{O})$ and $\mathfrak{A}_{\alpha_2}(\mathcal{O})$ are different. On the other hand, if the base of $\mathcal{O}$ is contained in Σ, then we conclude from the fact that the dynamics α has finite propagation speed and from Proposition 2.1.2:

Corollary 2.1.3 : *If $\mathcal{I} \subset \Sigma$ is the base of the double cone $\mathcal{O}$, then the algebra $\mathfrak{A}_\alpha(\mathcal{O})$ is independent of α. In particular,*

$$C^*(\mathfrak{M}) = C^*(\mathfrak{A}_\alpha) \quad .$$

2.2 Quantum Field Theory Models in $1+1$-Dimensions

A few remarks on non-trivial quantum field theory models are given in this section. We give a short description of the Cauchy data and the dynamics of $P(\phi)_2$- and Yukawa$_2$ models.

2.2.1 $P(\phi)_2$-Models

We denote by $\mathcal{H}_s$ the symmetrized Fock space over the Hilbert space $L_2(\mathbb{R})$ of complex-valued and square integrable functions, i.e.

$$\mathcal{H}_s := \bigoplus_{n=0}^{\infty} L_2(\mathbb{R})^{\otimes_s n}$$

where s stands for the symmetrization of the tensor product. The massive free Bose field ϕ and its canonically conjugate π act as operator valued and tempered distributions on $\mathcal{H}_s$, fulfilling canonical commutation relations (CCR). The net of Cauchy data of the $P(\phi)_2$ models is given by the CCR-algebras

$$\mathfrak{M}_s : \mathcal{I} \longmapsto \mathfrak{M}_s(\mathcal{I}) := \left\{ e^{i(\phi(f_1)+\pi(f_2))} \,\middle|\, f_j \in S_{\mathbb{R}}(\mathbb{R}) \ ; \ \mathrm{supp}(f_j) \subset \mathcal{I} \right\}'' \ .$$

According to J.Glimm and A.Jaffe [37], the Wick monomials $:\phi^n:$ of the time zero field exist. They are well defined operator valued distributions. The interacting part of a $P(\phi)_2$ Hamiltonian, regularized by an IR-cutoff $c > 0$, is given by

$$H_1(c) := \int \mathrm{d}\mathbf{x} : P(\phi(\mathbf{x})) : \ \chi_c(\mathbf{x})$$

where P is a polynomial of even degree $\deg(P) = 2n$ and χ_c is a smooth test function with the property:

$$\chi_c(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in (-c, c) \\ 0 & \text{if } \mathbf{x} \in (-c-\epsilon, c+\epsilon)\backslash\mathbb{R} \end{cases} \ .$$

It is well known that $H_1(c)$ is a self-adjoint operator which has a joint core with the free Hamiltonian H_0.

The family of operators $\{H_1(c); c > 0\}$ induces a dynamics $\alpha_1 \in \mathrm{dyn}(\mathfrak{M}_s)$ with zero propagation speed [37]. If $c_1 > c$ and $\mathcal{I} \subset (-c, c)$, then we have for each $a \in \mathfrak{M}_s(\mathcal{I})$ and for each $t \in \mathbb{R}$:

$$e^{itH_1(c)} \ a \ e^{-itH_1(c)} = e^{itH_1(c_1)} \ a \ e^{-itH_1(c_1)} \ .$$

Thus we may define α_1 by the uniform limit

$$a \longmapsto \alpha_{1,t}(a) := \lim_{c\to\infty} e^{itH_1(c)} \ a \ e^{-itH_1(c)} \ . \tag{2.1}$$

Since $H_1(c)$ has a joint core with the free Hamiltonian H_0, we are able to define the Trotter product of the automorphism groups α_0 and α_1 which is given for each local operator $a \in \mathfrak{M}_s(\mathcal{I})$ by the strong limit

$$\alpha_t(a) := (\alpha_0 \times \alpha_1)_t(a) = s - \lim_{n\to\infty} (\alpha_{0,t/n} \circ \alpha_{1,t/n})^n(a) \ .$$

Furthermore, the propagation speed is sub-additive with respect to the Trotter product [37], i.e.

$$\mathrm{ps}(\alpha_0 \times \alpha_1) \leq \mathrm{ps}(\alpha_0) + \mathrm{ps}(\alpha_1)$$

and we conclude that $\alpha \in \mathrm{dyn}(\mathfrak{M}_s)$ is a dynamics of $\mathfrak{M}_s$.

It has been shown by Glimm and Jaffe [37] that there exist vacuum states ω with respect to the interacting dynamics α. Due to Haag's Theorem (compare also [12]) there is *no* vector ψ in the Fock space $\mathcal{H}_s$ such that the linear functional

$$\omega_\psi : a \longmapsto \langle \psi, a\psi \rangle$$

is a vacuum state with respect to an interacting dynamics. But the vacuum states ω can be approximated by states which are induced by vectors in $\mathcal{H}_s$. The closure of the operator $H_0 + H_1(c)$ is self-adjoint and bounded from below. Let $E(c)$ be the infimum of the spectrum of $(H_0 + H_1(c))^{**}$, then

$$H(c) := (H_0 + H_1(c))^{**} - E(c)\mathbf{1}$$

is a positive operator. According to [37], there exists a sequence

$$\{\Omega(c_n) \in \ker(H(c_n)); n \in \mathbb{N}\} \;\; ; \;\; \lim_n c_n = \infty$$

such that the weak* limit

$$\omega := w^* - \lim_{n \to \infty} \langle \Omega(c_n), (\cdot)\Omega(c_n) \rangle \tag{2.2}$$

is a vacuum state with respect to the interacting dynamics α.

2.2.2 Yukawa$_2$ Models

Let us consider the Fock space

$$\mathcal{H}_a := \bigoplus_{n=0}^{\infty} L_2(\mathbb{R}, \mathbb{C}^2)^{\otimes_a n}$$

where a stands for anti-symmetrization of the tensor product and $L_2(\mathbb{R}, \mathbb{C}^2)$ is the Hilbert space of $\mathbb{C}^2$-valued and square integrable functions. The free Dirac spinor field ψ and its canonically conjugate $\bar{\psi}$ act as operator valued tempered distributions on $\mathcal{H}_a$, fulfilling canonical anti-commutation relations (CAR).

The field content of the Yukawa$_2$ model consists of one Bose field ϕ and one Dirac spinor field ψ. The net of Cauchy data of this model is given by the W*-tensor product

$$\mathfrak{M} : \mathcal{I} \longmapsto \mathfrak{M}(\mathcal{I}) = \mathfrak{M}_s(\mathcal{I}) \,\overline{\otimes}\, \mathfrak{M}_a(\mathcal{I})$$

where the net $\mathfrak{M}_a$ is given by the CAR-algebras

$$\mathfrak{M}_a : \mathcal{I} \longmapsto \mathfrak{M}_a(\mathcal{I}) := \left\{ \psi(f_1), \bar{\psi}(f_2) \;\middle|\; \mathrm{supp}(f_j) \subset \mathcal{I} \right\}'' \quad .$$

The unrenormalized Hamiltonian with IR-cutoff $c > 0$ is

$$H_{un}(c) := H_0 + \int \mathrm{d}\mathbf{x} \;\; : \bar{\Psi}(\mathbf{x})\Psi(\mathbf{x}) : \;\; \Phi(\mathbf{x}) \; \chi_c(\mathbf{x})$$

where H_0 is the free Hamiltonian and the fields Φ and Ψ are given by

$$\Phi := \phi \otimes \mathbf{1}_{\mathcal{H}_a} \quad \text{and} \quad \Psi := \mathbf{1}_{\mathcal{H}_s} \otimes \psi \quad .$$

The Hamiltonian $H_{un}(c)$ is defined in the sense of bilinear forms on $\mathcal{D} \times \mathcal{D}$, where $\mathcal{D}$ is a dense subspace in $\mathcal{H}_0 := \mathcal{H}_s \otimes \mathcal{H}_a$.

According to [37], the renormalized Hamiltonian can be constructed by introducing an UV-cutoff and adding appropriate counterterms. Then the UV-cutoff can be removed and we obtain an positive operator $H(c)$, depending on the IR-cutoff $c > 0$.

The dynamics α^Y of the Yukawa$_2$ model can be constructed from the family of Hamiltonians $\{H(c); c > 0\}$. It is given by

$$a \longmapsto \alpha^Y_t(a) := \lim_{c \to \infty} e^{itH(c)} \; a \; e^{-itH(c)} \quad .$$

However, the construction of the Yukawa$_2$ dynamics is more complicated than the construction of $P(\phi)_2$-dynamics and we postpone a detailed description of α^Y until Chapter 5 where more technical details are needed. The additional difficulties which arise here are due to the fact that α^Y can not be written as the Trotter product of the free dynamics α_0 and an interacting part α_1 with propagation speed $\mathrm{ps}(\alpha_1) = 0$.

Vacuum states with respect to the Yukawa$_2$ dynamics, whose existence have firstly been established by R. Schrader [75], can be obtained in the same manner as described for the $P(\phi)_2$ models (equation (2.2)).

3 Axiomatic Characterization of Kink States

3.1 Mathematical Description of Kinks

In this section, we give a precise definition of kink sectors in the framework of algebraic quantum field theory. Furthermore, we describe the construction of *kink homomorphisms* which are generalizations of the DHR and BF endomorphisms.

3.1.1 Kink Sectors

Let $\mathfrak{A}$ be a translationally covariant Haag-Kastler net and π be a representation of $\mathfrak{A}$, belonging to a sector $\theta \in \mathrm{sec}(\mathfrak{A})$. We define the restriction $\theta|_{\mathfrak{B}}$ of θ to a C*-sub-algebra $\mathfrak{B} \subset C^*(\mathfrak{A})$ as the unitary equivalence class of the restricted representation $\pi|_{\mathfrak{B}}$, i.e.

$$\theta|_{\mathfrak{B}} := [\pi|_{\mathfrak{B}}] \ .$$

Definiton 3.1.1 : A sector θ of $\mathfrak{A}$ is called a *kink sector*, which interpolates the vacuum sectors e_1, e_2, if it satisfies the conditions:

(a) θ fulfills the Borchers criterion (positivity of the energy). We refer to Section 1.1.1 for this notion.

(b) There is a left wedge region $W_1 \in \mathcal{W}_-$ and a right wedge region $W_2 \in \mathcal{W}_+$ such that:

$$\theta|_{C^*(\mathfrak{A},W_1)} = e_1|_{C^*(\mathfrak{A},W_1)} \tag{3.1}$$

$$\theta|_{C^*(\mathfrak{A},W_2)} = e_2|_{C^*(\mathfrak{A},W_2)} \tag{3.2}$$

A state (representation) is called a *kink state (representation)* if it belongs to a kink sector.

Of course, since a kink sector θ is translationally invariant, we conclude that the relation (3.1) holds for every left wedge region. Analogously, (3.2) holds for every right wedge region.

Notation: We denote the set of kink states, kink representations and kink sectors by $\mathfrak{S}(e_1, e_2)$, $\Delta(e_1, e_2)$ and $\sec(e_1, e_2)$ respectively. Furthermore, we shall call the right vacuum the *source* and the left vacuum the *range*.

We shall show that the existence of a kink sector $\theta \in \sec(e_1, e_2)$ implies that the vacuum representations π_1 and π_2, which belong to the sectors e_1 and e_2 respectively, are locally unitarily equivalent.

Proposition 3.1.2 : *Let $\theta \in \sec(e_1, e_2)$ be a kink sector. Then for each bounded double cone $\mathcal{O}$,*

$$\theta|_{\mathfrak{A}(\mathcal{O})} = e_1|_{\mathfrak{A}(\mathcal{O})} = e_2|_{\mathfrak{A}(\mathcal{O})} \quad .$$

Proof. Each double cone $\mathcal{O}$ can be written as an intersection $\mathcal{O} = \mathcal{O}_R \cap \mathcal{O}_L$ of a left wedge region $\mathcal{O}_L$ and a right wedge region $\mathcal{O}_R$. Let $\theta \in \sec(e_1, e_2)$ be a kink sector, then we obtain from equation (3.1) and (3.2):

$$\theta|_{C^*(\mathfrak{A},\mathcal{O}_L)} = e_1|_{C^*(\mathfrak{A},\mathcal{O}_L)} \quad \text{and} \quad \theta|_{C^*(\mathfrak{A},\mathcal{O}_R)} = e_2|_{C^*(\mathfrak{A},\mathcal{O}_R)} \quad . \tag{3.3}$$

Since $\mathfrak{A}(\mathcal{O}) \subset C^*(\mathfrak{A}, \mathcal{O}_L) \cap C^*(\mathfrak{A}, \mathcal{O}_R)$, we have:

$$\theta|_{\mathfrak{A}(\mathcal{O})} = e_1|_{\mathfrak{A}(\mathcal{O})} = e_2|_{\mathfrak{A}(\mathcal{O})} \quad .$$

The double cone $\mathcal{O}$ can be chosen arbitrarily which implies the result. $\square$

Remark: Proposition 3.1.2 states that local unitary equivalence of two vacuum representations π_1 and π_2 is a necessary condition for the existence of an interpolating kink state. Without loss of generality, we may assume the following conditions:

(i) The local algebras $\mathfrak{A}(\mathcal{O})$ are W*-algebras.

(ii) The states $\omega \in \mathfrak{S}(\mathfrak{A})$ under consideration are locally normal, i.e. their restrictions

$$\omega|_{\mathfrak{A}(\mathcal{O})}$$

are normal states of $\mathfrak{A}(\mathcal{O})$ for each double cone $\mathcal{O}$.

Proposition 3.1.3 : *Let $\pi \in \Delta(e_1, e_2)$ be a kink representation. Then each sub-representation $\hat{\pi} \subset \pi$ is a kink representation in $\Delta(e_1, e_2)$.*

Proof. We choose a projection E which is contained in the commutant of $\pi(C^*(\mathfrak{A}))$ and we consider the sub-representation $\hat{\pi}$ which is given by

$$\hat{\pi}(a) := \pi(a)E \quad .$$

We conclude from equation (3.1) that there is a vacuum representation π_1 with $[\pi_1] = e_1$ and

$$\pi|_{C^*(\mathfrak{A},W)} = \pi_1|_{C^*(\mathfrak{A},W)} \quad .$$

Since E is contained in the commutant of $\pi(C^*(\mathfrak{A}))$, we conclude that $E \in \mathfrak{A}_{\pi_1}(W')$. According to [21], the algebra $\mathfrak{A}_{\pi_1}(W)$ is a type III factor which implies that each sub-representation of $\pi_1|_{C^*(\mathfrak{A},W)}$ is unitarily equivalent to $\pi_1|_{C^*(\mathfrak{A},W)}$ itself. This yields:

$$\hat{\pi}|_{C^*(\mathfrak{A},W)} = E\pi_1|_{C^*(\mathfrak{A},W)} \cong \pi_1|_{C^*(\mathfrak{A},W)} \quad .$$

Analogously we show that

$$\hat{\pi}|_{C^*(\mathfrak{A},W')} = E\pi_2|_{C^*(\mathfrak{A},W')} \cong \pi_2|_{C^*(\mathfrak{A},W')}$$

for a suitable vacuum representation π_2 with $[\pi_2] = e_2$.

It remains to be proven that $\hat{\pi}$ satisfies the Borchers criterion. By a result of Borchers [3], the operators $U(x)$ which implement the translations in the representation π can be chosen in such a way that $U(x)$ is contained in $\pi(C^*(\mathfrak{A}))''$. Since the projection E is contained in $\pi(C^*(\mathfrak{A}))'$, we conclude that $\hat{U} : x \mapsto \hat{U}(x) := U(x)E$ is a unitary and strongly continuous representation which implements the translations in the representation $\hat{\pi}$. In particular the spectrum of $\hat{U}$ is also contained in the closed forward light cone. □

3.1.2 Kink (Soliton) Homomorphisms

A generalization of DHR and BF endomorphisms has been invented by K. Fredenhagen [23] in order to define the composition of kink sectors. We shall see that a kink sector, which interpolates vacuum sectors e_1, e_2, can be represented by two different types of C*-algebra homomorphisms

$$\rho : C^*(\mathfrak{A}^+_{e_1}) \longrightarrow C^*(\mathfrak{A}^+_{e_2}) \quad \text{and} \quad \rho' : C^*(\mathfrak{A}^-_{e_2}) \longrightarrow C^*(\mathfrak{A}^-_{e_1})$$

where $C^*(\mathfrak{A}^\pm_e)$ are extensions of the C*-algebra $C^*(\mathfrak{A})$, depending on a vacuum sector $e \in \mathrm{sec}_0(\mathfrak{A})$ and one of the space-like directions $\pm$-infinity. We briefly describe how kink homomorphisms can be constructed from a given kink sector.

Step 1: In the first step, we construct extensions of the C*-algebra $C^*(\mathfrak{A})$. Let $\omega \in \mathfrak{S}_0(\mathfrak{A})$ be a vacuum state and W a wedge region. Then we denote by

$$\mathfrak{A}_{[\omega]}(W) \tag{3.4}$$

the closure of the C*-algebra $C^*(\mathfrak{A}, W)$ with respect to the topology which is induced by semi norms:

$$||a||^T_\omega = |\mathrm{tr}(T\pi(a))| \ .$$

Here $(\mathcal{H}, \pi, \Omega)$ is the GNS triple of ω and T a trace class operator on $\mathcal{H}$ with trace $\mathrm{tr}(T) = 1$.

Obviously, the algebras $\mathfrak{A}_{[\omega]}(W)$ depend only on the sector $e = [\omega]$. The net

$$\mathfrak{A}^\pm_e : \mathcal{W}_\pm \ni W \longmapsto \mathfrak{A}_e(W)$$

is canonically isomorphic to the net

$$\mathfrak{A}^\pm_\pi : \mathcal{W}_\pm \ni W \longmapsto \mathfrak{A}_\pi(W) = \pi(C^*(\mathfrak{A}, W))''$$

where $\mathcal{W}_\pm$ denotes the set of right (left) wedge regions. Since the sets $\mathcal{W}_\pm$ are directed, the C*-inductive limits $C^*(\mathfrak{A}^\pm_e)$ and $C^*(\mathfrak{A}^\pm_\pi)$ can be constructed. It has been proven in [23, 70, 71] that the following statements hold:

(1) Each vacuum representation π_e, which belongs to the vacuum sector e, has unique extensions $\pi^\pm_e$ to the algebras $C^*(\mathfrak{A}^\pm_e)$. Moreover, if we assume that π_e is faithful, then

$$\pi^\pm_e : C^*(\mathfrak{A}^\pm_e) \longrightarrow C^*(\mathfrak{A}^\pm_{\pi_e})$$

are C*-algebra isomorphisms.

(2) If θ is a kink sector with source e_2 and range e_1, then each representation π, which belongs to the sector θ, has extensions π^+ to $C^*(\mathfrak{A}^+_{e_2})$ and π^- to $C^*(\mathfrak{A}^-_{e_1})$.

(3) If we assume that Haag duality holds in each vacuum sector e, then for each double cone $\mathcal{O}$ the intersection

$$\mathfrak{A}_e(\mathcal{O}_L) \cap \mathfrak{A}_e(\mathcal{O}_R) = \mathfrak{A}(\mathcal{O})$$

is independent of e.

Step 2: Let us consider a kink representation $(\mathcal{H}, \pi) \in \Delta(e_1, e_2)$ which interpolates the vacua e_1 and e_2. We choose vacuum representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ with $[\pi_1] = e_1$ and $[\pi_2] = e_2$. Then for each right wedge region $W \in \mathcal{W}_+$ there exist unitary operators $v_1^W : \mathcal{H} \to \mathcal{H}_1$ and $v_2^W : \mathcal{H} \to \mathcal{H}_2$ such that:

$$\mathrm{Ad}(v_1^W) \circ \pi|_{C^*(\mathfrak{A},W')} = \pi_1|_{C^*(\mathfrak{A},W')} \quad \text{and} \quad \mathrm{Ad}(v_2^W) \circ \pi|_{C^*(\mathfrak{A},W)} = \pi_2|_{C^*(\mathfrak{A},W)} \ .$$

We obtain representations

$$\tilde{\rho} := \mathrm{Ad}(v_1^W) \circ \pi^+ : C^*(\mathfrak{A}^+_{e_2}) \longrightarrow \mathfrak{B}(\mathcal{H}_1)$$

$$\tilde{\rho}' := \mathrm{Ad}(v_2^W) \circ \pi^- : C^*(\mathfrak{A}^-_{e_1}) \longrightarrow \mathfrak{B}(\mathcal{H}_2)$$

which are unitarily equivalent to π. It has been proven [23] that

$$\tilde{\rho}(C^*(\mathfrak{A}^+_{e_2})) \subset C^*(\mathfrak{A}^+_{\pi_1}) \quad \text{and} \quad \tilde{\rho}'(C^*(\mathfrak{A}^-_{e_1})) \subset C^*(\mathfrak{A}^-_{\pi_2})$$

and we finally define the *kink homomorphisms*:

$$\rho := (\pi_1^+)^{-1} \circ \tilde{\rho} : C^*(\mathfrak{A}^+_{e_2}) \longrightarrow C^*(\mathfrak{A}^+_{e_1})$$

$$\rho' := (\pi_2^-)^{-1} \circ \tilde{\rho}' : C^*(\mathfrak{A}^-_{e_1}) \longrightarrow C^*(\mathfrak{A}^-_{e_2}) \ .$$

Remark: As mentioned above, we can associate two different types of kink homomorphisms to each kink sector $\theta \in \mathrm{sec}(e_1, e_2)$, namely

(a) kink homomorphisms ρ which are localized in right wedge regions $W \in \mathcal{W}_+$, i.e.

$$\rho|_{C^*(\mathfrak{A},W')} = \mathrm{id}_{C^*(\mathfrak{A},W')}$$

(b) and kink homomorphisms ρ' which are localized in left wedge regions $W' \in \mathcal{W}_-$, i.e.

$$\rho'|_{C^*(\mathfrak{A},W)} = \mathrm{id}_{C^*(\mathfrak{A},W)} \quad .$$

Notation:

(i) We denote the set of all kink homomorphisms, which interpolate the vacua e_1, e_2, by $\Delta(q, e_1, e_2)$. The value $q \in \mathbb{Z}_2$ is called the *orientation* of a kink homomorphism where we set $q = 1$ for kink homomorphisms which are localized in right wedge regions and $q = -1$ for those which are localized in left wedge regions.

(ii) For each kink homomorphism ρ we choose one wedge region $\mathrm{supp}(\rho)$ in which ρ is localized.

3.2 Fusion, Subobjects, Direct Sums and Conjugation

The kink homomorphisms, which are discussed in the previous section, can be used to build the composition, direct sums and the conjugation of kink sectors. We shall see that the set of kink sectors is closed under these operations.

3.2.1 Composition of Kink Sectors

Let $\theta \in \mathrm{sec}(e_1, e_2)$ and $\hat{\theta} \in \mathrm{sec}(e_2, e_3)$ be kink sectors. Then we can choose either kink homomorphisms with orientation $q = 1$ or $q = -1$ to define their product.

(a) Let $\rho \in \Delta(1, e_1, e_2)$ and $\hat{\rho} \in \Delta(1, e_2, e_3)$ be kink homomorphisms such that $[\pi_1^+ \circ \rho] = \theta$ and $[\pi_2^+ \circ \hat{\rho}] = \hat{\theta}$. Then we define:

$$\theta \times_{(r)} \hat{\theta} := [\pi_1^+ \circ \rho\hat{\rho}] \quad . \tag{3.5}$$

(b) On the other hand, let $\rho' \in \Delta(-1, e_1, e_2)$ and $\hat{\rho}' \in \Delta(-1, e_2, e_3)$ be kink homomorphisms with $[\pi_2^- \circ \rho'] = \theta$ and $[\pi_3^- \circ \hat{\rho}'] = \hat{\theta}$. Then we define the product by:

$$\theta \times_{(l)} \hat{\theta} := [\pi_3^- \circ \hat{\rho}'\rho'] \quad . \tag{3.6}$$

Proposition 3.2.1 : *Let $\theta, \hat{\theta}$ be kink sectors. If the source $s(\hat{\theta})$ of $\hat{\theta}$ and the range $r(\theta)$ of θ coincide, then the following statements hold.*

(1) The products, which are given by equation (3.5) and (3.6), coincide and depend only on the sectors θ and $\hat{\theta}$, i.e.:

$$\theta\hat{\theta} := \theta \times_{(r)} \hat{\theta} = \theta \times_{(l)} \hat{\theta} \quad .$$

(2) The product $\theta\hat{\theta}$ is a kink sector with source

$$s(\theta\hat{\theta}) = s(\hat{\theta}) \quad \textit{and range} \quad r(\theta\hat{\theta}) = r(\theta) \quad .$$

Proof. The proof can be found in [23, 71]. □

Remark:

(i) For the investigation of the *fusion rules* for kink sectors, it is sufficient, according to Proposition 3.2.1, to consider products of kink homomorphisms with orientation $q = 1$.

(ii) The product of two kink sectors can only be defined if the source of one of the sectors coincides with the range of the other. This is different in the DHR case where any two sectors can be composed.

3.2.2 Subobjects and Direct Sums

In order to discuss subobjects and direct sums, we mention some important properties of operators which intertwine kink representations.

Remark:

(i) Let $\pi \in \Delta(e_1, e_2)$ and $\hat{\pi} \in \Delta(e_1', e_2')$ be two kink representations. Then we conclude from Proposition 3.1.3, that, if $e_1 \neq e_1'$ or $e_2 \neq e_2'$, then π and $\hat{\pi}$ are disjoint.

(ii) Let $\rho, \hat{\rho} \in \Delta(1, e_1, e_2)$ be kink homomorphisms, localized in W, and let π_1 be a vacuum representation, belonging to the sector e_1. Then each intertwiner $\tilde{v}$

$$\tilde{v}\ \pi_1^+ \circ \rho|_{C^*(\mathfrak{A})}(\cdot) = \pi_1^+ \circ \hat{\rho}|_{C^*(\mathfrak{A})}(\cdot)\ \tilde{v}$$

is contained in $\mathfrak{A}_{\pi_1}(W)$. Hence $v := (\pi_1^+)^{-1}(\tilde{v}) \in \mathfrak{A}_{e_1}(W)$ intertwines ρ and $\hat{\rho}$, i.e.:

$$v\,\rho(\cdot) = \hat{\rho}(\cdot)\,v \quad . \tag{3.7}$$

Thus it is sufficient to consider operators which intertwine kink homomorphisms to investigate reduction schemes for kink representations.

Notation: The vector space of operators, which intertwine the kink homomorphisms ρ and $\hat{\rho}$ (equation (3.7)), is denoted by $(\hat{\rho}|\rho)$.

Subobjects: A kink homomorphism ρ_1 is called a *subobject* of a kink homomorphism ρ, if there is an isometry $v \in (\rho|\rho_1)$ and we write $\rho_1 \prec \rho$.

Direct Sums: Conversely, we are able to define the *direct sum* $\rho_1 \oplus \rho_2$ for each pair of kink homomorphisms ρ_1, ρ_2 with $s(\rho_1) = s(\rho_2)$ and $r(\rho_1) = r(\rho_2)$. According to [6] we can find two isometries $v_1, v_2 \in \mathfrak{A}_e(W)$ with complementary range and we define $\rho_1 \oplus \rho_2$ as follows:

$$\rho(a) = (\rho_1 \oplus \rho_2)(a) := v_1\rho_1(a)v_1^* + v_2\rho_2(a)v_2^* \quad . \tag{3.8}$$

If W contains the localization regions of ρ_1 and ρ_2, then ρ is a kink homomorphism which is also localized in W. Note that the homomorphism $\rho_1 \oplus \rho_2$ depends on the choice of v_1, v_2, whereas the sector $[\rho_1 \oplus \rho_2]$ does not.

Proposition 3.2.2 : *The set of kink homomorphisms with orientation q is closed under multiplication, taking direct sums and subobjects.*

Proof. The statement of the proposition follows immediately from the discussion above. □

Remark:

(i) The set of kink homomorphisms possesses a category structure. The objects are vacuum sectors e_1, e_2 and the arrows are kink homomorphisms $\rho \in \Delta(1, e_1, e_2)$. The composition of arrows is given by the product of kink homomorphisms. The trivial arrow in $\Delta(1, e, e)$ is the identity id_e^+ of $C^*(\mathfrak{A}_e^+)$.

(ii) On the other hand, there is a second category structure, namely the objects are kink homomorphisms $\rho, \rho_1 \in \Delta(1, e_1, e_2)$ and the arrows are intertwiner $v \in (\rho_1|\rho)$.

Such a structure is also known as a 2-C^*-category. See [66] for this notion.

Given arrows $\rho_1, \rho_2 \in \Delta(1, e_1, e_2)$, $\rho \in \Delta(1, e_2, e_3)$ and $\hat{\rho} \in \Delta(1, e_0, e_1)$, then the following distributive laws are fulfilled:

$$(\rho_1 \oplus \rho_2)\rho \cong \rho_1\rho \oplus \rho_2\rho \quad \text{and} \quad \hat{\rho}(\rho_1 \oplus \rho_2) \cong \hat{\rho}\rho_1 \oplus \hat{\rho}\rho_2 \ .$$

Moreover, the sets of arrows $\Delta(1, e_1, e_2)$ possesses a partial order relation by the definition of subobjects.

3.2.3 Conjugation of Kink Sectors

A criterion for the existence of anti-kinks has been established in [71]. We briefly review the main facts which have been worked out there.

Theorem 3.2.3 : *Let* $\theta \in \mathrm{sec}(e_1, e_2)$ *be a kink sector. If there exists a kink sector* $\bar{\theta} \in \mathrm{sec}(e_2, e_1)$ *such that*

(a) $\theta\bar{\theta} \succ e_1$ *and* $\bar{\theta}\theta \succ e_2$*,*

then the following statements are true:

(1) $\bar{\theta}$ *is uniquely determined by (a) and the map*

$$j : \theta \in \mathrm{sec}(e_1, e_2) \longmapsto j(\theta) = \bar{\theta} \in \mathrm{sec}(e_2, e_1)$$

is a cofunctor.

(2) j *is an involution which respects direct sums and subobjects:*

(i) $j \circ j = \mathrm{id}$

(ii) $j(\theta_1 \oplus \theta_2) = j(\theta_1) \oplus j(\theta_2)$

(iii) If $\theta_1 \prec \theta$*, then* $j(\theta_1) \prec j(\theta)$*.*

(3) Denote by $\mathrm{sp}(\theta)$ *the spectrum of the implementation of the translation group in a kink representation which belongs to* θ*. Then*

$$\mathrm{sp}(\theta) = \mathrm{sp}(j(\theta)) \ .$$

Proof. The proof of the theorem can be found in [71]. The main tools are taken from the mathematical theory of sectors, which has been developed by R. Longo [57, 58] □

Remark:

(i) A kink sector $\theta \in \mathrm{sec}(e_1, e_2)$ fulfills condition *(a) if and only if* the index of the W*-inclusion

$$\rho(\mathfrak{A}_{e_2}(W)) \subset \mathfrak{A}_{e_1}(W)$$

is finite. Here $\rho \in \Delta(1, e_1, e_2)$ is a kink homomorphism which represents θ and which is localized in W. We refer to [57, 58, 71] for these notions.

(ii) Given two irreducible kink sectors $\theta_1 \in \mathrm{sec}(e_1, e_2)$ and $\theta_2 \in \mathrm{sec}(e_2, e_3)$ which fulfill condition *(a)*. Then their product is a finite direct sum of irreducible kink sectors, i.e.:

$$\theta_1\theta_2 = \bigoplus_{\theta} n^{\theta}_{\theta_1,\theta_2}\, \theta \ .$$

Here $n^{\theta}_{\theta_1,\theta_2}$ are natural numbers, called fusion coefficients, which are different from zero only for a finite number of irreducible kink sectors $\theta \in \mathrm{sec}(e_1, e_3)$. We conclude from *(1)* and *(2)* that the conjugation respects the fusion rules.

(iii) The statement *(3)* may be physically interpreted as the fact that the mass of a kink and its corresponding anti-kink coincide.

The anti-kink sector can be constructed by using the modular data of the wedge algebra $\mathfrak{A}_e(W_+)$ where the wedge regions $W_\pm$ are given by $\{x||x^0| \leq \pm x^1\}$. Let $(\mathcal{H}_e, \pi_e, \Omega_e)$ be the GNS-triple of a vacuum state which belongs to e. Then we denote by (J_e, Δ_e) the modular data with respect to the pair $(\mathfrak{A}_{\pi_e}(W), \Omega_e)$. For technical reasons, let us assume that the net $\mathfrak{A}$ possesses a PCT symmetry which is implemented in each vacuum representation π_e by the modular conjugation J_e.

Assumption: There is an anti-automorphism $j : C^*(\mathfrak{A}) \to C^*(\mathfrak{A})$ which reflects the translations, i.e. $j \circ \alpha_x = \alpha_{-x} \circ j$, and which maps $\mathfrak{A}(\mathcal{O})$ onto $\mathfrak{A}(-\mathcal{O})$. It is implemented in each vacuum representation π_e by the modular conjugation J_e, i.e.:

$$\pi_e(j(a)) = J_e \pi_e(a) J_e \quad .$$

We easily observe that for each vacuum sector e there are extensions $j_e^\pm$ of j:

$$j_e^\pm := (\pi_e^\mp)^{-1} \circ \mathrm{Ad}(J_e) \circ \pi_e^\pm : C^*(\mathfrak{A}_e^\pm) \longrightarrow C^*(\mathfrak{A}_e^\mp) \quad .$$

Let $\rho' \in \Delta(-1, e_1, e_2)$ be a kink homomorphism which belongs to the sector $\theta \in \sec(e_1, e_2)$. According to [71], a kink homomorphism $\bar\rho \in \Delta(1, e_2, e_1)$, which represents the conjugate sector $j(\theta)$ (Theorem 3.2.3), is given by

$$\bar\rho := j_{e_2}^- \circ \rho' \circ j_{e_1}^+ \in \Delta(1, e_2, e_1) \quad . \tag{3.9}$$

Remark:

(i) The map $\rho' \longmapsto \bar\rho$, which is given by equation (3.9), is the PCT transformation of the kink homomorphism ρ'. If ρ' is localized in a left wedge region W', then $\bar\rho$ is localized in the PT-reflected region $-W'$. In particular, the PCT transformation changes the orientation of a kink homomorphism.

(ii) Let ρ be a kink homomorphism, localized in a right wedge region W. We can choose a kink homomorphism ρ' which belongs to the same sector as ρ and which is localized in the PT-reflected region $-W$. Then the PCT transformation $\bar\rho$ of ρ' is a conjugate kink homomorphism for ρ which is also localized in W. The correspondence $\rho \longmapsto \rho'$ may be interpreted as a PT operation and thus $\rho \longmapsto \bar\rho$ is the composition of a PT and a PCT operation which is nothing else but the charge conjugation.

When Does a Theory Possess Kink States? 4

4.1 Interpolating Automorphisms

Interpolating automorphisms, which are also called soliton automorphisms, have been introduced by J. Fröhlich [31, Chapter 6] with the aim to give an axiomatic characterization of soliton sectors in the framework of algebraic quantum field theory. We shall give a precise definition of interpolating automorphisms in Section 4.1.1. In Section 4.1.2, we shall prove that a kink state corresponds to each pair which consists of a vacuum state and an interpolating automorphism.

4.1.1 Definitions and Notations

Let us consider a quantum field theory in two dimensions which is given by a translationally covariant Haag-Kastler net $\mathfrak{A}$ of W*-algebras.

Definiton 4.1.1 : An automorphism $\chi \in \mathrm{Aut}(C^*(\mathfrak{A}))$ is called a *symmetry* of the net $\mathfrak{A}$ if it preserves the net structure and commutes with the translations:

(1) $\chi(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O})$ for each double cone $\mathcal{O}$.

(2) $\chi \circ \alpha_x = \alpha_x \circ \chi$ for each $x \in \mathbb{R}^2$.

We shall denote the group of symmetries by $\mathrm{Sym}(\mathfrak{A})$.

Definiton 4.1.2 : Given a symmetry $\chi \in \mathrm{Sym}(\mathfrak{A})$, an automorphism ρ of $C^*(\mathfrak{A})$ is called χ*-interpolating* if it fulfills the conditions, listed below.

(1) There exists a bounded double cone $\mathcal{O}$, such that the relations

$$\rho|_{C^*(\mathfrak{A},\mathcal{O}_{LL})} = \mathrm{id}_{C^*(\mathfrak{A},\mathcal{O}_{LL})} \quad \text{and} \quad \rho|_{C^*(\mathfrak{A},\mathcal{O}_{RR})} = \chi|_{C^*(\mathfrak{A},\mathcal{O}_{RR})} \tag{4.1}$$

hold.

(2) There exists a strongly continuous map $\gamma_\rho : \mathbb{R}^2 \to C^*(\mathfrak{A})$ which fulfill the conditions:

(i)

$$\mathrm{Ad}(\gamma_\rho(x)) = \alpha_x \circ \rho \circ \alpha_{-x} \circ \rho^{-1} \ . \tag{4.2}$$

(ii) The operators fulfill the cocycle condition:

$$\gamma_\rho(x+y) = \alpha_x(\gamma_\rho(y))\gamma_\rho(x) \ . \tag{4.3}$$

Two interpolating automorphisms ρ_1, ρ_2 are called *equivalent* if $\rho_1\rho_2^{-1}$ is $C^*(\mathfrak{A})$-inner.

Notation:

(i) The set of all χ-interpolating automorphisms is denoted by $\mathrm{Aut}(\chi, \mathfrak{A})$. For each automorphism $\rho \in \mathrm{Aut}(\chi, \mathfrak{A})$ we choose one double cone $\mathrm{supp}_{\mathrm{I}}(\rho)$ which satisfies equation (4.1). We shall call $\mathrm{supp}_{\mathrm{I}}(\rho)$ the *interpolation region* of ρ.

(ii) We denote by $\mathrm{sec}(\chi, \mathfrak{A})$ the set of equivalence classes of χ- interpolating automorphisms and we shall call the elements of $\mathrm{sec}(\chi, \mathfrak{A})$ *interpolating sectors*.

(iii) A function γ_ρ which fulfills (4.2) and (4.3) is called a cocycle of ρ.

In [31], an additional property is assumed for the set of interpolating automorphisms $\mathrm{Aut}(\chi, \mathfrak{A})$. It is demanded that for a given symmetry χ, the set $\mathrm{sec}(\chi, \mathfrak{A})$ contains only one equivalence class.

4.1.2 Kink States, Induced by Interpolating Automorphisms

Proposition 4.1.3 : *Let* $\chi \in \mathrm{Sym}(\mathfrak{A})$ *be a symmetry and let* $\omega_0 \in \mathfrak{S}_0(\mathfrak{A})$ *be a vacuum state. For each automorphism* $\rho \in \mathrm{Aut}(\chi, \mathfrak{A})$*, the state* $\omega_0 \circ \rho$ *is a kink state in* $\mathfrak{S}([\omega_0], [\omega_0 \circ \chi])$.

We postpone the proof of the proposition above since we need some further results for preparation.

Observation: We observe that the GNS-representation $\pi_0 \circ \rho$ of the state $\omega_0 \circ \rho$ has the correct interpolation property. Indeed, for each interpolating automorphism $\rho \in \mathrm{Aut}(\chi, \mathfrak{A})$ with $\mathrm{supp}_{\mathrm{I}}(\rho) \subset \mathcal{O}$, the representation $\pi_\rho := \pi_0 \circ \rho$ fulfills the relations:

$$\pi_\rho|_{C^*(\mathfrak{A}, \mathcal{O}_{LL})} = \pi_0|_{C^*(\mathfrak{A}, \mathcal{O}_{LL})} \quad \text{and} \quad \pi_\rho|_{C^*(\mathfrak{A}, \mathcal{O}_{RR})} \cong \pi_\chi|_{C^*(\mathfrak{A}, \mathcal{O}_{RR})} \tag{4.4}$$

where π_χ denotes the GNS-representation of $\omega_0 \circ \chi$.

Thus it remains to be proven that $\omega_0 \circ \rho$ fulfills the Borchers criterion (positivity of the energy), which follows by applying the statement of [31, Theorem 7]. Since a detailed proof of [31, Theorem 7] is not given there, we shall present a complete proof of Proposition 4.1.3. In comparison to [31], we use methods of a completely model-independent analysis [18, 19].

Preparation of the Proof: In the first step, we shall prove that the representation π_ρ is translationally covariant.

Proposition 4.1.4 : *For each translationally covariant representation* $(\mathcal{H}, \pi)$*, the representation* $\pi \circ \rho$ *is translationally covariant.*

Proof. Let γ_ρ be a cocycle of ρ and U be a strongly continuous unitary representation which implements the translations in the representation π. Using equation (4.3), we obtain by a straight forward computation [18, 19]:

$$\pi \circ \rho \circ \alpha_x = \mathrm{Ad}(\, U(x)\pi(\gamma_\rho(-x))\,) \circ \pi \circ \rho \tag{4.5}$$

Thus

$$x \longmapsto U(x)\pi(\gamma_\rho(-x)) \tag{4.6}$$

is a strongly continuous unitary representation of the translation group, implementing the translations in the representation $\pi \circ \rho$. $\square$

Notation: Let $(\mathcal{H}_0, \pi_0)$ be a vacuum representation and let ρ be an interpolating automorphism. We define

$$U_\rho(x) := U_0(x)\pi_0(\gamma_\rho(-x)) \tag{4.7}$$

and denote the spectrum (of the generator) of U_ρ by $S(\rho)$ where U_0 implements the translations in the representation π_0.

In order to establish the statement of Proposition 4.1.3, we shall prove that $S(\rho)$ is contained in the closed forward light cone. For this purpose, we prove the additivity of the energy-momentum spectrum.

Lemma 4.1.5 : *Let $\{\rho_j \in \mathrm{Aut}(\chi_j, \mathfrak{A}); j = 1, 2\}$ be a pair of interpolating automorphisms, then the additivity of the energy-momentum spectrum holds:*

$$S(\rho_1) + S(\rho_2) \subset S(\rho_1\rho_2) \ .$$

Proof. The proof is standard and we can apply the methods which are used for the treatment of the DHR-framework [18, 19]. We also refer to [31]. The only difference which appears here consists of the fact that the representations ρ_j are localized in wedge regions and not in double cones. But for the proof it is sufficient that ρ_j acts as a symmetry on those observables which are localized in the right space-like complement of $\mathrm{supp}_{\mathrm{I}}(\rho)$.

Let f_j, $j = 1, 2$, be test functions with $\mathrm{supp}\tilde{f}_j \subset S(\rho_j)$ and let $a \in \mathfrak{A}(\mathcal{O})$ be a local operator. Then the operators

$$a_j := \int \mathrm{d}x \ f_j(x)\gamma_j(x)\alpha_x a$$

have energy-momentum transfer in $\mathrm{supp}\tilde{f}_j$ where γ_j is a cocycle of ρ_j.

This implies that

$$\Psi_1 := \pi_0(a_1)\Omega_0 \in \mathcal{H}_0$$

has energy-momentum support in $\mathrm{supp}\tilde{f}_1$ and that

$$\Psi_2 := \pi_0(\chi_1(a_2))\Omega_0 \in \mathcal{H}_0$$

has energy-momentum support in $\mathrm{supp}\tilde{f}_2$. Moreover,

$$\Psi := \pi_0(\rho_1(a_2)a_1)\Omega_0$$

has energy-momentum support in $\mathrm{supp}\tilde{f}_1 + \mathrm{supp}\tilde{f}_2$ which remains also true for

$$\Psi_y := \pi_0(\rho_1(a_2))U_{\rho_1}(y)\pi_0(a_1)\Omega_0 \quad .$$

Using the cluster theorem and the fact that a_1 and a_2 are almost local operators we conclude that

$$\begin{aligned}
\lim_{y^1-|y^0|\to-\infty} ||\Psi_y||^2 &= \lim_{y^1-|y^0|\to-\infty} \langle\Omega_0, \pi_0(\ a_1^*\ \rho_1(\alpha_{-y}(a_2^*a_2))\ a_1\ )\Omega_0\rangle \\
&= \lim_{y^1-|y^0|\to-\infty} \langle\Omega_0, \pi_0(\ a_1^*\ \alpha_{-y}\chi_1(a_2^*a_2)\ a_1\ )\Omega_0\rangle \\
&= \lim_{y^1-|y^0|\to-\infty} \langle\Omega_0, \pi_0(a_1^*a_1)\pi_0(\alpha_{-y}\chi_1(a_2^*a_2))\Omega_0\rangle \\
&= \lim_{y^1-|y^0|\to-\infty} \langle\Omega_0, \pi_0(a_1^*a_1)U_0(-y)\pi_0(\chi_1(a_2^*a_2))\Omega_0\rangle \\
&= \langle\Omega_0, \pi_0(a_1^*a_1)\Omega_0\rangle\langle\Omega_0, \pi_0(\chi_1(a_2^*a_2))\Omega_0\rangle \\
&= ||\Psi_1||^2\ ||\Psi_2||^2 \quad .
\end{aligned}$$

We conclude that for $||\Psi_j|| \neq 0$ there exists at least one $y \in \mathbb{R}^2$ such that $\Psi_y \neq 0$. The lemma follows since the vector Ψ_y is a non-zero vector in the representation Hilbert space of $\rho_1\rho_2$ with spectral support in $S(\rho_1) + S(\rho_2)$. $\square$

According to Section 3.2.3 (Theorem 3.2.3), there exists a conjugate $\bar{\rho} \in \mathrm{Aut}(\chi^{-1}, \mathfrak{A})$ for ρ which is given by

$$\bar{\rho} := j \circ \chi^{-1} \circ \rho \circ j \tag{4.8}$$

where we have assumed the existence of a PCT-symmetry j for technical reasons.

Lemma 4.1.6 : *The interpolating automorphisms ρ^{-1} and $\bar{\rho}$ are equivalent. In particular, $S(\rho) = S(\rho^{-1})$.*

Proof. Since ρ is an automorphism, we conclude from Theorem 3.2.3 and [36, 58, 71] that $\pi_\chi \circ \rho^{-1}$ and $\pi_\chi \circ \bar{\rho}$ are unitarily equivalent for each symmetry χ, i.e.

$$[\pi_\chi \circ \rho^{-1}] = [\pi_\chi \circ \bar{\rho}] \quad .$$

This implies

$$S(\rho^{-1}) = S(\bar{\rho}) = S(\rho)$$

and the proof is completed. □

Proposition 4.1.7 : *For each automorphism $\rho \in \mathrm{Aut}(\chi, \mathfrak{A})$, the representation*

$$\pi_0 \circ \rho$$

is a positive energy representation.

Proof. Using the additivity of energy-momentum spectrum (Lemma 4.1.5), we conclude that $S(\rho) + S(\rho^{-1}) \subset S(\pi_0)$. Since $S(\rho) = S(\rho^{-1})$ (Lemma 4.1.6), we finally obtain:

$$S(\rho) \subset \bar{V}_+ \quad . \tag{4.9}$$

□

Proof of Proposition 4.1.3: For each automorphism $\rho \in \mathrm{Aut}(\chi, \mathfrak{A})$, we conclude from Proposition 4.1.7 and equation (4.4) that $\pi_0 \circ \rho$ is a positive energy representation which satisfies the required interpolation property. □

We close this section by discussing the relation between interpolating automorphisms and kink homomorphisms. For this purpose, we repeat some notions of Chapter 3.

For each vacuum sector $e \in \mathrm{sec}_0(\mathfrak{A})$, we consider the nets

$$\mathfrak{A}_e^\pm : W \in \mathcal{W}_\pm \longmapsto \mathfrak{A}_e(W)$$

where $\mathfrak{A}_e(W)$ is the W*-completion of the C*-algebra $C^*(\mathfrak{A}, W)$, defined in Section 3.1.2. The corresponding C*-inductive limits $C^*(\mathfrak{A}_e^\pm)$ are extensions of the C*-algebra $C^*(\mathfrak{A})$. Furthermore, we denoted by $\mathrm{id}_e^\pm$ the identity of $C^*(\mathfrak{A}_e^\pm)$. Let π_0 be a vacuum representation which belongs to e, then we denote by e_χ the vacuum sector $[\pi_0 \circ \chi]$.

Proposition 4.1.8 : *Let $\rho \in \mathrm{Aut}(\chi, \mathfrak{A})$ be an interpolating automorphism and let $e \in \mathrm{sec}_0(\mathfrak{A})$ be a vacuum sector. Then there exists a unique kink homomorphism*

$$\rho_{(e,e_\chi)} : C^*(\mathfrak{A}_{e_\chi}^+) \longrightarrow C^*(\mathfrak{A}_e^+)$$

such that

$$\rho_{(e,e_\chi)}|_{C^*(\mathfrak{A})} = \rho \ . \tag{4.10}$$

Moreover, for each unitary operator $v \in C^*(\mathfrak{A})$ one has:

$$(\mathrm{Ad}(v) \circ \rho)_{(e,e_\chi)} = \mathrm{Ad}(v) \circ \rho_{(e,e_\chi)} \ . \tag{4.11}$$

Proof. Using the interpolation property (equation (4.1)) and the translation covariance of the representation $\pi_0 \circ \rho$ (Proposition 4.1.4), we conclude that there exist unique extensions $(\pi_0 \circ \rho)^+$ and $(\pi_0 \circ \rho)^-$ of the representation $\pi_0 \circ \rho$ to the C*-algebras $C^*(\mathfrak{A}^+_{e_\chi})$ and $C^*(\mathfrak{A}^-_e)$ respectively. According to Chapter 3, we obtain the desired C*-algebra homomorphism:

$$\rho_{(e,e_\chi)} := (\pi_0^+)^{-1} \circ (\pi_0 \circ \rho)^+ \ .$$

Here π_0^+ denotes the extension of π_0 to the algebra $C^*(\mathfrak{A}^+_e)$. Equation (4.11) is obviously fulfilled. □

4.2 A General Construction Scheme

In Section 4.2.1, we establish sufficient conditions for the existence of kink sectors. We use a general construction scheme which can also be applied if any two vacuum sectors are *not* related by a symmetry. Hence the results of the subsequent analysis go beyond those which have been presented in [31, 32].

We shall describe in Section 4.2.2 in which manner a multiple application of our construction scheme leads to states which can be interpreted as multi-kinks.

4.2.1 An Existence Criterion

We consider a Haag-Kastler net $\mathfrak{A}$ of W*-algebras which possesses at least two inequivalent vacuum states.

Notation: We denote by $\mathfrak{A} \overline{\otimes} \mathfrak{A}$ the net which is given by the twofold W*-tensor product of $\mathfrak{A}$, i.e.:

$$\mathfrak{A} \overline{\otimes} \mathfrak{A} : \mathcal{O} \longmapsto (\mathfrak{A} \overline{\otimes} \mathfrak{A})(\mathcal{O}) := \mathfrak{A}(\mathcal{O}) \overline{\otimes} \mathfrak{A}(\mathcal{O}) \tag{4.12}$$

Theorem 4.2.1 : *If there exists an interpolating automorphism* $\beta \in \mathrm{Aut}(\alpha_F, \mathfrak{A} \overline{\otimes} \mathfrak{A})$, *then for each pair of vacuum sectors* $e_1, e_2 \in \mathrm{sec}_0(\mathfrak{A})$ *there exists a kink state* $\omega \in \mathfrak{S}(e_1, e_2)$.

Proof. It is obvious that the flip automorphism α_F, which is given by exchanging the tensor factors, is a symmetry of $\mathfrak{A} \overline{\otimes} \mathfrak{A}$. Let $\omega_1, \omega_2 \in \mathfrak{S}_0(\mathfrak{A})$ be vacuum states and let $\beta \in \mathrm{Aut}(\alpha_F, \mathfrak{A} \overline{\otimes} \mathfrak{A})$ be an α_F-interpolating automorphism. We denote by $(\mathcal{H}_j, \pi_j, \Omega_j)$ the GNS-triple of ω_j $(j = 1, 2)$.

According to Proposition 4.1.7,

$$\pi_\beta := \pi_1 \otimes \pi_2 \circ \beta|_{C^*(\mathfrak{A}) \otimes \mathbf{1}}$$

is a positive energy representation. Using the Theorem of Reeh and Schlieder, we conclude that

$$\pi_\beta(C^*(\mathfrak{A}, \mathcal{O}_{LL} \cup \mathcal{O}_{RR}))\Omega_1 \otimes \Omega_2 = \pi_1(C^*(\mathfrak{A}, \mathcal{O}_{LL}))\Omega_1 \otimes \pi_2(C^*(\mathfrak{A}, \mathcal{O}_{RR}))\Omega_2$$

is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Thus the representation π_β is cyclic and therefore unitarily equivalent to the GNS-representation of the state

$$\omega = \omega_1 \otimes \omega_2 \circ \beta|_{C^*(\mathfrak{A}) \otimes \mathbf{1}} \quad . \tag{4.13}$$

We derive from the properties of β the relations:

$$\pi_\beta|_{C^*(\mathfrak{A}, \mathcal{O}_{LL})} \cong \pi_1 \otimes \mathbf{1}_{\mathcal{H}_2}|_{C^*(\mathfrak{A}, \mathcal{O}_{LL})} \cong_{quasi} \pi_1|_{C^*(\mathfrak{A}, \mathcal{O}_{LL})}$$

$$\pi_\beta|_{C^*(\mathfrak{A}, \mathcal{O}_{RR})} \cong \mathbf{1}_{\mathcal{H}_1} \otimes \pi_2|_{C^*(\mathfrak{A}, \mathcal{O}_{LL})} \cong_{quasi} \pi_2|_{C^*(\mathfrak{A}, \mathcal{O}_{RR})}$$

where the symbol $\cong_{quasi}$ means *quasi equivalent*. Using the fact that for each wedge region W the von Neumann algebras $\mathfrak{A}_{\pi_j}(W)$, $j = 1, 2$, are type III factors [21], we conclude by using standard arguments:

$$\pi_\beta|_{C^*(\mathfrak{A}, \mathcal{O}_{LL})} \cong \pi_1|_{C^*(\mathfrak{A}, \mathcal{O}_{LL})} \quad \text{and} \quad \pi_\beta|_{C^*(\mathfrak{A}, \mathcal{O}_{RR})} \cong \pi_2|_{C^*(\mathfrak{A}, \mathcal{O}_{RR})} \quad .$$

This implies that ω, given by equation (4.13), is a kink state which interpolates the vacua $e_1 = [\omega_1]$ and $e_2 = [\omega_2]$. □

4.2.2 Multi-Kink States

Motivated by the result of the previous section, we introduce the notion of the *interpolating product* which turns out to be of practical advantage for the discussion of multi-kink states.

Definiton 4.2.2 :

(1) Let $\beta \in \mathrm{Aut}(\alpha_F, \mathfrak{A} \overline{\otimes} \mathfrak{A})$ be an interpolating automorphism. We shall call the homomorphism

$$\Delta := \beta|_{C^*(\mathfrak{A})\otimes 1} : C^*(\mathfrak{A}) \longrightarrow C^*(\mathfrak{A} \overline{\otimes} \mathfrak{A}) \tag{4.14}$$

the *interpolating co-product* with respect to β.

(2) Let $\omega_1, \omega_2 \in \mathfrak{S}(\mathfrak{A})$ be locally normal states. The state

$$\mu_\beta(\omega_1 \otimes \omega_2) := \omega_1 \otimes \omega_2 \circ \Delta \tag{4.15}$$

is called the β*-interpolating product* of ω_1 and ω_2. Analogously, we define the β*-interpolating product* of two representations π_1, π_2.

Observation: We easily observe, that the sector $[\mu_\beta(\omega_1 \otimes \omega_2)]$ depends only on the sectors $[\omega_1]$, $[\omega_2]$ and the interpolating sector $[\beta] \in \sec(\alpha_F, \mathfrak{A} \overline{\otimes} \mathfrak{A})$. Hence we obtain a well defined map

$$\mu : \sec(\alpha_F, \mathfrak{A} \overline{\otimes} \mathfrak{A}) \times \sec(\mathfrak{A}) \times \sec(\mathfrak{A}) \longrightarrow \sec(\mathfrak{A}) \tag{4.16}$$

which is given by

$$(\iota, \theta_1, \theta_2) \longmapsto \mu_\iota(\theta_1 \otimes \theta_2) := [\mu_\beta(\omega_1 \otimes \omega_2)] \tag{4.17}$$

with $\theta_j = [\omega_j]$ and $\iota = [\beta]$.

Furthermore, we conclude from Proposition 4.1.4 that $\mu_\beta(\pi_1 \otimes \pi_2)$ is translationally covariant if both, π_1 and π_2, are translationally covariant.

According to the proofs of Proposition 4.1.3 and Theorem 4.2.1, we easily derive the following generalization:

Corollary 4.2.3 : *Let $\omega \in \mathfrak{S}(e_1, e_2)$ and $\hat{\omega} \in \mathfrak{S}(\hat{e}_1, \hat{e}_2)$ be kink states and let β be an α_F-interpolating automorphism. Then the β-interpolating product*

$$\mu_\beta(\omega \otimes \hat{\omega}) \tag{4.18}$$

is a kink state in $\mathfrak{S}(e_1, \hat{e}_2)$. In particular, let $e_1, \cdots, e_3$ be vacuum sectors and let ι_1, ι_2 be α_F-interpolating sectors. Then

$$\mu_{\iota_1}(e_1 \otimes \mu_{\iota_2}(e_2 \otimes e_3)) \tag{4.19}$$

is a kink sector in $\mathrm{sec}(e_1, e_3)$.

Since the right vacuum (source) of $\mu_{\iota_1}(e_1 \otimes e_2)$ and the left vacuum (range) of $\mu_{\iota_2}(e_2 \otimes e_3)$ coincide, i.e.

$$s(\mu_{\iota_1}(e_1 \otimes e_2)) = r(\mu_{\iota_2}(e_2 \otimes e_3)) = e_2 \quad , \tag{4.20}$$

we can build the fusion (see Section 3.2.1):

$$\mu_{\iota_1}(e_1 \otimes e_2)\mu_{\iota_2}(e_2 \otimes e_3) \in \mathrm{sec}(e_1, e_3) \quad . \tag{4.21}$$

We shall prove that the kink sector, which is given by equation (4.19), is nothing else but the product sector (equation (4.21)).

Theorem 4.2.4 : *Let $e_1, \cdots, e_3$ be vacuum sectors and let ι_1, ι_2 be α_F-interpolating sectors. Then*

$$\mu_{\iota_1}(e_1 \otimes \mu_{\iota_2}(e_2 \otimes e_3)) = \mu_{\iota_1}(e_1 \otimes e_2)\mu_{\iota_2}(e_2 \otimes e_3) \quad . \tag{4.22}$$

Let us establish another useful result, before we turn to the proof of Theorem 4.2.4.

Lemma 4.2.5 : *Let $e_1, e_2, e_3 \in \mathrm{sec}_0(\mathfrak{A})$ be vacuum sectors, let $\rho \in \Delta(1, e_2, e_3)$ be a kink homomorphism and let $\beta \in \mathrm{Aut}(\alpha_F, \mathfrak{A} \,\overline{\otimes}\, \mathfrak{A})$ be an interpolating automorphism. If the interpolation region $\mathrm{supp}_{\mathrm{I}}(\beta)$ is contained in the space-like complement of $\mathrm{supp}(\rho)$, then*

$$(\mathrm{id}^+_{e_1} \otimes \rho) \circ \beta = \beta_{(e_1 \otimes e_2, e_2 \otimes e_1)} \circ (\rho \otimes \mathrm{id}^+_{e_1}) \quad .$$

Here $\beta_{(e_1 \otimes e_2, e_2 \otimes e_1)}$ is the extension of β which is given due to Proposition 4.1.8.

Proof. Since $\mathrm{supp}_{\mathrm{I}}(\beta)$ and $\mathrm{supp}(\rho)$ are space-like separated, we can choose a right wedge region $W \in \mathcal{W}_+$ such that

$$\beta|_{C^*(\mathfrak{A}\,\overline{\otimes}\,\mathfrak{A},W)} = \alpha_F|_{C^*(\mathfrak{A}\,\overline{\otimes}\,\mathfrak{A},W)} \quad \text{and} \quad \rho|_{C^*(\mathfrak{A},W')} = \mathrm{id}_{C^*(\mathfrak{A},W')} \quad .$$

Let $\mathcal{O}$ be a double cone. Using the translation covariance of ρ and β, there are unitary intertwiners $\gamma_\rho \in C^*(\mathfrak{A}^+_{e_2})$ and $\gamma_\beta \in C^*(\mathfrak{A}\,\overline{\otimes}\,\mathfrak{A})$ such that the homomorphisms

$$\hat{\rho} := \mathrm{Ad}(\gamma_\rho^*) \circ \rho \quad \text{and} \quad \hat{\beta} := \mathrm{Ad}(\gamma_\beta^*) \circ \beta$$

have the localizing properties:

$$\hat{\beta}|_{C^*(\mathfrak{A}\,\overline{\otimes}\,\mathfrak{A},\mathcal{O}_{RR})} = \alpha_F|_{C^*(\mathfrak{A}\,\overline{\otimes}\,\mathfrak{A},\mathcal{O}_{RR})} \quad \text{and} \quad \hat{\rho}|_{C^*(\mathfrak{A},\mathcal{O}_{LL})} = \mathrm{id}_{C^*(\mathfrak{A},\mathcal{O}_{LL})} \quad .$$

This implies for each $a \in (\mathfrak{A}\,\overline{\otimes}\,\mathfrak{A})(\mathcal{O})$:

$$\begin{aligned}
(\mathrm{id}^+_{e_1} \otimes \rho) \circ \beta(a) &= \mathrm{Ad}(\mathbf{1} \otimes \gamma_\rho) \circ (\mathrm{id}^+_{e_1} \otimes \hat{\rho}) \circ \beta(a) \\
&= \mathrm{Ad}(\mathbf{1} \otimes \gamma_\rho) \circ \beta(a) \\
&= \mathrm{Ad}((\mathbf{1} \otimes \gamma_\rho)\gamma_\beta) \circ \hat{\beta}(a) \\
&= \mathrm{Ad}((\mathbf{1} \otimes \gamma_\rho)\gamma_\beta)(\alpha_F(a))
\end{aligned}$$

Keeping in mind that γ_β intertwines also the extensions $\beta_{(e_1 \otimes e_2, e_2 \otimes e_1)}$ and $\hat{\beta}_{(e_1 \otimes e_2, e_2 \otimes e_1)}$ (Proposition 4.1.8), we obtain on the other hand:

$$\begin{aligned}
\beta_{(e_1 \otimes e_2, e_2 \otimes e_1)} \circ (\rho \otimes \mathrm{id}^+_{e_1})(a) &= \mathrm{Ad}(\gamma_\beta) \circ \hat{\beta}_{(e_1 \otimes e_2, e_2 \otimes e_1)} \circ (\rho \otimes \mathrm{id}^+_{e_1})(a) \\
&= \mathrm{Ad}(\gamma_\beta) \circ (\mathrm{id}^+_{e_1} \otimes \rho)(\alpha_F(a)) \\
&= \mathrm{Ad}(\gamma_\beta(\mathbf{1} \otimes \gamma_\rho))(\alpha_F(a)) \ .
\end{aligned}$$

If the double cone $\mathcal{O}$ is large enough, i.e. $\mathcal{O}_{RR} \subset W$ and $\mathcal{O} \supset \mathrm{supp}_{\mathrm{I}}(\beta)$, then γ_ρ is contained in $\mathfrak{A}_{e_2}(W)$ and γ_β is contained in $C^*(\mathfrak{A}\,\overline{\otimes}\,\mathfrak{A}, W')$. Thus $\mathbf{1} \otimes \gamma_\rho$ and γ_β commute. Since $\mathcal{O}$ can be chosen arbitrarily large, the result follows. □

Proof of Theorem 4.2.4: Let β be an interpolating automorphism and let $\{\pi_j; j = 1, 2, 3\}$ be vacuum representations, such that $[\beta] = \iota_1$ and $[\pi_j] = e_j$. We choose a kink homomorphism $\rho \in \Delta(1, e_2, e_3)$ with $[\rho] = \mu_{\iota_2}(e_2 \otimes e_3)$ and $\mathrm{supp}(\rho) \subset \mathrm{supp}_{\mathrm{I}}(\beta)'$. We conclude from Lemma 4.2.5:

$$(\pi_1 \otimes (\pi_2^+ \circ \rho)) \circ \beta = (\pi_1 \otimes \pi_2)^+ \circ \beta_{(e_1 \otimes e_2, e_2 \otimes e_1)} \circ (\rho \otimes \mathrm{id}_{e_1}^+)$$

Let $\mu_\beta(\pi_1 \otimes \pi_2)^+$ be the extension of the kink representation $\mu_\beta(\pi_1 \otimes \pi_2)$ to the algebra $C^*(\mathfrak{A}_{e_2}^+)$, then we obtain:

$$(\pi_1 \otimes (\pi_2^+ \circ \rho)) \circ \beta|_{C^*(\mathfrak{A}) \otimes \mathbf{1}} = \mu_\beta(\pi_1 \otimes \pi_2)^+ \circ \rho|_{C^*(\mathfrak{A})} \ .$$

Finally, we get:

$$\begin{aligned} \mu_{\iota_1}(e_1 \otimes \mu_{\iota_2}(e_2 \otimes e_3)) &= [\mu_\beta(\pi_1 \otimes \pi_2)^+ \circ \rho|_{C^*(\mathfrak{A})}] \\ &= \mu_{\iota_1}(e_1 \otimes e_2)\mu_{\iota_2}(e_2 \otimes e_3) \ . \end{aligned}$$

□

According to Theorem 4.2.4, multiple products of kink sectors can be computed in a very convenient manner. To illustrate this, we choose a family of interpolating automorphisms

$$\beta_1, \cdots, \beta_n \in \mathrm{Aut}(\alpha_F, \mathfrak{A} \,\overline{\otimes}\, \mathfrak{A})$$

and vacuum states

$$\omega_1, \cdots, \omega_{n+1} \in \mathfrak{S}_0(\mathfrak{A})$$

such that $\mathrm{supp}_{\mathrm{I}}(\beta_j) \subset \mathrm{supp}_{\mathrm{I}}(\beta_{j+1})_{LL}$, i.e. the double cone $\mathrm{supp}_{\mathrm{I}}(\beta_j)$ is placed in the left space-like complement of $\mathrm{supp}_{\mathrm{I}}(\beta_{j+1})$. We conclude from Theorem 4.2.4 that the product sector

$$\hat{\theta} := [\mu_{\beta_1}(\omega_1 \otimes \omega_2)][\mu_{\beta_2}(\omega_2 \otimes \omega_3)] \cdots [\mu_{\beta_n}(\omega_n \otimes \omega_{n+1})]$$

can be represented by the state

$$\hat{\omega} := \mu_{\beta_1}(\omega_1 \otimes \mu_{\beta_2}(\omega_2 \otimes \cdots \mu_{\beta_n}(\omega_n \otimes \omega_{n+1}) \cdots))$$

which describes a configuration of n space-like separated kinks. To justify this interpretation, we shall prove the corollary below.

Corollary 4.2.6 : *Let $\hat{\omega}$ be the product state, defined above. Then*

$$\forall a \in \mathfrak{A}(\mathrm{supp}_{\mathrm{I}}(\beta_j)_{RR} \cap \mathrm{supp}_{\mathrm{I}}(\beta_{j+1})_{LL}) : \ \hat{\omega}(a) = \omega_j(a) \ . \tag{4.23}$$

Proof. We express $\hat{\omega}$ in terms of the interpolating co-products $\{\Delta_j = \beta_j|_{C^*(\mathfrak{A})\otimes\mathbf{1}}; j = 1, \cdots, n\}$:

$$\hat{\omega} = (\omega_1 \otimes \cdots \otimes \omega_{n+1}) \circ (\mathrm{id}^{\otimes n-1} \otimes \Delta_n) \circ \cdots \circ (\mathrm{id} \otimes \Delta_2) \circ \Delta_1 \tag{4.24}$$

Let a be an observable which is localized in $\mathrm{supp}_{\mathrm{I}}(\beta_j)_{RR} \cap \mathrm{supp}_{\mathrm{I}}(\beta_{j+1})_{LL}$, then we obtain:

$$\begin{aligned}
&(\mathrm{id}^{\otimes n-1} \otimes \Delta_n) \circ \cdots \circ (\mathrm{id} \otimes \Delta_2) \circ \Delta_1(a) \\
&= (\mathrm{id}^{\otimes n-1} \otimes \Delta_n) \circ \cdots \circ (\mathrm{id} \otimes \Delta_2)(\mathbf{1} \otimes a) \\
&= (\mathrm{id}^{\otimes n-1} \otimes \Delta_n) \circ \cdots \circ (\mathrm{id}^{\otimes j-1} \otimes \Delta_j)(\mathbf{1}^{\otimes(j-1)} \otimes a) \\
&= (\mathrm{id}^{\otimes n-1} \otimes \Delta_n) \circ \cdots \circ (\mathrm{id}^{\otimes j} \otimes \Delta_{j+1})(\mathbf{1}^{\otimes(j-1)} \otimes a \otimes \mathbf{1}) \\
&= \mathbf{1}^{\otimes(j-1)} \otimes a \otimes \mathbf{1}^{\otimes(n-j-1)} \quad .
\end{aligned}$$

Hence we have:

$$\hat{\omega}(a) = \omega_j(a) \quad . \tag{4.25}$$

□

4.3 Construction of Kink States via Universal Localizing Maps

In the previous section, the existence of an interpolating automorphisms is the crucial condition in order to construct kink states (Theorem 4.2.1). But how can we conclude whether the set of interpolating automorphisms is empty or not? It seems to be very difficult to decide this question by only assuming the Haag-Kastler axioms for the net $\mathfrak{A}$.

In order to formulate additional conditions for the net $\mathfrak{A}$ we give a brief discussion of standard split inclusions in Section 4.3.1. Assuming the conditions, formulated in Section 4.3.1, we prove in Section 4.3.2 the existence of interpolating automorphisms which implies the main result of this section: For each pair of vacuum states there exists an interpolating kink state (Corollary 4.3.7).

4.3.1 Standard Split Inclusions and Universal Localizing Map

We now describe a concept, invented by S. Doplicher and R. Longo [20], which will play a crucial role for our subsequent analysis.

We consider a pair of W*-algebras A, B which are faithfully represented on a separable Hilbert space $\mathcal{H}$.

Definiton 4.3.1 :

(a) If A is a sub-algebra of B, i.e. $A \subset B$, we call the pair (A, B) of W*-algebras a *W*-inclusion*. A vector $\psi \in \mathcal{H}$ is called a standard vector for the inclusion (A, B) if ψ is cyclic and separating for A, B and $A' \wedge B$.

(b) A W*-inclusion (A, B) is called *split* if there exists a type I factor N such that $A \subset N \subset B$. If $\psi \in \mathcal{H}$ is a standard vector for (A, B), the triple (A, B, ψ) is called a *standard split inclusion*.

Remark: We note that the group $\mathcal{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$ acts on the set of all standard split inclusions $\Lambda = (A, B, \psi)$. Indeed, let $u \in \mathcal{U}(\mathcal{H})$ be a unitary operator, then $u\Lambda := (uAu^*, uBu^*, u\psi)$ is also a standard split inclusion.

Proposition 4.3.2 : *Let $\Lambda = (A, B, \psi)$ be a standard split inclusion. Then the following statements hold:*

(1) There exists a unitary operator $w_\Lambda : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$ such that the W-isomorphism $\Psi_\Lambda := \mathrm{Ad}(w_\Lambda)$ satisfies the relation:*

$$\Psi_\Lambda(b' \otimes a) = b'a \quad \textit{for each } b' \in B' \textit{ and for each } a \in A$$

(2) There exists a canonical type I factor N_Λ between A and B, which is given by

$$A \subset \Psi_\Lambda(\mathbf{1}_{\mathcal{H}} \overline{\otimes} \mathfrak{B}(\mathcal{H})) = N_\Lambda \subset B \quad .$$

(3) Let u be a unitary operator in $\mathcal{U}(\mathcal{H})$. Then

$$\Psi_{u\Lambda} = \mathrm{Ad}(u) \circ \Psi_\Lambda \circ \mathrm{Ad}(u^* \otimes u^*) \quad .$$

Proof. The proof is standard and can be obtained from [20, 9]. For reasons of convenience, we cite the main steps of it here.

(1) Since Λ is a split inclusion, there is a (canonical) product vector $\phi_\Lambda \in \mathcal{H}$, such that

$$\langle \phi_\Lambda, b'a\phi_\Lambda \rangle = \langle \psi, b'\psi \rangle \cdot \langle \psi, a\psi \rangle \quad \text{for each } b' \in B' \text{ and for each } a \in A$$

and ϕ_Λ is cyclic for $B' \vee A$. Hence

$$w_\Lambda(b'a\phi_\Lambda) := b'\psi \otimes a\psi$$

defines a unitary operator with the desired property.

(2) We conclude from the construction of w_Λ:

$$\begin{aligned} A \quad &= \Psi_\Lambda(\mathbf{1}_\mathcal{H} \overline{\otimes} A) = \quad w_\Lambda(\mathbf{1}_\mathcal{H} \overline{\otimes} A)w_\Lambda^* \\ \\ B' \quad &= \Psi_\Lambda(B' \overline{\otimes} \mathbf{1}_\mathcal{H}) = \quad w_\Lambda(B' \overline{\otimes} \mathbf{1}_\mathcal{H})w_\Lambda^* \end{aligned}$$

and therefore

$$A \subset \Psi_\Lambda(\mathbf{1}_\mathcal{H} \overline{\otimes} \mathfrak{B}(\mathcal{H})) \subset B \quad .$$

(3) By a straight forward computation, we obtain

$$uw_\Lambda(u^* \otimes u^*) = w_{u\Lambda}$$

which implies the result.

□

Notation: We shall call the W*-isomorphism Ψ_Λ, which is given above, the *universal localizing map* of the standard split inclusion Λ.

Let $\mathfrak{A}$ be a translationally covariant Haag-Kastler net. Each pair $(\omega, \mathcal{O})$ which consists of a state ω and a double cone $\mathcal{O}$, can be identified with a W*-inclusion:

$$(\omega, \mathcal{O}) \longmapsto (\mathfrak{A}_\pi(\mathcal{O}_{RR}), \mathfrak{A}_\pi(\mathcal{O}_R))$$

where π is the GNS-representation of ω. [1]

[1] Alternatively, we can also choose the inclusion $(\mathfrak{A}_\pi(\mathcal{O}_{LL}), \mathfrak{A}_\pi(\mathcal{O}_L))$. For our purpose, it is sufficient to consider one of both possibilities.

Definiton 4.3.3 : Let $\omega \in \mathfrak{S}(\mathfrak{A})$ be a state and $\mathcal{O}$ be a double cone. We denote by $(\mathcal{H}, \pi, \Omega)$ the GNS-triple of ω. We shall call the pair $(\omega, \mathcal{O})$ a standard split inclusion if

$$\Lambda := (\mathfrak{A}_\pi(\mathcal{O}_{RR}), \mathfrak{A}_\pi(\mathcal{O}_R), \Omega)$$

is a standard split inclusion. Furthermore, we write $\Psi_{(\omega,\mathcal{O})}$ for the universal localizing map of the inclusion Λ.

An application of Proposition 4.3.2 gives:

Corollary 4.3.4 : *Let $\omega \in \mathfrak{S}(\mathfrak{A})$ be a translationally covariant state and let $\mathcal{O}$ be a double cone. If $\Lambda = (\omega, \mathcal{O})$ is a standard split inclusion, then the statements below hold.*

(1) $\forall x \in \mathbb{R}^2$: $\Lambda(x) := (\omega \circ \alpha_x, \mathcal{O} + x)$ is a standard split inclusion.

(2) Let U be the implementation of the translation group in the GNS-representation π of ω. Then the universal localizing map $\Psi_{(\omega,\mathcal{O})}$ is translationally covariant:

$$\Psi_{\Lambda(x)} = \mathrm{Ad}(U(x)) \circ \Psi_\Lambda \circ \mathrm{Ad}(U(-x) \otimes U(-x)) \ .$$

Proposition 4.3.5 : *Let $\omega, \omega_1, \omega_2 \in \mathfrak{S}_0(\mathfrak{A})$ be vacuum states. Then the statements, listed below, are true.*

(1) Let $\mathcal{O} \subset \hat{\mathcal{O}}$ be an inclusion of double cones. If $\Lambda = (\omega, \mathcal{O})$ is a standard split inclusion, then $\hat{\Lambda} = (\omega, \hat{\mathcal{O}})$ is a standard split inclusion.

(2) If $\Lambda_1 = (\omega_1, \mathcal{O}_1)$ and $\Lambda_2 = (\omega_2, \mathcal{O}_2)$ are standard split inclusions, then the tensor product

$$\Lambda_1 \otimes \Lambda_2 := (\omega_1 \otimes \omega_2, \mathcal{O}_1 \cup \mathcal{O}_2)$$

is a standard split inclusion.

Proof. Let $(\mathcal{H}, \pi, \Omega)$ and $(\mathcal{H}_j, \pi_j, \Omega_j)$ be the GNS-triples of ω and $\omega_j; j = 1, 2$ respectively.

(1) Using the Reeh-Schlieder property of the GNS-vector Ω, we conclude that for each double cone $\mathcal{O}$, Ω is a standard vector for the inclusion

$$(\mathfrak{A}_\pi(\mathcal{O}_{RR}), \mathfrak{A}_\pi(\mathcal{O}_R)) \ .$$

If the inclusion $(\mathfrak{A}_\pi(\mathcal{O}_{RR}), \mathfrak{A}_\pi(\mathcal{O}_R))$ is split, then the inclusion $(\mathfrak{A}_\pi(\hat{\mathcal{O}}_{RR}), \mathfrak{A}_\pi(\hat{\mathcal{O}}_R))$ is split for $\mathcal{O} \subset \hat{\mathcal{O}}$. Thus $\hat{\Lambda}$ is a standard split inclusion which implies *(1)*.

(2) By *(1)*, it is sufficient to prove *(2)* for the case $\mathcal{O}_1 = \mathcal{O}_2$. If $\Lambda_1 = (\omega_1, \mathcal{O})$ and $\Lambda_2 = (\omega_2, \mathcal{O})$ are standard split inclusions, then we conclude from [20, capter 9] that the tensor product $\Lambda_1 \otimes \Lambda_2 = (\omega_1 \otimes \omega_2, \mathcal{O})$ is also a standard split inclusion which implies *(2)*. Note that the relation $(\mathfrak{A}_{\pi_1} \overline{\otimes} \mathfrak{A}_{\pi_2})(W) = \mathfrak{A}_{\pi_1}(W) \overline{\otimes} \mathfrak{A}_{\pi_2}(W)$ holds for each wedge region W.

$\square$

4.3.2 Construction of Interpolating Automorphisms and Kink States

The results of the previous section can be used to formulate conditions for the existence of interpolating kink states. First, we shall prove the following result:

Theorem 4.3.6 : *Let $\chi \in \mathrm{Sym}(\mathfrak{A})$ be a symmetry, let $\omega \in \mathfrak{S}_0(\mathfrak{A})$ be a vacuum state and let $\mathcal{O}$ be a double cone such that:*

(a) The vacuum ω is χ-invariant: $\omega \circ \chi = \omega$.

(b) The pair $\Lambda = (\omega, \mathcal{O})$ is a standard split inclusion.

Then there exists a canonical χ-interpolating automorphism $\chi^\Lambda \in \mathrm{Aut}(\chi, \mathfrak{A})$ which depends on the inclusion Λ.

Before we prepare the proof, we want to discuss some consequences and applications of Theorem 4.3.6.

The statement of Theorem 4.2.1 tells us that the existence of an α_F-interpolating automorphism $\beta \in \mathrm{Aut}(\alpha_F, \mathfrak{A} \overline{\otimes} \mathfrak{A})$ of the twofold theory $\mathfrak{A} \overline{\otimes} \mathfrak{A}$ is a sufficient condition for the existence of interpolating kink states. We conclude from Theorem 4.3.6:

Corollary 4.3.7 : *If there exists a vacuum state $\omega_0 \in \mathfrak{S}_0(\mathfrak{A})$ such that $\Lambda = (\omega_0, \mathcal{O})$ is a standard split inclusion for a double cone $\mathcal{O}$, then for each pair of vacuum states $\omega_1, \omega_2 \in \mathfrak{S}_0(\mathfrak{A})$ there exists a kink state $\omega \in \mathfrak{S}(\omega_1, \omega_2)$.*

Proof. Let us suppose that $\Lambda = (\omega_0, \mathcal{O})$ is a standard split inclusion. Since the state $\omega_0 \otimes \omega_0$ is α_F-invariant and $\Lambda \otimes \Lambda$ is a standard split inclusion (Proposition 4.3.5), the conditions *(a)* and *(b)* of Theorem 4.3.6 are fulfilled. Hence there is a canonical α_F-interpolating automorphism

$$\beta^\Lambda = \alpha_F^{\Lambda\otimes\Lambda} \in \mathrm{Aut}(\alpha_F, \mathfrak{A} \,\overline{\otimes}\, \mathfrak{A})$$

associated with Λ. Finally, we conclude from Theorem 4.2.1 that the interpolating product

$$\mu_{\beta^\Lambda}(\omega_1 \otimes \omega_2) \in \mathfrak{S}(\omega_1, \omega_2)$$

is a kink state for each pair of vacuum states $\omega_1, \omega_2 \in \mathfrak{S}_0(\mathfrak{A})$. □

Preparation of the proof of Theorem 4.3.6: We assume that ω is a χ-invariant vacuum state and that there is a double cone $\mathcal{O}$ such that $\Lambda = (\omega, \mathcal{O})$ is a standard split inclusion. First, we shall show that the pair (χ, Λ) induces an automorphism χ^Λ of the C*-algebra $C^*(\mathfrak{A})$.

Observation: Since ω is χ-invariant, there is a unitary operator u which implements χ in the GNS-representation π of ω, i.e.:

$$\pi \circ \chi = \mathrm{Ad}(u) \circ \pi \tag{4.26}$$

Let Ψ_Λ be the universal localizing map which corresponds to the standard split inclusion $\Lambda = (\omega, \mathcal{O})$, then

$$u_\Lambda := \Psi_\Lambda(\mathbf{1}_\mathcal{H} \otimes u) \tag{4.27}$$

is a unitary operator which is contained in $\mathfrak{A}_\pi(\mathcal{O}_R)$. Moreover, we have for each $a \in C^*(\mathfrak{A}, \mathcal{O}_{RR})$ and for each $a' \in C^*(\mathfrak{A}, \mathcal{O}_{LL})$:

$$u_\Lambda \pi(a) = \pi(\chi(a)) u_\Lambda \quad \text{and} \quad u_\Lambda \pi(a') = \pi(a') u_\Lambda \ . \tag{4.28}$$

Lemma 4.3.8 : *Let u_Λ be the unitary operator, given above. If the double cone $\hat{\mathcal{O}}$ contains $\mathcal{O}$, then*

$$\mathrm{Ad}(u_\Lambda)(\mathfrak{A}(\hat{\mathcal{O}})) \subset \mathfrak{A}_\pi(\hat{\mathcal{O}}) \ .$$

Proof. We choose an operator $b \in C^*(\mathfrak{A}, \hat{\mathcal{O}}_{RR})$ and we obtain by using equation (4.28):

$$u_\Lambda \pi(a) u_\Lambda^* \pi(b) = u_\Lambda \pi(a\chi(b)) u_\Lambda^* = u_\Lambda \pi(\chi(b)a) u_\Lambda^* = \pi(b) u_\Lambda \pi(a) u_\Lambda^*$$

This implies that the operator

$$u_\Lambda \pi(a) u_\Lambda^*$$

is contained in $\mathfrak{A}_\pi(\hat{\mathcal{O}}_{RR})' = \mathfrak{A}_\pi(\hat{\mathcal{O}}_L)$. On the other hand, $u_\Lambda \pi(a) u_\Lambda^*$ is also contained in $\mathfrak{A}_\pi(\hat{\mathcal{O}}_R)$. By using Haag duality for the net $\mathfrak{A}_\pi$, we finally conclude that

$$u_\Lambda \pi(a) u_\Lambda^* \in \mathfrak{A}_\pi(\hat{\mathcal{O}}_R) \wedge \mathfrak{A}_\pi(\hat{\mathcal{O}}_L) = \mathfrak{A}_\pi(\mathcal{O})$$

which completes the proof. □

Observation: According to Lemma 4.3.8, the automorphism $\mathrm{Ad}(u_\Lambda)$ maps local algebras into local algebras. Since π is a faithful representation of $C^*(\mathfrak{A})$, it follows that

$$\chi^\Lambda : a \longmapsto \pi^{-1}(u_\Lambda \pi(a) u_\Lambda^*)$$

is a well defined automorphism of $C^*(\mathfrak{A})$.

Lemma 4.3.9 : *Let χ^Λ be the automorphism, given above. Then there exists a strongly continuous map $\gamma_\Lambda : \mathbb{R}^2 \to C^*(\mathfrak{A})$ with the properties:*

(1)

$$\mathrm{Ad}(\gamma_\Lambda(x)) = \alpha_x \circ \chi^\Lambda \circ \alpha_{-x} \circ (\chi^\Lambda)^{-1} \quad . \tag{4.29}$$

(2) The operators $\gamma_\Lambda(x)$ fulfill the cocycle condition:

$$\gamma_\Lambda(x+y) = \alpha_x(\gamma_\Lambda(y))\gamma_\Lambda(x) \quad . \tag{4.30}$$

Proof. The translation covariance of the universal localizing map implies $u_{\Lambda(x)} = U(x) u_\Lambda U(-x)$, where U implements the translations in the vacuum representation π. This implies for $\mathcal{O}_{RR} \supset \mathcal{O}_{RR} + x$ and $a \in C^*(\mathfrak{A}, \mathcal{O}_{RR} + x)$:

$$\pi(\chi(a)) = u_\Lambda \pi(a) u_\Lambda^* = u_{\Lambda(x)} \pi(a) u_{\Lambda(x)}^* \quad .$$

Hence $u_{\Lambda(x)}u_\Lambda^*$ is contained in $\mathfrak{A}_\pi(\mathcal{O}_L+x)$. On the other hand $u_{\Lambda(x)}u_\Lambda^*$ is contained in $\mathfrak{A}_\pi(\mathcal{O}_R)$ and we obtain by using Haag duality:

$$u_{\Lambda(x)}u_\Lambda^* \in \mathfrak{A}_\pi(\mathcal{O}_R \cap (\mathcal{O}_L + x)) \quad .$$

For the case $\mathcal{O}_{RR} \subset \mathcal{O}_{RR} + x$, we obtain analogously:

$$u_{\Lambda(x)}u_\Lambda^* \in \mathfrak{A}_\pi((\mathcal{O}_R + x) \cap \mathcal{O}_L) \quad .$$

Writing $x \in \mathbb{R}^2$ as a sum of space-like vectors $x = x_1 + x_2$, $x_1 \in W_+$ and $x_2 \in W_-$, we conclude that for each x the operator $u_{\Lambda(x)}u_\Lambda^*$ is contained in $\mathfrak{A}_\pi(\mathcal{O}_x)$, where $\mathcal{O}_x$ is a sufficiently large double cone. For each x we define the unitary operator

$$\gamma_\Lambda(x) := \pi^{-1}(u_{\Lambda(x)}u_\Lambda^*) \in \mathfrak{A}(\mathcal{O}_x) \quad . \tag{4.31}$$

This implies the relation

$$\alpha_x \circ \chi^\Lambda \circ \alpha_{-x} \circ (\chi^\Lambda)^{-1} = \pi^{-1} \circ \mathrm{Ad}(u_{\Lambda(x)}u_\Lambda^*) \circ \pi = \mathrm{Ad}(\gamma_\Lambda(x)) \quad .$$

It remains to be proven that γ_Λ fulfills the cocycle condition:

$$\begin{aligned}
\gamma_\Lambda(x+y) &= \pi^{-1}(u_{\Lambda(x+y)}u_\Lambda^*) \\
&= \pi^{-1}(u_{\Lambda(x+y)}u_{\Lambda(x)}^* u_{\Lambda(x)}u_\Lambda^*) \\
&= \pi^{-1}(U(x)u_{\Lambda(y)}u_\Lambda^* U(-x) u_{\Lambda(x)}u_\Lambda^*) \\
&= \pi^{-1}(U(x)u_{\Lambda(y)}u_\Lambda^* U(-x))\pi^{-1}(u_{\Lambda(x)}u_\Lambda^*) \\
&= \alpha_x(\gamma_\Lambda(y))\gamma_\Lambda(x)
\end{aligned}$$

which completes the proof. □

Proof of Theorem 4.3.6. According to Lemma 4.3.8 and Lemma 4.3.9, we conclude that χ^Λ is a χ-interpolating automorphism whose interpolation region is localized in $\mathcal{O}$, i.e.

$$\chi^\Lambda|_{C^*(\mathfrak{A},\mathcal{O}_{RR})} = \chi|_{C^*(\mathfrak{A},\mathcal{O}_{RR})} \quad \text{and} \quad \chi^\Lambda|_{C^*(\mathfrak{A},\mathcal{O}_{LL})} = \mathrm{id}_{C^*(\mathfrak{A},\mathcal{O}_{LL})} \quad .$$

□

4.4 Miscellaneous Results

In the present section, we investigate the question how large the class of kink sectors is which can be obtained via our construction scheme. Furthermore, assuming that the constructed kink representations contain massive one-particle representations, we establish a lower bound for the corresponding kink masses.

4.4.1 A Completeness Theorem for Simple Kink Sectors

According to Theorem 4.2.1, a kink sector can be constructed if a pair of vacuum sectors and an α_F-interpolating automorphism is given. But how large is the set of kink sectors which can be obtained in such a way? We shall see that this set contains, for instance, simple kink sectors (see Definition 4.4.2 below).

First, let us establish a preliminary result. We consider a standard split inclusion $\Lambda = (A, B, \Omega)$ on a Hilbert space $\mathcal{H}$ and denote by $\Psi_{\Lambda\otimes\Lambda}$ the universal localizing map with respect to the inclusion $\Lambda \otimes \Lambda$. Moreover, let u_F be the unitary operator, acting on $\mathcal{H} \otimes \mathcal{H}$ by exchanging the factors in the tensor product, i.e.:

$$u_F : \psi_1 \otimes \psi_2 \longmapsto \psi_2 \otimes \psi_1 \quad .$$

The unitary operator

$$\theta_\Lambda := \Psi_{\Lambda\otimes\Lambda}(\mathbf{1} \otimes u_F) \tag{4.32}$$

is contained in $B \overline{\otimes} B$ and implements the flip on $A \overline{\otimes} A$.

Lemma 4.4.1 : *Let θ_Λ be the unitary operator, given by equation (4.32). Then*

$$B' \overline{\otimes} A \subset \theta_\Lambda(\mathfrak{B}(\mathcal{H}) \otimes \mathbf{1})\theta_\Lambda \subset A' \overline{\otimes} B \quad .$$

Proof. If $c \in \mathfrak{B}(\mathcal{H})$, $a \in A$ and $b' \in B'$, then we have

$$\theta_\Lambda(c \otimes \mathbf{1})\theta_\Lambda \; (a \otimes b') = \theta_\Lambda(c \otimes ab')\theta_\Lambda = (a \otimes b') \; \theta_\Lambda(c \otimes \mathbf{1})\theta_\Lambda$$

which implies:

$$\theta_\Lambda(\mathfrak{B}(\mathcal{H}) \otimes \mathbf{1})\theta_\Lambda \subset A' \overline{\otimes} B \quad .$$

On the other hand, we obtain analogously:

$$\theta_\Lambda(\mathbf{1} \otimes \mathfrak{B}(\mathcal{H}))\theta_\Lambda \subset B \overline{\otimes} A'$$

This finally gives:

$$B' \overline{\otimes} A \subset [\theta_\Lambda(\mathbf{1} \otimes \mathfrak{B}(\mathcal{H}))\theta_\Lambda]' = \theta_\Lambda(\mathfrak{B}(\mathcal{H}) \otimes \mathbf{1})\theta_\Lambda \quad .$$

□

Definiton 4.4.2 : We shall call a kink sector $\theta \in \mathrm{sec}(e_1, e_2)$ *simple* if wedge duality holds in a kink representation π which belongs to θ.

Theorem 4.4.3 : *Let $\omega \in \mathfrak{S}(e_1, e_2)$ be a kink state which belongs to a simple sector. If $\Lambda = (\omega, \mathcal{O})$ is a standard split inclusion for a double cone $\mathcal{O}$, then there exists an α_F-interpolating sector $\iota \in \mathrm{sec}(\alpha_F, \mathfrak{A} \overline{\otimes} \mathfrak{A})$ such that the interpolating product*

$$\mu_\iota(e_1 \otimes e_2)$$

is a countably infinite multiple of $[\omega]$.

Proof. Let $(\mathcal{H}, \pi, \Omega)$ be the GNS triple of ω and define $A := \mathfrak{A}_\pi(\mathcal{O}_{RR})$ and $B := \mathfrak{A}_\pi(\mathcal{O}_R)$. According to Lemma 4.4.1, we obtain:

$$\theta_\Lambda(\pi(a) \otimes \mathbf{1})\theta_\Lambda \in \mathfrak{A}_\pi(\mathcal{O}_L) \overline{\otimes} \mathfrak{A}_\pi(\mathcal{O}_R) \quad .$$

Since duality holds in the representation π, we conclude that

$$\beta := (\pi \otimes \pi)^{-1} \circ \mathrm{Ad}(\theta_\Lambda) \circ (\pi \otimes \pi)$$

is a well defined α_F-interpolating automorphism (compare also Theorem 4.3.6). We can choose unitary operators v_1 and v_2, such that

$$\mathrm{Ad}(v_1) \circ \pi|_{C^*(\mathfrak{A}, \mathcal{O}_L)} = \pi_1|_{C^*(\mathfrak{A}, \mathcal{O}_L)} \quad \text{and} \quad \mathrm{Ad}(v_2) \circ \pi|_{C^*(\mathfrak{A}, \mathcal{O}_R)} = \pi_2|_{C^*(\mathfrak{A}, \mathcal{O}_R)}$$

and we obtain by a straight forward computation:

$$\mathrm{Ad}(\theta_\Lambda(v_1^* \otimes v_2^*)) \circ \mu_\beta(\pi_1 \otimes \pi_2) = \pi \otimes \mathbf{1}_\mathcal{H} \quad .$$

□

4.4.2 A Lower Bound Estimate for the Kink Mass

In order to discuss the mass of kink (soliton) states, we consider theories, given by a Haag-Kastler net $\mathfrak{A}$, which possess *massive vacuum states*. Furthermore, we assume the existence of at least one α_F-interpolating automorphism $\beta \in \mathrm{Aut}(\alpha_F, \mathfrak{A} \overline{\otimes} \mathfrak{A})$.

We briefly give the definition of massive vacuum and massive one-particle states here (see also Figure 4.1 for illustration).

Let ω be a translationally covariant state and denote by U an implementation of the translation group in the GNS-representation of ω.

(1) The state ω is called a *massive vacuum state* if the spectrum of (the generator of) U consists of $\{0\}$ and a subset of $C_\mu := \{p \in \mathbb{R}^2 : p^2 > \mu^2; p_0 \geq 0\}$ where $\mu > 0$ is a positive real number, called the *mass gap* of ω. We denote the set of all massive vacuum states with mass gap μ by $\mathfrak{S}_0(\mu)$.

(2) If the spectrum of (the generator of) U consists of the mass shell $H_m := \{p \in \mathbb{R}^2 : p^2 = m^2; p_0 \geq 0\}$ and a subset of $C_{\mu+m}$, then we call ω a *massive one-particle state* with mass $m > 0$.

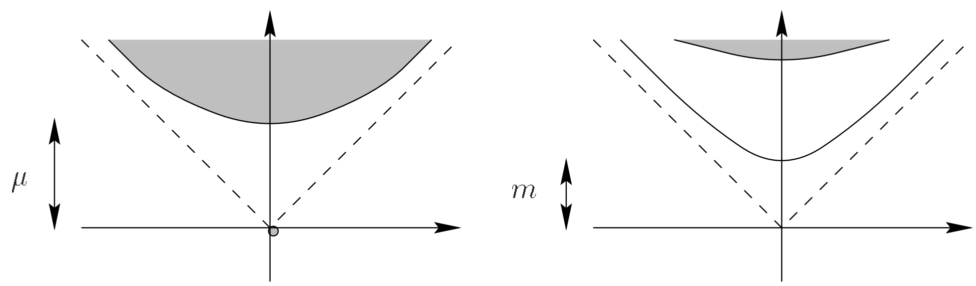

Figure 4.1: The left picture shows the spectrum of a massive vacuum state and the picture on the right side shows the spectrum of a massive one-particle state.

It has been shown [10, 23, 70], that for each massive one-particle state ω (in two space-time dimensions), there exist two massive vacuum states ω_1, ω_2, such that ω interpolates $\omega_1 \in \mathfrak{S}_0(\mu_1)$ and $\omega_2 \in \mathfrak{S}_0(\mu_2)$.

Conversely, if two inequivalent massive vacuum states $\omega_1 \in \mathfrak{S}_0(\mu_1)$ and $\omega_2 \in \mathfrak{S}_0(\mu_2)$ are given, then there exists a kink state ω which interpolates ω_1 and ω_2, namely

$$\omega = \mu_\beta(\omega_1 \otimes \omega_2) \quad .$$

We denote by $S(\pi)$ the spectrum of $U_\pi(x)$, where U_π is a strongly continuous representation of the translation group which implements α_x in the GNS-representation π of ω. Note that, if the vacuum states ω_1 and ω_2 are inequivalent, then it follows that $0 \notin S(\pi)$.

From Proposition 4.1.7, we conclude that $S(\pi)$ is a subset of the closed forward light cone which does not contain the point $k = 0$. Hence we have

$$S(\pi) \subset \frac{1}{2}(S(\pi_1) + S(\pi_2)) \tag{4.33}$$

and obtain the estimate:

$$\inf(S(\pi)) \geq \frac{1}{2}\min(\mu_1, \mu_2) \ . \tag{4.34}$$

Here the infimum $\inf(S(\pi))$ is defined as the the infimum of the spectrum of the mass operator $M = (P_\mu P^\mu)^{1/2}$, where P is the generator of the translation group U_π.

Finally, we conclude that, if the kink sector $[\mu_\beta(\omega_1 \otimes \omega_2)]$ contains a massive one-particle sector, then its corresponding mass m fulfills the lower bound estimate:

$$m \geq \frac{1}{2}\min(\mu_1, \mu_2) \ .$$

5 Kink States in Quantum Field Theory Models

5.1 Interpolating Automorphisms in Quantum Field Theory Models

As we can see from the discussion of Chapter 4, interpolating automorphisms are a key tool in order to construct kink states. In this section we establish sufficient conditions for the existence of interpolating automorphisms for a concrete quantum field theory model.

5.1.1 Preliminary Results

Let us consider a net of Cauchy data $\mathfrak{M}$ which is represented on a separable Hilbert space $\mathcal{H}_0$ (see Section 2.1). According Proposition 2.1.2, a translationally covariant Haag-Kastler net, which we shall denote by $\mathfrak{A}_\alpha$, can be constructed from a given net of Cauchy data $\mathfrak{M}$ and a dynamics $\alpha \in \mathrm{dyn}(\mathfrak{M})$.

Definiton 5.1.1 : We denote by $G(\mathfrak{M})$ the group of unitary operators $u \in \mathfrak{B}(\mathcal{H}_0)$ whose adjoint actions $\chi_u := \mathrm{Ad}(u)$ commute with the spatial translations, i.e.:

$$\chi_u \circ \alpha_{\mathbf{x}} = \alpha_{\mathbf{x}} \circ \chi_u \quad .$$

Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a dynamics of the net $\mathfrak{M}$. Then we define the following subgroup of $G(\mathfrak{M})$:

$$G(\alpha, \mathfrak{M}) := \{u \in G(\mathfrak{M}) | \chi_u \circ \alpha_t = \alpha_t \circ \chi_u \text{ for each } t \in \mathbb{R}.\}$$

Remark: Each operator $u \in G(\alpha, \mathfrak{M})$ induces a symmetry of the Haag-Kastler net $\mathfrak{A}_\alpha$.

We make the following assumptions for the net of Cauchy data $\mathfrak{M}$:

Assumption:

(a) The net $\mathfrak{M}$ fulfills duality, i.e.

$$\mathfrak{M}(\mathcal{I})' = \mathfrak{M}(\mathcal{I}_{LL}) \vee \mathfrak{M}(\mathcal{I}_{RR}) \tag{5.1}$$

(b) There exists a dynamics α_0 and a normalized vector Ω_0 in $\mathcal{H}_0$, such that

$$\omega_0 = \langle \Omega_0, (\cdot)\Omega_0 \rangle$$

is a vacuum state with respect to the dynamics α_0.

(c) For each bounded interval $\mathcal{I}$, the inclusion

$$(\mathfrak{M}(\mathcal{I}_{RR}), \mathfrak{M}(\mathcal{I}_R))$$

is split.

According to our assumption, we conclude form the Theorem of Reeh and Schlieder that Ω_0 is a standard vector for the inclusion $(\mathfrak{M}(\mathcal{I}_{RR}), \mathfrak{M}(\mathcal{I}_R))$ which implies that

$$\Lambda(\mathcal{I}) := (\mathfrak{M}(\mathcal{I}_{RR}), \mathfrak{M}(\mathcal{I}_R), \Omega_0) \tag{5.2}$$

is a standard split inclusion for each interval $\mathcal{I}$. We denote by $\Psi_{\mathcal{I}}$ the universal localizing map of the inclusion $\Lambda(\mathcal{I})$.

Remark: We shall make a few remarks on the assumptions given above.

(i) The results, which we shall establish in the following, remain to be correct if the net of Cauchy data fulfills *twisted duality* instead of duality [62, 78].

(ii) For the application of our analysis to quantum field theory models, like $P(\phi)_2$- or Yukawa$_2$ models, we can choose as Cauchy data tensor products of the time-zero algebras of the massive free Bose or Fermi field. The time-zero algebras of the massive free Bose field fulfill the assumptions *(a)* [63] and *(b)* and we shall proof in Appendix A (compare also [8]) that *(c)* is also fulfilled. Replacing duality by twisted duality, the assumptions *(a)* to *(c)* hold for the massive free Fermi field, too [78].

(iii) The state ω_0 plays the role of a free massive vacuum state, called the bare vacuum.

Proposition 5.1.2 : *Let $u \in G(\mathfrak{M})$ be an operator and let $\mathcal{I}$ be a bounded interval. Then there exists a canonical automorphism $\chi_u^{\mathcal{I}}$ with the properties:*

(1) The relations

$$\chi_u^{\mathcal{I}}|_{C^*(\mathfrak{M},\mathcal{I}_{LL})} = \mathrm{id}_{C^*(\mathfrak{M},\mathcal{I}_{LL})} \quad \textit{and} \quad \chi_u^{\mathcal{I}}|_{C^*(\mathfrak{M},\mathcal{I}_{RR})} = \chi_u|_{C^*(\mathfrak{M},\mathcal{I}_{RR})} \tag{5.3}$$

hold.

(2) There exists a strongly continuous map $\gamma^1_{(u,\mathcal{I})} : \Sigma \to C^*(\mathfrak{M})$ such that:

(i)

$$\mathrm{Ad}(\gamma^1_{(u,\mathcal{I})}(\mathbf{x})) = \alpha_{\mathbf{x}} \circ \chi_u^{\mathcal{I}} \circ \alpha_{-\mathbf{x}} \circ (\chi_u^{\mathcal{I}})^{-1} \ .$$

(ii) The cocycle condition is fulfilled:

$$\gamma^1_{(u,\mathcal{I})}(\mathbf{x}+\mathbf{y}) = \alpha_{\mathbf{x}}(\gamma^1_{(u,\mathcal{I})}(\mathbf{y}))\gamma^1_{(u,\mathcal{I})}(\mathbf{x}) \ .$$

Proof.

(1) In the same manner as in the proof of Lemma 4.3.8, we show that

$$\mathrm{Ad}(\Psi_{\mathcal{I}}(\mathbf{1} \otimes u))(\mathfrak{M}(\hat{\mathcal{I}})) \subset \mathfrak{M}(\hat{\mathcal{I}})$$

if the interval $\hat{\mathcal{I}}$ contains $\mathcal{I}$. This implies that

$$\chi_u^{\mathcal{I}} := \mathrm{Ad}(\Psi_{\mathcal{I}}(\mathbf{1} \otimes u))$$

is a well defined automorphism of $C^*(\mathfrak{M})$. By using the properties of the universal localizing map $\Psi_{\mathcal{I}}$, we conclude that $\chi_u^{\mathcal{I}}$ fulfills equation (5.3).

(2) Using the proof of Lemma 4.3.9, we conclude that the statement *(2)* holds where $\gamma^1_{(u,\mathcal{I})}(\mathbf{x})$ is given by:

$$\gamma^1_{(u,\mathcal{I})}(\mathbf{x}) = \Psi_{\mathcal{I}+\mathbf{x}}(\mathbf{1} \otimes u)\Psi_{\mathcal{I}}(\mathbf{1} \otimes u^*) \ .$$

□

5.1.2 A Non-Local Extension of the Net of Cauchy Data

The automorphisms $\chi_u^{\mathcal{I}}$, which have been constructed in the previous section, seem to be good candidates for χ_u-interpolating automorphisms. To decide whether $\chi_u^{\mathcal{I}}$ is an interpolating automorphism, we have to investigate how $\chi_u^{\mathcal{I}}$ is transformed under the action of a dynamics α.

Let α be a dynamics and $G \subset G(\alpha, \mathfrak{M})$ be a *finite* subgroup. By using the universal localizing map $\Psi_{\mathcal{I}}$, we obtain for each bounded interval $\mathcal{I}$ a unitary representation of G

$$U_{\mathcal{I}} : G \ni g \longmapsto U_{\mathcal{I}}(g) := \Psi_{\mathcal{I}}(\mathbf{1} \otimes g) \in \mathfrak{M}(\mathcal{I}_R) \ .$$

In the previous section it has been shown that $U_{\mathcal{I}}(g)$ implements an automorphism $\chi_g^{\mathcal{I}}$ which is covariant under spatial translations (Proposition 5.1.2). For a dynamics $\alpha \in \mathrm{dyn}(\mathfrak{M})$, we wish to construct a cocycle $\gamma_{(g,\mathcal{I})}$ in order to show that $\chi_g^{\mathcal{I}}$ is an interpolating automorphism. The *formal* operator

$$\gamma_{(g,\mathcal{I})}(t, \mathbf{x}) := \alpha_{(t,\mathbf{x})}(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$$

seems to be a useful Ansatz since it formally implements the automorphism

$$\alpha_{(t,\mathbf{x})} \circ \chi_g^{\mathcal{I}} \circ \alpha_{(-t,-\mathbf{x})} \circ (\chi_g^{\mathcal{I}})^{-1} \ .$$

Unfortunately, the operators $U_{\mathcal{I}}(g)$ are *not* contained in $C^*(\mathfrak{M})$ and the term $\alpha_{(t,\mathbf{x})}(U_{\mathcal{I}}(g))$ has no well defined mathematical meaning. To get a well defined solution for $\gamma_{(g,\mathcal{I})}$, we construct an extension of the net $\mathfrak{M}$ which contains the operators $U_{\mathcal{I}}(g)$ (compare also [62]).

Definiton 5.1.3 : Let $G \subset G(\mathfrak{M})$ be a compact sub-group. The net $\mathfrak{M} \rtimes G$ is defined by the assignment

$$\mathfrak{M} \rtimes G : \mathcal{I} \mapsto (\mathfrak{M} \rtimes G)(\mathcal{I}) := \mathfrak{M}(\mathcal{I}) \vee U_{\mathcal{I}}(G)'' \ .$$

To investigate the properties of the net $\mathfrak{M} \rtimes G$, we briefly explain the notion of *crossed products* of von Neumann algebras by compact groups. A detailed description can be found in text books, for example [7]. We also refer to [46].

(i) Let us consider a W*-algebra A, represented on a separable Hilbert space $\mathcal{H}$, and a compact group G, acting by automorphisms $\chi_g \in \mathrm{Aut}(A)$ on A. We

denote by $K(G,\mathcal{H})$ the vector space of all $\mathcal{H}$-valued continuous functions on G. Consider the inner product

$$\langle \xi | \zeta \rangle := \int \mathrm{d}\mu(g) \, \langle \xi(g), \zeta(g) \rangle$$

where ξ and ζ are functions in $K(G,\mathcal{H})$ and μ denotes the normalized Haar measure on G. We denote by $L_2(G,\mathcal{H})$ the completion of $K(G,\mathcal{H})$ with respect to the inner product $\langle \cdot | \cdot \rangle$.

(ii) On the Hilbert space $L_2(G,\mathcal{H})$ we define representations π of A and U of G as follows:

$$(\pi(a)\xi)(g) := \chi_g^{-1}(a)\xi(g) \ \ , \ \ a \in A$$

$$(U(g_1)\xi)(g) := \xi(g_1^{-1}g)$$

Denote by 1 the unit of G and by $\mathbf{1}$ the unit of A. Obviously, $U(1) = \pi(\mathbf{1})$ is the identity operator on $L_2(G,\mathcal{H})$. The *crossed product* $A \rtimes G$ of A by G is the von Neumann algebra on $L_2(G,\mathcal{H})$ which is generated by $\pi(A)$ and $U(G)$.

For a compact subgroup $G \subset G(\mathfrak{M})$ and a bounded interval $\mathcal{I}$, G acts by automorphisms on $\mathfrak{M}(\mathcal{I})$ via the group homomorphism

$$\chi^{\mathcal{I}} : G \ni g \longmapsto \chi_g^{\mathcal{I}} \in \mathrm{Aut}(\mathfrak{M}(\mathcal{I}))$$

and we can construct the crossed product $\mathfrak{M}(\mathcal{I}) \rtimes G$.

Proposition 5.1.4 : *Let $\mathcal{I}$ be a bounded interval, then the map*

$$\pi^{\mathcal{I}} : \mathfrak{M}(\mathcal{I}) \rtimes G \ni a \cdot g \longmapsto a \, U_{\mathcal{I}}(g) \in \mathfrak{M}(\mathcal{I}) \vee U_{\mathcal{I}}(G)''$$

is a faithful representation of the crossed product $\mathfrak{M}(\mathcal{I}) \rtimes G$.

Proof. First, we easily observe that $\pi^{\mathcal{I}}$ is a well defined representation of $\mathfrak{M}(\mathcal{I}) \rtimes G$. According to [46, Theorem 2.2, Corollary 2.3], we conclude that the crossed product $\mathfrak{M}(\mathcal{I}) \rtimes G$ is isomorphic to the von Neumann algebra $\mathfrak{M}(\mathcal{I}) \vee U_{\mathcal{I}}(G)''$ and $\pi^{\mathcal{I}}$ is a W*-isomorphism. □

Definiton 5.1.5 : A one parameter automorphism group α, which satisfies the conditions, listed below, is called a G-dynamics of the extended net $\mathfrak{M} \rtimes G$.

(a) α is a dynamics of the net $\mathfrak{M} \rtimes G$ (Definition 2.1.1).

(b) The automorphisms α_t commute with the automorphisms χ_g, i.e.

$$\alpha_t \circ \chi_g = \chi_g \circ \alpha_t \ ; \text{ for each } t \in \mathbb{R} \text{ and for each } g \in G.$$

The set of all G-dynamics of $\mathfrak{M} \rtimes G$ is denoted by $\mathrm{dyn}_G(\mathfrak{M} \rtimes G)$.

Proposition 5.1.6 : *Let $\alpha \in \mathrm{dyn}_G(\mathfrak{M} \rtimes G)$ be a G-dynamics and $\mathcal{I}$ be a bounded interval. Then the operator*

$$\gamma^0_{(g,\mathcal{I})}(t) := \alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$$

is contained in $\mathfrak{M}(\mathcal{I}_{|t|})$ where $\mathcal{I}_{|t|}$ denotes the enlarged interval $\mathcal{I} + (-|t|, |t|)$ and the operator

$$\gamma_{(g,\mathcal{I})}(t, \mathbf{x}) := \alpha_{(t,\mathbf{x})}(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$$

fulfills the cocycle condition of Definition 4.1.2.

Proof. For $a \in C^*(\mathfrak{M}, \mathcal{I}_{|t|,RR})$, the operator $\alpha_{-t}(a)$ is contained in $C^*(\mathfrak{M}, \mathcal{I}_{RR})$ which implies

$$\begin{aligned} a\ \alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^* &= \alpha_t(\alpha_{-t}(a)U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^* \\ &= \alpha_t(U_{\mathcal{I}}(g)\chi_g\alpha_{-t}(a))U_{\mathcal{I}}(g)^* \\ &= \alpha_t(U_{\mathcal{I}}(g)\alpha_{-t}\chi_g(a))U_{\mathcal{I}}(g)^* \\ &= \alpha_t(U_{\mathcal{I}}(g))\chi_g(a)U_{\mathcal{I}}(g)^* \\ &= \alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*\ a \end{aligned}$$

and we conclude:

$$\alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^* \in C^*(\mathfrak{M}, \mathcal{I}_{|t|,RR})' = \mathfrak{M}(\mathcal{I}_{|t|,L})$$

By a similar argument, $\alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$ is contained in $\mathfrak{M}(\mathcal{I}_{|t|,R})$ and we conclude from duality that it is contained in $\mathfrak{M}(\mathcal{I}_{|t|})$. The cocycle condition for $\gamma_{(g,\mathcal{I})}$ is obviously fulfilled and the proposition follows. □

5.1.3 A Criterion for the Existence of Interpolating Automorphisms

Definiton 5.1.7 : Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a dynamics and $G \subset G(\mathfrak{M})$ be a compact subgroup. We shall call α *G-extendible* if there exists a G-dynamics $\hat{\alpha}$ of the extended net $\mathfrak{M} \rtimes G$, such that

$$\hat{\alpha}_t|_{C^*(\mathfrak{M})} = \alpha_t$$

for each $t \in \mathbb{R}$.

We are now prepared to formulate a criterion for the existence of interpolating automorphisms.

Theorem 5.1.8 : *If $\alpha \in \mathrm{dyn}(\mathfrak{M})$ is a G-extendible dynamics, then the automorphism $\chi_g^{\mathcal{I}}$, which can be constructed by Proposition 5.1.2, is a χ_g-interpolating automorphism of $\mathfrak{A}_\alpha$.*

Proof. As postulated, there exists an extension $\hat{\alpha} \in \mathrm{dyn}_G(\mathfrak{M} \rtimes G)$ of α. We show that for each $g \in G$ the operator

$$\gamma^0_{(g,\mathcal{I})}(t) := \hat{\alpha}_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$$

implements the automorphism

$$\alpha_t \circ \chi_g^{\mathcal{I}} \circ \alpha_{-t} \circ (\chi_g^{\mathcal{I}})^{-1}$$

on $C^*(\mathfrak{M})$. Indeed, we have for each $a \in C^*(\mathfrak{M})$:

$$\begin{aligned}
\mathrm{Ad}(\gamma^0_{(g,\mathcal{I})}(t))a &= \hat{\alpha}_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*\, a\, U_{\mathcal{I}}(g)\hat{\alpha}_t(U_{\mathcal{I}}(g))^* \\
&= \hat{\alpha}_t(U_{\mathcal{I}}(g))\,(\chi_g^{\mathcal{I}})^{-1}(a)\,\hat{\alpha}_t(U_{\mathcal{I}}(g))^* \\
&= \hat{\alpha}_t\Big(U_{\mathcal{I}}(g)\alpha_{-t}\Big((\chi_g^{\mathcal{I}})^{-1}(a)\Big)U_{\mathcal{I}}(g)^*\Big) \\
&= \alpha_t\Big(U_{\mathcal{I}}(g)\alpha_{-t}\Big((\chi_g^{\mathcal{I}})^{-1}(a)\Big)U_{\mathcal{I}}(g)^*\Big) \\
&= \alpha_t \circ \chi_g^{\mathcal{I}} \circ \alpha_{-t} \circ (\chi_g^{\mathcal{I}})^{-1}(a)
\end{aligned}$$

Finally we conclude from Proposition 5.1.6 that $\chi_g^{\mathcal{I}}$ is a χ_g-interpolating automorphism. □

The dynamics α of $P(\phi)_2$- and Yukawa$_2$ models are locally implementable by unitary operators. More precisely, for each bounded interval $\mathcal{I}$ and for each positive number $\tau > 0$, there exists a unitary operator $u(\mathcal{I}, \tau|t)$ with the properties:

(1) If $|t_1|, |t_2|, |t_1 + t_2| < \tau$, then we have

$$u(\mathcal{I}, \tau|t_1 + t_2) = u(\mathcal{I}, \tau|t_1)u(\mathcal{I}, \tau|t_2) \ .$$

(2) For $|t| < \tau$, the operator $u(\mathcal{I}, \tau|t)$ implements α_t on $\mathfrak{M}(\mathcal{I})$, i.e.:

$$\alpha_t(a) = u(\mathcal{I}, \tau|t)\ a\ u(\mathcal{I}, \tau|t)^* \ ; \ \text{ for each } a \in \mathfrak{M}(\mathcal{I}). \tag{5.4}$$

Let $G \subset G(\alpha, \mathfrak{M})$ be a compact sub-group. In order to show that α is G-extendible, it is sufficient to prove that the operators

$$u(\mathcal{I}_1, \tau|t)U_{\mathcal{I}}(g)u(\mathcal{I}_1, \tau|t)^* \ ,$$

which are the obvious candidates for $\hat{\alpha}(U_I(g))$, are independent of $\mathcal{I}_1$ for $\mathcal{I}_1 \supset \mathcal{I}$ and $|t| \leq \tau$.

Lemma 5.1.9 : *If for each $\mathcal{I} \subset \mathcal{I}_1$, for each $\tau < \tau_1$ and for each $g \in G$ the equation*

$$u(\mathcal{I}, \tau|t)U_{\mathcal{I}}(g)u(\mathcal{I}, \tau|t)^* = u(\mathcal{I}_1, \tau_1|t)U_{\mathcal{I}}(g)u(\mathcal{I}_1, \tau_1|t)^* \tag{5.5}$$

holds, then the dynamics α is G-extendible. Here $u(\mathcal{I}, \tau|t)$ are unitary operators which fulfill equation (5.4).

Proof. Let $(\mathcal{I}_n, \tau_n)_{n\in\mathbb{N}}$ be a sequence, such that $\lim_n \mathcal{I}_n = \mathbb{R}$ and $\lim_n \tau_n = \infty$. We conclude from our assumption (equation (5.5)) that the uniform limit

$$\hat{\alpha}_t(a) := \lim_{n\to\infty} \mathrm{Ad}(u(\mathcal{I}_n, \tau_n|t))(a)$$

exists. Thus $\hat{\alpha} : t \longmapsto \hat{\alpha}_t$ is a well defined one-parameter automorphism group, extending the dynamics α. It remains to be proven that $\hat{\alpha}$ has propagation speed

$\mathrm{ps}(\hat{\alpha}) \leq 1$. Since $\hat{\alpha}$ is an extension of α and $\mathrm{ps}(\alpha) \leq 1$, we conclude for each $a \in C^*(\mathfrak{M}, \mathcal{I}_{|t|,RR})$ and for each $b \in C^*(\mathfrak{M}, \mathcal{I}_{|t|,LL})$:

$$\begin{aligned} ab\, \hat{\alpha}_t(U_{\mathcal{I}}(g)) &= \hat{\alpha}_t(\alpha_{-t}(a)\alpha_{-t}(b)U_{\mathcal{I}}(g)) \\ &= \hat{\alpha}_t(U_{\mathcal{I}}(g)\alpha_{-t}\chi_g(a)\alpha_{-t}(b)) \\ &= \hat{\alpha}_t(U_{\mathcal{I}}(g))\, \chi_g(a)b \end{aligned}$$

Thus the operator $\hat{\alpha}_t(U_{\mathcal{I}}(g))$ is contained in $\mathfrak{M}(\mathcal{I}_{|t|,R})$ and implements χ_g on $\mathfrak{M}(\mathcal{I}_{|t|,RR})$. This finally implies:

$$\hat{\alpha}_t(U_{\mathcal{I}}(g))U_{\mathcal{I}_{|t|}}(g)^* \in \mathfrak{M}(\mathcal{I}_{|t|})$$

and the lemma follows. □

5.1.4 A Criterion for the Existence of Interpolating Kink States

Let us consider the twofold W*-tensor product of the net of Cauchy data, i.e.:

$$\mathfrak{M} \overline{\otimes} \mathfrak{M} : \mathcal{I} \longmapsto \mathfrak{M}(\mathcal{I}) \overline{\otimes} \mathfrak{M}(\mathcal{I})$$

Observation:

(i) If the net $\mathfrak{M}$ fulfills the conditions *(a)* to *(c)* of Section 5.1.1, then the net $\mathfrak{M} \overline{\otimes} \mathfrak{M}$ fulfills them, too.

(ii) Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a dynamics of $\mathfrak{M}$, then $\alpha^{\otimes 2}$ is a dynamics of $\mathfrak{M} \overline{\otimes} \mathfrak{M}$. Note that the flip operator u_F, which is given by

$$u_F : \mathcal{H}_0 \otimes \mathcal{H}_0 \longrightarrow \mathcal{H}_0 \otimes \mathcal{H}_0 \;\; ; \;\; \psi_1 \otimes \psi_2 \longmapsto \psi_2 \otimes \psi_1$$

is contained in $G(\alpha^{\otimes 2}, \mathfrak{M} \overline{\otimes} \mathfrak{M})$. Hence u_F induces an embedding of $\mathbb{Z}_2$ into $G(\alpha^{\otimes 2}, \mathfrak{M} \overline{\otimes} \mathfrak{M})$.

(iii) According to Section 5.1.2, we can construct a non-local extension

$$\hat{\mathfrak{M}} := (\mathfrak{M} \overline{\otimes} \mathfrak{M}) \rtimes \mathbb{Z}_2$$

of the twofold net $\mathfrak{M} \overline{\otimes} \mathfrak{M}$. Let $\Psi_{\mathcal{I}}$ be the universal localizing map of the standard split inclusion

$$\Lambda(\mathcal{I}) \otimes \Lambda(\mathcal{I}) = (\mathfrak{M}(\mathcal{I}_{RR})^{\overline{\otimes}\, 2}, \mathfrak{M}(\mathcal{I}_R)^{\overline{\otimes}\, 2}, \Omega_0 \otimes \Omega_0)$$

and define $\theta_{\mathcal{I}} := \Psi_{\mathcal{I}}(\mathbf{1} \otimes u_F)$. Then the algebra $\hat{\mathfrak{M}}(\mathcal{I})$ is simply given by

$$\hat{\mathfrak{M}}(\mathcal{I}) = ((\mathfrak{M} \overline{\otimes} \mathfrak{M}) \rtimes \mathbb{Z}_2)(\mathcal{I}) = (\mathfrak{M}(\mathcal{I}) \overline{\otimes} \mathfrak{M}(\mathcal{I})) \vee \{\theta_{\mathcal{I}}\}'' \quad .$$

(iv) By Proposition 5.1.2, there exists a canonical automorphism

$$\beta^{\mathcal{I}} := \mathrm{Ad}(\theta_{\mathcal{I}}) \ , \tag{5.6}$$

associated with the pair $(u_F, \mathcal{I})$.

Notation: Let α be a dynamics of $\mathfrak{M}$. In the sequel, we shall call α *extendible* if $\alpha^{\otimes 2}$ is $\mathbb{Z}_2$-extendible.

Theorem 5.1.10 : *Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be an extendible dynamics, then for each pair of vacuum states $\omega_1, \omega_2 \in \mathfrak{S}_0(\mathfrak{A}_\alpha)$, the state*

$$\omega = \mu_{\beta^{\mathcal{I}}}(\omega_1 \otimes \omega_2)$$

is a kink state.

Proof. We conclude from an application of Theorem 5.1.8 that $\beta^{\mathcal{I}}$ (equation (5.6)) is an α_F-interpolating automorphism and the statement follows from Theorem 4.2.1. $\square$

5.2 Application to Quantum Field Theory Models

We show that a sufficient condition for the existence of interpolating automorphisms, i.e. the extendibility of the dynamics, is satisfied for the $P(\phi)_2$, the Yukawa$_2$ and special types of Wess-Zumino models.

5.2.1 Kink States in $P(\phi)_2$-Models

We shall show that the dynamics of $P(\phi)_2$-models are extendible. As described in Section 2.2 the dynamics of a $P(\phi)_2$-model consists of two parts.

(1) The first part is given by the free dynamics α_0, with propagation speed $\mathrm{ps}(\alpha_0) = 1$,

$$\alpha_{0,t}(a) = e^{iH_0 t} a e^{-iH_0 t}$$

where $(H_0, D(H_0))$ is the free Hamiltonian which is a self-adjoint operator on the domain $D(H_0) \subset \mathcal{H}_0$.

(2) The second part is a dynamics α_1 with propagation speed $\mathrm{ps}(\alpha_1) = 0$, i.e. $\alpha_{1,t}$ maps each local algebra $\mathfrak{M}(\mathcal{I})$ onto itself. The interaction part of the full Hamiltonian is given by a Wick polynomial of the time-zero field ϕ:

$$H_1(\mathcal{I}) = H_1(\chi_{\mathcal{I}}) = \int \mathrm{dx} \ :P(\phi(\mathbf{x})): \ \chi_{\mathcal{I}}(\mathbf{x})$$

where $\chi_{\mathcal{I}}$ is a smooth test function which is one on $\mathcal{I}$ and zero on the complement of a slightly lager region $\hat{\mathcal{I}} \supset \mathcal{I}$. The unitary operator $\exp(itH_1(\mathcal{I}))$ implements the dynamics α_1 locally, i.e. for each $a \in \mathfrak{M}(\mathcal{I})$ we have:

$$\alpha_{1,t}(a) := e^{iH_1(\mathcal{I})t} a e^{-iH_1(\mathcal{I})t} \quad .$$

Definiton 5.2.1 : An operator valued distribution $v : S(\mathbb{R}) \to L(\mathcal{H}_0)$ is called an *ultra local interaction*, if the following conditions are fulfilled:

(1) For each real valued test function $f \in S(\mathbb{R})$, $v(f)$ is self-adjoint and has a common core with H_0.

(2) Let $f \in S(\mathbb{R})$ be a real valued test function with support in a bounded interval $\mathcal{I}$, then the spectral projections of $v(f)$ are contained in $\mathfrak{M}(\mathcal{I})$.

(3) For each pair of test functions $f_1, f_2 \in S(\mathbb{R})$, the spectral projections of $v(f_1)$ commute with the spectral projections of $v(f_2)$.

Remark: It has been proven in [37], that the Wick polynomials of the time zero fields are ultra local interactions. Furthermore, each ultra local interaction v induces a dynamics $\alpha^v \in \mathrm{dyn}(\mathfrak{M})$ with propagation speed $\mathrm{ps}(\alpha^v) = 0$. Let $\mathcal{I}$ be a bounded interval and let $\chi_{\mathcal{I}} \in S(\mathbb{R})$ be a positive test function with $\chi_{\mathcal{I}}(\mathbf{x}) = 1$ for each $x \in \mathcal{I}$. Indeed, by an application of J. Glimm's and A. Jaffe's analysis [37], we conclude that the automorphisms

$$\alpha_t^v : \mathfrak{M}(\mathcal{I}) \longrightarrow \mathfrak{M}(\mathcal{I}) \ ; \ a \longmapsto \mathrm{Ad}(\exp(itv(\chi_{\mathcal{I}})))a$$

define a dynamics with zero propagation speed. In the sequel, we shall call a dynamics α^v *ultra local* if it is induced by an ultra local interaction v.

In order to prove that a dynamics α, which is given by the Trotter product

$$\alpha = \alpha_0 \times \alpha^v$$

of a free and an ultra local dynamics, is extendible, we show that each part of the dynamics can be extended separately.

Since the free part of the dynamics can be extended to the algebra $\mathfrak{B}(\mathcal{H}_0)$ of all bounded operators on the Fock space $\mathcal{H}_0$, it is obvious that α_0 is extendible. Thus it remains to be proven the following:

Lemma 5.2.2 : *Each ultra local dynamics $\alpha^v \in \mathrm{dyn}(\mathfrak{M})$ is extendible.*

Proof. Let us consider any ultra local interaction v. For each test function $f \in S(\mathbb{R})$, we introduce the unitary operator

$$u(f|t) := e^{itv(f)} \otimes e^{itv(f)} \quad .$$

Let $\mathcal{I}$ be a bounded interval and denote by $\mathcal{I}_\epsilon$, $\epsilon > 0$, the enlarged interval $\mathcal{I} + (-\epsilon, \epsilon)$. We choose test functions $\chi^{(\mathcal{I},\epsilon)} \in S(\mathbb{R})$ such that

$$\chi^{(\mathcal{I},\epsilon)}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \mathcal{I} \\ 0 & \mathbf{x} \in \mathcal{I}_\epsilon^c = \mathcal{I}_\epsilon \backslash \mathbb{R} \end{cases} \quad .$$

For an interval $\hat{\mathcal{I}} \supset \mathcal{I}_\epsilon$, the region $\hat{\mathcal{I}}_\epsilon \backslash \mathcal{I}_\epsilon$ consists of two connected components $(\hat{\mathcal{I}}_\epsilon \backslash \mathcal{I}_\epsilon)_\pm$ and there exist test functions $\chi^\pm \in S(\mathbb{R})$ with

$$\begin{aligned} &\mathrm{supp}(\chi^-) \subset (\hat{\mathcal{I}}_\epsilon \backslash \mathcal{I}_\epsilon)_- \subset \mathcal{I}_{LL} \\ &\mathrm{supp}(\chi^+) \subset (\hat{\mathcal{I}}_\epsilon \backslash \mathcal{I}_\epsilon)_+ \subset \mathcal{I}_{RR} \\ &\chi^{(\hat{\mathcal{I}},\epsilon)} - \chi^{(\mathcal{I},\epsilon)} = \chi^+ + \chi^- \quad . \end{aligned}$$

Let us write

$$u(\mathcal{I}, \epsilon|t) := u(\chi^{(\mathcal{I},\epsilon)}|t) \ \text{ and } \ u_\pm(t) := u(\chi^\pm|t).$$

Since we have $[u(f_1|t), u(f_2|t)] = 0$ for any pair of test functions $f_1, f_2 \in S(\mathbb{R})$, we obtain for each $\epsilon > 0$ and for $\mathcal{I}_\epsilon \subset \hat{\mathcal{I}}$:

$$u(\hat{\mathcal{I}}, \epsilon|t) = u(\mathcal{I}, \epsilon|t)u_-(t)u_+(t) \tag{5.7}$$

If we make use of the fact that $u_+(t)$ is α_F-invariant and localized in $\mathcal{I}_{RR}$, we conclude that $\theta_{\mathcal{I}}$ and $u_\pm(t)$ commute. Thus we obtain

$$\mathrm{Ad}(u(\hat{\mathcal{I}}, \epsilon|t))\theta_{\mathcal{I}} = \mathrm{Ad}(u(\mathcal{I}, \epsilon|t))\theta_{\mathcal{I}} \tag{5.8}$$

which depends only of the localization interval $\mathcal{I}$ since $\epsilon > 0$ can be chosen arbitrarily small. According to Lemma 5.1.9, the automorphisms

$$\hat{\alpha}_t^v : \hat{\mathfrak{M}}(\mathcal{I}) \ni a \longmapsto \mathrm{Ad}(u(\mathcal{I}, \epsilon|t))a \in \hat{\mathfrak{M}}(\mathcal{I})$$

define a dynamics of $\hat{\mathfrak{M}}$ whose restriction to $\mathfrak{M} \mathbin{\overline{\otimes}} \mathfrak{M}$ is $\alpha^v \otimes \alpha^v$. Thus α^v is extendible. □

If $\hat{\alpha}_0$ denotes the natural extension of the free dynamics $\alpha_0^{\otimes 2}$ to $\hat{\mathfrak{M}}$ and let $\hat{\alpha}^v$ be the extension of the ultra local dynamics $\alpha^v \otimes \alpha^v$ then, by using the Trotter product, we conclude that the dynamics

$$\hat{\alpha} := \hat{\alpha}_0 \times \hat{\alpha}^v$$

is an extension of the dynamics $(\alpha_0 \times \alpha^v)^{\otimes 2}$ to $\hat{\mathfrak{M}}$. This leads to the following result:

Proposition 5.2.3 : *Each dynamics of a $P(\phi)_2$-model is extendible.*

Proof. The statement follows from Lemma 5.2.2 and from the fact that each dynamics of a $P(\phi)_2$-model is a Trotter product of the free dynamics α_0 and an ultra local dynamics α_1. □

The existence of interpolating kink states in $P(\phi)_2$-models is an immediate consequence of Proposition 5.2.3.

Corollary 5.2.4 : *Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a dynamics of a $P(\phi)_2$-model. Then for each pair of vacuum states $\omega_1, \omega_2 \in \mathfrak{S}_0(\alpha, \mathfrak{M})$ there exists an interpolating kink state $\omega \in \mathfrak{S}(\omega_1, \omega_2)$.*

Proof. By Proposition 5.2.3 each dynamics of a $P(\phi)_2$-model is extendible and we can apply Theorem 5.1.8 which implies the result. □

5.2.2 The Dynamics of the Yukawa$_2$ Model

Since the dynamics of a Yukawa$_2$-like model can not be written as a Trotter product which consists of a free and an ultra local dynamics, it is a bit more complicated to show that these dynamics are extendible. We briefly summarize here the construction of the Yukawa$_2$ dynamics which has been carried out by J. Glimm and A. Jaffe [37]. We also refer to the work of R. Schrader [74, 75].

Let $\mathfrak{M}_s$ and $\mathfrak{M}_a$ be the nets of Cauchy data for the free Bose and Fermi field, represented on the Fock spaces $\mathcal{H}_s$ and $\mathcal{H}_a$ respectively. The Cauchy data of the Yukawa$_2$ model are given by the W*-tensor product $\mathfrak{M} := \mathfrak{M}_s \overline{\otimes} \mathfrak{M}_a$ of the nets $\mathfrak{M}_s$ and $\mathfrak{M}_a$. Moreover, we set $\mathcal{H}_0 := \mathcal{H}_s \otimes \mathcal{H}_a$.

Step 1: In the first step, a Hamiltonian, which is regularized by an UV-cutoff $c_0 > 0$ and an IR-cutoff $c_1 > 1$, $c_0 << c_1$, is constructed. For this purpose, one chooses test functions $\delta_{c_0}, \chi_{c_1} \in S(\mathbb{R})$ with the properties:

(a)

$$\mathrm{supp}(\delta_{c_0}) \subset (-c_0, c_0) \quad \text{and} \quad \int \mathrm{dx}\; \delta_{c_0}(\mathbf{x}) = 1$$

(b)

$$\mathrm{supp}(\chi_{c_1}) \subset (-c_1 - 1, c_1 + 1) \quad \text{and} \quad \chi_{c_1}(\mathbf{x}) = 1 \quad \text{for each } \mathbf{x} \in (-c_1, c_1).$$

The UV-regularized fields are given by

$$\phi(c_0, \mathbf{x}) := (\phi * \delta_{c_0})(\mathbf{x}) \quad \text{and} \quad \psi(c_0, \mathbf{x}) := (\psi * \delta_{c_0})(\mathbf{x}) \tag{5.9}$$

where ϕ is a massive free Bose field and ψ a free Dirac spinor field at $t = 0$. The fields, defined by equation (5.9), act on $\mathcal{H}_0$ via the operators

$$\Phi(c_0, \mathbf{x}) := \phi(c_0, \mathbf{x}) \otimes \mathbf{1}_{\mathcal{H}_a} \quad \text{and} \quad \Psi(c_0, \mathbf{x}) := \mathbf{1}_{\mathcal{H}_s} \otimes \psi(c_0, \mathbf{x}) \ .$$

The regularized Hamiltonian $H(c_0, c_1)$ can be written as a sum of three parts:

(1) The free Hamiltonian H_0 which is given by

$$H_0 = H_{0,s} \otimes \mathbf{1}_{\mathcal{H}_a} + \mathbf{1}_{\mathcal{H}_s} \otimes H_{0,a}$$

where $H_{0,s}$ and $H_{0,a}$ are the free Hamilton operators of the Bose and the Fermi field respectively.

(2) The regularized Yukawa interaction term:

$$H_Y(c_0, c_1) = \int \mathrm{d}\mathbf{x}\ \chi_{c_1}(\mathbf{x})\ \Phi(c_0, \mathbf{x})\ :\bar{\Psi}(c_0, \mathbf{x})\Psi(c_0, \mathbf{x}):$$

(3) The counterterms:

$$H_C(c_0, c_1) = \sum_{n=0}^{N} z_n(c_0) \int \mathrm{d}\mathbf{x}\ \chi_{c_1}(\mathbf{x})\ :\Phi(\mathbf{x})^n:$$

where $z_n(c_0)$ are suitable renormalization constants.

The following statement has been established by J.Glimm and A.Jaffe [37, 39]:

Theorem 5.2.5 : *The counterterms $H_C(c_0, c_1)$ can be chosen in such a way that*

(1) the cutoff Hamiltonian $H(c_0, c_1) = (H_0 + H_Y(c_0, c_1) + H_C(c_0, c_1))^{**}$ is a positive and self adjoint operator with domain $D(H_0)$.

(2) The uniform limit

$$R(c_1, \zeta) = \lim_{c_0 \to 0} (H(c_0, c_1) - \zeta)^{-1}$$

is the resolvent of a self adjoint operator $H(c_1)$.

(3) $H(c_1)$ is the limit of $H(c_0, c_1)$ in the strong graph topology.

Notation: In the sequel, we shall use the following notation:

$$u(c_0, c_1, t) := \exp(itH(c_0, c_1)) \quad \text{and} \quad u(c_1, t) := \exp(itH(c_1))\ .$$

Remark: The aim is to show that $H(c_1)$ induces a dynamics of $\mathfrak{M}$, given locally by the equation

$$\alpha_t|_{\mathfrak{M}(\mathcal{I})} = \mathrm{Ad}(\ u(c_1,t)\ ) \quad \text{for}\ \mathcal{I}_{|t|} := \mathcal{I} + (-|t|,|t|) \subset (-c_1,c_1).$$

However, in comparison to the $P(\phi)_2$-models, there are some more technical difficulties which have to be overcome.

(i) The Hamiltonian $H(c_1)$ is only defined as a limit of the Hamiltonians $H(c_0,c_1)$ and it has no mathematical meaning when written as a sum

$$H_0 + H_Y(c_1) + H_C(c_1) \quad .$$

Thus the construction scheme for a dynamics, as it has been used for $P(\phi)_2$-models, does not apply.

(ii) On the other hand, one might try to apply $P(\phi)_2$-like methods to the Hamiltonian $H(c_0,c_1)$, for which the UV-cutoff is not removed. For this purpose, one wishes to write $H(c_0,c_1)$ as a sum $H(c_0,c_1) = H_1(c_0,c_1) + H_2(c_0,c_1)$ where $H_1(c_0,c_1)$ induces a dynamics α_1 with propagation speed $\mathrm{ps}(\alpha_1) \leq 1$ and $H_2(c_0,c_1)$ induces a dynamics α_2 with propagation speed $\mathrm{ps}(\alpha_2) = 0$.

The difficulty with writing such a decomposition for $H(c_0,c_1)$ arises from the fact that the Yukawa interaction term $H_Y(c_0,c_1)$ induces an automorphism group with infinite propagation speed.

Step 2: In the next step, one introduces test functions $\chi_{(\mathcal{I},s,c_0)}$ (see Figure 5.1), depending on a bounded interval $\mathcal{I}$, a real number $s > 0$ and the UV-cutoff c_0, fulfilling the conditions

$$\begin{aligned} &\mathrm{supp}(\chi_{(\mathcal{I},s,c_0)}) \subset \mathcal{I}_{2c_0+|s|+\epsilon} \backslash \mathcal{I}_{|s|-\epsilon} \quad \text{and} \\ &\chi_{(\mathcal{I},s,c_0)}(\mathbf{x}) = 1 \quad \text{if}\ \mathbf{x} \in \mathcal{I}_{2c_0+|s|} \backslash \mathcal{I}_{|s|}. \end{aligned} \tag{5.10}$$

Here $\epsilon << c_0$ is any sufficiently small positive number. The Hamiltonian $H(c_0,c_1)$ is replaced by the operator

$$H(\mathcal{I},s,c_0,c_1) := H_0 + H_C(c_0,c_1) + H_Y(\mathcal{I},s,c_0,c_1) \tag{5.11}$$

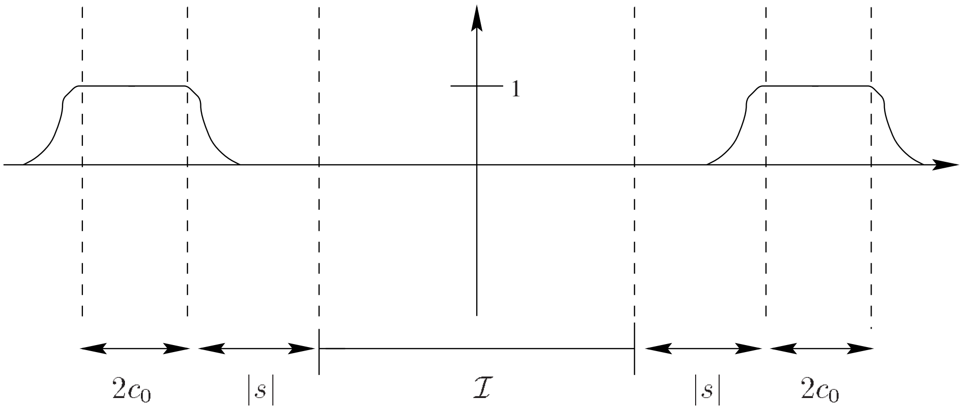

Figure 5.1: The figure shows the graph of the function $\chi_{(\mathcal{I},s,c_0)}$.

depending additionally on $\mathcal{I}$ and s, where $H_Y(\mathcal{I}, s, c_0, c_1)$ is given by

$$H_Y(\mathcal{I}, s, c_0, c_1) := \int \mathrm{d}\mathbf{x}\ \Phi(c_0, \mathbf{x})\ :\bar{\Psi}(c_0, \mathbf{x})\Psi(c_0, \mathbf{x}):\ (\ \chi_{c_1}(\mathbf{x}) - \chi_{(\mathcal{I},s,c_0)}(\mathbf{x})\ )\quad .$$

In order to construct from these data a c_1-independent approximation of the dynamics which maps $\mathfrak{M}(\mathcal{I})$ onto $\mathfrak{M}(\mathcal{I}_{|t|})$, one defines the unitary operators

$$w(\mathcal{I}, c_0, c_1, t) := \prod_{j=1}^{n} \exp\Big(i\frac{t}{n}H(\ \mathcal{I}, (n-j)n^{-1}t, c_0, c_1\ )\Big)$$

where n is equal to the integral part of $|c_0^{-1}t|$. The lemma, given below, has been established in [37].

Lemma 5.2.6 : [37, Lemma 9.1.2] *The adjoint action of* $w(\mathcal{I}, c_0, c_1, t)$ *induces an automorphism*

$$\alpha_t^{(\mathcal{I},c_0)} := \mathrm{Ad}(w(\mathcal{I}, c_0, c_1, t)) : \mathfrak{M}(\mathcal{I}) \longrightarrow \mathfrak{M}(\mathcal{I}_{|t|})$$

which is independent of c_1.

Step 3: For technical reasons, to control convergence as c_0 tends to zero, the length of time propagation is scaled, and one defines for $\lambda \in [0, 1]$ the c_1-independent automorphism

$$\alpha_t^{(\mathcal{I},c_0,\lambda)} := \mathrm{Ad}(w(\mathcal{I}, c_0, c_1, \lambda, t)) : \mathfrak{M}(\mathcal{I}) \longrightarrow \mathfrak{M}(\mathcal{I}_{|t|})$$

where $w(\mathcal{I}, c_0, c_1, \lambda, t)$ is given by

$$w(\mathcal{I}, c_0, c_1, \lambda, t) := \prod_{j=1}^{n} \exp\Big(i \frac{\lambda \cdot t}{n} H(\mathcal{I}, (n-j)n^{-1}t, c_0, c_1) \Big) \quad .$$

The final approximation is given by averaging over λ:

$$\alpha_t^{(\mathcal{I}, c_0, \ell)}(a) := \int \mathrm{d}\lambda \; f_\ell(\lambda) \; \alpha_t^{(\mathcal{I}, c_0, \lambda)}(a)$$

where f_ℓ is a positive continuous function such that

$$\int \mathrm{d}\lambda \; f_\ell(\lambda) = 1 \quad \text{and} \; \mathrm{supp}(f_\ell) \subset [1-\ell, 1], \ell \leq 1.$$

Finally, J. Glimm and A. Jaffe have established the result:

Theorem 5.2.7 : [37, Theorem 9.1.3] *There exists a function* $c : \ell \mapsto c_\ell$ *with* $\lim_{\ell \to 0} c_\ell = 0$ *such that*

$$\alpha_t^Y(a) := w - \lim_{\ell \to 0} \alpha_t^{(\mathcal{I}, c_\ell, \ell)}(a) = u(c_1, t) \; a \; u(c_1, t)^* \tag{5.12}$$

for each $a \in \mathfrak{M}(\mathcal{I})$ *and for each sufficiently large* c_1.

5.2.3 Kink States in Models with Yukawa$_2$ Interaction

We shall use an analogous strategy as above (step 1- step 3) in order to show that the dynamics α^Y, which is given due to Theorem 5.2.7 is extendible.

Theorem 5.2.8 : *The dynamics* α^Y *of the Yukawa*$_2$ *model is extendible.*

Let us prepare the proof. First, we give a few comments on the notation to be used.

Notation:

(a) In the sequel, we write $\hat{w}(\cdots) = w(\cdots)^{\otimes 2}$ and $\hat{u}(\cdots) = u(\cdots)^{\otimes 2}$ for the corresponding quantities of the twofold theory. As in step 3 above, we also define the automorphism

$$\hat{\alpha}_t^{(\mathcal{I}, c_0, \lambda)} := \mathrm{Ad}(\hat{w}(\mathcal{I}, c_0, c_1, \lambda, t))$$

and the average

$$\hat{\alpha}_t^{(\mathcal{I}, c_0, \ell)}(a) = \int \mathrm{d}\lambda \; f_\ell(\lambda) \; \hat{\alpha}_t^{(\mathcal{I}, c_0, \lambda)}(a) \quad .$$

(b) Let ω_0 be the vacuum state with respect to the free dynamics which is induced by H_0. We denote by $\Psi_{\mathcal{I}}$ the universal localizing map of the standard split inclusion $\Lambda(\mathcal{I}) \otimes \Lambda(\mathcal{I})$ and we define $\theta_{\mathcal{I}} := \Psi_{\mathcal{I}}(\mathbf{1} \otimes u_F)$.

Lemma 5.2.9 : *The adjoint action of $\hat{w}(\mathcal{I}, c_0, c_1, t)$ induces an automorphism*

$$\hat{\alpha}_t^{(\mathcal{I},c_0)} : \hat{\mathfrak{M}}(\mathcal{I}) \longrightarrow \hat{\mathfrak{M}}(\mathcal{I}_{|t|})$$

which is independent of c_1.

Proof. By Lemma 5.2.6, it is sufficient to prove that

$$\mathrm{Ad}(\hat{w}(\mathcal{I}, c_0, c_1, t))\theta_{\mathcal{I}}$$

is c_1-independent. Indeed, following the arguments in the proof of Proposition 5.2.3, we conclude that

$$\theta'_{\mathcal{I}} := \exp(i\tau H(\mathcal{I}, s, c_0, c_1))^{\otimes 2} \; \theta_{\mathcal{I}} \; \exp(-i\tau H(\mathcal{I}, s, c_0, c_1))^{\otimes 2}$$

is c_1-independent for $|\tau| \leq c_0$ and that $\theta'_{\mathcal{I}}$ is contained in $\hat{\mathfrak{M}}(\mathcal{I}_{|s|+|\tau|})$. Composing n such maps, we obtain the lemma. □

In complete analogy to Theorem 5.2.7 we have:

Lemma 5.2.10 :

$$\hat{\alpha}^Y(a) := w - \lim_{\ell \to 0} \hat{\alpha}_t^{(\mathcal{I},c_\ell,\ell)}(a) = \hat{u}(c_1, t) \; a \; \hat{u}(c_1, t)^*$$

For each $a \in \hat{\mathfrak{M}}(\mathcal{I})$ and for each sufficiently large c_1.

Proof. By Theorem 5.2.7, we conclude that the lemma holds for each $a \in \mathfrak{M}(\mathcal{I}) \overline{\otimes} \mathfrak{M}(\mathcal{I})$. Hence it remains to be proven that

$$w - \lim_{\ell \to 0} \hat{\alpha}_t^{(\mathcal{I},c_\ell,\ell)}(\theta_{\mathcal{I}}) = \hat{u}(c_1, t) \; \theta_{\mathcal{I}} \; \hat{u}(c_1, t)^* \ .$$

The Corollary 9.1.9 of [37] states:

$$w - \lim_{\ell \to 0} \int \mathrm{d}\lambda \; (\; \hat{w}(\mathcal{I}, c_\ell, c_1, \lambda, t) \; - \; \hat{u}(c_\ell, c_1, \lambda t) \;) f_\ell(\lambda) = 0 \ .$$

We define

$$\theta_{\mathcal{I}}(\ell,t) := \hat{\alpha}_t^{(\mathcal{I},c_\ell,\ell)}(\theta_{\mathcal{I}}) \quad \text{and} \quad \bar{\theta}_{\mathcal{I}}(\ell,t) := \int \mathrm{d}\lambda \; f_\ell(\lambda) \; \mathrm{Ad}(\; \hat{u}(c_\ell, c_1, \lambda t) \;)\theta_{\mathcal{I}} \quad .$$

The Schwarz's inequality implies for each $\psi \in \mathcal{H}_0 \otimes \mathcal{H}_0$:

$$\begin{aligned} &|\langle \psi, \theta_{\mathcal{I}}(\ell,t) - \bar{\theta}_{\mathcal{I}}(\ell,t)\psi\rangle| \\ &\leq 2||\psi|| \cdot \Big(\textstyle\int \mathrm{d}\lambda \; f_\ell(\lambda) \; ||(\; \hat{w}(\mathcal{I}, c_\ell, c_1, \lambda, t) \; - \; \hat{u}(c_\ell, c_1, \lambda t) \;)\psi||^2 \Big)^{1/2} \end{aligned}$$

Since $||(v-u)\psi||^2 = 2 \cdot \mathrm{Re}(\langle (v-u)\psi, u\psi\rangle)$, we obtain:

$$\begin{aligned} &|\langle \psi, \theta_{\mathcal{I}}(\ell,t) - \bar{\theta}_{\mathcal{I}}(\ell,t)\psi\rangle| \\ &\leq 4||\psi|| \cdot \Big(\textstyle\int \mathrm{d}\lambda \; f_\ell(\lambda) \; \mathrm{Re}\Big\langle \; (\hat{w}(\mathcal{I}, c_\ell, c_1, \lambda, t) \; - \; \hat{u}(c_\ell, c_1, \lambda t) \;)\psi, \hat{u}(c_\ell, c_1, \lambda t)\psi \Big\rangle \Big)^{1/2} \end{aligned}$$

In order to conclude the proof we establish the following sublemma:

Sublemma: Let $u_\ell : \lambda \mapsto u_\ell(\lambda)$ and $w_\ell : \lambda \mapsto w_\ell(\lambda)$, $\ell > 0$, be strongly continuous functions with values in the group of unitary operators on $\mathcal{H}_0$ such that

$$w - \lim_{\ell \to 0} \int \mathrm{d}\lambda \; f_\ell(\lambda) \; (w_\ell(\lambda) - u_\ell(\lambda)) = 0 \tag{5.13}$$

and

$$s - \lim_{\ell \to 0} u_\ell(1) = u \quad .$$

Then we have

$$\lim_{\ell \to 0} \left| \int \mathrm{d}\lambda \; f_\ell(\lambda) \; \langle \; (w_\ell(\lambda) - u_\ell(\lambda))\psi, u_\ell(\lambda)\psi \rangle \right| = 0 \quad .$$

Proof of Sublemma. Since u_ℓ is strongly continuous and f_ℓ has support in $[1-\ell, 1]$ we conclude that for each $\epsilon > 0$ there is a number $l > 0$ such that for each $\ell < l$ we have:

$$\left| \int \mathrm{d}\lambda \; f_\ell(\lambda) \; \langle \; (w_\ell(\lambda) - u_\ell(\lambda))\psi, (u_\ell(\lambda) - u_\ell(1))\psi \rangle \right| < \frac{\epsilon}{4}$$

and

$$\left| \int \mathrm{d}\lambda \; f_\ell(\lambda) \; \langle \; (w_\ell(\lambda) - u_\ell(\lambda))\psi, (u_\ell(1) - u)\psi \rangle \right| < \frac{\epsilon}{4}$$

which implies:

$$\left| \int d\lambda \; f_\ell(\lambda) \left\langle \, (w_\ell(\lambda) - u_\ell(\lambda))\psi, (u_\ell(\lambda) - u)\psi \right\rangle \right| < \frac{\epsilon}{2} \; .$$

According to equation (5.13), we obtain on the other hand:

$$\left| \int d\lambda \; f_\ell(\lambda) \left\langle \, (w_\ell(\lambda) - u_\ell(\lambda))\psi, u\psi \right\rangle \right| < \frac{\epsilon}{2} \; .$$

This implies finally the desired result:

$$\left| \int d\lambda \; f_\ell(\lambda) \left\langle \, (w_\ell(\lambda) - u_\ell(\lambda))\psi, u_\ell(\lambda)\psi \right\rangle \right| < \epsilon \; .$$

□

An application of the sublemma gives:

$$\lim_{\ell \to 0} |\langle \psi, \theta_{\mathcal{I}}(\ell, t) - \bar{\theta}_{\mathcal{I}}(\ell, t)\psi \rangle| = 0$$

which proves the lemma. □

Proof of Theorem 5.2.8: We conclude from Lemma 5.2.10 and Lemma 5.1.9 that the automorphism group $\hat{\alpha}^Y$ is a dynamics of the extended net $\hat{\mathfrak{M}}$ whose restriction to $\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}$ is $\alpha^Y \otimes \alpha^Y$. Thus α^Y is extendible. □

Remark: According to [75], each dynamics α^{Y+P} of a quantum field theory model with Yukawa$_2$ plus $P(\phi)_2$ boson self-interaction is extendible.

Finally, we conclude from Theorem 5.2.8:

Corollary 5.2.11 : *Let α^{Y+P} be a dynamics of a quantum field theory model with Yukawa$_2$ plus $P(\phi)_2$ boson self-interaction. For each pair ω_1, ω_2 of vacuum states with respect to α^{Y+P}, there exists a kink state ω in $\mathfrak{S}([\omega_1], [\omega_2])$.*

5.2.4 Wess-Zumino Models

One interesting class of quantum field theory models which possess more than one vacuum sector are the $N = 2$ Wess-Zumino models in two-dimensional space-time. Their properties have been studied in several papers [47, 50, 51, 48, 49] and we summarize the main results which are established there in order to setup our subsequent analysis.

The field content of these models with a finite volume cutoff $c > 0$ consists of one *complex* Bose field ϕ_c and one Dirac spinor field ψ_c, acting as operator valued distributions on the Fock spaces

$$\begin{aligned} \mathcal{H}_a(c) &:= \bigoplus_{n=0}^{\infty} L_2(T_c, \mathbb{C}^2)^{\otimes a} \\ \mathcal{H}_s(c) &:= \bigoplus_{n=0}^{\infty} L_2(T_c, \mathbb{C})^{\otimes s} \end{aligned}$$

where a, s stands for symmetrization or anti-symmetrization of the tensor product and $L_2(T_c, \mathbb{C}^k)$ $(k = 1, 2)$ denotes the Hilbert space of $\mathbb{C}^k$-valued and square integrable functions, living on the circle T_c of length c. The net of Cauchy data for the finite volume theory is given by

$$\mathfrak{M}_c : (-c, c) \supset \mathcal{I} \longmapsto \mathfrak{M}_c(\mathcal{I}) = \mathfrak{M}_{c,s}(\mathcal{I}) \,\overline{\otimes}\, \mathfrak{M}_{c,a}(\mathcal{I})$$

where the nets $\mathfrak{M}_{c,s}$ and $\mathfrak{M}_{c,a}$ are defined by the assignments:

$$\mathfrak{M}_{c,s} : (-c, c) \supset \mathcal{I} \longmapsto \mathfrak{M}_{c,s}(\mathcal{I}) := \left\{ e^{i(\phi_c(f_1)+\pi_c(f_2))} \,\middle|\, \mathrm{supp}(f_j) \subset \mathcal{I} \right\}''$$

$$\mathfrak{M}_{c,a} : (-c, c) \supset \mathcal{I} \longmapsto \mathfrak{M}_{c,a}(\mathcal{I}) := \left\{ \psi_c(f_1), \bar{\psi}_c(f_2) \,\middle|\, \mathrm{supp}(f_j) \subset \mathcal{I} \right\}''$$

where π_c is the canonically conjugate of ϕ_c.

Let $\mathfrak{M} := \mathfrak{M}_{c=\infty}$ be the net of Cauchy data in the infinite volume limit, then the map

$$\iota_c : \begin{pmatrix} \phi(f_{11}) & \pi(f_{12}) \\ \psi(f_{21}) & \bar{\psi}(f_{22}) \end{pmatrix} \longmapsto \begin{pmatrix} \phi_c(f_{11}) & \pi_c(f_{12}) \\ \psi_c(f_{21}) & \bar{\psi}_c(f_{22}) \end{pmatrix} \;\; ; \;\; \mathrm{supp}(f_{ij}) \subset (-c, c)$$

is a W*-isomorphism which identifies the nets $\mathfrak{M}$ and $\mathfrak{M}_c$ for those regions $\mathcal{I}$ which are contained in $(-c, c)$.

The interaction part of the *formal* Hamiltonian consists of two parts.

(a) A $P(\phi)_2$-like part:

$$H_P(v, c) = \int_{T_c} \mathrm{dx} \;\; : |v'(\Phi_c)|^2 : \; - \; : |\Phi_c|^2 :$$

(b) A Yukawa$_2$-like part:

$$H_Y(v,c) := \int_{T_c} \mathrm{d}\mathbf{x} \ : \bar{\Psi}_c \begin{pmatrix} v''(\Phi_c) - 1 & 0 \\ 0 & v''(\Phi_c)^* - 1 \end{pmatrix} \Psi_c :$$

where v is a polynomial of degree $\deg(v) = n$, called *superpotential*, and the fields Φ_c and Ψ_c are given by

$$\Phi_c := \phi_c \otimes \mathbf{1}_{\mathcal{H}_a(c)} \quad \text{and} \quad \Psi_c := \mathbf{1}_{\mathcal{H}_s(c)} \otimes \psi_c \quad .$$

According to the results of [47, 49, 50, 51], it has been shown that, there is a self-adjoint Fredholm operator $Q(v,c)$, called *supersymmetry generator*, on $\mathcal{H}_0(c) := \mathcal{H}_s(c) \otimes \mathcal{H}_a(c)$. The *Fredholm index* of $Q(v,c)$

$$\mathrm{ind}(Q(v,c)) = \dim\ker(Q(v,c)) - \dim\mathrm{coker}(Q(v,c))$$

has been computed in [50]. The result is

$$|\mathrm{ind}(Q(v,c))| = \deg(v) - 1 \quad .$$

The space $\mathcal{H}_0(c)$ may be decomposed $\mathcal{H}_0(c) = \mathcal{H}_+(c) \oplus \mathcal{H}_-(c)$ into the eigenspaces of the fermion parity operator $\Gamma := (-1)^{N_a}$, where N_a is the fermion number operator. With respect to this decomposition, the operator $Q(v,c)$ has the form

$$Q(v,c) = \begin{pmatrix} 0 & Q_+(v,c) \\ Q_-(v,c) & 0 \end{pmatrix} \quad .$$

The full Hamiltonian of the finite volume model is given by

$$H(v,c) = Q(v,c)^2$$

which implies:

$$\dim\ker(H(v,c)) = |\dim\ker(Q_+(v,c)) - \dim\ker(Q_-(v,c))| = \deg(v) - 1$$

The Hamiltonian $H(v,c)$ induces a dynamics $\alpha^{(v,c)}$ of the finite volume model and we conclude from the results of [47]:

Theorem 5.2.12 : [47, Theorem 1] *There exists at least* $\deg(v)-1$ *vacuum sectors with respect to the dynamics* $\alpha^v := \alpha^{(v,c=\infty)}$ *of the model in the infinite volume limit.*

5.2.5 Kink States in Wess-Zumino Models

In order to prove the existence of kink sectors, we now apply the results which have been established in Section 5.2.1 and Section 5.2.3 to $N = 2$ Wess-Zumino Models.

The case $\deg(v) = 3$*:* Let us have a closer look at the simplest non-trivial case $\deg(v) = 3$. We let

$$v'(z) = \lambda_2 z^2 + \lambda_1 z + \lambda_0 \quad .$$

As in the previous sections (equation (5.9)), we introduce the UV-regularized fields:

$$\Phi(c_0, \mathbf{x}) := (\Phi * \delta_{c_0})(\mathbf{x}) \quad \text{and} \quad \Psi(c_0, \mathbf{x}) := (\Psi * \delta_{c_0})(\mathbf{x})$$

where δ_{c_0} is a smooth test function with support in $(-c_0, c_0)$. We obtain for the $P(\phi)_2$-like part of the regularized interaction Hamiltonian

$$H_P(v; c_0, c_1) = \int \mathrm{d}\mathbf{x}\ \chi_{c_1}(\mathbf{x})\ (:|\lambda_2 \Phi(c_0, \mathbf{x})^2 + \lambda_1 \Phi(c_0, \mathbf{x}) + \lambda_0|^2 : \ - \ :|\Phi(c_0, \mathbf{x})|^2 :)$$

and for the Yukawa$_2$-like part:

$$H_Y(v; c_0, c_1) = \int \mathrm{d}\mathbf{x}\ \chi_{c_1}(\mathbf{x}) \Big[$$

$$: \bar{\Psi}(c_0, \mathbf{x}) \begin{pmatrix} 2\lambda_2 \Phi(c_0, \mathbf{x}) + \lambda_1 - 1 & 0 \\ 0 & 2\bar{\lambda}_2 \Phi(c_0, \mathbf{x})^* + \bar{\lambda}_1 - 1 \end{pmatrix} \Psi(c_0, \mathbf{x}) : \Big]$$

Using the same techniques as in Section 5.2.1 and Section 5.2.3, we obtain the corollary (see also Corollary 5.2.11):

Corollary 5.2.13 : *Let* v *be a superpotential of degree* $\deg(v) = 3$*. Then the following statements are true:*

(1) The dynamics $\alpha^v \in \mathrm{dyn}(\mathfrak{M})$ *of the model in the infinite volume limit is extendible.*

(2) There exists two different vacuum sectors $e_1, e_2 \in \sec_0(\alpha^v, \mathfrak{M})$ *and two different kink sectors* $\theta \in \sec(e_1, e_2)$*,* $\bar{\theta} \in \sec(e_2, e_1)$*.*

The case $\deg(v) > 3$*:* We close this section by discussing the remaining case.

In order to show the extendibility of $\alpha^v \in \mathrm{dyn}(\mathfrak{M})$, we can try to proceed in the same manner as for the case $\deg(v) = 3$. According to Section 5.2.3 (*Step 2* and *Step 3*), we construct an approximation

$$\mathfrak{M}(\mathcal{I}) \,\overline{\otimes}\, \mathfrak{M}(\mathcal{I}) \ni a \longmapsto \hat{\alpha}_t^{(v;\mathcal{I},c_0,\ell)}(a) := \int \mathrm{d}\lambda \; f_\ell(\lambda) \; \hat{\alpha}_t^{(v;\mathcal{I},c_0,\lambda)}(a)$$

of the dynamics $\alpha^v \otimes \alpha^v$ of the twofold theory. Provided that the corresponding result of Lemma 5.2.9 is true for the case $\deg(v) > 3$ also, the linear maps $\hat{\alpha}_t^{(v;\mathcal{I},c_0,\ell)}$ can be extended to the algebra $\hat{\mathfrak{M}}(\mathcal{I})$.

For the generalization of Theorem 5.2.8, it seems that the most difficult part is to show that there exists a function $c : \ell \mapsto c_\ell$ with $\lim_{\ell \to 0} c_\ell = 0$ such that

$$\alpha_t^v(a) := w - \lim_{\ell \to 0} \alpha_t^{(v;\mathcal{I},c_\ell,\ell)}(a) \; . \tag{5.14}$$

The regularized Yukawa-like part of the Hamilton density contains terms of the form

$$: \Psi^{(i)}(c_0, \mathrm{x}) \Psi^{(j)}(c_0, \mathrm{x}) : \; : \Phi(c_0, \mathrm{x})^k : \; ; \quad i, j \in \{0, 1\}, i \neq j \text{ and } k \leq \deg(v) - 2,$$

where $\Psi^{(j)}$ denotes the j-component of the Dirac spinor field Ψ. Since there are contributions with $k > 1$, the proof of Theorem 5.2.7 does not directly apply.

Provided that for each superpotential v the dynamics α^v is extendible, we conclude that for each pair of vacuum sectors $e_1, e_2 \in \mathrm{sec}_0(\alpha^v, \mathfrak{M})$ there exists a kink state $\omega \in \mathfrak{S}(e_1, e_2)$. Then the model possesses at least $\deg(v)(\deg(v) - 1)$ different non-trivial kink sectors.

6 Collision Theory for Kinks and Solitons

6.1 Technical Preliminaries

This section is destined to introduce technical notions which are needed for the construction and analysis of kink fields. In order to investigate multi-kink representations and statistics relations, we briefly repeat the notion of one-kink sectors and introduce a method to compose kink representations which generalizes the composition of kink homomorphisms, given in Section 3.2.1.

6.1.1 One-Kink Sectors

The properties of massive one-particle sectors are closely related to the properties of kinks. As already mentioned in Section 4.4.2, for each massive one-particle state ω, there are vacuum sectors $e_1, e_2 \in \mathrm{sec}_0(\mathfrak{A})$ such that ω is a kink state which interpolates e_1 and e_2 [10, 22, 71].

Notation: In the sequel, we shall call a state (sector) *one-kink* state (sector) if it is a pure massive one-particle state (irreducible massive one-particle sector). The corresponding kink homomorphisms are called *one-kink homomorphisms* and we shall denote by $\Delta_1(q, e_1, e_2)$ the set of all one-kink homomorphisms which interpolate the vacua $e_1, e_2 \in \mathrm{sec}_0(\mathfrak{A})$ and have orientation q.

6.1.2 From One-Kink Homomorphisms to Multi-Kink Representations

Definiton 6.1.1 :

(a) Let $n \in \mathbb{N}$ be a natural number. We shall write $\underline{n}$ for the set $\{1, \cdots, n\}$. We call a subset $u \subset \underline{n}$ *connected* if it is of the form

$$u = \{k, k+1, k+2, \cdots, k+l\} \quad .$$

(b) Let $P_l(n)$ be the set of all partitions κ of $\underline{n}$ into l connected non-empty subsets

$$\kappa = (\kappa(1) \cdots \kappa(l)) \quad .$$

(c) Let $\underline{e} = (e_{-m}, \cdots, e_0, \cdots, e_n)$, $n, m \in \mathbb{N}$,be a family of $n+m+1$ vacuum sectors. We denote by $T(n, m, \underline{e})$ the set of all families of one-kink homomorphisms

$$\underline{\rho} := (\rho_j, \rho^i)_{j \in \underline{n}}^{i \in \underline{m}}$$

such that and $\rho_j \in \Delta_1(1, e_{j-1}, e_j)$ and $\rho^i \in \Delta_1(-1, e_{-i}, e_{-i+1})$.

Definiton 6.1.2 : Let $\underline{\rho} \in T(n, m, \underline{e})$ be a family of one-kink homomorphisms. We define for each pair of partitions

$$\kappa = (\kappa_+, \kappa_-) \in P_l(n) \times P_l(m)$$

a representation $\underline{\rho}_\kappa$ in the following way:

(1) We parameterize the partitions κ_k, $k = \pm$:

$$\begin{aligned} \kappa_k = (&\{1, \cdots, \kappa_k(1)\}, \{\kappa_k(1)+1, \cdots, \kappa_k(2)\}, \cdots \\ &\cdots, \{\kappa_k(l-1)+1, \cdots, \kappa_k(l)\}) \end{aligned} \tag{6.1}$$

(2) The representation $\underline{\rho}_\kappa$ is defined by

$$\begin{aligned} \underline{\rho}_\kappa := \Bigg(\cdots \Bigg(\Bigg(\Big(\pi_{e_0} \circ \rho_1 \cdots \rho_{\kappa_+(1)}| \Big)^{-} \rho^1 \cdots \rho^{\kappa_-(1)}| \Bigg)^{+} \rho_{\kappa_+(1)+1} \cdots \rho_{\kappa_+(2)}| \Bigg)^{-} \\ \cdots \rho_{\kappa_+(l-1)+1} \cdots \rho_{\kappa_+(l)}| \Bigg)^{-} \rho^{\kappa_-(l-1)+1} \cdots \rho^{\kappa_-(l)}| \end{aligned}$$

where the vertical bar $|$ denotes the restriction to the algebra $C^*(\mathfrak{A})$ and the symbol $^\pm$ denotes the extension to the algebra $C^*(\mathfrak{A}_e^\pm)$ where e is a suitable vacuum sector.

(3) We denote by $\Delta(e, f; e_0)$ the set of all kink representations $\underline{\rho}_\kappa$ which correspond to a family $\underline{\rho} \in T(n, m, \underline{e})$ with $e_{-m} = e$, $e_n = f$, and partitions $\kappa = (\kappa_+, \kappa_-) \in P_l(n) \times P_l(m)$.

Example: In order to illustrate the definitions above, we compute $\underline{\rho}_\kappa$ for a simple example. Consider a family

$$\underline{\rho} = (\rho_1, \rho_2, \rho^1, \rho^2) \in T(2, 2; e_{-2}, e_{-1}, e_0, e_1, e_2)$$

of kink homomorphisms and a pair of partitions $\kappa = (\kappa_+, \kappa_-)$

$$\kappa_+ = \kappa_- = (\{1\}, \{2\}) \in P_2(2) \ \ .$$

The representation

$$\pi_1^- := (\pi_{e_0} \circ \rho_1|)^-$$

maps $C^*(\mathfrak{A}_{e_0}^-)$ into $\mathfrak{B}(\mathcal{H}_{e_0})$ and

$$\pi_2 := \pi_1^- \circ \rho^1|$$

is a well defined kink representation whose left vacuum is e_{-1} and whose right vacuum is e_1. Thus the representation π_2^+ maps $C^*(\mathfrak{A}_{e_1}^+)$ into $\mathfrak{B}(\mathcal{H}_{e_0})$ and we obtain a further kink representation:

$$\pi_3 := \pi_2^+ \circ \rho_2| \ \ .$$

Since the left vacuum of π_3 is e_{-1}, the representation π_3^- maps $C^*(\mathfrak{A}_{e_{-1}}^-)$ into $\mathfrak{B}(\mathcal{H}_{e_0})$ and we obtain finally:

$$\underline{\rho}_\kappa = \pi_3^- \circ \rho^2| \ \ .$$

Remark:

(i) We easily verify that for a family $\underline{\rho} \in T(n,m;\underline{e})$ of kink representations and for a pair of partitions κ, the sector $[\underline{\rho}_\kappa]$ is contained in $\mathrm{sec}(e_{-m}, e_n)$. The representation $\underline{\rho}_\kappa$ describes the creation of $n+m$ kink charges out of the vacuum e_0. The partitions $\kappa = (\kappa_+, \kappa_-)$ describe the order in which the $n+m$ kink charges are created and we shall call a pair $\kappa = (\kappa_+, \kappa_-)$ an *arrangement*.

(ii) The set $T(n,0,\underline{e})$ contains only families of one-kink homomorphisms with orientation $q=1$ and $T(0,m,\underline{e})$ contains only families of one-kink homomorphisms with orientation $q=-1$. For a family of kink homomorphisms $\underline{\rho} \in T(n,0,\underline{e})$ there is only one possibility to build a product representation, namely

$$\pi^+_{e_0} \circ \rho_1 \cdots \rho_n| \ . \tag{6.2}$$

The same holds for a family of kink homomorphisms $\underline{\sigma} \in T(0,m,\underline{e})$. Here we obtain the representation:

$$\pi^-_{e_0} \circ \sigma^1 \cdots \sigma^m| \ . \tag{6.3}$$

We shall call a kink representation *oriented* if it is of the form (6.2) or (6.3).

(iii) Let $\tau \in \Delta(e,f;g)$ be a kink representation. Then τ can be composed with oriented kink representations $\sigma \in \Delta(e_1,e;e)$ and $\rho \in \Delta(f,f_1;f)$:

$$\begin{aligned} \tau^\# \rho &:= \tau^+ \circ (\pi^+_f)^{-1} \circ \rho| \in \Delta(e,f_1;g) \\ \tau^\# \sigma &:= \tau^- \circ (\pi^-_e)^{-1} \circ \sigma| \in \Delta(e_1,f;g) \ . \end{aligned}$$

(iv) Each kink representation $\tau \in \Delta(e,f;g)$ is a finite composition of oriented kink representations $\rho_j \in \Delta(f_{j-1}, f_j; f_{j-1})$ and $\sigma_j \in \Delta(e_j, e_{j-1}; e_{j-1})$, $j \in \underline{n}$, with $e_0 = f_0 = g$, $e_n = e$ and $f_n = f$:

$$\tau = \rho_1{}^\# \sigma_1{}^\# \rho_2{}^\# \sigma_2 \cdots \rho_n{}^\# \sigma_n \ . \tag{6.4}$$

(v) To each representation of the form $\underline{\rho}_\kappa$, a finite $\mathbb{Z}_2$-valued sequence can be associated, namely:

$$\begin{aligned} \underline{\rho}_\kappa \longmapsto (&+_1 \cdots +_{\kappa_+(1)}, -_1 \cdots -_{\kappa_-(1)}, +_{\kappa_+(1)+1} \cdots +_{\kappa_+(2)}, \cdots \\ &\cdots +_{\kappa_+(l-1)+1} \cdots +_{\kappa_+(l)}, -_{\kappa_-(l-1)+1} \cdots -_{\kappa_-(l)}) \ . \end{aligned}$$

6.2 Quasi-Statistics, Kink Fields and Cluster Properties

In order to prove the existence of multi-kink collision states and to analyze their properties, we discuss in this section the statistics properties of kinks and develop an adequate field bundle formalism. Furthermore, we establish some useful cluster properties for correlation functions of kink fields.

6.2.1 Quasi-Statistics

In the DHR case [18, 19], the notion of statistics is related to the exchange of the order of factors in a product of charged DHR endomorphisms. If we consider DHR endomorphisms $(\rho_1, \cdots, \rho_n)$, then for each permutation $p \in S_n$ there exists a unitary intertwiner $\epsilon_p(\rho_1, \cdots, \rho_n)$, called statistics operator, which intertwines the product representations $\rho_1 \cdots \rho_n$ and $\rho_{p(1)} \cdots \rho_{p(n)}$.

In case of kink representations the situation is more complicated which is due to the fact that kink representations can not arbitrarily be composed. Let us consider one-kink homomorphisms $\rho_1 \in \Delta(1, e'', e)$ and $\rho_2 \in \Delta(1, e, e')$. Then the product $\rho_1\rho_2$ is well defined, but for $e'' \neq e'$ the expression $\rho_2\rho_1$ has no mathematical meaning. On the other hand, we shall see that for a representation $\tau \in \Delta(e_1, e_2; e_0)$ and oriented representations $\rho \in \Delta(e_2, e_2'; e_2)$, $\sigma \in \Delta(e_1', e_1; e_1)$, the representations $\tau^\# \rho^\# \sigma$ and $\tau^\# \sigma^\# \rho$ (see equation (6.4)) are unitarily equivalent.

These facts lead to the notion of *quasi-statistics* where the word *quasi* emphasizes the fact that the exchange of two one-kink representations within a product only makes sense if their orientations are different. In comparison to the DHR and BF situation, we are faced with substituting the permutation group by the set of pairs

$$G(n, m) := P(n, m) \times P(n, m) \quad .$$

This can be justified by the following statement:

Theorem 6.2.1 : *Let $\underline{\rho} \in T(n, m, \underline{e})$ be a family of one-kink homomorphisms. For each pair of arrangements $(\kappa, \hat{\kappa}) \in G(n, m)$, there exists a unitary operator $\epsilon_{(\kappa,\hat{\kappa})}(\underline{\rho})$, called quasi-statistics operator, which intertwines the representations $\underline{\rho}_\kappa$ and $\underline{\rho}_{\hat{\kappa}}$:*

$$\underline{\rho}_\kappa(a)\, \epsilon_{(\kappa,\hat{\kappa})}(\underline{\rho}) = \epsilon_{(\kappa,\hat{\kappa})}(\underline{\rho})\, \underline{\rho}_{\hat{\kappa}}(a) \quad \forall a \in C^*(\mathfrak{A}). \tag{6.5}$$

The intertwiner $\epsilon_{(\kappa,\hat{\kappa})}(\underline{\rho})$ is defined in terms of auxiliary kink homomorphisms an charge transporters, but does not depend on their choice.

Proof. The proof of the theorem is rather technical and therefore placed in Appendix B.2. □

Remark: To illustrate the statement of Theorem 6.2.1, we compare it with the DHR situation. Given a family $\underline{\rho} = (\rho_1, \cdots, \rho_n)$ of DHR endomorphisms and an element of the permutation group $p \in S_n$. We define

$$\underline{\rho}_p := \rho_{p(1)} \cdots \rho_{p(n)} \quad .$$

By using the statistics operator $\epsilon_p(\rho_1 \cdots \rho_n)$, we obtain:

$$\underline{\rho}_p = \mathrm{Ad}(\epsilon_p(\underline{\rho})) \circ \underline{\rho}_{\mathrm{id}} = \mathrm{Ad}(\epsilon_p(\underline{\rho})\epsilon_q(\underline{\rho})^*) \circ \underline{\rho}_q$$

We conclude that, in the DHR case, the analogue of the quasi-statistics relation (6.5) is given by

$$\underline{\rho}_p = \mathrm{Ad}(\epsilon_{(p,q)}(\underline{\rho})) \circ \underline{\rho}_q$$

where $\epsilon_{(p,q)}(\underline{\rho}) = \epsilon_p(\underline{\rho})\epsilon_q(\underline{\rho})^*$ is a product of ordinary statistics operators.

6.2.2 Kink Field Operators

Each kink representation $\tau \in \Delta(e_1, e_2; e_0)$ is a representation of the observable algebra $C^*(\mathfrak{A})$ on the Hilbert space $\mathcal{H}_{e_0}$. To each pair

$$\psi := \{\tau, \psi_\tau\} \in \Delta(e_1, e_2; e_0) \times \mathcal{H}_{e_0} \quad ,$$

a state ω_ψ can be associated, i.e.:

$$\omega_\psi(a) := ||\psi_\tau||^{-2} \cdot \langle \psi_\tau, \tau(a)\psi_\tau \rangle \quad .$$

Thus each pair $\psi := \{\tau, \psi_\tau\}$ represents a state in the sector $[\tau]$. Instead of $\psi := \{\tau, \psi_\tau\}$ we can choose another pair $\hat{\psi} = \{\hat{\tau}, \psi_{\hat{\tau}}\}$ which also induces ω_ψ. Let u be a unitary operator which intertwines τ and $\hat{\tau}$. Then the vectors $u\psi := \{\hat{\tau}, u\psi_\tau\}$ and ψ induce the same state.

The idea is to describe a kink field by a map

$$\psi = \{\tau, \psi_\tau\} \longmapsto \hat{\psi} = \{\tau^{\#}\rho, \hat{\psi}_{\tau^{\#}\rho}\} \quad .$$

We shall introduce here an appropriate *field bundle formalism* for kinks which generalizes the field bundle formalism used for the treatment of DHR and BF sectors [18, 19, 10].

Definiton 6.2.2 : The *state bundle* is defined by

$$\underline{\mathcal{H}} := \bigcup_{e_0, e_1, e_2} \Delta(e_1, e_2; e_0) \times \mathcal{H}_{e_0}$$

where the union is taken over all vacuum sectors e_0, e_1, e_2. Let $\tau \in \Delta(e_1, e_2; e_0)$ be a kink representation, then we denote by

$$\mathcal{H}_\tau := \{\tau\} \times \mathcal{H}_{e_0}$$

the fiber space over τ.

We have to distinguish different types of actions on $\underline{\mathcal{H}}$.

The action of observables on $\underline{\mathcal{H}}$*:* Given an operator $a \in C^*(\mathfrak{A})$ and a vector $\psi = \{\tau, \psi_\tau\} \in \underline{\mathcal{H}}$. Then a acts on ψ as follows:

$$a\psi := \{\tau, \tau(a)\psi_\tau\} \quad .$$

The action of the translation group: Since each kink representation τ is translationally covariant we define a representation of the translation group on the bundle $\underline{\mathcal{H}}$, namely

$$x \longmapsto U(x) : \psi \longmapsto U(x)\psi := \{\tau, U_\tau(x)\psi_\tau\}$$

where U_τ implements the translation group in the representation τ.

The action of intertwiners: Each operator v, which intertwines kink representations τ and $\hat{\tau}$, induces a linear map between the fiber spaces $\mathcal{H}_\tau$ and $\mathcal{H}_{\hat{\tau}}$.

$$v : \psi \in \mathcal{H}_\tau \longmapsto v\psi = \{\hat{\tau}, v\psi_\tau\} \in \mathcal{H}_{\hat{\tau}} \quad .$$

We shall denote the set of all these maps by $(\hat{\tau}|\tau)$.

The action of kink fields: Let $\rho \in \Delta_1(q, e_1, e_2)$ be a one-kink homomorphism. Then the vector space of *one-kink fields* $F(q, \rho)$ with respect to ρ is given by

$$F(1, \rho) := \{\rho\} \times C^*(\mathfrak{A}^+_{e_1})$$

$$F(-1, \rho) := \{\rho\} \times C^*(\mathfrak{A}^-_{e_2})$$

Given two one-kink fields $\mathbf{a} \in F(1, \rho)$, $\rho \in \Delta_1(1, e_1, \hat{e}_1)$, and $\mathbf{b} \in F(-1, \sigma)$, $\sigma \in \Delta_1(-1, \hat{e}_2, e_2)$. Then their action is defined on

$$\mathcal{H}(e_1, e_2; e_0) := \Delta(e_1, e_2; e_0) \times \mathcal{H}_{e_0} \subset \underline{\mathcal{H}}$$

in the following way:

$$\mathbf{a}\psi = (\rho, a)\{\tau, \psi_\tau\} := \{\tau^+\rho, \tau^+(a)\psi_\tau\}$$

$$\mathbf{b}\psi = (\sigma, b)\{\tau, \psi_\tau\} := \{\tau^-\sigma, \tau^-(b)\psi_\tau\}$$

Indeed, the action of one-kink fields can be interpreted as the creation of an additional kink charge.

Remark: Let $\hat{\rho}, \rho \in \Delta_1(q, e_1, e_2)$ be one-kink homomorphisms. Then each intertwiner $v \in (\hat{\rho}|\rho)$ induces a linear map from $F(q, \rho)$ to $F(q, \hat{\rho})$, i.e.

$$v : \mathbf{a} = (\rho, a) \in F(q, \rho) \longmapsto v\mathbf{a} := (\hat{\rho}, va) \in F(q, \hat{\rho}) \tag{6.6}$$

To compute correlation functions of kink fields, we are interested in vectors $\psi \in \underline{\mathcal{H}}$ which can be obtained by a multiple application of one-kink fields to a vacuum vector $\Omega_e = \{\pi_e, \Omega_e\}$. It can be seen from the definition of one-kink fields that their action is not defined on each fiber of $\underline{\mathcal{H}}$. This makes the situation more complicated as in the DHR and BF case [18, 19, 10]. For this purpose, we introduce special families of one-kink fields.

Definiton 6.2.3 : Let $\underline{\rho} \in T(n, m, \underline{e})$ be a family of one-kink homomorphisms. We denote by $F(n, m, \underline{\rho})$ the vector space

$$F(n, m, \underline{\rho}) := \Big(\bigotimes_{j \in \underline{n}} F(1, \rho_j) \Big) \otimes \Big(\bigotimes_{i \in \underline{m}} F(-1, \rho^i) \Big)$$

and we shall call the elements of $F(n, m, \underline{\rho})$ *multi-kink fields*.

Proposition 6.2.4 : *Let $\underline{\rho} \in T(n, m, \underline{e})$ be a family of one-kink homomorphisms. Then for each arrangement $\kappa \in P(n, m)$ there exists a canonical linear map which maps $F(n, m, \underline{\rho})$ into $\mathcal{H}_{\underline{\rho}_\kappa}$.*

$$F(n, m, \underline{\rho}) \longrightarrow \mathcal{H}_{\underline{\rho}_\kappa} \ ; \ \mathbf{a} \longmapsto \mathbf{a}_\kappa \Omega_{e_0} \ .$$

Proof. Let $\mathbf{a} \in F(n, m, \underline{\rho})$ be of the form

$$\mathbf{a} := \Big(\bigotimes_{j \in \underline{n}} \mathbf{a}_j \Big) \otimes \Big(\bigotimes_{i \in \underline{m}} \mathbf{a}^i \Big) \ .$$

Then the prescription

$$\mathbf{a}_\kappa \Omega_{e_0} := \mathbf{a}^m \cdots \mathbf{a}^{\kappa_-(l)+1} \mathbf{a}_n \cdots \mathbf{a}_{\kappa_+(l)+1} \cdots \mathbf{a}^{\kappa_-(1)} \cdots \mathbf{a}^1 \mathbf{a}_{\kappa_+(1)} \cdots \mathbf{a}_1 \Omega_{e_0}$$

defines a linear function which maps $F(n, m, \underline{\rho})$ into $\mathcal{H}_{\underline{\rho}_\kappa}$ where the arrangement $\kappa = (\kappa_+, \kappa_-)$ is parameterized by equation (6.1). $\square$

In order to discuss quasi-statistics for kink fields we need an additional definition.

Definiton 6.2.5 : Given two families of one-kink homomorphisms $\underline{\rho}, \underline{\hat{\rho}} \in T(n, m, \underline{e})$. Then we define the vector space of intertwiner

$$(\underline{\hat{\rho}} | \underline{\rho}) := \Big(\bigotimes_{j \in \underline{n}} (\hat{\rho}_j | \rho_j) \Big) \otimes \Big(\bigotimes_{i \in \underline{m}} (\hat{\rho}^i | \rho^i) \Big) \ .$$

We shall say that the elements of $(\underline{\hat{\rho}} | \underline{\rho})$ intertwine the families $\underline{\rho}$ and $\underline{\hat{\rho}}$.

Remark: Note that, according to equation (6.6), each intertwiner $v \in (\underline{\hat{\rho}} | \underline{\rho})$ can be canonically identified with a linear map

$$v : F(n, m, \underline{\rho}) \longrightarrow F(n, m, \underline{\hat{\rho}}) \ .$$

Proposition 6.2.6 : *For each arrangement $\kappa \in P(n, m)$, there exists a canonical linear embedding*

$$v \in (\underline{\hat{\rho}} | \underline{\rho}) \hookrightarrow v_\kappa \in (\underline{\hat{\rho}}_\kappa | \underline{\rho}_\kappa)$$

such that the following relation is fulfilled for each $\mathbf{a} \in F(n, m, \underline{\rho})$:

$$(v\mathbf{a})_\kappa \Omega_{e_0} = v_\kappa \cdot \mathbf{a}_\kappa \Omega_{e_0}$$

Proof. The proof of the proposition is given in Appendix B.1. □

In order to formulate the conditions for quasi-statistics relation, we introduce a local structure on the space of multi-kink fields. (Compare also [18, 19, 10] for this notion.)

Definiton 6.2.7 : Given a wedge region W and a one-kink homomorphism $\rho \in \Delta_1(q, e_1, e_2)$. We shall say that a one-kink field $\mathbf{a} = (\rho, a) \in F(q, \rho)$ *is localized in* W if there exists a kink homomorphism $\hat{\rho} \in \Delta_1(q, e_1, e_2)$ and a unitary intertwiner $u \in (\hat{\rho}, \rho)$ such that:

(a) $\hat{\rho}$ is localized in W.

(b) The operator ua is localized in W.

The localization region of a one-kink field does not depend on the choice of the intertwiner u and it is covariantly transformed under space-time translations:

Proposition 6.2.8 : *Let* $\mathbf{a} = (\rho, a) \in F(q, \rho)$ *be a one-kink field which is localized in* W*. Then the following statements hold:*

(1) For each unitary intertwiner $v \in (\rho_1|\rho)$ *for which* ρ_1 *is localized in* W*, the operator* va *is localized in* W*.*

(2) For each $x \in \mathbb{R}^2$*,*

$$\mathbf{a}(x) := U(x) \circ \mathbf{a} \circ U(-x)$$

is a one kink field which is localized in $W + x$*.*

Proof.

(1) Let us consider a one-kink field $\mathbf{a} = (\rho, a)$ which is localized in W. By definition, there exists a unitary intertwiner $u \in (\hat{\rho}|\rho)$ such that $\hat{\rho}$ and ua are localized in W. If $v \in (\rho_1|\rho)$ is a unitary intertwiner such that ρ_1 is also localized in W, then va is localized in W since $uv^* \in (\hat{\rho}|\rho')$ is localized in W.

(2) Let γ_ρ be a cocycle of charge transporters of ρ. Then we obtain:

$$\mathbf{a}(x) = (\rho, \gamma_\rho(x)^* \alpha_x(a)) \ .$$

If $u \in (\hat{\rho}|\rho)$ is a unitary intertwiner such that $\hat{\rho}$ and ua are localized in W, then the operator

$$w = \alpha_x(u)\gamma_\rho(x)^*$$

is a unitary intertwiner in $(\alpha_x \circ \hat{\rho} \circ \alpha_{-x}|\rho)$. Since $\alpha_x \circ \hat{\rho} \circ \alpha_{-x}$ and $\alpha_x(ua)$ are localized in $W + x$, the statement *(2)* follows.

□

We introduce a further technical definition:

Definiton 6.2.9 :

(1) We denote by $F(q, \rho; W)$ the linear subspace of all one-kink fields which are localized in W.

(2) Let $\underline{W} = (W_j, W^i)_{j \in \underline{n}}^{i \in \underline{m}}$ be a family of wedge regions where the W_j's are right wedge regions and the W^i's are left wedge regions. We define

$$F(n, m, \underline{\rho}; W) := \Big(\bigotimes_{j \in \underline{n}} F(1, \rho_j; W_j) \Big) \otimes \Big(\bigotimes_{i \in \underline{m}} F(-1, \rho^i; W^i) \Big)$$

and we shall say that the multi-kink fields in $\mathbf{a} \in F(n, m, \underline{\rho}; \underline{W})$ are localized in $\underline{W}$.

We are now prepared to formulate the quasi-statistics relations for kink fields.

Theorem 6.2.10 : *Let $\mathbf{a} \in F(n, m, \underline{\rho}; \underline{W})$ be a multi-kink field. If W_j and W^i are space-like separated for each pair (j, i), then for each pair $(\kappa, \hat{\kappa}) \in G(n, m)$ the equation*

$$\mathbf{a}_\kappa \Omega_{e_0} = \epsilon_{(\kappa, \hat{\kappa})}(\rho) \ \mathbf{a}_{\hat{\kappa}} \Omega_{e_0}$$

holds.

Proof. (Compare also [18, 19, 10]) We shall show in Appendix B.2 that there exists a unitary intertwiner $u \in (\underline{\rho}, \underline{\hat{\rho}})$ such that

$$(u\mathbf{a})_\kappa \Omega_{e_0} = (u\mathbf{a})_{\hat{\kappa}} \Omega_{e_0}$$

for each pair $(\kappa, \hat{\kappa}) \in G(n, m)$. This is due to the localization property of $\mathbf{a}$. Thus we obtain from Proposition 6.2.4:

$$\mathbf{a}_\kappa \Omega_{e_0} = u_\kappa^* u_{\hat{\kappa}} \, \mathbf{a}_{\hat{\kappa}} \Omega_{e_0} \quad .$$

In Appendix B.2, the quasi-statistics operator is defined by

$$u_\kappa^* u_{\hat{\kappa}} := \epsilon_{(\kappa,\hat{\kappa})}(\rho)$$

and is shown to be independent of the choice of the intertwiner u. □

6.2.3 Cluster Properties for Kink Fields

In order to prove the existence of kink collision states, we shall study the cluster property of products of one-kink fields.

In the first step, we shall establish cluster properties for the simplest case, namely for products of two one-kink fields. We consider one-kink homomorphisms $(\rho_1, \rho^1) \in T(1, 1; e_{-1}, e_0, e_1)$ and wedge regions W_1, W^1, $W_1 \subset (W^1)'$ with space-like distance

$$\tau_0 := \sup\{|t| \mid W_1 + (t, 0) \subset (W^1)'\} \quad .$$

Then we have:

Lemma 6.2.11 : *Let* $\mathbf{a}_1 \in F(1, \rho_1, W_1)$ *and* $\mathbf{a}^1 \in F(-1, \rho^1, W^1)$ *be one-kink fields. Denote by* μ_0 *the value of the mass gap in* $\mathrm{sp}(U_{e_0})$*. Then there exists a constant* $K > 0$*, depending only on* $||\mathbf{a}_1||$ *and* $||\mathbf{a}^1||$*, such that the estimate*

$$\left| \, ||\mathbf{a}_1 \mathbf{a}^1 \Omega_{e_0}||^2 \; - \; ||\mathbf{a}_1 \Omega_{e_0}||^2 ||\mathbf{a}^1 \Omega_{e_0}||^2 \, \right| \;\leq\; K \, e^{-\mu_0 \tau_0}$$

holds.

Proof. The lemma follows directly from a result which has been established by K.Fredenhagen [25]. Compare also [28]. □

For establishing cluster properties for arbitrary products of kink fields, it seems to be necessary to assume stronger localizing properties for kink homomorphisms and kink fields.

The definition, given below, and the following proposition are formulated only for kink homomorphisms with orientation $q = 1$. Since the case $q = -1$ can simply be discussed in the same manner, we omit it here.

Definiton 6.2.12 : Given a kink homomorphism $\rho \in \Delta(1, e_1, e_2)$, we shall say that the *interpolation region* of ρ is contained in the double cone $\mathcal{O}$, if the following conditions are fulfilled.

(a) The homomorphism ρ is localized in $\mathcal{O}_R$, i.e. $\mathrm{supp}(\rho) \subset \mathcal{O}_R$.

(b) There exists a symmetry $\chi_\rho \in \mathrm{Sym}(\mathfrak{A})$ such that:

$$\rho|_{C^*(\mathfrak{A}, \mathcal{O}_{RR})} = \chi_\rho|_{C^*(\mathfrak{A}, \mathcal{O}_{RR})} \quad .$$

Remark: Note that kink homomorphisms, which are induced by interpolating automorphisms fulfill the conditions *(a)* and *(b)*.

Proposition 6.2.13 : *Let $\rho_1, \rho_2 \in \Delta(1, e_1, e_2)$ be kink homomorphisms whose interpolation regions are contained in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. If $\chi_{\rho_1} = \chi_{\rho_2}$, then each intertwiner $v \in (\rho_1|\rho_2)$ is contained in $\mathfrak{A}(\mathcal{O}_{12})$ where $\mathcal{O}_{12}$ denotes the smallest double cone which contains the union $\mathcal{O}_1 \cup \mathcal{O}_2$.*

Proof. It is sufficient to prove the proposition for $q = 1$. The case $q = -1$ can be treated analogously. We obtain for each observable a which is localized in $(\mathcal{O}_{12})_{LL}$:

$$v\rho_1(a) = va = \rho_2(a)v = av \quad .$$

We define $\chi := \chi_{\rho_1} = \chi_{\rho_2}$, and we obtain for each observable b which is localized in $(\mathcal{O}_{12})_{RR}$:

$$v\rho_1(b) = v\chi(b) = \rho_2(b)v = \chi(b)v \quad .$$

Since χ is a symmetry, we conclude by using Haag duality:

$$v \in \mathfrak{A}_{e_1}((\mathcal{O}_{12})_{LL})' \wedge \mathfrak{A}_{e_1}((\mathcal{O}_{12})_{RR}))' = \mathfrak{A}_{e_1}((\mathcal{O}_{12})_R) \wedge \mathfrak{A}_{e_1}((\mathcal{O}_{12})_L)) = \mathfrak{A}(\mathcal{O}_{12})$$

which completes the proof. □

Definiton 6.2.14 : Given a one-kink field $\mathbf{a} = (\rho, a)$. We shall say that the *interpolation region* of $\mathbf{a}$ is contained in the double cone $\mathcal{O}$ if there exists a unitary intertwiner $u \in (\hat{\rho}, \rho)$ such that:

(a) The interpolation region of $\hat{\rho}$ is contained in $\mathcal{O}$.

(b) The operator ua is localized in $\mathcal{O}$.

We shall see later that this localization property is sufficient for controlling the clustering of correlation functions of multi-kink fields.

Remark:

(i) Note that such one-kink fields exist. Let us choose a one-kink homomorphism ρ with interpolation region in $\mathcal{O}$ and an operator $a \in \mathfrak{A}(\mathcal{O})$, then the interpolation region of $\mathbf{a} = (\rho, a)$ is contained in $\mathcal{O}$. In particular, each one-kink field $\mathbf{a}$ whose interpolation region is contained in $\mathcal{O}$ is localized in $\mathcal{O}_R$, for $q = 1$, and $\mathcal{O}_L$, for $q = -1$.

(ii) In complete analogy to the proof of Proposition 6.2.8, it can be verified that the interpolation region of the translated one-kink field $\mathbf{a}(x)$ is contained in $\mathcal{O} + x$ if the interpolation region of $\mathbf{a}$ is contained in $\mathcal{O}$.

In order to formulate the main result of this section, we consider the subspace

$$F(n, m; \underline{\rho}, \underline{\mathcal{O}}) \subset F(n, m; \underline{\rho})$$

which is spanned by multi-kink fields of the form

$$\mathbf{a} = \Big(\bigotimes_{j \in \underline{n}} \mathbf{a}_j \Big) \otimes \Big(\bigotimes_{i \in \underline{m}} \mathbf{a}^i \Big) \in F(n, m; \underline{\rho}) \tag{6.7}$$

such that the interpolation region of $\mathbf{a}_j$ is contained in a double cone $\mathcal{O}_j$ and the interpolation region of $\mathbf{a}^i$ is contained in a double cone $\mathcal{O}^i$.

Notation: We write for the vacuum vectors

$$\Omega_{j-1} = \Omega_{e_{j-1}} \quad \text{and} \quad \Omega^{i-1} = \Omega_{e_{-i+1}}$$

and for the implementations of the translation group in the corresponding vacuum representations:

$$U_{j-1}(x) = U_{e_{j-1}}(x) \quad \text{and} \quad U^{i-1}(x) = U_{e_{-i+1}}(x) \ .$$

The value of the mass gap in $\mathrm{sp}(U_j)$ and $\mathrm{sp}(U^i)$ is denoted by μ_j and μ^i respectively. We also define the space-like distance (see Figure 6.1):

$$\tau_0 = \tau^0 := \sup\{|t| \mid (\mathcal{O}^1)_R \supset (\mathcal{O}_1)_{RR} + (t,0)\}$$

$$\tau_j := \sup\{|t| \mid (\mathcal{O}_j)_R \supset (\mathcal{O}_{j+1})_{RR} + (t,0)\}$$

$$\tau^i := \sup\{|t| \mid (\mathcal{O}^{i+1})_R \supset (\mathcal{O}^i)_{RR} + (t,0)\} \ .$$

Moreover, we introduce a partial ordering $\prec$ on the set of double cones. We write $\mathcal{O} \prec \hat{\mathcal{O}}$ if $\mathcal{O}_L \subset \hat{\mathcal{O}}_{LL}$.

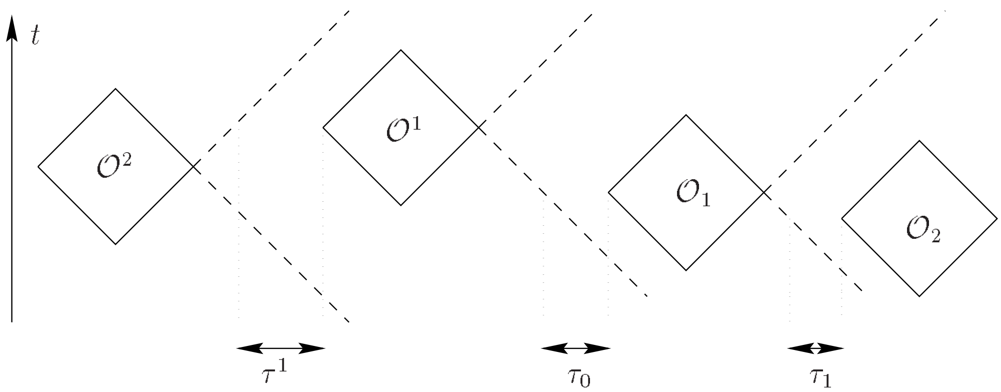

Figure 6.1: This figure shows the positions of the double cones $\underline{\mathcal{O}} = (\mathcal{O}_j, \mathcal{O}^i)$ for $i, j \in \{1, 2\}$.

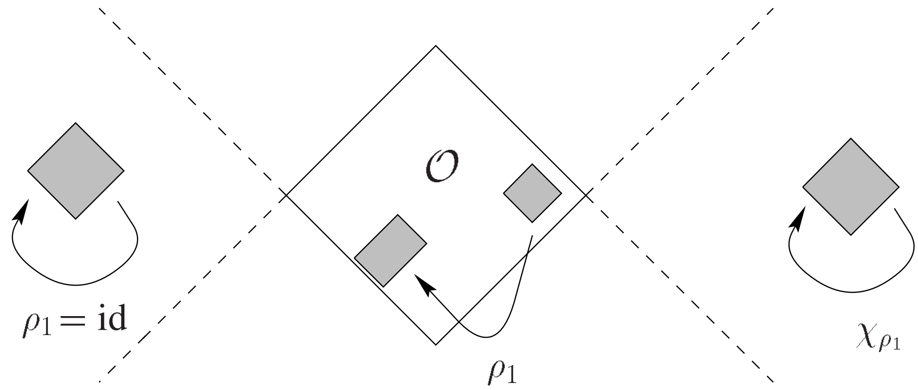

Figure 6.2: The figure above illustrates how the kink homomorphism ρ_1 changes the localization region of local operators. This property encodes that the energy density of the kink is strictly localized in $\mathcal{O}_1$.

Proposition 6.2.15 : *Let* $\mathbf{a} \in F(n, m, \underline{\rho}, \underline{\mathcal{O}})$ *be a multi-kink field which is a tensor product as given by equation (6.7). If the double cones* $\underline{\mathcal{O}} = (\mathcal{O}_j, \mathcal{O}^i)$ *are placed in such a way that*

$$\mathcal{O}^m \prec \mathcal{O}^{m-1} \prec \cdots \prec \mathcal{O}_1 \prec \cdots \prec \mathcal{O}_n \quad .$$

Then there exists a constant $K > 0$*, depending only on* $||\mathbf{a}_j||$ *and* $||\mathbf{a}^i||$*, such that for each arrangement* $\kappa \in P(n, m)$ *the following estimate holds:*

$$\left| \ ||\mathbf{a}_\kappa \Omega_0||^2 \ - \ \prod_{j=1}^{n} ||\mathbf{a}_j \Omega_{j-1}||^2 \prod_{i=1}^{m} ||\mathbf{a}^i \Omega^{i-1}||^2 \ \right|$$

$$\leq \ K \left(\sum_{j=0}^{n-1} e^{-\mu_j \tau_j} + \sum_{i=0}^{m-1} e^{-\mu^i \tau^i} + e^{-\mu_0 \tau_0} \right)$$

Proof. From Theorem 6.2.10, we conclude that

$$||\mathbf{a}_\kappa \Omega_0||^2 = ||\mathbf{a}^m \cdots \mathbf{a}^1 \mathbf{a}_n \cdots \mathbf{a}_1 \Omega_0|| \ .$$

Since $\mathbf{a}^m, \cdots, \mathbf{a}^1$ are localized in $(\mathcal{O}^1)_L$ and $\mathbf{a}_n, \cdots, \mathbf{a}_1$ are localized in $(\mathcal{O}_1)_R$, an straight forward generalization of Lemma 6.2.11 gives:

$$\left| \ ||\mathbf{b}^1 \mathbf{b}_1 \Omega_0||^2 \ - \ ||\mathbf{b}_1 \Omega_0||^2 ||\mathbf{b}^1 \Omega_0||^2 \ \right| \ \leq \ K \ e^{-\mu_0 \tau_0} \tag{6.8}$$

where $\mathbf{b}_1$ and $\mathbf{b}^1$ are given by

$$\mathbf{b}_1 := \mathbf{a}_n \circ \cdots \circ \mathbf{a}_1 \quad \text{and} \quad \mathbf{b}^1 = \mathbf{a}^m \circ \cdots \circ \mathbf{a}^1$$

and $K > \max_{i,j}(||\mathbf{a}_j||, ||\mathbf{a}^i||)^{2(n+m)}$ is a sufficiently large constant.

Keeping in mind, how kink fields act on the state bundle $\underline{\mathcal{H}}$, we conclude that $||\mathbf{b}_1\Omega_0||^2$ can be written as

$$||\mathbf{b}_1\Omega_0||^2 = \omega_0(a_1^*\rho_1(b^*b)a_1)$$

where b is localized in $(\mathcal{O}_2)_R$. The interpolation region of ρ_1 is contained in $\mathcal{O}_1$ and hence $\rho_1(b^*b)$ is also localized in $(\mathcal{O}_2)_R$ (see Figure 6.2). Since a_1 is localized in $(\mathcal{O}_1)_L$, we conclude from [25] and the fact that $\omega_0(\rho_1(b^*b)) = \omega_1(b^*b)$:

$$|\omega_0(a_1^*\rho_1(b^*b)a_1) - \omega_1(b^*b)\omega_0(a^*a)| \ \leq \ K\, e^{-\mu_0\tau_0}$$

Let $\mathbf{b}_2 := \mathbf{a}_n \circ \cdots \circ \mathbf{a}_2$. Then we have $||\mathbf{b}_2\Omega_1||^2 = \omega_1(b^*b)$ and we obtain the estimate:

$$\Big| \ ||\mathbf{b}_1\Omega_0||^2 \ - \ ||\mathbf{b}_2\Omega_1||^2||\mathbf{a}_1\Omega_0||^2 \ \Big| \ \leq \ K\, e^{-\mu_0\tau_0} \tag{6.9}$$

In a similar way, we derive:

$$\Big| \ ||\mathbf{b}_2\Omega_1||^2 \ - \ ||\mathbf{b}_3\Omega_2||^2||\mathbf{a}_2\Omega_1||^2 \ \Big| \ \leq \ K\, e^{-\mu_1\tau_1} \tag{6.10}$$

with $\mathbf{b}_3 = \mathbf{a}_n \circ \cdots \circ \mathbf{a}_3$. Inserting equation (6.10) into equation (6.9) gives:

$$\Big| \ ||\mathbf{b}_1\Omega_0||^2 \ - \ ||\mathbf{b}_3\Omega_2||^2||\mathbf{a}_2\Omega_1||^2||\mathbf{a}_1\Omega_0||^2 \ \Big| \tag{6.11}$$

$$\leq \ K\left(e^{-\mu_0\tau_0} + e^{-\mu_1\tau_1}\right)$$

By induction it follows:

$$\Big| \ ||\mathbf{b}_1\Omega_0||^2 \ - \ \prod_{j=1}^{n} ||\mathbf{a}_j\Omega_{j-1}||^2 \ \Big| \ \leq \ K\sum_{j=1}^{n} e^{-\mu_{j-1}\tau_{j-1}} \tag{6.12}$$

Analogously we obtain:

$$\Big| \ ||\mathbf{b}^1\Omega_0||^2 \ - \ \prod_{i=1}^{m} ||\mathbf{a}^i\Omega^{i-1}||^2 \ \Big| \ \leq \ K\sum_{i=1}^{n} e^{-\mu^{i-1}\tau^{i-1}} \tag{6.13}$$

Finally, the proposition follows by inserting equation (6.12) and (6.13) into equation (6.8). □

6.3 The Structure of Scattering States

In this section we introduce one-kink creation operators. We apply the results of the previous Section 6.2 to products of one-kink creation operators in order to prove the existence of the Haag-Ruelle collision states. We also refer to [44, 68, 18, 19, 10]. The cluster property for kink fields is one of the crucial facts to carry through our subsequent analysis.

6.3.1 Creation Operators for Massive One-Kink States

To describe a stable one-particle state, we use one-kink fields $\mathbf{a} = (\rho, a)$, such that the vector $\psi = \{\rho, a\Omega_e\}$ has spectral support on the mass shell H_m, i.e. it is an eigenstate of the mass operator:

$$P_\mu P^\mu \psi = m^2 \psi$$

Indeed, there are such fields. If we choose a energy-momentum distribution $f \in S(\mathbb{R}^2)$ such that $\mathrm{supp}(f) \cap \mathrm{sp}(U_\rho) \subset H_m$, then the one-kink field

$$\mathbf{a}(f) = \int \mathrm{d}x \; f(x) \; \mathbf{a}(x)$$

has the desired property.

Since f has compact support in momentum space, the field operator $\mathbf{a}(f)$ is only an almost local operator, i.e. the kink field $\mathbf{a}(f)$ can be approximated in norm by fields which are localized in wedge regions.

We shall show that for a momentum distribution $f \in S(\mathbb{R}^2)$ and a one-kink field $\mathbf{a}$ the operator $\mathbf{a}(f)$ can be approximated in norm by one-kink fields with bounded interpolation region.

Proposition 6.3.1 **:** *Let $f \in S(\mathbb{R}^2)$ be a momentum distribution and $\mathbf{a} \in F(q, \rho)$ a one-kink field whose interpolation region is contained in $\mathcal{O}$. Then for each $\epsilon > 0$ there exists a compact region $G \subset \mathbb{R}^2$ such that*

$$||\mathbf{a}(f) - \mathbf{a}_G(f)|| < \epsilon \tag{6.14}$$

where the interpolation region of the field operator

$$\mathbf{a}_G(f) := \int_G \mathrm{d}x \; f(x) \; \mathbf{a}(x) \tag{6.15}$$

is contained in a bounded double cone.

Proof. It is sufficient to prove the proposition for kink fields with orientation $q = 1$. The case $q = -1$ can be treated in the same manner. To establish equation (6.14), we consider the following inequality:

$$\begin{aligned} ||\mathbf{a}(f) - \mathbf{a}_G(f)|| &= \left|\left| \int_{\mathbb{R}^2 \backslash G} \mathrm{d}x \; f(x) \; \mathbf{a}(x) \right|\right| \\ &\leq \int_{\mathbb{R}^2 \backslash G} \mathrm{d}x \; |f(x)| \; ||\mathbf{a}|| \end{aligned}$$

Since f is of fast decrease, equation (6.14) follows.

It remains to be proven that the kink field $\mathbf{a}_G(f)$ has a bounded interpolation region. We choose $\mathbf{a} = (\rho, a)$ such that ρ has interpolation region in $\mathcal{O}$ and a is contained in $\mathfrak{A}(\mathcal{O})$. The translated one-kink field operator

$$\mathbf{a}(x) = (\rho, \gamma_\rho(x)^* \alpha_x(a))$$

has interpolation region in $\mathcal{O} + x$. We choose a point $x_G \in \mathbb{R}^2$ such that

$$\rho_G := \mathrm{Ad}(\gamma_\rho(x_G)) \circ \rho$$

is localized in the right space-like complement of

$$\mathcal{O}_G := \bigvee_{x \in G} \mathcal{O} + x \;\; .$$

Here γ_ρ is a cocycle of charge transporters of ρ. By Proposition 6.2.13, the operator

$$\gamma_\rho(x_G)\gamma_\rho(x)^* \in (\rho_G | \alpha_x \circ \rho \circ \alpha_{-x})$$

is localized in $\mathcal{O}_G$ and we conclude:

$$\gamma_\rho(x_G) \int_G \mathrm{d}x \; f(x) \; \mathbf{a}(x) = \Big(\rho_G, \; \int_G \mathrm{d}x \; f(x) \; \gamma_\rho(x_G)\gamma_\rho(x)^* \alpha_x(a)\Big) \;\; .$$

Since the operator

$$\int_G \mathrm{d}x \; f(x) \; \gamma_\rho(x_G)\gamma_\rho(x)^* \alpha_x(a)$$

is contained in $\mathfrak{A}(\mathcal{O}_G)$ and ρ_G has kink region in $\mathcal{O}_G$, the proposition follows. □

The one-kink field $\mathbf{a}(f) \in F(q, \rho)$ describes the creation of a kink with energy-momentum distribution f. We are interested in those momentum distributions which describes the free propagation of a massive particle. For this purpose, we assign to each one-kink homomorphism $\rho \in \Delta_1(q, e_1, e_2)$ a linear subspace $L_\rho \subset S(\mathbb{R}^2)$ of energy-momentum distributions which is given by the condition:

(a) The support $\mathrm{supp}(\tilde{f})$ is compact and $\mathrm{supp}(\tilde{f}) \cap \mathrm{sp}(U_\rho) \subset H_{m_\rho}$ where m_ρ is the value of the mass which corresponds to the one-kink homomorphism ρ.

The *velocity support* $v_r(f)$, $r > 0$, of an energy-momentum distribution $f \in L_\rho$ is given by

$$v_r(f) := \{v_\rho(p) | p \in B(r) + k \text{ for some } k \in \mathrm{supp}(\tilde{f}).\}$$

where $v_\rho(p_0, p_1) = p_1 \cdot (p_1^2 + m_\rho^2)^{1/2}$ is the velocity with respect to p in the Lorenz frame which is given by the x^0-coordinate and $B(r)$ is the closed ball with radius r and center $k = 0$.

Given a one kink field $\mathbf{a} \in F(q, \rho)$ and an energy-momentum distribution $f \in L_\rho$. We define the *time-depending* field operator

$$\mathbf{a}(f, t) := \mathbf{a}(D_t f)$$

where the operator D_t is given by the kernel

$$D_t(x - y) := \int \mathrm{d}p \; e^{i(p_0 - w_\rho(p))t} \; e^{-i(x-y)} \quad .$$

If we apply $a(f, t)$ to the corresponding vacuum Ω_e, then we obtain a vector which is independent of t and represents a stable massive particle. Indeed, we obtain by a straight forward computation

$$\mathbf{a}(f, t)\Omega_e = 2\pi \cdot \{\rho, \tilde{f}(P_\rho)\pi_e(a)\Omega_e\}$$

which is obviously independent of t. Here P_ρ denotes the generator of the translations $U_\rho(x)$. (Compare also [10, 18, 19, 28].)

Given an energy-momentum distribution $f \in L_\rho$. We consider the compact region

$$G_r(f, t) := \{x \in \mathbb{R}^2 | \; x^0 \in [t - r, t + t] \text{ and } t^{-1}x^1 \in v_r(f).\}$$

and the one-kink field

$$\mathbf{a}_r(f, t) := \mathbf{a}_{G_r(f,t)}(D_t f)$$

By Proposition 6.3.1, we conclude that the interpolation region of $\mathbf{a}_r(f, t)$ is contained in a bounded double cone. It is well known [10, 18, 19, 28] that the norm of the difference of $\mathbf{a}_r(f, t)$ and $\mathbf{a}(f, t)$ is of fast decrease in t, i.e. a fast decreasing function $h \in S(\mathbb{R})$ exists such that:

$$||\mathbf{a}_r(f, t) - \mathbf{a}(f, t)|| \; \leq \; h(t) \tag{6.16}$$

Let us have a closer look to the kink-region of $\mathbf{a}_r(f, t)$.

Lemma 6.3.2 : *Let* $\mathbf{a}$ *be a one-kink field with interpolation region in* $\mathcal{O}$ *and an energy-momentum distribution* $f \in L_\rho$ *and define* $v_L := \inf(v_r(f))$ *and* $v_R := \sup(v_r(f))$*. Then the interpolation region of the one-kink field* $\mathbf{a}_r(f,t)$ *is contained in*

$$\mathcal{O}_r(f,t) := \mathcal{O}_L + (t, t \cdot v_L) \cap \mathcal{O}_R + (t, t \cdot v_R)$$

Proof. From the definition of the velocity support we obtain:

$$\mathcal{O}_r(f,t) = \bigvee_{x \in G_r(f,t)} \mathcal{O} + x$$

We choose a charge transporter $u \in (\hat{\rho}, \rho)$ such that $\hat{\rho}$ is localized in $G_r(f,t)_R$. Then, for each $x \in \mathcal{G}_r(f,t)$, the operator $u\gamma_\rho(x)^*$ is contained in $(\hat{\rho}, \alpha_x \circ \rho \circ \alpha_{-x})$. Therefore, the operator

$$\int_{G_r(f,t)} \mathrm{d}x \; D_t f(x) \; u\gamma_\rho(x)^* \alpha_x(a)$$

is localized in $\mathcal{O}_r(f,t)$. Since the interpolation region of $\hat{\rho}$ is contained in $\mathcal{O}_r(f,t)$, the lemma follows. $\square$

6.3.2 Construction of the Haag-Ruelle Collision States

We consider energy-momentum distributions f for with velocity support in I, i.e.

$$f \in L_\rho(I) := \{f_1 \in L_\rho | v_r(f_1) \subset I \text{ for at least one } r > 0\} \; .$$

Energy-momentum distributions of multi-kink configurations are described by test functions which are contained in

$$L(n,m,\underline{\rho},\underline{I}) := \left(\bigotimes_{j \in \underline{n}} L_{\rho_j}(I_j) \right) \otimes \left(\bigotimes_{i \in \underline{m}} L_{\rho^i}(I^i) \right) \; \subset \; S(\mathbb{R}^2)^{\otimes(n+m)}$$

where $\underline{\rho}$ is contained in $T(n,m,\underline{e})$.

Given a multi-kink field $\mathbf{a} \in F(n,m,\underline{\rho})$ and an energy momentum distribution $f \in L(n,m,\underline{\rho},\underline{I})$. Then the multi-kink fields $\mathbf{a}(f,t)$ are analogously defined to the one-kink case: Let

$$f = \left(\bigotimes_{j \in \underline{n}} f_j \right) \otimes \left(\bigotimes_{i \in \underline{m}} f^i \right) \quad \text{and} \quad \mathbf{a} = \left(\bigotimes_{j \in \underline{n}} \mathbf{a}_j \right) \otimes \left(\bigotimes_{i \in \underline{m}} \mathbf{a}^i \right) \tag{6.17}$$

then $\mathbf{a}(f,t)$ is given by

$$\mathbf{a}(f,t) = \Big(\bigotimes_{j\in\underline{n}} \mathbf{a}_j(f_j,t) \Big) \otimes \Big(\bigotimes_{i\in\underline{m}} \mathbf{a}^i(f^i,t) \Big) \quad . \tag{6.18}$$

The operators $\mathbf{a}(f,t)$ describe configurations of n right-moving and m left-moving kinks with velocities $v_j \in I_j$ and $v^i \in I^i$. For each arrangement $\kappa \in P(n,m)$, the vector

$$\psi(f,t) := \mathbf{a}(f,t)_\kappa \Omega_0$$

represents the corresponding multi-kink state.

To obtain asymptotically stable configurations of $n+m$ free kinks, we consider multi-kink fields $\mathbf{a}$ and energy-momentum distributions f which fulfill the following conditions:

(a) $\mathbf{a} \in F(n,m,\underline{\rho},\underline{\mathcal{O}})$ such that:

$$\mathcal{O}^m \prec \mathcal{O}^{m-1} \prec \cdots \prec \mathcal{O}_1 \prec \cdots \prec \mathcal{O}_n$$

(b) The energy-momentum distributions $f \in L(n,m,\underline{\rho},\underline{I})$ is *velocity ordered*i.e.:

$$I^m < I^{m-1} < \cdots < I_1 < \cdots < I_n \quad .$$

(c) The multi-kink field $\mathbf{a}$ and the energy-momentum distribution $f \in L(n,m,\underline{\rho},\underline{I})$ are tensor products as given by equation (6.17).

Proposition 6.3.3 : *Let* $\mathbf{a}$ *be a multi-kink field* $\mathbf{a}$ *and let f be an energy-momentum distribution such that the conditions (a) to (c) are fulfilled. Then there exists a rapidly decreasing function h such that, for each arrangement $\kappa \in P(n,m)$, the following inequality holds for each $t > 0$:*

$$\Big| \, ||\mathbf{a}(f,t)_\kappa \Omega_0||^2 \; - \; \prod_{j=1}^{n} ||\psi(f_j)||^2 \prod_{i=1}^{m} ||\psi(f^i)||^2 \, \Big| \;\leq\; h(t)$$

Here $\psi(f_j) := \mathbf{a}_j(f_j)\Omega_{j-1}$ and $\psi(f^i) := \mathbf{a}(f^i)\Omega^{i-1}$ are t-independent vectors, representing one-kink states.

Proof. Since the energy-momentum distribution f is velocity ordered we conclude from Lemma 6.3.2 that there are positive numbers $T > 0$ and $R > 0$ such that for each $t > T$ and for each $r < R$ the interpolation regions $\mathcal{O}_j(t)$ and $\mathcal{O}^i(t)$ of the one-kink fields $\mathbf{a}_{j,r}(f_j,t)$ and $\mathbf{a}^i_r(f^i,t)$ are placed as follows:

$$\mathcal{O}^m(t) \prec \mathcal{O}^{m-1}(t) \prec \cdots \prec \mathcal{O}_1(t) \prec \cdots \prec \mathcal{O}_n(t)$$

Moreover, the space-like distances (compare Proposition 6.2.13) of the interpolation regions $(\mathcal{O}_j(t), \mathcal{O}^i(t))$ increase like $|t|$ if t tends to infinity.

We conclude from equation (6.16) that there is a rapidly decreasing function h_1 such that

$$||\mathbf{a}_{j,r}(f_j,t) - \mathbf{a}_j(f_j,t)|| \leq h_1(t) \quad \text{and} \quad ||\mathbf{a}^i_r(f^i,t) - \mathbf{a}^i(f^i,t)|| \leq h_1(t) \quad .$$

Thus the proposition follows from Proposition 6.2.15. □

Corollary 6.3.4 : *There exists a rapidly decreasing function h, such that*

$$\left|\left| \ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(f,t)_\kappa \Omega_0 \ \right|\right| \leq \ h(t)$$

for each arrangement κ.

Proof. By using Proposition 6.3.3 and applying the Leibniz rule, we obtain the result. □

Lemma 6.3.5 : *Let $\mathbf{a} \in F(n,m,\underline{\rho},\underline{\mathcal{O}})$ be a multi-kink field and let $f \in L(n,m,\underline{\rho},\underline{I})$ be a energy-momentum distribution. If the conditions (a) to (c) are fulfilled, then the following statements are true:*

(1) *For each arrangement $\kappa \in P(n,m)$ the strong limits*

$$s - \lim_{t\to\infty} \mathbf{a}(f,t)_\kappa \Omega_0 = \psi^{\mathrm{out}}_\kappa(f) \in \mathcal{H}_{\underline{\rho}_\kappa}$$

exist.

(2) *Let $\psi(f_j) := \mathbf{a}_j(f_j)\Omega_{j-1}$ and $\psi(f^i) := \mathbf{a}(f^i)\Omega^{i-1}$, then the norm of $\psi_\kappa(f)$ is*

$$||\psi^{\mathrm{out}}_\kappa(f)|| = \prod_{j=1}^{n} ||\psi(f_j)|| \prod_{i=1}^{m} ||\psi(f^i)|| \quad .$$

Proof.

(1) For each $t_1, t_2 > 0$, we obtain from Corollary 6.3.4 the estimate:

$$\left\| \ \mathbf{a}(f,t_1)_\kappa\Omega_0 - \ \mathbf{a}(f,t_2)_\kappa\Omega_0 \ \right\| \ = \ \left\| \ \int_{t_2}^{t_1} \mathrm{d}t \ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(f,t)_\kappa\Omega_0 \ \right\|$$
$$\leq \ \int_{t_2}^{t_1} \mathrm{d}t \ \left\| \ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(f,t)_\kappa\Omega_0 \ \right\|$$
$$\leq \ \int_{t_2}^{t_1} \ h(t)$$

Since h is rapidly decreasing, we conclude

$$\lim_{t_1,t_2\to\infty} \left\| \ \mathbf{a}(f,t_1)_\kappa\Omega_0 - \ \mathbf{a}(f,t_2)_\kappa\Omega_0 \ \right\| \ = \ 0$$

which implies *(1)*.

(2) The statement *(2)* follows directly from Proposition 6.3.3.

□

Given a one-kink homomorphism $\rho \in \Delta_1(q,e_1,e_2)$. Then we define the one-kink Hilbert space

$$\mathcal{H}_1(\rho) := \{\psi \in \mathcal{H}_\rho | P_\mu P^\mu \psi = m_\rho^2 \psi\} \quad .$$

It is well known, that

$$\mathcal{D}_\rho := \{\psi = \mathbf{a}(f)\Omega_e | \ \mathbf{a} \in F(q,\rho) \text{ and } f \in L_\rho.\}$$

is a dense subspace of $\mathcal{H}_1(\rho)$.

Let $\underline{\rho} \in T(n,m,\underline{e})$ be a family of one-kink homomorphisms. We consider for each $\kappa \in P(n,m)$ the Hilbert space

$$\mathcal{H}(n,m,\underline{\rho})_\kappa := \mathcal{H}_1(\rho^m) \otimes \cdots \otimes \mathcal{H}_1(\rho^{\kappa_-(l)+1}) \otimes \mathcal{H}_1(\rho_n) \otimes \cdots \mathcal{H}_1(\rho_{\kappa_+(l)+1}) \otimes \cdots$$
$$\cdots \otimes \mathcal{H}_1(\rho^{\kappa_-(1)}) \otimes \cdots \otimes \mathcal{H}_1(\rho^1) \otimes \mathcal{H}_1(\rho_{\kappa_+(1)}) \otimes \cdots \otimes \mathcal{H}_1(\rho_1)$$

where the arrangement $\kappa = (\kappa_+, \kappa_-)$ is parameterized by equation (6.1).

Given a pair of arrangements $(\kappa, \hat{\kappa}) \in G(n, m)$, then we denote by

$$u_{(\kappa,\hat{\kappa})} : \mathcal{H}(n, m, \underline{\rho})_{\hat{\kappa}} \longrightarrow \mathcal{H}(n, m, \underline{\rho})_{\kappa}$$

the unitary operator, canonically given by exchanging the tensor factors in $\mathcal{H}(n, m, \underline{\rho})_{\hat{\kappa}}$ in an appropriate way.

Moreover, let $U_\kappa \longmapsto U_\kappa(x)$ be the representation of the translation group on $\mathcal{H}(n, m, \underline{\rho})_{\hat{\kappa}}$, induced by the translations $U : x \longmapsto U(x)$ which act on the state bundle $\underline{\mathcal{H}}$.

The elements of $\mathcal{H}(n, m, \underline{\rho})_\kappa$ describe a configuration of $n + m$ freely moving kinks which do not interact. We close this section by showing that the asymptotic scattering states $\Psi^{\mathrm{out}}(f)$ can be interpreted in the same manner.

Theorem 6.3.6 : *For each arrangement $\kappa \in P(n, m)$, there are isometries*

$$\Psi^{\mathrm{ex}}_\kappa : \mathcal{H}_{\underline{\rho}_\kappa} \longrightarrow \mathcal{H}(n, m, \underline{\rho})_\kappa$$

$\mathrm{ex} = \mathrm{out}, \mathrm{in}$*, such that the following statements hold:*

(1) Let $\epsilon_{(\kappa,\hat{\kappa})}(\underline{\rho})$ be the quasi statistics operator with respect to $(\kappa, \hat{\kappa}) \in G(n, m)$, then one has:

$$u_{(\kappa,\hat{\kappa})} \circ \Psi^{\mathrm{ex}}_{\hat{\kappa}} = \Psi^{\mathrm{ex}}_\kappa \circ \epsilon_{(\kappa,\hat{\kappa})}(\underline{\rho})$$

(2) The isometries Ψ^{ex} are translationally covariant, i.e.:

$$U_\kappa(x) \circ \Psi^{\mathrm{ex}}_\kappa = \Psi^{\mathrm{ex}}_\kappa \circ U(x)$$

Proof. The existence of the isometries $\Psi^{\mathrm{ex}}_\kappa$ follows directly from Lemma 6.3.5.

(1) The statement *(1)* can be obtained by an application of the quasi-statistics theorem (Theorem 6.2.10) and an application of Proposition 6.3.3.

(2) Let $f \in L_\rho$ be an energy-momentum distribution. Then one has for each $\mathbf{a} \in F(q, \rho)$:

$$U(x)\mathbf{a}(f)\Omega_e = \mathbf{a}(\tau_x f)\Omega_e$$

where $\tau_x f$ denotes the translated test function f by x. Thus we conclude for an arrangement $\kappa \in P(n, m)$:

$$U(x)\mathbf{a}(f, t)_\kappa \Omega_0 = \mathbf{a}(\tau_x f, t)_\kappa \Omega_0$$

and *(2)* follows from Proposition 6.3.3.

□

7 Conclusion and Outlook

7.1 Remarks on Open Problems

In Chapter 4 and Chapter 5, a construction scheme for kink sectors has been developed which can be applied to a large class of quantum field theory models. Most of the techniques which are used, except those in the proof of the extendibility of the dynamics, concern operator algebraic methods. They are model independent in the sense that they can be derived from first principles. There are still some interesting open problems and we shall make a few remarks on them here.

7.1.1 Some Further Remarks on Kink States, Induced by Interpolating Automorphisms

Let us consider a quantum field theory model $(P(\phi)_2, Y_2)$, possessing vacua ω_1, ω_2 which are related by a symmetry χ. According to Theorem 5.1.8, there exists a χ-interpolating automorphism $\chi^{\mathcal{I}}$ which induces a kink state $\omega = \omega_1 \circ \chi^{\mathcal{I}}$. Note that ω is a pure state in this case.

Alternatively, we obtain a kink state $\hat{\omega}$ by passing to the two folded tensor product of the theory with itself first and then by restricting the α_F-interpolating automorphism $\beta^{\mathcal{I}}$, whose existence follows also from Theorem 5.1.8, to the first tensor factor, i.e.:

$$\hat{\omega} = \omega_1 \otimes \omega_2 \circ \beta_{C^*(\mathfrak{A})\otimes \mathbf{1}} \quad .$$

It would be interesting to know in which way both sectors $[\omega]$ and $[\hat{\omega}]$ are related to each other.

7.1.2 The Problem of Reducibility

The problem of reducibility arises if the vacua under consideration are not related by a symmetry since then our construction scheme leads to kink representations of the form

$$\pi = \pi_1 \otimes \pi_2 \circ \beta|_{C^*(\mathfrak{A})\otimes \mathbf{1}}$$

where $\beta \in \mathrm{Aut}(\alpha_F, \mathfrak{A} \overline{\otimes} \mathfrak{A})$ is an interpolating automorphism and π_1, π_2 are vacuum representations. The representation π is not irreducible and whether π can be decomposed into irreducible sub-representations is still an open problem. Some of our results (Theorem 4.4.3) suggest that π is, in non exceptional cases, an infinite multiple of a finite direct sum of irreducible components.

7.2 Application to other Topics

Some of the ideas and techniques which we have used in Chapter 4 and Chapter 5 can also be applied to topics which have not been considered in the present work. In this section we shall briefly mention some examples.

7.2.1 Further Applications in $1+1$-Dimensional Quantum Field Theories

We consider a net of field algebras $\mathfrak{F} : \mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$ in $1+1$-dimensional quantum field theory with an compact internal symmetry group G, acting by automorphisms α_g on $C^*(\mathfrak{F})$. Furthermore, we assume that $\mathfrak{F}$ is faithfully and irreducibly represented by a vacuum representation on some Hilbert space $\mathcal{H}$ and that the inclusion

$$\Lambda(\mathcal{O}) = (\mathfrak{F}(\mathcal{O}_{RR}), \mathfrak{F}(\mathcal{O}_R), \Omega)$$

is standard split, where Ω is the vacuum vector in $\mathcal{H}$. Furthermore, we denote by $\Psi_{\mathcal{O}}$ is the universal localizing map with respect to $\Lambda(\mathcal{O})$.

The fix-point net $\mathfrak{A}$ under the action of G does not fulfill Haag duality and we are interested in the dual net $\mathfrak{A}^d$. Recently, an explicit construction of the dual net $\mathfrak{A}^d$ has been carried out by M. Müger [62]. He applied for his purposes, similar techniques as we have used in Chapter 4 and Chapter 5. We shall briefly sketch the main ideas. In the same manner as in Chapter 5, he constructs a non-local extension $\hat{\mathfrak{F}}$ of the net $\mathfrak{F}$:

$$\hat{\mathfrak{F}}(\mathcal{O}) := \mathfrak{F}(\mathcal{O}) \vee U_{\mathcal{O}}(G)''$$

where the representation $U_{\mathcal{O}}$ is given by

$$U_{\mathcal{O}}(g) := \Psi_{\mathcal{O}}(\mathbf{1} \otimes U(g)) \ .$$

Here U is a unitary representation of G on $\mathcal{H}$ which implements the action of G in the vacuum representation.

The action of G on $C^*(\mathfrak{F})$ can be lifted to the extension $C^*(\hat{\mathfrak{F}})$ in a natural way. Hence one can build the fix-point net $\hat{\mathfrak{A}}$ of $\hat{\mathfrak{F}}$ under the action of G. According to [62], the net $\hat{\mathfrak{A}}$ is nothing else but the dual net. Finally, we illustrate this result by the following diagram:

$$\begin{tikzcd}
\mathfrak{F}(\mathcal{O}) \arrow[r, hook] & \hat{\mathfrak{F}}(\mathcal{O}) \\
\mathfrak{A}(\mathcal{O}) \arrow[r, hook] \arrow[u, hook] & \hat{\mathfrak{A}}(\mathcal{O}) = \mathfrak{A}^d(\mathcal{O}) \arrow[u, hook]
\end{tikzcd}$$

7.2.2 Kink Sectors in $d > 1+1$ Dimensions

It would be desirable to apply our program to quantum field theories in higher dimensions. Let us suppose a theory, given by a net of W*-algebras $\mathfrak{A}$, possesses two locally normal vacuum states ω_1, ω_2.

As a sensible generalization of a kink states to $d > 1+1$, we propose to consider locally normal states ω which fulfill the interpolation condition:

$$\omega|_{C^*(\mathfrak{A},S_1)} = \omega_1|_{C^*(\mathfrak{A},S_1)} \quad \text{and} \quad \omega|_{C^*(\mathfrak{A},S_2')} = \omega_2|_{C^*(\mathfrak{A},S_2')} \tag{7.1}$$

where S_1, S_2, $S_1 \subset S_2$, are space-like cones. The state ω describes the coexistence of two phases which are separated by the *phase boundary* $\partial S := S_1' \cap S_2$.

Let us assume duality for space-like cones in the vacuum representations under consideration. Furthermore, we assert that the inclusion

$$\Lambda = (\mathfrak{A}_{\pi_1}(S_1), \mathfrak{A}_{\pi_1}(S_2), \Omega_1)$$

is standard split. Here $(\mathcal{H}_1, \pi_1, \Omega_1)$ is a GNS-triple with respect to ω_1.

Unfortunately, for $d > 1+1$ the phase boundary ∂S is not compact and therefore our construction scheme can not directly be generalized to higher dimensions.

In order to overcome this difficulties, we consider a sequence of standard split inclusions

$$\Lambda_n := (\mathfrak{A}_{\pi_1}(\mathcal{O}_{1n}), \mathfrak{A}_{\pi_1}(\mathcal{O}_{2n}), \Omega_1)$$

where $\mathcal{O}_{1n} \subset\subset \mathcal{O}_{2n}$ are bounded double cones such that $\mathcal{O}_{jn}$ tends to S_j for $n \to \infty$.

As in the $1+1$-dimensional case we pass now to the two folded tensor product of the theory with itself. Denote by $\Psi_{\Lambda_n \otimes \Lambda_n}$ the universal localizing map with respect to the inclusion $\Lambda_n \otimes \Lambda_n$. Since the operators

$$\theta_n := \Psi_{\Lambda_n \otimes \Lambda_n}(\mathbf{1} \otimes u_F)$$

are localized in a bounded region, we may define the following automorphisms of $C^*(\mathfrak{A})$:

$$\beta_n := (\pi_1 \otimes \pi_1)^{-1} \circ \beta_n \circ (\pi_1 \otimes \pi_1) \ .$$

We obtain a sequence of states $\{\omega_n, n \in \mathbb{N}\}$ where ω_n is given by:

$$\omega_n := \omega_1 \otimes \omega_2 \circ \beta_n|_{C^*(\mathfrak{A}) \otimes \mathbf{1}} \ .$$

For large n the states ω_n have *almost* the correct interpolation property, namely for each pair of local observables a, b where a is localized S_2' and b is localized in S_1, there exists a sufficiently large N such that

$$\omega_n(a) = \omega_1(a) \quad \text{and} \quad \omega_n(b) = \omega_2(b)$$

for each $n > N$. Note that each state ω_n fulfills the Borchers criterion since ω_n belongs to the vacuum sector $[\omega_1]$.

In order to obtain generalized kink states, we propose to investigate weak*-limit points of the sequence $\{\omega_n, n \in \mathbb{N}\}$. Note that each weak*-limit ω_ι point of the sequence $\{\omega_n, n \in \mathbb{N}\}$ fulfills the interpolation condition (7.1). It remains to be proven that the weak*-limit points are locally normal.

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A Remarks on the Split Property for Massive Free Scalar Fields

A.1 Preliminaries:

We work here with the self-dual CCR-algebra in d spatial dimensions. Therefore, we need some technical definitions.

Definiton A.1.1 : For the vector space $K = S(\mathbb{R}^d) \oplus S(\mathbb{R}^d)$, we denote by Γ the complex conjugation in K, $\Gamma f = \bar{f}$. Moreover, we introduce the following sesquilinear form γ on K:

$$\gamma(f,g) = \left(f, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} g\right) \tag{A.1}$$

where $(\cdot,\cdot)$ denotes the ordinary scalar-product in $L_2(\mathbb{R}^d) \oplus L_2(\mathbb{R}^d)$. The *self-dual CCR-algebra* $\mathfrak{A}(K,\gamma,\Gamma)$ is the *-algebra which is generated by the set of symbols $\{b(f) : f \in K\}$ modulo the following relations:

(1) The map $b : f \in K \mapsto b(f) \in \mathfrak{A}(K,\gamma,\Gamma)$ is linear.

(2) We have the following *-relation: $b(f)^* = b(\Gamma f)$.

(3) We have the commutator relation $[b(f)^*, b(g)] = \gamma(f,g)\mathbf{1}$.

For a region $G \subset \mathbb{R}^d$ we consider the CCR-algebra $\mathfrak{A}(G) := \mathfrak{A}(K(G),\gamma,\Gamma)$ where $K(G)$ is defined by $K(G) := \mu^{1/2} S(G) \oplus \mu^{-1/2} S(G)$. Here μ is the pseudo differential operator which is given by kernel

$$\mu(\mathbf{x}-\mathbf{y}) := \int \mathrm{d}\mathbf{p} \, (\mathbf{p}^2 + m^2)^{1/2} \, e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \ . \tag{A.2}$$

We now define the quasi-free vacuum functional ω_0 on $\mathfrak{A}(K, \gamma, \Gamma)$ by

$$\omega_0(b(f)^* b(g)) := 1/2\gamma(f, g) \quad .$$

where the functions f, g are contained in K.

A.2 Product States:

Let us consider now two regions $G_1, G_2 \subset \mathbb{R}^d$ with non vanishing distance. In the sequel we write $G := G_1 \cup G_2$ for their union.

We denote by $\mathfrak{A}(G_1) \vee \mathfrak{A}(G_2)$ the algebra which is given by all finite sums $\sum a_n b_n$ with $a_n \in \mathfrak{A}(G_1)$ and $b_n \in \mathfrak{A}(G_2)$. Since G_1 and G_2 have non vanishing distance we conclude that $\mathfrak{A}(G_1) \vee \mathfrak{A}(G_2) = \mathfrak{A}(G)$. We define now a *product* state ω on $\mathfrak{A}(G)$ by

$$\omega(\sum a_n b_n) := \sum \omega_0(a_n)\omega_0(b_n) \quad . \tag{A.3}$$

Since ω_0 is quasi-free, ω is also a quasi-free state on $\mathfrak{A}(G)$.

We are now interested in a criterion which give us the possibility to decide for which regions G_1, G_2 with non vanishing distance the GNS-representations with respect to the states ω and ω_0 are unitarily equivalent on $\mathfrak{A}(G)$.

We are going to use a criterion which is proven by H. Araki. To formulate this criterion, let us consider the following two scalar products on the space $K(G_1 \cup G_2)$:

(1) $(f, g)_0 := \omega_0(b(f)^* b(g)) + \omega_0(b(\Gamma f)^* b(\Gamma g))$

(2) $(f, g)_p := \omega(b(f)^* b(g)) + \omega(b(\Gamma f)^* b(\Gamma g))$

Here ω is the product state, induced by ω_0. The completion of $K(G)$ with respect to the norm $||\cdot||_0 = (\cdot, \cdot)_0$ (resp. $||\cdot||_p = (\cdot, \cdot)_p$) is denoted by $K(G)_0$ (resp. $K(G)_p$).

Moreover, denote by s_0 (resp. s_p) a positive operator, bounded by 1, with the property $(f, s_0 g)_0 = \omega_0(b(f)^* b(g))$ (resp. $(f, s_p g)_p = \omega(b(f)^* b(g))$).

Criterion: The GNS-representations with respect to ω_0 and ω are unitarily equivalent if the following conditions hold:

(1) The values $0, 1/2$ are not eigenvalues of s_0 (resp. s_p) in $K(G)_0$ (resp. $K(G)_p$).

(2) The norms $||\cdot||_0$ and $||\cdot||_p$ are equivalent on $K(G)$.

(3) The following operators are of Hilbert-Schmidt class in $K(G)_0 = K(G)_p$:

$$(s_0 - s_p)(\mathbf{1} - 2s_0)^{-1} \quad \text{and} \quad (s_0(\mathbf{1} - s_0))^{1/2} - (s_p(\mathbf{1} - s_p))^{1/2}$$

The following analysis can be done in complete analogy to those of D. Buchholz [8] who has proven that ω and ω_0 are unitarily equivalent on $\mathfrak{A}(G)$, in the case where $G_1 = O_1$ is a compact region and $G_2 = O_2$ the complement of a slightly larger compact region in $\mathbb{R}^3$. The only argument in this analysis which depends on the spatial dimension is contained in the proof of condition *(2)* ([8, Lemma 3.2]). The necessary generalization is given in the next paragraph.

If one carries through the analysis of [8], we obtain the following criterion: Consider two regions $\hat{G}_j \supset G_j \;;\; j = 1, 2$ such that $\hat{G}_1$ and $\hat{G}_2$ have also non vanishing distance and let χ_{G_1}, χ_{G_2} be two C^∞-functions with $\mathrm{supp}(\chi_{G_j}) \subset \hat{G}_j$ and $\chi_{G_j}(\mathbf{x}) = 1$ for $\mathbf{x} \in G_j$. Then we obtain:

Proposition A.2.1 : *The states ω and ω_0 are unitarily equivalent on $\mathfrak{A}(G)$ if the integral-kernel*

$$\chi_{G_1}(\mathbf{x})\mu(\mathbf{x} - \mathbf{y})\chi_{G_2}(\mathbf{y}) \tag{A.4}$$

is an element of $S(\mathbb{R}^{2d})$.

A.3 Equivalence of Norms:

For convenience, we cite now the proof of [8, Lemma 3.2] by making the necessary changes to show that the result is independent of the spatial dimension.

Lemma A.3.1 : *Let (G_1, G_2) be any pair of regions with non-vanishing distance, then the norms $||\cdot||_0$ and $||\cdot||_p$ are equivalent on $K(G_1 \cup G_2)$.*

Proof. Let $t > 0$ be the distance between G_1 and G_2. Moreover, let s, be a function in S with support in $B_d(t/2)$ and Fourier transform $\hat{s}$, such that $\hat{s}(\mathbf{p}) \geq 0$ for all $p \in \mathbb{R}^d$. A function with these properties exists and can be obtained by using the convolution theorem. Hence there are constants $c > a > 0$ such that $c > (\mathbf{p}^2 + m^2)^{1/2}(\hat{s}(\mathbf{p}) + a) \geq a > 0$. This implies

$$\begin{aligned} |(\mathbf{p}^2 + m^2)^{-1/2} - c^{-1}(\hat{s}(\mathbf{p}) + a)| &\leq ac^{-1}(\mathbf{p}^2 + m^2)^{-1/2} \\ |(\mathbf{p}^2 + m^2)^{1/2} - c^{-1}(\mathbf{p}^2 + m^2)(\hat{s}(\mathbf{p}) + a)| &\leq ac^{-1}(\mathbf{p}^2 + m^2)^{-1/2} \end{aligned} \tag{A.5}$$

We consider now the following operators which are diagonal in momentum space:

$$\begin{aligned} w_1(\mathbf{p}) &= c^{-1}(\hat{s}(\mathbf{p}) + a) \\ w_2(\mathbf{p}) &= c^{-1}(\mathbf{p}^2 + m^2)(\hat{s}(\mathbf{p}) + a) \end{aligned} \tag{A.6}$$

For any element $g \in S(G_2)$ one has

$$\begin{aligned} (w_1 g)(\mathbf{x}) &= c^{-1}(s * g(\mathbf{x}) + ag(\mathbf{x})) \\ (w_2 g)(\mathbf{x}) &= c^{-1}(\partial_\alpha \partial^\alpha + m^2)(s * g + ag)(\mathbf{x}) \quad , \quad \alpha = 1,2,3 \end{aligned} \tag{A.7}$$

and hence $\mathrm{supp} w_j g \cap G_1 = \emptyset$. Thus one gets $(f, w_j g) = 0$ for each $f \in K(G_1)$ and each $g \in K(G_2)$. Now we compute:

$$\begin{aligned} |(f, \mu^{-1} g)| &= |(f, \mu^{-1} g - w_1 g)| \\ &\leq \int d\mathbf{p}\, |(\mathbf{p}^2 + m^2)^{-1/2} - c^{-1}(\hat{s}(\mathbf{p}) + a)||\hat{f}(\mathbf{p})||\hat{g}(\mathbf{p})| \\ &\leq \; ac^{-1} \int d\mathbf{p}\, (\mathbf{p}^2 + m^2)^{-1/2} |\hat{f}(\mathbf{p})||\hat{g}(\mathbf{p})| \\ &\leq \; ac^{-1}(f, \mu^{-1} f)^{1/2} (g, \mu^{-1} g)^{1/2} \end{aligned} \tag{A.8}$$

Analogously we obtain the estimate $|(f, \mu g)| \leq \; ac^{-1}(f, \mu f)^{1/2}(g, \mu g)^{1/2}$. Keeping in mind that $ac^{-1} < 1$, the equivalence of the norms $|| \cdot ||_0$ and $|| \cdot ||_p$ can be obtained by using the same arguments as in [8]. $\square$

A.4 Application of the Criterion:

In this paragraph, we discuss the application of Proposition A.2.1 with respect to the possible cases for G_1 and G_2.

Denote by $S(\mathbb{R}^d; 0)$ the space of functions f such that $\chi f \in S(\mathbb{R}^d)$ for each test function $\chi \in S(\mathbb{R}^d)$ with $0 \notin \mathrm{supp}(\chi)$.

It turns out that the problem can be reduced to the following question:

Let f be a function in $S(\mathbb{R}^d; 0)$. For which pairs of regions $G_1, G_2 \subset \mathbb{R}^d$ is the function

$$f_{(G_1, G_2)} : (\mathbf{x}, \mathbf{y}) \mapsto \chi_{G_1}(\mathbf{x}) f(\mathbf{x} - \mathbf{y}) \chi_{G_2}(\mathbf{y}) \tag{A.9}$$

contained in $S(\mathbb{R}^{2d})$?

Since f may be singular at $\mathrm{x} = 0$, one has to require that G_1 and G_2 have non vanishing distance.

Definiton A.4.1 : A pair of regions $G_1, G_2 \subset \mathbb{R}^d$ with non vanishing distance is called *admissible* if there exists a constant $k > 0$ such that for each $r > 0$ the set

$$G(r) := \{(\mathbf{x}_1, \mathbf{x}_2)|\mathbf{x}_1 \in G_1, \mathbf{x}_2 \in G_2 \; ; \; \mathbf{x}_1 - \mathbf{x}_2 \in B_d(r)\}$$

is contained in $B_{2d}(kr)$, where $B_d(r)$ denotes the closed ball in $\mathbb{R}^d$ with radius r.

Lemma A.4.2 : *If (G_1, G_2) is a pair of regions in $\mathbb{R}^d$ which is admissible, then the function $f_{(G_1,G_2)}$ is contained in $S(\mathbb{R}^{2d})$.*

Proof. Since the pair (G_1, G_2) is admissible, the region $G(k^{-1}r) := \{(\mathbf{x}, \mathbf{y})|\mathbf{x} \in G_1, \mathbf{y} \in G_2 \; ; \; \mathbf{x} - \mathbf{y} \in B_d(k^{-1}r)\}$ is contained in the closed ball $B_{2d}(r)$ for a constant $k > 0$. This implies that for each $m \in \mathbb{N}$ one has

$$\begin{aligned} |\chi_{G_1}(\mathbf{x}) f(\mathbf{x} - \mathbf{y}) \chi_{G_2}(\mathbf{y})| &< \mathrm{const.} \cdot |\mathbf{x} - \mathbf{y}|^{-m} \\ &\leq \mathrm{const.} \cdot k^m r^{-m} \leq \mathrm{const.} \cdot |(\mathbf{x}, \mathbf{y})|^{-m} \quad . \end{aligned} \tag{A.10}$$

Hence we conclude that $f_{(G_1,G_2)}$ is of fast decrease and thus contained in $S(\mathbb{R}^{2d})$. □

Corollary A.4.3 : *If the pair of regions (G_1, G_2) is admissible, then the states ω_0 and ω are unitarily equivalent on $\mathfrak{A}(G)$.*

Proof. The function

$$f : \mathbf{x} \in \mathbb{R}^d \backslash \{0\} \mapsto f(\mathbf{x}) = \int \mathrm{d}\mathbf{p} \; (\mathbf{p}^2 + m^2)^{1/2} \; e^{i\mathbf{p}\mathbf{x}} \tag{A.11}$$

is contained in $S(\mathbb{R}^d; 0)$. An application of Proposition A.2.1 and Lemma A.4.2 implies the result. □

Let us now discuss the cases for witch the pair (G_1, G_2) is admissible. To carry through this analysis, we have to give a few more definitions. Let $e \in \mathbb{R}^d$ be a vector of unit length and $s \in (0, 1)$, then we define the convex cone $C(e, s) := \mathbb{R}_+ \cdot (B_d(s) + e)$. The complement of $C(e, s)$ in $\mathbb{R}^d$ is denoted by $C'(e, s)$.

Lemma A.4.4 : *Let $s_1, s_2 \in (0, 1)$ with $s_1 < s_2$ and e a unit vector, then for each $\epsilon > 0$ the pair $(C(e, s_1) + \epsilon e, C'(e, s_2))$ is admissible.*

Proof. Let us consider the set $C(e, s_2)\backslash C(e, s_1) = C(e, s_2, s_1)$. For $s_2 > s_1$, there exists a convex cone $C(e', s_3)$ which is contained in $C(e, s_2, s_1)$. Hence for each $\mathbf{x} \in \partial C(e, s_1)$ exists $r > 0$, such that $B_d(r) + \mathbf{x} \subset C(e, s_2)$. Moreover, we have the following relation between $\mathbf{x}$ and r:

$$|\mathbf{x}| \geq \sin(\varphi_2 - \varphi_1)^{-1} \cdot r \tag{A.12}$$

Here $\varphi_j = \arcsin(s_j)$ is the opening angle of $C(e, s_j)$. We set $t := \sin(\varphi_2 - \varphi_1)^{-1}$ and conclude for each $\mathbf{x} \in B_d(tr)' \cap C(e, s_1)$

$$B_d(r) \subset C(e, s_2) + \mathbf{x} \quad . \tag{A.13}$$

Hence for each $\mathbf{x} \in B_d(tr)' \cap C(e, s_1)$ there is no $\mathbf{y} \in C'(e, s_2)$ such that $\mathbf{x} + \mathbf{y} \in B_d(r)$. Since for each $\epsilon > 0$ the set $C(e, s_1) + \epsilon e$ is contained in $C(e, s_1)$, we obtain that

$$G(r) := \{(\mathbf{x}, \mathbf{y})|\mathbf{x} \in C(e, s_1) + \epsilon e, \mathbf{y} \in C'(e, s_2)\, ;\ \mathbf{x} + \mathbf{y} \in B_d(r)\} \tag{A.14}$$

is contained in $B_d(tr) \times C'(e, s_2)$. On the other hand, for each $r > 0$ there exists $\mathbf{y} \in \partial C(e, s_2)$ such that $B_d(r) \cap C(e, s_1) = \emptyset$. We have the following relation for $\mathbf{y}$ and r:

$$|\mathbf{y}| \geq \sin(\varphi_2 - \varphi_1)^{-1} \cdot r \tag{A.15}$$

Thus with the same argument as above we conclude finally that there exists a constant $k > 0$, such that

$$G(r) \subset B_d(tr) \times B_d(tr) \subset B_{2d}(kr) \tag{A.16}$$

which implies the result. □

We see that for $d > 1$ the arguments in the proof of Lemma A.4.4 fails for cones with the same opening angle, i.e. the pair $(C(e, s) + \epsilon e, C'(e, s))$ is *not* admissible.

On the other hand, for $d = 1$ the pair $((-\infty, 0], [\epsilon, \infty))$ is indeed admissible.

A.5 The Split Property:

To discuss the split property, we briefly describe the construction of the local v.Neumann algebras for the free massive scalar field in the vacuum representation. Denote by

$(\mathcal{H}_0, \pi_0, \Omega_0)$ the GNS-triple of ω_0. We define for each $f \in K_\Gamma := \{g \in K : \Gamma g = g\}$, the field operator $b_0(f) := \pi_0(b(f))$ which is essentially self-adjoint on $\pi_0(\mathfrak{A}(K, \gamma, \Gamma))\Omega_0$. For a region $G \subset \mathbb{R}^d$ we denote by $\mathfrak{M}(G)$ the v.Neumann algebra which is given by $\mathfrak{M}(G) := \{e^{i\pi_0(b(f))} : f \in K_\Gamma(G)\}''$, where $''$ denotes the double commutant in $\mathcal{B}(\mathcal{H}_0)$.

Let us consider a pair of admissible regions (G_1, G_2), then by Corollary 3.1 we know that the vacuum state ω_0 and its induced product state ω are unitarily equivalent on $\mathfrak{A}(G_1 \cup G_2)$. Hence the product state ω induces a normal state on $\mathfrak{M}(G_1) \vee \mathfrak{M}(G_2)$ which is given by a vector $\eta \in \mathcal{H}_0$, where η is cyclic for $\mathfrak{M}(G_1) \vee \mathfrak{M}(G_2)$. Thus we have for $a_1 \in \mathfrak{M}(G_1)$ and $a_2 \in \mathfrak{M}(G_2)$

$$\langle \eta, a_1 a_2 \eta \rangle = \langle \Omega_0, a_1 \Omega_0 \rangle \, \langle \Omega_0, a_2 \Omega_0 \rangle \tag{A.17}$$

By standard arguments [8], we conclude that for a pair of admissible regions (G_1, G_2) the inclusion

$$\mathfrak{M}(G_1)' \subset \mathfrak{M}(G_2) \tag{A.18}$$

is a split inclusion.

Example: We close the appendix by discussing the $1+1$-dimensional case briefly. We consider the regions $(0, \infty)$ and $(-\infty, 0)$. For $\mathrm{x} \in (0, \infty)$ the pair $((\mathrm{x}, \infty), (-\infty, 0))$ is admissible (see Lemma A.4.4). Keeping in mind that the net of the free field $\mathcal{I} \mapsto \mathfrak{M}(\mathcal{I})$ satisfies wedge duality we obtain that the inclusion

$$\mathfrak{M}(\mathrm{x}, \infty) \subset \mathfrak{M}(0, \infty) \tag{A.19}$$

is standard split. Hence the massive free scalar field in $1+1$ dimensions satisfies the split property for wedge regions.

B The Proofs for Chapter 6

B.1 The Proof of Proposition 6.2.6

In order to compute products of one-kink fields, we introduce the following notation: Let $\underline{\rho} \in T(n, m, \underline{e})$ and consider a family of operators $a = (a_j, a^i)_{j\in\underline{n}}^{i\in\underline{m}}$ with $a_j \in C^*(\mathfrak{A}^+_{e_{j-1}})$ and $a^i \in C^*(\mathfrak{A}^-_{e_{1-i}})$.

(1) From the above data, we define operators in $C^*(\mathfrak{A}^+_{e_0})$ by:

$$a_1 \times \cdots \times a_n := a_1\rho_1(a_2)\cdots(\rho_1\cdots\rho_{n-2})(a_{n-1})(\rho_1\cdots\rho_{n-1})(a_n)$$

(2) Analogously, we define operators in $C^*(\mathfrak{A}^-_{e_0})$:

$$a^1 \times \cdots \times a^m := a^1\rho^1(a^2)\cdots(\rho^1\cdots\rho^{m-2})(a^{m-1})(\rho^1\cdots\rho^{m-1})(a^m)$$

(3) Denote by $\mathbf{1}_j$ and $\mathbf{1}^i$ the unit in $C^*(\mathfrak{A}^+_{e_j})$ and $C^*(\mathfrak{A}^-_{e_{-i}})$ respectively. For $\kappa \in P(n, m)$ we define:

$$a_\kappa := \pi^+_{e_0}(a_1 \times \cdots \times a_{\kappa_+(1)})\pi^-_{e_0}(a^1 \times \cdots \times a^{\kappa_-(1)})\cdots$$

$$\cdots\pi^+_{e_0}(\mathbf{1}_1 \times \cdots \times \mathbf{1}_{\kappa_+(l)} \times a_{\kappa_+(l)+1} \times \cdots \times a_n)$$

$$\pi^-_{e_0}(\mathbf{1}^1 \times \cdots \times \mathbf{1}^{\kappa_-(l)} \times a^{\kappa_-(l)+1} \times \cdots \times a^m)$$

which is an operator in $\mathfrak{B}(\mathcal{H}_{e_0})$. where $\kappa = (\kappa_+, \kappa_-)$ is parameterized by equation (6.1). Furthermore, we let

$$a^\#_\kappa := (a^*)_\kappa{}^*$$

where we have set: $a^* := (a_j^*, a^{i*})_{j\in\underline{n}}^{i\in\underline{m}}$.

Now, let $\mathbf{a} \in F(n, m, \underline{\rho})$ be a kink field of the form

$$\mathbf{a} = \Big(\bigotimes_{j \in \underline{n}} \mathbf{a}_j \Big) \otimes \Big(\bigotimes_{i \in \underline{m}} \mathbf{a}^i \Big) \quad .$$

Then it is not hard to prove that for each $\kappa \in P(n, m)$ we have:

$$\mathbf{a}_\kappa \Omega_{e_0} = \{\underline{\rho}_\kappa, a_\kappa^{\#} \Omega_{e_0}\} \tag{B.1}$$

In order to complete the proof of Proposition 6.2.6, we are now prepared to establish the lemma below.

Lemma B.1.1 : *Let* a *be the kink field above and let* $v \in (\hat{\underline{\rho}}, \rho)$ *be an intertwiner of the form*

$$v = \Big(\bigotimes_{j \in \underline{n}} v_j \Big) \otimes \Big(\bigotimes_{i \in \underline{m}} v^i \Big)$$

with $v_j \in (\hat{\rho}_j, \rho_j)$ *and* $v^i \in (\hat{\rho}^i, \rho^i)$*. Then for each* $\kappa \in P(n, m)$ *the following holds:*

$$(v\mathbf{a})_\kappa \Omega_{e_0} = v_\kappa \mathbf{a}_\kappa \Omega_{e_0} \quad .$$

Proof. An application of equation (B.1) gives:

$$(v\mathbf{a})_\kappa \Omega_{e_0} = \{\hat{\underline{\rho}}_\kappa, (va)_\kappa^{\#} \Omega_{e_0}\}$$

where the family va is given by $(v_j a_j, v^i a^i)_{j \in \underline{n}}^{i \in \underline{m}}$. Let $\kappa = (\kappa_+, \kappa_-)$ be parameterized by equation (6.1). Then we have for $k \leq l$:

$$a^*_{\kappa_n(k)+1} v^*_{\kappa_n(k)+1} \times \cdots \times a^*_{\kappa_n(k+1)+1} v^*_{\kappa_n(k+1)+1}$$

$$= a^*_j v^*_j \hat{\rho}_j (a^*_{j+1} v^*_{j+1}) \cdots (\hat{\rho}_j \cdots \hat{\rho}_{j+i-1})(a^*_{j+i} v^*_{j+i})$$

$$= a^*_j \rho_j (a^*_{j+1}) \cdots (\rho_j \cdots \rho_{j+i-1})(a^*_{j+i}) v^*_j \hat{\rho}_j (v^*_{j+1}) \cdots (\hat{\rho}_j \cdots \hat{\rho}_{j+i-1})(v^*_{j+i})$$

$$= (a^*_{\kappa_n(k)+1} \times \cdots \times a^*_{\kappa_n(k+1)+1})(v_{\kappa_n(k)+1} \times \cdots \times v_{\kappa_n(k+1)+1})^*$$

Here we have set $j = \kappa_+(k) + 1$ and $j + i = \kappa_+(k+1)$. Analogously, we find:

$$a^{\kappa_-(k)+1^*} v^{\kappa_-(k)+1^*} \times \cdots \times a^{\kappa_-(k+1)+1^*} v^{\kappa_-(k+1)+1^*}$$

$$= (a^{\kappa_-(k)+1^*} \times \cdots \times a^{\kappa_-(k+1)+1^*})(v^{\kappa_-(k)+1} \times \cdots \times v^{\kappa_-(k+1)+1})^*$$

By inserting the above relation into the expression for $(va)_\kappa \Omega_{e_0}$ and $v_\kappa a_\kappa \Omega_{e_0}$, the lemma follows. □

B.2 The Proofs of Theorem 6.2.1 and Theorem 6.2.10

In order to prove Theorem 6.2.1 we establish some useful lemmas.

Lemma B.2.1 : *Let $\tau \in \Delta(e_1, e_2; e)$ be a kink representation and let $\rho \in \Delta(e_1, \hat{e}_1; e_1)$, $\sigma \in \Delta(\hat{e}_2, e_2; e_2)$ be oriented kink representations such that ρ and σ are localized in space-like separated regions. Then one has:*

$$\tau^{\#} \rho^{\#} \sigma = \tau^{\#} \sigma^{\#} \rho \in \Delta(\hat{e}_1, \hat{e}_2; e)$$

Proof. Since ρ and σ are localized in space-like separated regions, there exist a right wedge W such that ρ is localized in W and σ is localized in W'.

Let $\mathcal{O}$ be any sufficiently large double cone and let us choose oriented kink representations $\hat{\rho} \in \Delta(e_1, \hat{e}_1; e_1)$ and $\hat{\sigma} \in \Delta(\hat{e}_2, e_2; e_2)$ where $\hat{\rho} \cong \rho$ is localized in $\mathcal{O}_{RR} \subset W$ and $\hat{\sigma} \cong \sigma$ is localized in $\mathcal{O}_{LL} \subset W'$. Furthermore, there exists an oriented kink representation $\hat{\tau} \in \Delta(e_1, e_2; e_1)$, $\hat{\tau} \cong \tau$, which is localized in W. We choose unitary intertwiner $w \in (\hat{\tau}, \tau)$, $u \in (\hat{\rho}, \rho)$ and $v \in (\hat{\sigma}, \sigma)$. Furthermore, we define for a kink representation $\eta \in \Delta(e_1, e_2, e)$:

$$\eta^{\#}(a) := \eta \circ (\pi_{e_1}^-)^{-1}(a) \quad \text{for } a \in \pi_{e_1}^-(C^*(\mathfrak{A}_{e_1}^-)).$$

$$\eta^{\#}(b) := \eta \circ (\pi_{e_2}^+)^{-1}(b) \quad \text{for } b \in \pi_{e_2}^+(C^*(\mathfrak{A}_{e_2}^+)).$$

The localization properties of $\hat{\rho}, \rho$ and $\hat{\sigma}, \sigma$ imply that u is localized in W and that v is localized in W'. Thus we obtain for each $a \in \mathfrak{A}(\mathcal{O})$:

$$\begin{aligned}\tau^{\#}\sigma^{\#}\rho(a) &= w\hat{\tau}^{\#}(v)\hat{\tau}^{\#}(u)\hat{\tau}(a)\hat{\tau}^{\#}(u^*)\hat{\tau}^{\#}(v^*)w^* \\ &= wv\hat{\tau}^{\#}(u)\hat{\tau}(a)\hat{\tau}^{\#}(u^*)v^*w^* \\ &= w\hat{\tau}^{\#}(u)v\hat{\tau}(a)v^*\hat{\tau}^{\#}(u^*)w^* \\ &= \tau^{\#}(u)\tau^{\#}(v)\tau(a)\tau^{\#}(v^*)\tau^{\#}(u^*)\end{aligned}$$

On the other hand, we have:

$$\tau^{\#}\rho^{\#}\sigma(a) = \tau^{\#}(u)\tau^{\#}(v)\tau(a)\tau^{\#}(v^*)\tau^{\#}(u^*)$$

which implies the result. □

Lemma B.2.2 : *Let $\tau \in \Delta(e_1, e_2; e)$ be a kink representation and let $\rho \in \Delta(e_1, \hat{e}_1; e_1)$, $\sigma \in \Delta(\hat{e}_2, e_2; e_2)$ be oriented kink representations. Then there exists a unitary intertwiner $\epsilon_\tau(\rho,\sigma)$*

$$\epsilon_\tau(\rho,\sigma)\tau^{\#}\rho^{\#}\sigma(a) = \tau^{\#}\sigma^{\#}\rho(a)\epsilon_\tau(\rho,\sigma)$$

which depends only on τ, ρ, σ.

Proof. We choose unitary intertwiner $u \in (\hat{\rho}, \rho)$ and $v \in (\hat{\sigma}, \sigma)$ such that $\hat{\rho}$ is localized in a right wedge W and $\hat{\sigma}$ is localized in W'. By Lemma B.2.1 we conclude:

$$\begin{aligned}\tau^{\#}\sigma^{\#}\rho(a) &= \tau^{\#}(v\hat{\sigma}^{\#}\rho(a)v^*) \\ &= \tau^{\#}(v)\tau^{\#}\hat{\sigma}^{\#}\rho(a)\tau^{\#}(v^*) \\ &= \mathrm{Ad}(\tau^{\#}(v)(\tau^{\#}\hat{\sigma})^{\#}(u))\tau^{\#}\hat{\sigma}^{\#}\hat{\rho}(a) \\ &= \mathrm{Ad}(\tau^{\#}(v)(\tau^{\#}\hat{\sigma})^{\#}(u))\tau^{\#}\hat{\rho}^{\#}\hat{\sigma}(a) \\ &= \mathrm{Ad}((\tau^{\#}\sigma)^{\#}(u)\tau^{\#}(v)\tau^{\#}(u^*)(\tau^{\#}\rho)^{\#}(v^*))\tau^{\#}\rho^{\#}\sigma(a)\end{aligned}$$

We define quasi-statistics operator $\epsilon_\tau(\rho,\sigma)$ by:

$$\epsilon_\tau(\rho,\sigma) := (\tau^\#\sigma)^\#(u)\tau^\#(v)\tau^\#(u^*)(\tau^\#\rho)^\#(v^*)$$

Let us choose intertwiner $u_1 \in (\rho_1,\rho)$ and $v_1 \in (\sigma_1,\sigma)$ such that ρ_1 is localized in a right wedge $W_1 \subset W$ and σ_1 is localized in $W_2 \subset W'$. By substituting $u \mapsto uu_1$ and $v \mapsto vv_1$, we obtain an alternative quasi-statistics operator $\epsilon'_\tau(\rho,\sigma)$. In order to prove that $\epsilon_\tau(\rho,\sigma)$ depends only on τ,ρ,σ, we establish that $\epsilon'_\tau(\rho,\sigma) = \epsilon_\tau(\rho,\sigma)$. Let $w \in (\tau_1,\tau)$ be a unitary intertwiner where τ_1 is oriented and localized in W_1.

$$\begin{aligned} w^*\epsilon'_\tau(\rho,\sigma)w &= (\tau_1{}^\#\sigma)^\#(uu_1)\tau_1{}^\#(vv_1)\tau_1{}^\#(u_1^*u^*)(\tau_1{}^\#\rho)^\#(v_1^*v^*) \\ &= \tau_1{}^\#(v)(\tau_1{}^\#\hat\sigma)^\#(uu_1)v_1\tau_1{}^\#(u_1^*)v_1^*(\tau_1{}^\#\hat\rho)^\#(v^*)\tau_1{}^\#(u^*) \\ &= \tau_1{}^\#(v)(\tau_1{}^\#\hat\sigma)^\#(u)\tau_1{}^\#(u_1)v_1\tau_1{}^\#(u_1^*)v_1^*(\tau_1{}^\#\hat\rho)^\#(v^*)\tau_1{}^\#(u^*) \\ &= \tau_1{}^\#(v)(\tau_1{}^\#\hat\sigma)^\#(u)(\tau_1{}^\#\hat\rho)^\#(v^*)\tau_1{}^\#(u^*) \\ &= \epsilon_{\tau_1}(\rho,\sigma) = w^*\epsilon_\tau(\rho,\sigma)w \end{aligned}$$

Here we have used the fact that v_1 is localized in W' and that $\tau_1{}^\#(u_1)$ is localized in W. $\square$

Proof of Theorem 6.2.1: Let $\underline{\rho} \in T(n,m,\underline{e})$. We define oriented kink representations:

$$\underline{\rho}_n := \pi^+_{e_0} \circ \rho_1 \cdots \rho_n \quad \text{and} \quad \underline{\rho}^m := \pi^-_{e_0} \circ \rho^1 \cdots \rho^m \quad .$$

Let $u \in (\hat{\underline{\rho}},\underline{\rho})$ be a unitary intertwiner and let us choose $\hat{\underline{\rho}}$ in such a way that $\hat\rho^j$ is localized in W and that $\hat\rho^i$ is localized in W'. We conclude from Lemma B.2.1:

$$\hat{\underline{\rho}}_\kappa = \hat{\underline{\rho}}_n{}^\#\hat{\underline{\rho}}^m$$

for each $\kappa \in P(n,m)$. By Lemma B.1.1 we obtain for each $\kappa \in P(n,m)$:

$$\underline{\rho}_\kappa = \mathrm{Ad}(u_\kappa) \circ \hat{\underline{\rho}}_\kappa$$

Thus we have for each pair $(\kappa, \hat{\kappa}) \in G(n,m)$:

$$\underline{\rho}_{\kappa} = \mathrm{Ad}(u_{\kappa}) \circ \underline{\hat{\rho}}_{n} \# \underline{\hat{\rho}}^{m} = \mathrm{Ad}(u_{\kappa}) \circ \underline{\hat{\rho}}_{\hat{\kappa}} = \mathrm{Ad}(u_{\kappa} u_{\hat{\kappa}}^{*}) \circ \underline{\rho}_{\hat{\kappa}} \quad .$$

Finally we define the quasi statistics operator:

$$\epsilon_{(\kappa,\hat{\kappa})}(\underline{\rho}) := u_{\kappa} u_{\hat{\kappa}}^{*}$$

In the same manner as in the proof of Lemma B.2.2 it can be shown that $\epsilon_{(\kappa,\hat{\kappa})}(\underline{\rho})$ does not depend on the auxiliary family $\underline{\hat{\rho}}$. □

Proof of Theorem 6.2.10: Let $\mathbf{a} \in F(n, m; \underline{\rho})$ be a multi-kink field of the form

$$\mathbf{a} = \left(\bigotimes_{j \in \underline{n}} \mathbf{a}_{j} \right) \otimes \left(\bigotimes_{i \in \underline{m}} \mathbf{a}^{i} \right) \quad .$$

Since for each pair (j, i) the one-kink fields $\mathbf{a}_{j} = (\rho_{j}, a_{j})$ and $\mathbf{a}^{i} = (\rho^{i}, a^{i})$ are localized in space-like separated regions, there exists unitary intertwiners $u_{j} \in (\hat{\rho}_{j}, \rho_{j})$ and $u^{i} \in (\hat{\rho}^{i}, \rho^{i})$ such that $u_{j} a_{j}$ and $\hat{\rho}_{j}$ are localized in a right wedge W and $u^{i} a^{i}$ and $\hat{\rho}^{i}$ are localized in W'. This implies

$$(ua)_{\kappa} = (ua)_{\hat{\kappa}}$$

for each pair $(\kappa, \hat{\kappa})$ and therefore:

$$u_{\kappa} \mathbf{a}_{\kappa} \Omega_{e_0} = u_{\hat{\kappa}} \mathbf{a}_{\hat{\kappa}} \Omega_{e_0} \quad .$$

Finally, Theorem 6.2.10 follows from the identity

$$\epsilon_{(\kappa,\hat{\kappa})}(\underline{\rho}) := u_{\kappa} u_{\hat{\kappa}}^{*} \quad .$$

□

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