# The Free Maxwell Field in Curved Spacetime

Diploma Thesis

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### Abstract

The aim of this thesis is to discuss quantizations of the free Maxwell field in flat and curved spacetimes. First we introduce briefly some notions from tensor analysis and the causal structure of spacetime. As an introduction to the main topic, we review some aspects of the two axiomatic quantum field theories, Wightman theory and algebraic quantum field theory. We also give an introduction into concepts of the quantization of fields on curved spacetime backgrounds. Then the wave equation and quantization of the Maxwell field in flat spacetimes is discussed. It follows a review of J. DIMOCK's quantization of the Maxwell field on curved spacetimes and then we come to our main result:

We show explicitly that the Maxwell field, defined by $\mathrm{d}\mathscr{F} = 0$ and $\delta\mathscr{F} = 0$, has a well posed initial value formulation on arbitrary globally hyperbolic spacetime manifolds. We prove the existence and uniqueness of fundamental solutions without employing a vector potential. Thus our solution is also applicable to spacetimes not satisfying the Poincaré lemma and should lead to a quantization of the Maxwell field on non-trivial spacetime backgrounds. This in turn provides the opportunity to investigate physical states on non-trivial spacetime-topologies and could lead to the discovery of new quantum phenomena.



### Zusammenfassung

Das Ziel dieses Arbeit ist es, die Quantisierung des Maxwell-Feldes in flacher und gekrümmter Raumzeit zu diskutieren. Wir führen zunächst in aller Kürze einige Begriffe aus der Tensor-Analysis und zur kausalen Struktur der Raumzeit ein. Als Einführung in das Haupthema wiederholen wir einige Aspekte der zwei axiomatischen Feldtheorien, der Wightman-Theorie und der algebraischen Quantenfeldtheorie. Wir geben ebenfalls eine Einführung in die Konzepte der Feldquantisierung in gekrümmter Raumzeit. Danach diskutieren wir die Wellengleichung und die Quantisierung des Maxwell Feldes in flacher Raumzeit. Es folgt eine Besprechung der Ergebnisse von J. DIMOCK zur Quantisierung des Maxwell-Feldes in gekrümmter Raumzeit. Dann kommen wir zu unserem Hauptresultat:

Wir zeigen explizit, daß das Maxwell-Feld, definiert durch $\mathrm{d}\mathscr{F} = 0$ und $\delta\mathscr{F} = 0$, ein wohldefiniertes Anfangswertproblem auf einer beliebigen global hyperbolischen Raumzeit darstellt. Wir beweisen die Existenz und Eindeutigkeit der Fundamentallösungen, ohne Zuhilfenahme eines Vektorpotentials. Daher ist unsere Lösung auch gültig im Raumzeiten, die das Poincaré-Lemma nicht erfüllen. Dieses wiederum erlaubt die Analyse von physikalische Zuständen auf nicht-trivialen Raumzeit-Topologien und könnte zur Entdeckung neuer Quantenphänomene führen.

# Contents

# Introduction

This work is concerned with the quantization of the Maxwell field in curved spacetimes within the framework of algebraic quantum field theory.

Quantum field theory in curved spacetimes is the appropriate theory for the analysis of quantum phenomena, where the effects of gravitation have to be taken into account, but where the quantum nature of gravitation itself can be neglected. Gravitation is thus described by a classical, curved spacetime, as in general relativity. This leads to a theory where quantum fields propagate on a curved background manifold. A similar approach has already proved valuable in quantum electrodynamics, where in the early days many calculations were made regarding the electromagnetic field as a background field interacting with quantized matter. Some of the results obtained by this approximation are in complete accordance with the full theory of quantum electrodynamics. Quantum mechanical effects in the theory of gravitation are expected to be relevant near the Planck scale, $10^{-35}$m, whereas the present standard model, describing elementary particles, is appropriate for scales $\geq 10^{-19}$m. Thus the range of validity of quantum field theory in curved spacetimes is quite large and should include a wide variety of interesting phenomena. The most popular example is certainly the Hawking radiation; Hawking discovered that particle creation should occur in the vicinity of Black holes. Another field of research within this theory are phenomena occurring in the very early universe.

There is another reason [AS80], why the inclusion of gravitational effects into quantum theory might be interesting. One could argue that the gravitational coupling constant is so weak, that it should provide only minute corrections to the quantitative results of Minkowskian quantum field theory, if one allows spacetime to be curved. However, gravitational interaction might also introduce qualitatively new features into quantum theory, which are measurable already at laboratory scale. We know that special relativity is a theory of high velocities, comparable to the velocity of light. But, the theoretical prediction of antiparticles by special relativity introduced a qualitatively new feature into physics, which is not at all connected to the velocity of the involved particles. Why shouldn't general relativity have a similar impact on laboratory physics?

The structure of spacetime in general relativity is that of a 4-dimensional manifold $\mathcal{M}$ with Lorentzian metric $g_{\mu\nu}$. The metric determines almost every property of spacetime and thus is the central object of general relativity. Minkowski spacetime, the spacetime of standard quantum field theory, is included in the theory as a special spacetime without curvature. In particular it has a symmetry group, the Poincaré group, consisting of Lorentz transformations, i.e. four-dimensional rotations, and translations.

Quantum field theory in Minkowski space relies heavily on plane wave expansion. The field is assumed to be an infinite collection of decoupled, time-independent, harmonic oscillators. The Poincare Group plays a key role in the definition of a preferred vacuum state and the particle interpretation of the theory. However, the Poincaré group is not a symmetry group of general spacetimes. No analogue of either a plane wave basis or a positive frequency subspace is available in a general curved spacetime. The particle interpretation, though under appropriate circumstances available, does not play a fundamental role anymore.

The primary difference between a quantum system of particles and quantum field theory is that the latter has infinitely many degrees of freedom. In case of finite degrees of freedom, the Stone-von Neumann theorem assures, that the canonical commutation relations for position and momentum operators completely determine a choice of Hilbert space $\mathcal{H}$ and a choice of selfadjoint operators on $\mathcal{H}$ corresponding to position and momentum observables. Unfortunately the Stone-von Neumann theorem does not hold for systems with infinitely many degrees of freedom. In fact, infinitely many unitarily inequivalent, irreducible representations for the canonical commutation relations exist in quantum field theory.

As already indicated, the Poincaré symmetry of Minkowski space manifests itself in the possibility to choose a preferred vacuum state and particle notion. These properties get lost in the transition to curved spacetimes. On some non-compact spacetimes, it is still possible to define a particle notion in the asymptotic future and in the asymptotic past; but the representations of the canonical commutation relations for these regions will in general be unitarily inequivalent.

The apparent incompatibility of quantum field theory and curved spacetimes dissolves if ones uses the algebraic approach to quantum field theory. Algebraic quantum field theory allows one to consider simultaneously all unitarily inequivalent Hilbert space constructions of the physical system. This in turn, leads to a mathematically rigorous construction of quantum field theory without the need to pick out a preferred representation of the canonical commutation relations, in particular, without the need of an a priori particle notion. This suggests that quantum field theory has to be treated truly as a field theory and not as a particle theory in disguise.

The preferred model of quantum field theory in curved spacetime is, of course, the scalar (Klein-Gordon) field. Almost all investigations regarding the structure of the theory are carried out using this model. This is mainly justified by the mathematical simplicity of the scalar field, compared to the Dirac or Maxwell field. However, a series of papers by J. DIMOCK is concerned with the quantization of all of these fields on curved manifolds [Dim80, Dim82, Dim92]. We shall explicitly have to deal with [Dim92], in which a rigorous quantization of the Maxwell field on arbitrary globally hyperbolic Lorentzian manifolds with a compact Cauchy surface is developed. A similar line is followed by E. FURLANI [Fur99]. Both papers introduce a globally defined electromagnetic vector potential and thus restrict their result to simply-connected portions of spacetime. The purpose of this thesis is to start an extension of this theory which is suitable for topologically nontrivial manifolds. The existence of a globally definable vector potential is only guaranteed, if the spacetime is contractible. This is already not the case for Schwarzschild-Kruskal spacetime, which is certainly an interesting physical spacetime. A. ASHTEKAR and A. SEN [AS80] discuss source free Maxwell fields on Schwarzschild-Kruskal spacetime and show that there exists a two-parameter family of unitarily inequivalent representations of the canonical commutation relations. This is a consequence of the nontrivial topology of Schwarzschild-Kruskal spacetime.

**Layout of the work.** The course of this thesis runs as follows. In the first chapter we introduce some basic notions of general relativity. Manifolds and tensors are introduced as well as differential forms and some operations on differential forms. Curvature and the causal structure of spacetime is discussed briefly. The 3+1-splitting of spacetime as a method to investigate dynamics is explained.

The second chapter begins with some generalities of axiomatic quantum field theory and then provides a very short introduction into the concepts of algebraic quantum field theory to the readers not familiar with it. We give some mathematical definitions needed to understand this approach. It follows the general quantization scheme of quantum field theory in curved spacetime for a hermitian real scalar field. Furthermore a review of the symplectic structure of classical dynamics is given and it is investigated how this approach leads to what we call 'symplectic quantization'. This method is then applied to the scalar field.

In chapter three we review basic facts from classical Maxwell theory on flat spacetime. We introduce the Maxwell tensor and the field equations. The standard treatment of the source-free wave equation in Minkowski spacetime leads to explicit solutions for electromagnetic waves. We demonstrate then the Minkowski space quantization of the Maxwell equations, which is an example of a field theory with indefinite metric. We shall not use the Fock space approach here. This somewhat non-standard in the usual quantum field theory literature.

Chapter four deals with the Maxwell field in curved spacetimes and demonstrates its quantization, as carried out by J. DIMOCK. This quantization is necessarily carried out in the spirit of chapter four and is rather different from the indefinite metric quantization in Minkowski space.

In the second part of this chapter we address the Cauchy problem for the Maxwell field and prove it's well posedness using the methods of chapter one. This is our contribution to the quantization of the Maxwell field. At last we give an outlook on possible future projects that could follow.

# Chapter 1

# Curved Spacetimes

In this chapter we review the mathematical framework of Einstein's general relativity and also some results of this theory, which we will need in later chapters. First we define the manifolds representing spacetime and introduce tensors and differential forms on manifolds as the basic objects of general relativity. We equip the manifold with a Lorentzian metric and discuss the resulting causal structure of spacetime. Then we state an important result from differential geometry, the Poincaré lemma. The last section is a review of a special method used to analyze the time evolution of physical objects in general relativity, known as the 3+1-splitting.

## 1.1 Manifolds and Tensor Fields

In the framework of general relativity, spacetime is described as a 4-dimensional differentiable manifold. Basically a manifold is a topological space, that locally looks like $\mathbb{R}^n$. The existence of infinitely differentiable coordinate systems makes the manifold differentiable. Contrary to a Euclidean space, on a curved manifold $\mathcal{M}$ the naive notion of a vector as an arrow pointing from one point to another does not make sense anymore. Thus, one defines a vector to be a tangent vector to to a curve in $\mathcal{M}$. This definition leads to the theory of fiber bundles, which has proved valuable in many theories in physics, such as general relativity and gauge theory.

### 1.1.1 Vectors on a manifold

A *differentiable Manifold* $\mathcal{M}$ is a paracompact Hausdorff space with an atlas [Fred]. We could abandon paracompactness at this stage, since we will equip the manifold with a Lorentzian metric and a Hausdorff manifold with a Lorentzian metric is always paracompact [HE]. On the manifold we need smooth maps in order to define smooth curves. A map from a subset $M \subset \mathbb{R}^m$ into a subset $N \subset \mathbb{R}^n$, with $p$-times continuously differentiable components $\phi_i(x), (i = 1, 2, \ldots, m)$, is said to be of class $C^p$. The infinitely differentiable maps $C^\infty$ are called *smooth*. By the existence of an *atlas*, we can define smooth functions on any differentiable manifold. Let us denote the space of all smooth functions $f$ on $\mathcal{M}$, i.e. $C^\infty$-functions $f : \mathcal{M} \to \mathbb{R}^m$ by $\mathcal{E}(\mathcal{M})$.

Now consider a *smooth curve* $\gamma(t)$ on $\mathcal{M}$, i.e. a continuous map from an interval on the real line $\mathbb{R}$ into the Manifold $\mathcal{M}$, such that $f \circ \gamma$ is a $C^\infty$ map for any $f \in \mathcal{E}(\mathcal{M})$. At each point $p = \gamma(t)$ the curve has a *tangent vector*

$$\dot{\gamma} = \frac{d\gamma}{dt} \,. \tag{1.1}$$

We identify the tangent vector $\dot{\gamma}$ at $p$ with the Operator

$$\begin{aligned} X : \mathcal{E}(\mathcal{M}) &\to \mathbb{C} \\ f &\mapsto \frac{d}{dt}(f \circ \gamma(t)) \,. \end{aligned} \tag{1.2}$$

Due to the product rule, $X$ has the property $X(fg) = (Xf)g + f(Xg)$ in common with all derivatives. Conversely one can show that all linear maps $X : \mathcal{E}(\mathcal{M}) \to \mathbb{C}$, satisfying this equation, are represented as tangent vectors in $p$.

The collection of all tangent vectors to all possible curves passing through a given point $p$ in $\mathcal{M}$ is called the *tangent space* to $\mathcal{M}$ at $p$, and is denoted by $T_p\mathcal{M}$. On an $n$-dimensional manifold $T_p\mathcal{M}$ is an $n$-dimensional vector space with $\partial_\mu = \frac{\partial}{\partial x^\mu}$ providing a *coordinate basis* of $\mathcal{M}$. Hence we can write any tangent vector as $X = X^\mu \partial_\mu$.

The collection of all tangent spaces to all points $p$ in $\mathcal{M}$ is called the *tangent bundle* of $\mathcal{M}$, denoted by

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M} \,. \tag{1.3}$$

The individual tangent spaces are called the *fibers* of the bundle. The tangent bundle $T\mathcal{M}$ of a smooth $n$-dimensional manifold $\mathcal{M}$ is a smooth manifold of dimension $2n$.

To any vector space $V$ we call the vector space of all linear functionals $f : V \to \mathbb{R}$ the *dual space* $V^*$ of $V$. The dual space of a tangent space $T_p\mathcal{M}$ is called the *cotangent space* and is denoted $T_p^*\mathcal{M}$. The *cotangent bundle* $T^*\mathcal{M}$ is defined as in (1.3). An appropriate basis for the cotangent spaces is provided by the gradients $\mathrm{d}x^\mu$ of the coordinate functions. (The exterior derivative $\mathrm{d}$ will be introduced later). Hence, any cotangent vector $X$ is written $X = X_\mu \mathrm{d}x^\mu$. In the finite dimensional case, the dual spaces are isomorphic to each other. But since there is in general no natural isomorphism, it makes sense to distinguish between the spaces. As it is common we denote tangent vectors by lower indices, $V_\mu$, and cotangent vectors by upper indices, $V^\mu$.

Consider a set of vectors with exactly one at each point in a space. We call this a *vector field*. An *integral curve* of a vector field $V$ is a smooth parameterized curve $\gamma(t) = \gamma^\mu(t)\partial_\mu$ whose tangent vector at any point coincides with the value of $V$ at that same point

$$\frac{\gamma(t)}{dt} = V|_{\gamma(t)} \,. \tag{1.4}$$

This means the integral curve $\gamma(t)$ of $V$ can be obtained as the solution to the autonomous system of ordinary differential equations

$$\frac{d\gamma^\mu}{dt} = V^\mu \,. \tag{1.5}$$

### 1.1.2 Tensors

In this section we introduce tensors as straightforward generalizations of vectors and dual vectors. However, tensors on curved spacetime bring about the question of how to define a derivative that is independent from the local coordinate system. The covariant derivative is a tool that serves this need. But one has to be careful, since in general covariant derivatives do not commute. Furthermore we note that covariant derivatives depend on 'connections', of which a particular interesting one is the Levi-Civita connection.

The definitions given here, can be found in any book treating tensor analysis, e.g. [Nak], and in almost every book on general relativity, e.g. [Wala, Wei, MTW] (See also [Car97] for a non-rigorous introduction). A classic text is also [HE].

**The notion of a tensor.** As noted before, a tensor is the straight forward generalization of vectors and dual vectors. A *tensor* of type $(k,l)$ is a multilinear map on Cartesian products of vector spaces and dual vector spaces:

$$T : \underbrace{V^* \times \cdots \times V^*}_{k-\text{times}} \times \underbrace{V \times \cdots \times V}_{l-\text{times}} \to \mathbb{R} \; . \tag{1.6}$$

Scalar functions are treated as tensors of type $(0,0)$. The collection $\mathcal{T}(k,l)$ of all tensors of type $(k,l)$ has the structure of a vector space with dimension $(k+l)$. Consequently the multilinearity property allows to specify a tensor by giving it's values in a basis $\{v_{(\mu)}\}$ of $V$ and it's dual basis $\{v^{(\nu)}\}$ of $V^*$.

Given a Tensor $T$ of type $(k,l)$ and a tensor $T'$ of type $(k',l')$ one can construct a new tensor $T \otimes T'$ of type $(k+k', l+l')$ by demanding that

$$\begin{aligned} &T \otimes T'(V^{(1)}, \ldots, V^{(k+k')}, V_{(1)}, \ldots, V_{(l+l')}) \\ &= T(V^{(1)}, \ldots, V^{(k)}, V_{(1)}, \ldots, V_{(l)}) T'(V^{(k+1)}, \ldots, V^{(k')}, V_{(l+1)}, \ldots, V_{(l')}) \; . \end{aligned} \tag{1.7}$$

The operation $\otimes$ is known as the *tensor product*. Every tensor $T$ of type $(k,l)$ can be expressed in the form

$$T = T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \; v_{(\mu_1)} \otimes v_{(\mu_k)} \otimes v^{(\nu_1)} \otimes v^{(\nu_l)} \; , \tag{1.8}$$

where the coefficients $T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}$ are the components of $T$ in the given basis. The transformation law for a tensor under a coordinate transformation is

$$T^{\mu'_1 \ldots \mu'_k}{}_{\nu'_1 \ldots \nu'_l} = \frac{\partial x^{\mu'_1}}{x^{\mu_1}} \ldots \frac{\partial x^{\mu'_k}}{x^{\mu_k}} \frac{\partial x^{\nu_1}}{x^{\nu'_1}} \ldots \frac{\partial x^{\nu_l}}{x^{\nu'_l}} T^{\mu_1 \ldots \mu_k}{}_{\mu_1 \ldots \mu_l} \; . \tag{1.9}$$

**Abstract index notation.** Throughout this text we will use the so called 'abstract index notation', which is a slight modification of the usual component notation for tensors. The idea is to work without specification of any basis, but to use a notation that mirrors the expressions for the basis components, if we had introduced one. A tensor of type $(k,l)$ is denoted by a letter followed by $k$ upper (contravariant) and $l$ lower (covariant) indices, e.g. $T^{\rho\sigma}{}_{\lambda\mu\nu}$ stands for a $(2,3)$ tensor. We also accept the widespread convention, that indices written with Greek letters take values $0, 1, 2, 3$ and indices with Roman letters go from 1 to 3. Furthermore we employ the summation convention. This means indices appearing twice, once as an upper index and once as a lower index, denote summation over the given index range; e.g. $T^{\rho\lambda}{}_{\lambda\mu\nu}$ denotes the $(1,2)$ tensor obtained by calculating the *contraction* $\sum_{\lambda=0}^{3} T^{\rho\lambda}{}_{\lambda\mu\nu}$. Further discussion of this notation can be found in [Wala].

**Covariant differentiation.** When we use Cartesian coordinates in a flat spacetime, the partial derivative operator $\partial_\mu$ is is a map from from $(k,l)$ tensor fields to $(k,l+1)$ tensor fields. Curved spacetimes are represented by manifolds with nonzero curvature. On such manifolds the partial derivative no longer maps tensors to tensors anymore. (This is already the case with non-Cartesian coordinates in a flat spacetime). Hence, a *covariant derivative* for contravariant vectors is defined by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \ . \tag{1.10}$$

The $\Gamma^\nu_{\mu\lambda}$ are $n \times n$ matrices, where $n$ is the dimension of the manifold considered and are called *connection coefficients*. The associated covariant derivative for a covariant vector, using the the same connection coefficients, is given by

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \ . \tag{1.11}$$

We also give a general expression for the covariant derivative of arbitrary rank tensors:

$$\begin{aligned} \nabla_\sigma T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = {} & \partial_\sigma T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \\ & + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1 \lambda \mu_3 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \dots \\ & - \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1 \dots \mu_k}{}_{\lambda\nu_2 \dots \nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \lambda \nu_3 \dots \nu_l} - \dots \ . \end{aligned} \tag{1.12}$$

The $\Gamma^\nu_{\mu\lambda}$ are $n \times n$ matrices, where $n$ is the dimension of the manifold considered and are called *connection coefficients*.

As required for any derivative, the covariant derivative is linear $\nabla(T+S) = \nabla T + \nabla S$, and satisfies the Leibnitz rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$.

**Commutation relation for covariant derivatives.** The use of covariant derivatives instead of partial derivatives poses the problem of commutativity of derivatives. We know that partial derivatives commute, $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$, but covariant derivatives in general do not. For an arbitrary tensor field $X^{\mu_1 \cdots \mu_p}{}_{\nu_1 \dots \nu_q}$ the commutator is

$$\begin{aligned} [\nabla_\rho, \nabla_\sigma] X^{\mu_1 \cdots \mu_p}{}_{\nu_1 \dots \nu_q} = {} & -T_{\rho\sigma}{}^{\lambda} \nabla_\lambda X^{\mu_1 \cdots \mu_p}{}_{\nu_1 \dots \nu_q} \\ & + R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda\mu_2 \dots \mu_p}{}_{\nu_1 \dots \nu_q} + R^{\mu_2}{}_{\lambda\rho\sigma} X^{\mu_1 \lambda \dots \mu_p}{}_{\nu_1 \dots \nu_q} + \dots \\ & - R^{\lambda}{}_{\nu_1\rho\sigma} X^{\mu_1 \cdots \mu_p}{}_{\lambda\nu_2 \dots \nu_q} - R^{\lambda}{}_{\nu_2\rho\sigma} X^{\mu_1 \cdots \mu_p}{}_{\nu_1 \lambda \dots \nu_q} - \dots \end{aligned} \tag{1.13}$$

Here $T_{\rho\sigma}{}^{\lambda}$ is the torsion tensor as before and $R^\rho{}_{\sigma\mu\nu}$ is the *Riemann tensor*:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \ . \tag{1.14}$$

We note some key properties of the Riemann tensor with lower indices, $R_{\rho\sigma\mu\nu} = g_{\rho\rho'}R^{\rho'}{}_{\sigma\mu\nu}$:

- $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$,
- $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$,
- $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$.

**Levi-Civita connection.** To any given connection one can associate a *torsion tensor* defined by

$$T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \; . \tag{1.15}$$

A connection which is symmetric in its lower indices, that is if $T^{\lambda}_{\mu\nu} = 0$, is called *torsion free*. A further property of some connections is *metric compatibility*. In section 1.2 we will introduce the metric tensor $g_{\mu\nu}$. The metric $g_{\mu\nu}$ is naturally used to raise or lower Tensor indices: $g_{\mu\nu}T^{\nu\lambda} = T_{\nu}{}^{\lambda}$. For now it is sufficient to note, that metric compatibility means that the covariant derivative associated with a connection satisfies

$$\nabla_{\lambda} g_{\mu\nu} = 0 \; . \tag{1.16}$$

An important property of metric-compatible covariant derivatives is, that they commute with the raising and lowering of indices:

$$g_{\mu\nu}\nabla_{\lambda}V^{\nu} = \nabla_{\lambda}V_{\mu} \; . \tag{1.17}$$

A connection, which is torsion-free and metric-compatible, exists on any manifold and it can be proved that it has the components:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\rho\mu} + \partial_{\rho}g_{\mu\nu}) \; . \tag{1.18}$$

The unique torsion-free, metric compatible connection defined above is known as the *Levi-Civita connection* and the associated connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ are called the *Christoffel symbols*.

The *covariant divergence* of a contravariant antisymmetric tensor $A^{\mu\nu}$, with respect to the Christoffel connection, can be expressed in terms of the metric determinant $g = \det(g_{\mu\nu})$ [Wei]:

$$\nabla_{\mu}A^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_{\mu}(\sqrt{|g|}A^{\mu\nu}) \; . \tag{1.19}$$

### 1.1.3 (Anti-) Symmetrization

A tensor which remains the same after permutation of some of its indices is called *symmetric* in the chosen set of indices. If the tensor changes sign under odd permutations of indices but remains the same under even permutations it is *antisymmetric* in this set of indices.

Any given tensor can be symmetrized in any number of indices by the following procedure: take the sum of the permutations of the relevant indices and divide by the number of terms.

$$T_{(\mu_1\dots\mu_p)} = \frac{1}{p!}(T_{\mu_1\dots\mu_p} + \text{sum over permutations of } \mu_1\dots\mu_p)\ . \tag{1.20}$$

To get an anti-symmetrized tensor any permutation which is the result of an odd number of index exchanges is subtracted (this is called the alternating sum):

$$T_{[\mu_1\dots\mu_p]} = \frac{1}{p!}(T_{\mu_1\dots\mu_p} + \text{alternating sum over permutations of } \mu_1\dots\mu_p)\ . \tag{1.21}$$

We use round brackets for symmetric and square brackets for anti-symmetrization of tensors. If we want to exclude indices we put these indices between vertical bars: $T_{(\mu|\lambda|\nu)\rho}{}^{\sigma}$ means it is not summed over $\lambda, \rho$ and $\sigma$.

The definition of anti-symmetrization, as it is given above, is not very handy when it comes to explicit calculations. A more compact notation for anti-symmetrization is:

$$T_{[\mu_1\dots\mu_p]} = \frac{1}{p!}\delta^{\nu_1\dots\nu_p}_{\mu_1\dots\mu_p}T_{\nu_1\dots\nu_p} \tag{1.22}$$

where we used the *generalized Kronecker-$\delta$* symbol:

$$\delta^{\nu_1\dots\nu_p}_{\mu_1\dots\mu_p} = \begin{cases} +1 & \text{if } (\mu_1\dots\mu_p) \text{ is an even permutation of } (\nu_1\dots\nu_p) \\ -1 & \text{if } (\mu_1\dots\mu_n) \text{ is an odd permutation of } (\nu_1\dots\nu_p) \\ 0 & \text{otherwise .} \end{cases} \tag{1.23}$$

This definition is equivalent to

$$\delta^{\nu_1\dots\nu_p}_{\mu_1\dots\mu_p} = \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_1}_{\mu_2} & \dots & \delta^{\nu_1}_{\mu_p} \\ \delta^{\nu_2}_{\mu_1} & \delta^{\nu_2}_{\mu_2} & \dots & \delta^{\nu_2}_{\mu_p} \\ \dots & \dots & \dots & \dots \\ \delta^{\nu_p}_{\mu_1} & \delta^{\nu_p}_{\mu_2} & \dots & \delta^{\nu_p}_{\mu_p} \end{vmatrix} \tag{1.24}$$

where $\delta^{\mu}_{\nu}$ is the simple Kronecker-$\delta$ symbol and the right side is a determinant.

### 1.1.4 The Levi-Civita tensor

Some important objects, that look line tensors at first sight, do not transform like tensors. An Object $g$ that transforms according to the law $g(x^{\mu'}) = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|^{-w} g(x^{\mu})$ is called a *tensor density* of *weight $w$*. There are some important objects that are tensor densities. On of them is the *Levi-Civita symbol* (or completely antisymmetric symbol):

$$\tilde{\epsilon}_{\mu_1\dots\mu_n} = \begin{cases} +1 & \text{if } \mu_1\dots\mu_n \text{ is an even permutation of } 0\ 1\dots(n-1) \\ -1 & \text{if } \mu_1\dots\mu_n \text{ is an odd permutation of } 0\ 1\dots(n-1) \\ 0 & \text{otherwise.} \end{cases} \tag{1.25}$$

The Levi-Civita symbol is of weight 1. It has the same components in any coordinate system. One can also define a Symbol with upper indices, which has the same components as the one with lower indices:

$$\tilde{\epsilon}^{\mu_1\dots\mu_n} = \tilde{\epsilon}_{\mu_1\dots\mu_n} \; . \tag{1.26}$$

Note that we could also have defined:

$$\tilde{\epsilon}_{\mu_1\dots\mu_n} = \delta^{1\dots\dots n}_{\mu_1\dots\mu_n} \qquad \text{and} \qquad \tilde{\epsilon}^{\mu_1\dots\mu_n} = \delta^{\mu_1\dots\mu_n}_{1\dots\dots n} \; . \tag{1.27}$$

Another example for a tensor density is the determinant of the metric $g = \det(g_{\mu\nu})$, which is a density of weight 2.

Tensor densities become honest tensors when multiplied by $|g|^{w/2}$, where $w$ is again the weight of the density, and $|g|$ is the absolute value of the metric determinant. Now we can define the *Levi-Civita tensor* by

$$\epsilon_{\mu_1\dots\mu_n} = \sqrt{|g|}\; \tilde{\epsilon}_{\mu_1\dots\mu_n} \; . \tag{1.28}$$

These are the components of the Levi-Civita tensor in an arbitrary basis $(e_0, e_1, e_2, e_3)$, with $e_0$ pointing towards the future and $e_1, e_2, e_3$ right-handed. A manifold on which the Levi-Civita tensor can be defined unambiguously is *orientable*.

We collect some more useful relations of the introduced quantities. Note that

$$\epsilon^{\mu_1\dots\mu_n} = \frac{1}{g}\epsilon_{\mu_1\dots\mu_n} \tag{1.29}$$

and

$$\epsilon^{\mu_1\dots\mu_n} = \frac{(-1)^s}{\sqrt{|g|}}\tilde{\epsilon}^{\mu_1\dots\mu_n} \; , \tag{1.30}$$

where $s$ is the number of minuses appearing in the signature of the metric (The signature is introduced in section 1.2).

The contraction of two Levi-Civita symbols can be calculated in terms of the generalized Kronecker delta from

$$\tilde{\epsilon}^{\mu_1\dots\mu_j\mu_{j+1}\dots\mu_n}\tilde{\epsilon}_{\mu_1\dots\mu_j\nu_{j+1}\dots\nu_n} = (-1)^s j!\; \delta^{\mu_{j+1}}_{\nu_{j+1}}\dots\delta^{\mu_n}_{\nu_n} \; . \tag{1.31}$$

From this it is easily seen that the Levi-Civita symbol is subject to the normalization condition

$$\tilde{\epsilon}^{\mu_1\dots\mu_n}\tilde{\epsilon}_{\mu_1\dots\mu_n} = (-1)^s n! \, . \tag{1.32}$$

On a four dimensional Lorentzian manifold (see section 1.2) equation (1.31) is equivalent to [LL]:

$$\tilde{\epsilon}^{\kappa\lambda\mu\nu}\tilde{\epsilon}_{\pi\rho\sigma\tau} = -\begin{vmatrix} \delta^\kappa_\pi & \delta^\kappa_\rho & \delta^\kappa_\sigma & \delta^\kappa_\tau \\ \delta^\lambda_\pi & \delta^\lambda_\rho & \delta^\lambda_\sigma & \delta^\lambda_\tau \\ \delta^\mu_\pi & \delta^\mu_\rho & \delta^\mu_\sigma & \delta^\mu_\tau \\ \delta^\nu_\pi & \delta^\nu_\rho & \delta^\nu_\sigma & \delta^\nu_\tau \end{vmatrix}, \quad \tilde{\epsilon}^{\kappa\lambda\mu\nu}\tilde{\epsilon}_{\pi\rho\sigma\nu} = -\begin{vmatrix} \delta^\kappa_\pi & \delta^\kappa_\rho & \delta^\kappa_\sigma \\ \delta^\lambda_\pi & \delta^\lambda_\rho & \delta^\lambda_\sigma \\ \delta^\mu_\pi & \delta^\mu_\rho & \delta^\mu_\sigma \end{vmatrix}, \tag{1.33}$$

$$\tilde{\epsilon}^{\kappa\lambda\mu\nu}\tilde{\epsilon}_{\pi\rho\mu\nu} = -2(\delta^\mu_\pi\delta^\nu_\rho - \delta^\mu_\rho\delta^\nu_\pi), \qquad \tilde{\epsilon}^{\kappa\lambda\mu\nu}\tilde{\epsilon}_{\pi\lambda\mu\nu} = -6\;\delta^\kappa_\pi, \tag{1.34}$$

$$\tilde{\epsilon}^{\kappa\lambda\mu\nu}\tilde{\epsilon}_{\kappa\lambda\mu\nu} = -24 \; . \tag{1.35}$$

Since a contraction with the Levi-Civita tensor has the effect of anti-symmetrization, the complete contraction with an an already anti-symmetrized tensor gives:

$$\tilde{\epsilon}^{\mu_1 \dots \mu_n} A_{[\mu_1 \dots \mu_n]} = \tilde{\epsilon}^{\mu_1 \dots \mu_n} A_{\mu_1 \dots \mu_n} \ . \tag{1.36}$$

### 1.1.5 Differential forms

A completely antisymmetric tensor of type $(0, p)$ is called a *differential p-form*, or just $p$-form. The smooth differential $p$-forms on $\mathcal{M}$ are denoted $\Omega^p(\mathcal{M})$, or $\Omega^p_c(\mathcal{M})$ in the case of compact support.

**Wedge Product.** The *wedge product* maps a $p$-form $A$ and a $q$-form $B$ to a $(p+q)$-form $(A \wedge B)$:

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]} \ . \tag{1.37}$$

The wedge product is not commutative but it is possible to alter the order of the factors if one is careful with signs:

$$A \wedge B = (-1)^{pq} (B \wedge A) \ . \tag{1.38}$$

**Exterior derivative.** The *exterior derivative* d maps $p$-forms to $(p+1)$-forms:

$$(\mathrm{d}A)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]} \ . \tag{1.39}$$

The simplest example of an exterior derivative is the gradient, which is the exterior derivative of a 0-form:

$$(\mathrm{d}\phi)_\mu = \partial_\mu \phi \ . \tag{1.40}$$

From the commutation property of partial derivatives, $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$, it follows that

$$\mathrm{d}\mathrm{d}A = 0 \tag{1.41}$$

for any $p$-form $A$.

**Star operator.** On an $n$-dimensional manifold the *(Hodge) star operator* $*$ sends $p$-forms to $(n-p)$-forms:

$$(*A)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \dots \nu_p}{}_{\mu_1 \dots \mu_{n-p}} A_{\nu_1 \dots \nu_p} \ . \tag{1.42}$$

Note that this is a very natural operation, since the number of linearly independent $p$-forms on an $n$-dimensional vector space is $n!/(p!(n-p)!)$ and thus the space $\Omega^p(\mathcal{M})$ is isomorphic to the space $\Omega^{(n-p)}(\mathcal{M})$. Contrary to the exterior derivative and the wedge product, the star operator is metric dependent. The metric enters the definition via the Levi-Civita-Tensor $\epsilon^{\nu_1 \dots \nu_p} = \sqrt{|g|}\ \tilde{\epsilon}_{\mu_1 \dots \mu_n}$ (1.28). Applying the $*$-operator twice returns the original form or its negative:

$$* * A = (-1)^{s + p(n-p)} A \ . \tag{1.43}$$

Here $s$ is again the number of minuses in the signature of the metric and $n$ is the dimension of the manifold.

**Inner product.** The *inner product* on $\Omega^1_c(\mathcal{M})$ is defined by

$$\langle \omega, \theta \rangle = \int_{\mathcal{M}} \omega \wedge *\theta = \int_{\mathcal{M}} \omega_\mu \theta^\mu |g|^{1/2} \,. \tag{1.44}$$

**Coderivative.** The *codifferential (inner derivative)*

$$\delta = *\mathrm{d}* \tag{1.45}$$

maps $p$-forms to $(p-1)$-forms. The operator $\delta$ is the generalization of the divergence and in a sense can be treated as the adjoint of $\mathrm{d}$ since $\langle \delta\omega, \theta \rangle = \langle \omega, \mathrm{d}\theta \rangle$.

### 1.1.6 Pullbacks and pushforwards

Consider two manifolds $\mathcal{M}$ and $\mathcal{N}$ with coordinate systems $x^\mu$ and $y^\alpha$ respectively. If we have a map $\phi : \mathcal{M} \to \mathcal{N}$ and a function $f : \mathcal{N} \to \mathbb{R}$ we can compose $\phi$ with $f$ to get a map $(f \circ \phi) : \mathcal{M} \to \mathbb{R}$. This special map is called the *pullback* of $f$ by $\phi$:

$$\phi_* f = (f \circ \phi) \,. \tag{1.46}$$

We need the pullback of a function to define the pullback and pushforward of tensors in general. If $V(p)$ is a vector at a point $p$ on $\mathcal{M}$ the *pushforward* of $V$ at the point $\phi(p)$ is

$$(\phi^* V)(f) = V(\phi_* f) \,. \tag{1.47}$$

The pullback of a 1-form on is given by

$$(\phi_* \omega)(V) = \omega(\phi^* V) \,. \tag{1.48}$$

In coordinate based language these operations can be expressed by the matrix of partial derivatives:

$$\begin{aligned} (\phi^* V)^\alpha &= (\phi^*)^\alpha{}_\mu V^\mu \\ (\phi_* \omega)_\mu &= (\phi_*)_\alpha{}^\mu \omega_\alpha \\ (\phi^*)^\alpha{}_\mu = (\phi_*)_\alpha{}^\mu &= \frac{\partial y^\alpha}{\partial x^\mu} \,. \end{aligned} \tag{1.49}$$

The next step is to define the pullback and pushforward of completely covariant or contravariant tensors:

$$\begin{aligned} (\phi_* T)(V^{(1)}, V^{(2)}, \ldots, V^{(l)}) &= T_*(\phi^* V^{(1)}, \phi^* V^{(2)}, \ldots, \phi^* V^{(l)}) \\ (\phi^* T)(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(l)}) &= T^*(\phi_* \omega^{(1)}, \phi_* \omega^{(2)}, \ldots, \phi_* \omega^{(l)}) \,. \end{aligned} \tag{1.50}$$

If $\phi$ is a *diffeomorphism*, that means if $\phi$ is invertible and the inversion $\phi^{-1}$ is smooth, we can pullback or pushforward tensors of any type:

$$\begin{aligned} &(\phi^* T)(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) = \\ &\qquad T(\phi_* \omega^{(1)}, \ldots, \phi_* \omega^{(k)}, [\phi^{-1}]^* V^{(1)}, \ldots, [\phi^{-1}]^* V^{(l)}) \,. \end{aligned} \tag{1.51}$$

## 1.2 Spacetime Structure

In modern physics spacetime is assumed to be a differentiable manifold equipped with a Lorentzian metric. The metric imposes a particular causal structure on the spacetime, which we shall review briefly. We are especially interested in globally hyperbolic spacetimes, since they allow the existence of well posed initial value formulations for wave equations. One global property of manifolds is their contractibility. The assumption of non-contractibility of the considered spacetimes is one key element of this thesis. The most important difference between contractible and non-contractible spaces is characterized by the Poincaré lemma. In the last part we introduce the 3+1-splitting of a spacetime, and collect important equations needed later on.

### 1.2.1 The metric tensor

Let $\mathcal{M}$ be any differentiable manifold. A *metric tensor* $g_{\mu\nu}$ at a point $p \in \mathcal{M}$ is a symmetric tensor of type (0,2) at $p$. The metric assigns to each pair of vectors $V, W \in T_p\mathcal{M}$ a scalar $g(V, W) = g_{\mu\nu}V^\mu W^\nu$. This is the natural generalization of the Euclidean scalar product to metric manifolds. The metric is *non-degenerate* if there is no nonzero vector $V \in T_p\mathcal{M}$ such that $g(V, W) = 0$ for all $W \in T_p\mathcal{M}$. A non-degenerate metric is called a *Riemannian metric*. If the metric is non-degenerate we can define a unique inverse of the metric-tensor by

$$g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu \ . \tag{1.52}$$

The metric and it's inverse provide an isomorphism between the covariant and contravariant components of any tensor. This means one can raise and lower indices with the help of $g^{\mu\nu}$ and $g_{\mu\nu}$ [HE].

A result from the theory of quadratic forms says, that a real quadratic form $g_{\mu\nu}$, at a point $p$, can be put into a canonical form, which has the property of being diagonal and only having components $+1$ and $-1$. This is done by diagonalization and subsequent scaling of the basis vectors [Nak]. After that, the diagonal of the matrix $(g_{\mu\nu})$ at $p$ looks like $(+\cdots+-\cdots-)$. The number of plus and minus signs accounts to the number of positive and negative eigenvalues of $(g_{\mu\nu})$. The *signature* of $g_{\mu\nu}$ at $p$ is the sum of the signs of the eigenvalues of $(g_{\mu\nu})$. Sometimes one refers to the $n$-tuple of characterizing signs $(+\cdots+-\cdots-)$ as the signature.

A metric with only positive eigenvalues, i.e. a metric with signature $n$, where $n$ is the dimension of the underlying manifold, is a positive definite metric. A metric with signature $(n-2)$ or $(n+2)$, is called a *Lorentz metric*. A pair $(\mathcal{M}, g_{\mu\nu})$ consisting of a differentiable manifold $\mathcal{M}$ and a Lorentzian metric $g_{\mu\nu}$ is called a *Lorentzian manifold*.

We adopt the conventional metric in quantum field theory, which is a Lorentzian metric with signature $(+---)$. In the absence of gravitational fields, i.e. in flat spacetime, the metric becomes

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{1.53}$$

The metric $\eta_{\mu\nu}$ is called Minkowski metric and consequently the spacetime manifold $\mathbb{M} = (\mathcal{M}, \eta_{\mu\nu})$ is known as Minkowski space.

### 1.2.2 The causal structure of spacetime

Let $(\mathcal{M}, g_{\mu\nu})$ be a Lorentzian manifold. Thanks to the Lorentzian metric, the elements of $T_x\mathcal{M}$, that is the vectors $v$ at any point $x \in \mathcal{M}$, are divided into three classes as follows:

- *timelike* $\Leftrightarrow g(v, v) > 0$
- *lightlike* $\Leftrightarrow g(v, v) = 0$
- *spacelike* $\Leftrightarrow g(v, v) < 0$.

All lightlike vectors in in $x$ define the *open lightcone* $V(x) = V^+(x) \cup V^-(x)$. Here $V_+$ comprises the future directed and $V_-$ the past directed lightlike vectors. A curve that is timelike or lightlike in every point $x$ is called *causal*. For a given point $x \in \mathcal{M}$ we define the *causal future* $J^+(x)$ of $x$ to be the set of all points which can be reached by a future directed causal curve through $x$. All points which can be reached by a past directed causal curve define the *causal past* $J^-(x)$. Restricting ourselves to timelike curves we define the *chronological future* $I^+(x)$ and the *chronological past* $I^-(x)$ of $x \in \mathcal{M}$ in a similar way.

The *future domain of dependence* $D^+(\mathcal{O})$ of a connected subset $\mathcal{O} \in \mathcal{M}$, is the set of all points $p \in \mathcal{M}$ such that every past-moving, causal , inextendible curve through $p$ must intersect $\mathcal{O}$. The *past domain of dependence* $D^-(\mathcal{O})$ is defined similarly by changing past moving to future moving.

**Definition 1.1** A possibly curved spacetime $(\mathcal{M}, g_{\mu\nu})$ with a hypersurface $\Sigma$ such that every inextendible curve in $\mathcal{M}$ intersects $\Sigma$ precisely once is called *globally hyperbolic*. The hypersurfaces $\Sigma$ are called *Cauchy surfaces*.

Inextendible means that the curves do not end in a finite point.

The topological structure of globally hyperbolic spacetimes allows the choice of a global time coordinate:

**Theorem 1.2** *Let $(\mathcal{M}, g_{\mu\nu})$ be a globally hyperbolic spacetime. On $\mathcal{M}$ there exists a (non-unique) smooth global time coordinate* t $\in \mathbb{R}$, *such that each $\{t\} \times \Sigma$ is a Cauchy surface.*

Or to put it in other words: topologically one has the identity $\mathcal{M} = \mathbb{R} \times \Sigma$. On a globally hyperbolic spacetime the entire history of the universe can be predicted from conditions at the instant of time represented by the Cauchy surface $\Sigma$. This promotes causality and is the reason why most of the papers on quantum field theory in curved spacetime assume global hyperbolicity of spacetime. So do we. However one should note that there exist exact solutions to Einstein's field equations which do not admit Cauchy surfaces [HE].

### 1.2.3 Poincaré group

A point $x^\mu = (t, \mathbf{x})$ on the spacetime manifold $\mathcal{M}$ corresponds to a specific time $t$ and a specific location $\mathbf{x}$ and thus deserves the name *event*. One introduces the *spacetime interval* $s$ between two points (or events) $x^\mu, x^\nu$ by

$$s^2 = g_{\mu\nu} x^\mu x^\nu \; . \tag{1.54}$$

The set of all transformations $x^\mu \to x^{\mu\prime}$ that leaves the interval $s$ between any pair of points of the Minkowski space $\mathbb{M} = (\mathcal{M}, \eta_{\mu\nu})$ invariant is called the *Poincaré Group* $P$. A general *Poincaré transformation* $(\Lambda, a)$ of a point $x = x^\mu$ has the form

$$x \to x' = \Lambda x + a \tag{1.55}$$

where $\Lambda = \Lambda^\mu{}_\nu$ are four-dimensional transformation matrices and $a = a^\mu$ is a fixed translation vector. The group operation of the Poincaré group is

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) \; . \tag{1.56}$$

The requirement that $s$ has to be left invariant by the transformation imposes the condition

$$\eta = \Lambda^T \eta \Lambda \tag{1.57}$$

on the transformation matrices $\Lambda$. The matrices $\Lambda$ which satisfy (1.57) are known as the *Lorentz transformations* and they form the *Lorentz group* $L$ which is a subgroup of $P$.

The Lorentz group is disconnected; It consist of four connected components distinguished by the properties $\det \Lambda = \pm 1$ and $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq 1$. Only $L_+^\uparrow$, i.e. the component with $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$, is a subgroup of the Lorentz group (since it contains the identity element). $L_+^\uparrow$ is called the *proper Lorentz group*. The corresponding subgroup $P_+^\uparrow$ of the Poincaré group is considered to be the correct symmetry group in relativistic field theory. Later we will see that the lack of Poincaré symmetry is the main problem of quantum field theory in curved spacetimes.

### 1.2.4 The Poincaré lemma

A $p$-form $A$ is called *closed* if $\mathrm{d}A = 0$. It is called *exact* if $A = \mathrm{d}B$ for some $(p-1)$-form $B$. Since $\mathrm{d}^2 = 0$, an exact form is always closed but the converse is not necessarily true. Dealing with the Maxwell field we need the converse statement, that a closed form is exact, in order to define a vector potential $A$ by the formula $F = \mathrm{d}A$, where $F$ is the field strength tensor. The situation in which we can propose the existence of such a potential is provided by the Poincaré lemma. A necessary condition for the validity of the Poincaré lemma is contractibility of the spacetime manifold:

**Definition 1.3** A topological space $X$ is *contractible*, if there exists a point $x_0 \in X$ and a smooth map $\Phi : X \times [0,1] \to X$, such that $\Phi(x, 0) = x_0$ and $\Phi(x, 1) = x$ for all $x \in X$.

Contractible spaces are *simply connected*, i.e. on such spaces every closed curve can be continuously deformed to a point. Not all admissible spaces are simply connected. There are solutions for Einstein's equations which allow spacetime to be *multiply connected*. On contractible manifolds the following theorem holds:

**Theorem 1.4 (Poincaré lemma)** *If a coordinate neighborhood $O$ of a manifold $M$ is contractible to a point $x_0 \in M$, any closed p-form on $O$ is also exact.*

A nice proof of this can be given within the theory of deRham cohomology [Nak].

On multiply connected spaces one still has the statement that any closed form is exact at least locally. However this makes patching arguments necessary when a global property has to be proved.

### 1.2.5 Splitting spacetime into space and time

Sometimes it is desirable to split the 4-dimensional spacetime into three spatial and one time dimension. Then the metric $\gamma_{ij}$ of the spatial part becomes a dynamical variable changing with time. But of course this procedure can be used for any quantity on the Cauchy surfaces changing with time, like e.g. the components of the Maxwell tensor. We will need the 3+1 splitting, in the analysis of the Cauchy problem for the field strength tensor. For a deeper discussion on lapse and shift vectors see [MTW] or [Wala]. Our notation is compatible with that of [DK82] and [Wip98].

**The induced spatial metric.** Let $\Sigma_t$ be a Cauchy surface in $(\mathcal{M}, g_{\mu\nu})$. We introduce unit normal vector fields $n^\mu$ to $\Sigma_t$. The 4-dimensional metric $g_{\mu\nu}$ with signature $(+ - --)$ induces a 3-dimensional positive definite metric $\gamma_{\mu\nu}$, i.e. a metric with signature $(+ + +)$, on $\Sigma_t$ by

$$g_{\mu\nu} = n_\mu n_\nu - \gamma_{\mu\nu} \; . \tag{1.58}$$

The spatial components of these metrics are related to each other by

$$g_{ij} = -\gamma_{ij} \; . \tag{1.59}$$

Global hyperbolicity and thus the existence of a global time coordinate $t$ suggests the definition of a vector field $t^\mu$ by

$$t^\mu \nabla_\mu t = 1 \; . \tag{1.60}$$

This vector field represents the time evolution of the Cauchy surfaces $\Sigma_t$. It joins infinitesimally distant Cauchy surfaces to each other.

**Shift and lapse functions.** We are interested in the relation between the vector fields $t^\mu$ and $n^\mu$. It is evident that the vector field $t^\mu$ can be decomposed into normal and tangential parts with respect to $\Sigma_t$:

$$t^\mu = N n^\mu + N^\mu \; . \tag{1.61}$$

The *shift function* $N$ is the projection of $t^\mu$ onto $n^\mu$ and the *lapse functions* $N^i$ are the projections of $t^\mu$ onto $\Sigma_t$. Next we introduce adapted coordinates $x^\mu = (t, x^i)$, satisfying $t^\mu \nabla_\mu x^i = 0$, so that [Wip98]:

$$t^\mu \nabla_\mu = \partial_t \quad \text{and} \quad N^\mu \partial_\mu = N^i \partial_i \; . \tag{1.62}$$

In this coordinate system one finds the following relations for the shift function:

$$N^2 = \frac{1}{g^{00}} = -\frac{g}{\gamma} \; . \tag{1.63}$$

The determinant of the Lorentz metric $g_{\mu\nu}$ is negative and thus this equation reads

$$N = \frac{\sqrt{|g|}}{\sqrt{\gamma}} \; . \tag{1.64}$$

The lapse functions are given as

$$N^i = -\frac{g^{i0}}{g^{00}} \quad \text{and} \quad N_i = -g_{0i} \tag{1.65}$$

where the lapse functions with upper and lower indices are related by the spatial metric:

$$N_i = \gamma_{ij} N^j \; . \tag{1.66}$$

Of course there are also direct relations between lapse and shift functions:

$$g_{00} = N^2 - N^i N_i \tag{1.67}$$

$$g^{ij} = -\gamma^{ij} + \frac{N^i N^j}{N^2} \; . \tag{1.68}$$

# Chapter 2

# Local Quantum Physics

In this chapter we shall begin with some remarks on axiomatic quantum field theory and its application to curved spacetimes. This is followed by the axioms of quantum field theory on curved spacetimes. We explain the transition between usual quantum mechanics and the algebraic approach, and mention the problem of constructing physical states. The practical procedure of quantization is addressed next. In the last section we show how a coordinate independent Poisson bracket formulation leads to commutation relations for the fields in curved spacetime.

## 2.1 Axiomatic Quantum Field Theory

Quantum field theory (QFT) is the basic theory of relativistic quantum systems. It describes the whole range of elementary particles and to some extent their interactions. It is a synthesis of quantum theory and special relativity, supplemented by the principle of locality and the spectral condition.

While constructive QFT and perturbation theory works on the explanation and prediction of experiments, axiomatic QFT is the attempt to analyze the abstract structures of relativistic quantum systems. Axiomatic QFT is interested in the qualitative features and the fundamental theorems underlying the theory.

Two main lines of axiomatic theories evolved in quantum field theory, both to some extend complementary, the *Wightman theory* in the fifties and *algebraic quantum field theory* (AQFT) in the sixties. The central object of Wightman theory are quantum fields in a Hilbert space, whereas algebraic quantum field theory emphasizes the role of algebraic relations between the observables, represented as selfadjoint operators on a Hilbert space. The universality of the algebraic language allows deeper insights into the universal principles of physics. However the Wightman approach is usually preferred when it comes to explicit calculations. The transition between the two theories is still investigated, but it is known for sure, that the algebraic approach allows quantum systems which do not support actual fields. This is why the name 'local quantum physics' is more appropriate than algebraic quantum field theory [FR].

The foundations of AQFT were laid by R. HAAG and D. KASTLER, based on earlier work of J. VON NEUMANN and I. SEGAL, and culminated in their famous 1964 paper [HK64]. Significant further developments came, among others, by H. J. BORCHERS and H. ARAKI. Originally formulated in Minkowski spacetime, J. DIMOCK proposed a curved spacetime version of AQFT in 1980 [Dim80]. The contemporary reference for local quantum physics is [Haa].

**Quantum field theory in curved spacetimes**

A unification of quantum theory and general relativity into a theory of 'quantum gravity' is still not constructed in a satisfying manner. But it is possible to analyze phenomena where quantum and gravitational effects are important within the semiclassical theory of quantum field theory on curved spacetime. The realm of such a theory lies between the scale where quantum effects become important, i.e. the Planck scale $\approx 10^{-35}\mathrm{m}$, and the appropriate scale for the standard model $\geq 10^{-19}\mathrm{m}$. Within this range we can work with a theory of quantum fields propagating on a classical curved spacetime background.

The transition from flat Minkowski space to a curved space leads to some serious problems in QFT. Curved spacetimes lack Poincaré symmetry. Thus there is no (global) Fourier transformation available anymore. Field expansions into positive and negative frequency solutions are no longer possible. Connected to this is the problem that no unique vacuum state can be defined in a curved spacetime. This happens because the vacuum state in the Wightman formalism is explicitly defined by it's Poincaré invariance and the spectrum condition. Furthermore, having no vacuum state we cannot establish a unique Fock space, and thus have no means to describe particle creation and annihilation.

It turns out that these curved spacetime problems are best addressed within the algebraic approach to QFT. Since the theory is based on the abstract algebras of the quantum system, one can treat all representations of the system on an equal footing. All states in all possible (unitarily inequivalent) Hilbert space constructions are treated simultaneously. Later on one can single out the physical states by a spectrum condition. This leads to the definition of Hadamard states.

We refer the readers interested in quantum field theory in curved spacetimes to the textbooks [Walb, BD, Ful].

## 2.2 The Algebraic Approach

Usually one describes quantum mechanics in terms of a Hilbert space on which the vectors correspond to physical states. Self adjoint operators represent the observables of the system. The operators act in the Hilbert space and provide the physically relevant quantities. In the algebraic approach, quantum mechanics is described by $C^*$-algebras. For some applications the use of a $*$-algebra would be sufficient. But the representation of a $*$-algebra on a Hilbert space is not necessarily bounded, hence not defined everywhere.

The use of $C^*$-algebras is not only restricted to quantum mechanics and quantum field theory. It has also been very fruitful in the study of statistical mechanics and pure mathematics (see [Lan92] and references therein).

**Some fundamentals of operator algebras**

Before we state the axioms of local quantum physics in curved spacetimes, we list some related, non-rigorous definitions from the theory of operator algebras.

- An *algebra* $\mathfrak{A}$ over $\mathbb{C}$ is a vector space over $\mathbb{C}$ with a multiplication $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ which is bilinear and associative.
- An *involution* on a complex algebra is an anti-linear map $^* : \mathfrak{A} \to \mathfrak{A}$ satisfying $A^{**} = A$ and $(AB)^* = B^*A^*$ for all $A, B \in \mathfrak{A}$. By anti-linearity we mean that $(\lambda A)^* = \overline{\lambda} A^*$ for all $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$ (the overline denotes complex conjugation). $A^*$ is called the *adjoint* of $A$.
- A $^*$*-algebra* is a complex algebra with involution.
- An algebra is called a *normed algebra* if it possesses a vector space norm $|| \ ||$ which satisfies $||AB|| \leq ||A|| \cdot ||B||$ for all $A, B \in \mathfrak{A}$.
- A normed and complete algebra is called a *Banach algebra*.
- If an algebra $\mathfrak{A}$ is both a $^*$-algebra and a Banach algebra and if we have $||A^*|| = ||A||$ for all $A \in \mathfrak{A}$ then $\mathfrak{A}$ is called a *Banach* $^*$*-algebra*.
- Finally a Banach $^*$-algebra which also satisfies $||A^*A|| = ||A||^2$ for all $A \in \mathfrak{A}$ is a $C^*$*-algebra*.

### 2.2.1 The axioms of local quantum physics on curved spacetimes

The essential feature of QFT is the principle of locality. This principle states that physical effects have to propagate somehow through time and space. Effects only propagate from one point to a neighboring point. The concept of fields is the appropriate tool to put this principle into mathematics.

Of course locality must be implemented into the algebraic approach somehow. First we associate to the observables, elements of a $C^*$-algebra $\mathfrak{A}$. Locality says that it is meaningful to talk of observables, which can only be measured in a specific spacetime region. Hence we do not only need one single $C^*$-algebra $\mathfrak{A}$, but also distinguished subalgebras $\mathfrak{A}(\mathcal{O})$ corresponding to local regions $\mathcal{O}$ of spacetime. These algebras are generated by all $\phi(f)$, the fields smeared out with appropriate test functions $f$ having their support in $\mathcal{O}$ [Haa].

The fundamental object needed to describe the observables is a *net of local algebras*, which assigns to each bounded open set $\mathcal{O} \subset \mathcal{M}$ a $C^*$-algebra $\mathfrak{A}(\mathcal{O})$. The closed set-theoretic union of all $\mathfrak{A}(\mathcal{O})$,

$$\mathfrak{A} = \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})} \,, \tag{2.1}$$

forms a $C^*$-algebra, which is called the *algebra of local observables*. So far these are purely mathematical requirements. If we want the algebra $\mathfrak{A}$ to describe a physical theory in curved spacetimes, it has to fulfill the following five axioms [Dim80]:

1. Isotony: If $\mathcal{O} \subset \mathcal{O}'$ then $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')$.
2. Primitivity: There is a faithful irreducible representation of $\mathfrak{A}$.
3. Locality: If $\mathcal{O}$ is spacelike separated from $\mathcal{O}'$ then $[\mathfrak{A}(\mathcal{O}), \mathfrak{A}(\mathcal{O}')] = 0$.
4. Causality: If $\mathcal{O}$ is causally dependent on $\mathcal{O}'$ then $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')$.
5. Covariance: For any isometry $\kappa : (\mathcal{M}, g) \to (\widehat{\mathcal{M}}, \widehat{g})$ there is an isomorphism $\alpha_\kappa : \mathfrak{A} \to \widehat{\mathfrak{A}}$ such that $\alpha_\kappa[\mathfrak{A}(\mathcal{O})] = \widehat{\mathfrak{A}}\big(\kappa(\mathcal{O})\big)$. Furthermore one requires $\alpha_{\mathrm{id}} = \mathrm{id}$ and $\alpha_{\kappa_1} \circ \alpha_{\kappa_2} = \alpha_{\kappa_1 \circ \kappa_2}$.

The first three of these axioms are essentially the Haag-Kastler axioms [HK64]. The last two were added to the system by J. DIMOCK in order to incorporate curved spacetimes into the framework.

### 2.2.2 Quantum mechanics in terms of $C^*$-algebras

Suppose we have a $C^*$-algebra $\mathfrak{A}$. We assume that the observables correspond to the self-adjoint elements of the algebra. The states then are defined to be the normalized, positive, bounded, linear forms on $\mathfrak{A}$. This means a state $\omega$ is any expectation functional which assigns to each $A \in \mathfrak{A}$ a number $\omega(A)$ such that

- $\omega(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 \omega(A_1) + \lambda_2 \omega(A_2)$ (linearity),
- $\omega(A^* A) \geq 0$ (positivity),
- $\omega(\mathbb{1}) = 1$ (normalization).

Now what is the connection between the usual approach to quantum fields and the algebraic one? Two propositions explain the transition between the different formulations:

**Proposition 2.1** *Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{A}$ a subalgebra of the $C^*$-algebra $\mathcal{L}(\mathcal{H})$ of bounded linear maps on $\mathcal{H}$. Any density matrix state $\rho$ on $\mathcal{H}$ gives rise to an algebraic state $\omega(A) = \mathrm{tr}(\rho A)$ for all $A \in \mathfrak{A}$.*

Conversely we can produce a representation from a state by:

**Proposition 2.2 (GNS-Construction)** *Let $\omega$ be a state on a $C^*$-algebra $\mathfrak{A}$. Then there exists a GNS-triple $(\mathcal{H}, \pi, \Psi)$, consisting of a Hilbert space $\mathcal{H}$, a vector $\Psi \in \mathcal{H}$ and a representation $\pi$ of $\mathfrak{A}$ by means of bounded operators on $\mathfrak{A}$, such that*

*1.* $\langle \Psi \mid \pi(A) \mid \Psi \rangle = \omega(A) \quad \forall A \in \mathfrak{A}$.

*2.* $\pi(A)\Psi$ *is dense in $\mathcal{H}$, that is $\Psi$ is cyclic.*

*The GNS-triple is unique up to unitary equivalence.*

This means every state in the algebraic sense corresponds to a state in the usual sense in some Hilbert space construction.

### 2.2.3 Hadamard States

In section 2.4.1 we will discuss the scalar field on a globally hyperbolic spacetime. However it turns out, that not all algebraic states constructed in this theory are physically acceptable. One needs a criterion to pick out the sensible ones. In Minkowski space the crucial point is the *spectral condition*, i.e. the existence of a strongly unitary representation of the translations in $\mathbb{M}$, with generators $P_\mu$ which satisfy

$$\mathrm{sp}\, P_\mu \subset \overline{V}^+ \,. \tag{2.2}$$

This is equivalent to the statement that there exists a complete system of states with non-negative energies, in the corresponding Hilbert space [BLOT].

Two different approaches to the characterization of physical states are available, that demand similar local properties for the curved spacetime solutions to the Minkowski space solutions. The first is due to L. PARKER, who introduced the *adiabatic vacua*, that minimize the particle production in an expanding universe. The second approach goes back to B. DE WITT and R. BREHME [DB60]. It restricts physically nonsingular states in curved spacetimes by the requirement that their two-point function satisfy the Hadamard condition. The Hadamard condition insures that the ultra-violet behaviour of the state is similar to that of the vacuum state in Minkowski spacetime, and that the expected stress energy tensor in the state is finite [Wal95]. The first mathematical precise definition of the *Hadamard states* was given in [KW91]. W. JUNKER showed that the adiabatic vacua are indeed Hadamard states [Jun96], and thus the two approaches are compatible.

The construction of Hadamard states, even with the methods of [KW91], is cumbersome. It was a great breakthrough for the theory, when M. RADZIKOWSKI found a characterization of Hadamard states for scalar quantum fields on a 4-dimensional globally hyperbolic spacetime in terms of a specific form of the wavefront set of their 2-point functions [Rad96, Rad]. This so called 'wavefront set spectrum condition' initiated a major progress in the development of quantum field theory in curved spacetime. An extension of this result from scalar fields to vector fields by H. SAHLMANN and R. VERCH can be found in [SV00].

We shall not go into detail here, since we will not get so far to characterize Hadamard states of the Maxwell field. We refer the interested reader to a forthcoming paper by W. JUNKER and F. LLEDO on the Hadamard states for the vector potential on a manifold [JL].

## 2.3 Quantization Axioms in Curved Spacetime

After L. GÅRDING and A. WIGHTMAN the fields $\phi(f)$ of a physical theory should be subject to a system of axioms [GA64], usually called the *Wightman axioms*. (See [SW] for a thorough discussion of the axioms. Their consequences within the algebraic approach are discussed in [Haa] and [BLOT]). We shall not give the Wightman axioms here, but the requirements on a quantized, hermitian scalar field $\phi(f)$ on a curved manifold $\mathcal{M}$ derived from them.

Quantizing a classical field $\phi$ means to construct a Hilbert space $\mathcal{H}$ of physical states and an operator valued distribution $\phi(f)$ acting on $\mathcal{H}$ which describes the field observables localized in $\mathrm{supp}\, f \subset \mathcal{M}$. ($f \in \mathcal{D}(\mathcal{M})$ is a test function, i.e. a smooth function with compact support on $\mathcal{M}$, where the support, $\mathrm{supp}\, f$, is the closure of all points on $\mathcal{M}$ where $f$ does not vanish). The quantized field is subject to the following axioms [Jun97]:

1. Operator-valued distribution: For all $f \in \mathcal{D}(\mathcal{M})$ the distribution $\phi(f)$ is a linear (unbounded) closable operator on $\mathcal{H}$ with dense domain $D \subset \mathcal{H}$ such that $\phi(f)^* \supset \phi(\overline{f})$ (hermiticity) and $\phi(f)D \subset D$.

2. *$n$-point functions*: For all $\psi \in D$ one defines
$$\omega_n(f_1, \ldots, f_n) = \langle \psi, \phi(f_1) \ldots \phi(f_n)\psi \rangle \in \mathcal{D}'(\mathcal{M}^n) \; . \tag{2.3}$$
$\mathcal{D}'(\mathcal{M}^n)$ denotes the space of distributions over $\mathcal{M}^n$. The $n$-point functions are sometimes called *Wightman functions*.

3. Commutation relations: For all $f_1, f_2 \in \mathcal{D}(\mathcal{M})$ we require
$$[\phi(f_1), \phi(f_2)] := -i\langle f_1, Ef_2 \rangle \; , \tag{2.4}$$
where $E := E^+ - E^-$ is the fundamental solution (see 3.2) of the field equations and $E^+, E^-$ are the retarded and advanced solutions.

4. Field equations: For all $f \in \mathcal{D}(\mathcal{M})$ the field equations have to be satisfied:
$$\phi((\Box + m^2)f) = 0 \; . \tag{2.5}$$

5. Poincaré-covariance: In Minkowski space one demands that the translations $T_a$ : $x^\mu \to x^\mu + a^\mu$ can be implemented in $\mathcal{H}$ by a strongly continuous unitary group $U(a) = \exp(ia_\mu P^\mu)$ whose generator has spectrum in the positive forward light cone $\overline{V^+}$. The *vacuum state* is the unique eigenstate $\psi_0 \in D \subset \mathcal{H}$ of $P^\mu$ to the eigenvalue 0:
$$U(a, \Lambda)\psi_0 = \psi_0 \; . \tag{2.6}$$

In early formulations of field theory the field has been treated as an operator-valued function $\phi(x)$ acting on Hilbert space vectors. Soon it became apparent (by the analysis of field measurements in quantum electrodynamics) that the fields $\phi(x)$ are very singular objects, thus suggesting a distributional treatment. This becomes particularly apparent in the commutation relations where $\delta$-functions are involved. To make this statement more precise, we note that there is a physical as well as a mathematical reason why the quantum field at a point cannot be an honest observable. From the physical point of view, a measurement at a point would require infinite energy. Mathematically the field $\phi(x)$ is not an honest operator on the Hilbert space $\mathcal{H}$. One considers $\phi(x)$ as a sesquilinear form on some dense domain $D \in \mathcal{H}$, i.e. the matrix element $\langle \psi_2 | \phi(x) | \psi_1 \rangle$ is a finite number if $\psi_1, \psi_2 \in D$ and depends linearly on $\psi_1$ and conjugate linearly on $\psi_2$.

To obtain an operator defined on the vectors in $D$ one has to smear out $\phi$ with a smooth function f:

$$\phi(f) = \int \phi(x) f(x) d^4x \ . \tag{2.7}$$

If $f$ belongs to the test function space then $\phi(f)$ is an (unbounded) operator acting on $\mathcal{H}$ defined on $D$ [Haa]. For most purposes the test function space is chosen to be the space of fastly decreasing smooth functions (Schwartz space) $\mathcal{S}(\mathcal{M}) := \{\phi \in C^\infty(\mathcal{M}) : \sup|x^\beta \partial^\alpha \phi(x)| < \infty \ \forall$ multi-indices $\alpha, \beta\}$ rather than the space of all test functions $\mathcal{D}(\mathcal{M}) = C_c^\infty(\mathcal{M})$. The objects obtained by smearing out with functions $f \in \mathcal{S}$ are called *temperate distributions*. The reason for this choice is the apparatus of Fourier transform, which is easier to handle in the space of temperate distributions.

The second axiom can be understood by a principal theorem of axiomatic quantum field theory, the *reconstruction theorem*. It asserts, that the theory is completely characterized, up to unitary equivalence, by the *vacuum expectation values*, i.e. the $n$-point functions of the vacuum state. In H. BORCHERS alternative formulation of Wightman's theory the reconstruction theorem becomes just a natural modification of the GNS-theorem.

The first four conditions stated here can equally well be formulated on an arbitrary spacetime manifold $\mathcal{M}$. However this is not straightforward with the last axiom, since it depends essentially on the special global structure of Minkowski space.

Connected to this is the fact, that the vacuum state defined here does in general not exist in a curved spacetime. It has been the main problem of quantum field theory on curved spacetimes to find a substitute for the last condition. In order to study this question one considers only *quasifree* states $\psi \in D$. A quasifree state is, by definition, completely characterized by its two point-function $\omega_2$. The odd $n$-point functions of a quasifree state vanish and the even $n$-point functions of higher order can be decomposed into two-point functions.

$$\omega_{2n} = \sum_\sigma \prod_{j=1}^n \omega_2(f_{\sigma(j)}, f_{\sigma(j+n)}) \ , \qquad n \in \mathbb{N} \ , \tag{2.8}$$

where the sum goes over all permutations $\sigma$ of $\{1, \ldots, 2n\}$ with $\sigma(1) < \sigma(2) < \ldots \sigma(n)$ and $\sigma(j) < \sigma(j+n), j = 1, \ldots, n$. The restriction to quasifree states is a priori not physically motivated but by the fact that they form a class of states that is easily tractable. This class comprises the usual vacuum state on stationary spacetimes as well as the 'frequency splitting vacua' obtained by mode decomposition of the field operators [BD]. But it also contains all sorts of unphysical states that have to be removed by the Hadamard condition.

## 2.4 Symplectic Quantization

Quantization of a classical system is usually obtained by 'quantizing' the Poisson brackets of the classical system. The main problem on curved spacetimes is the coordinate dependency of the usual Poisson brackets. It is not clear what to do with the global canonical coordinates when we quantize on a curved spacetime. A reformulation of the Poisson brackets in a coordinate independent manner is necessary. This is in fact possible for linear dynamical systems with a Lagrangian formulation.

A very good pedagogical introduction into this matter is [Walb] on which we base the following treatment. A brief introduction is given in [Wal95]. R. WALD treats the scalar field as the simplest example of a linear field. A. CORICHI gives an analogous treatment of the Maxwell field [Cor98]. We shall stick to the scalar field, in order to clarify the structure of the theory.

### Symplectic quantization of a classical system

In classical dynamics a system with finitely many degrees of freedom $n$ is described by generalized coordinates $(q^1, \dots, q^n)$ and conjugated momenta $(p_1, \dots, p_n)$ in a *phase space* $\mathscr{P} = (q^1, \dots, q^n, p_1, \dots, p_n)$ of dimension $2n$. Observables are smooth, real valued functions $f : \mathscr{P} \to \mathbb{R}$ on the phase space. An outstanding observable is the Hamilton function $H(q^i, p_i) = p_i q^i - L(q^i, \dot{q}^i)$, ($L$ is the Lagrangian), since the whole dynamical evolution of a classical physical system can be obtained from the *Hamiltonian equations*:

$$\dot{q}^\mu = \frac{\partial H}{\partial p_\mu} \,, \qquad \dot{p}_\mu = -\frac{\partial H}{\partial q^\mu} \,. \tag{2.9}$$

In the absence of constraining forces and time dependence $H$ is the total energy of the system.

From a more abstract point of view, emphasizing the geometrical aspects of the theory, the configuration variables $(q^1, \dots, q^n)$ are coordinates on an $n$-dimensional manifold $\mathcal{Q}$. The phase space $\mathscr{P}$ of the dynamical system then corresponds to the cotangent bundle of $\mathcal{Q}$, that is $\mathscr{P} = T^*\mathcal{Q}$. The crucial object one needs to describe physical dynamics is a symplectic form:

**Definition 2.3** A *symplectic form* on a manifold $M$ of dimension $2n$ is a (weakly) non-degenerate, antisymmetric closed two-form $\sigma$ on $M$. A pair $(M, \sigma)$ is called a *symplectic manifold*.

In local coordinates $(q^1, \dots, q^n)$ one may identify the canonical momentum variables $(p_1, \dots, p_n)$ with the cotangent vectors in the coordinate basis of $(q^1, \dots, q^n)$. Then, by the Darboux theorem, a symplectic form on $\mathscr{P}$ is given by

$$\sigma = \sum_{\mu=1}^{n} \mathrm{d}p_\mu \wedge \mathrm{d}q^\mu \,, \tag{2.10}$$

and the pair $(\mathscr{P}, \sigma)$ is a symplectic manifold. That $\sigma$ is non-degenerate means, that for any tangent vector $V \in \mathscr{P}$ we have $\sigma_{\mu\nu} V^\nu = 0$ if and only if $V^\nu = 0$. This also implies the existence of a unique inverse $\sigma^{\mu\nu}$ of the symplectic form, satisfying $\sigma^{\mu\nu}\sigma_{\nu\lambda} = \delta^\mu_\lambda$.

The observables of the system are (smooth) real-valued functions $f : \mathscr{P} \to \mathbb{R}$. The set of all observables, denoted $O$, is a vector space and becomes a Poisson-algebra if we define as a product the *Poisson bracket*

$$\{f, g\} = \sigma^{\mu\nu} \nabla_\mu f \nabla_\nu g \; . \tag{2.11}$$

The canonical coordinates $q^\mu, p_\nu$ are fundamental in the sense that all other observables are functions of them. Their Poisson brackets are

$$\{q^\mu, p_\nu\} = \delta_{\mu\nu} \tag{2.12}$$

$$\{q^\mu, q^\nu\} = 0 = \{p_\mu, p_\nu\} \; . \tag{2.13}$$

Since $\mathscr{P}$ is assumed to be a vector space we can identify the tangent space at any point $y \in \mathscr{P}$ with $\mathscr{P}$. In this case the symplectic form, $\sigma_{ab}$, becomes a bilinear function, $\sigma : \mathscr{P} \times \mathscr{P} \to \mathbb{R}$, on $\mathscr{P}$. This map, $\sigma$, is independent of the choice of $y$ and is referred to as a *symplectic structure* on $\mathscr{P}$ and the pair $(\mathscr{P}, \sigma)$ is called a *symplectic vector space*. For any two points $y, \tilde{y} \in \mathscr{P}$ this symplectic form is given explicitly by

$$\sigma(y, \tilde{y}) = \sum_{\mu=1}^{n} (p_\mu \tilde{q}^\mu - \tilde{p}_\mu q^\mu) \; . \tag{2.14}$$

For a fixed point $y \in \mathscr{P}$ we define a linear function, $\sigma(y, \cdot) : \mathscr{P} \to \mathbb{R}$, on $\mathscr{P}$. If we choose $y = (0, \ldots, 0, q^\mu = 1, 0, \ldots, 0)$ we have $\sigma(y, \cdot) = p_\mu$. Similarly for $y = (0, \ldots, 0, p_\mu = 1, 0, \ldots, 0)$ we have $\sigma(y, \cdot) = q^\mu$. Thus any relation involving linear combinations of the coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ can be rewritten in terms of the functions $\sigma(y, \cdot)$. The fundamental Poisson brackets (2.12) are then equivalent to the single relation:

$$\{\sigma(y_1, \cdot), \sigma(y_2, \cdot)\} = -\sigma(y_1, y_2) \; . \tag{2.15}$$

We have made no particular choice of coordinates on $\mathscr{P}$ to formulate this equation, therefore this is a coordinate independent formulation of the Poisson brackets. But unlike (2.12) this relation generalizes straightforwardly to curved spacetimes, in which case $\mathscr{P}$ is infinite dimensional.

On linear dynamical systems the Hamiltonian $H$ is a quadratic function on $\mathscr{P}$. Then the equations of motion are linear in the canonical coordinates and the manifold of solutions $\mathscr{S}$ acquires a natural vector space structure. Let $y_1(t), y_2(t) \in \mathscr{S}$ be two solutions of the equations of motion. Define the *symplectic product* $s = \sigma(y_1(t), y_2(t))$. Then we have $\frac{ds}{dt} = 0$ (proof in [Walb]), i.e. the symplectic product of two solutions is conserved. As a consequence the symplectic structure $\sigma : \mathscr{P} \times \mathscr{P} \to \mathbb{R}$ on $\mathscr{P}$ gives rise to a symplectic structure $\sigma : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ on $\mathscr{S}$.

As pointed out already, the quantization of a classical field means finding a Hilbert space $\mathcal{H}$ and self adjoint operators $\widehat{f}_i$ corresponding to the classical observables $f_i$. It is far from obvious how to define the correspondence map $\widehat{\ }: O \to \widehat{O}$ from the space of classical observables, $O$, into the set of quantum observables, $\widehat{O}$. However, the comparison of classical and quantum dynamics guides us in this question. In classical mechanics the rate of change of an observable $f$ is given by the Poisson bracket with the Hamiltonian H:

$$\frac{\partial f}{\partial t} = \{f, H\} \; . \tag{2.16}$$

On the other hand, in the Heisenberg representation of quantum mechanics we have

$$\frac{\partial \widehat{f}}{\partial t} = -i[\widehat{f}, \widehat{H}] . \tag{2.17}$$

This suggests a correspondence map $\widehat{\ }$, such that for any pair of classical observables $f, g$ we have

$$[\widehat{f}, \widehat{g}] = i\widehat{\{f, g\}} . \tag{2.18}$$

The Poisson bracket $\{\cdot, \cdot\}$ defines an algebra on the space of classical observables $O$ and the commutator $[\cdot, \cdot]$ defines a similar algebra on on the set of quantum observables $\widehat{O}$.

It is well known that in general no map $\widehat{\ }$ exists, which implements (2.18) on all observables. However, in case of $\mathscr{P} = T^*\mathcal{Q}$ it is always possible to choose $\mathcal{H}$ and a map $\widehat{\ }$ such that for the canonical coordinates on $\mathscr{P}$ we have

$$[\widehat{q}^\mu, \widehat{p}_\nu] = i\widehat{\{q^\mu, p_\nu\}} = i\delta_{\mu\nu}\mathbb{1} \tag{2.19}$$

$$[\widehat{q}^\mu, \widehat{q}^\nu] = 0 = [\widehat{p}_\mu, \widehat{p}_\nu] . \tag{2.20}$$

Inserting (2.15) yields

$$[\widehat{\sigma}(y_1, \cdot), \widehat{\sigma}(y_2, \cdot)] = -i\sigma(y_1, y_2)\mathbb{1} . \tag{2.21}$$

Some potential technical difficulties arise when we attempt to work with this relation. The operator $\widehat{\sigma}(y, \cdot)$ should be unbounded and hence can only be defined on a dense domain. Therefore compositions, for example commutators, are not automatically well defined. These difficulties are dealt with by working with an exponentiated version of equation (2.21). For each $y$ we define the classical observable $W(y)$ by

$$W(y) = \exp\bigl(-i\sigma(y, \cdot)\bigr) \tag{2.22}$$

Then equation (2.21) together with the self adjointness of $\sigma(y, \cdot)$ is formally equivalent to the *Weyl relations*:

$$\widehat{W}(y_1)\widehat{W}(y_2) = \exp\left(\frac{1}{2}i\sigma(y_1, y_2)\right)\widehat{W}(y_1 + y_2) \tag{2.23}$$

$$\widehat{W}^\dagger(y) = \widehat{W}(-y) . \tag{2.24}$$

The Weyl relations can be viewed as a precise statement of the Poisson bracket relations which avoids operator domain problems.

### 2.4.1 Application to the real scalar field

Using the methods of the last section we can easily construct the quantum field theory of the Klein-Gordon scalar field in a globally hyperbolic spacetime $(\mathcal{M}, g_{\mu\nu})$.

First of all one needs a well posed initial value formulation. In [Dim80] the Cauchy problem for the Klein-Gordon equation is addressed and a proof for the existence and uniqueness of solutions on arbitrary globally hyperbolic manifolds and arbitrary Cauchy surface is given. We quote the theorems as stated in [Dim92]. Let $(\mathcal{M}, g_{\mu\nu})$ be a globally hyperbolic manifold with a compact Cauchy surface $\Sigma$. The Klein-Gordon equation for a scalar field $\phi \in \Omega^0(\mathcal{M})$ on $\mathcal{M}$ is

$$(\Box + m^2)\phi = 0 \; . \tag{2.25}$$

Let $i : \Sigma \to \mathcal{M}$ be the identity map. One defines two operators $\rho_0(\phi), \rho_1(\phi) \in \Omega^0(\Sigma)$ by

$$\rho_0(\phi) = i^*(\phi) \tag{2.26}$$
$$\rho_1(\phi) = (-1) * i^*(*\mathrm{d}\phi) \; . \tag{2.27}$$

The operator $\rho_0(\phi)$ is the restriction of $\phi$ to the Cauchy surface $\Sigma$ and $\rho_1(\phi)$ is the forward normal derivative of $\phi$ on $\Sigma$. If $(e_1, e_2, e_3)$ is an orthonormal basis at a point on $\Sigma$ and n is the forward normal to $\Sigma$ at that point, then $(n, e_1, e_2, e_3)$ is an oriented orthonormal basis for the tangent space at that point. One finds

$$\begin{aligned}\rho_1(\phi) &= i^*(*\mathrm{d}\phi)(e_1, e_2, e_3) \\ &= (*\mathrm{d}\phi)(e_1, e_2, e_3) \\ &= \mathrm{d}\phi(n) \; .\end{aligned} \tag{2.28}$$

**Proposition 2.4 (Cauchy problem)** *Let $(\mathcal{M}, g)$ be globally hyperbolic with compact Cauchy surface $\Sigma$. For any $\alpha, \beta \in \Omega^0(\Sigma)$ there exists a unique $\phi \in \Omega^0(\mathcal{M})$ such that $\Box\phi = 0, \quad \rho_0(\phi) = \alpha, \quad \rho_1(\phi) = \beta$.*

Here $\alpha$ and $\beta$ serve as initial data for the wave equation.

**Proposition 2.5 (Fundamental solutions)** *There are operators $E^{\pm} : \Omega^0_c(\mathcal{M}) \to \Omega^0(\mathcal{M})$ such that*

$$\Box E^{\pm} = E^{\pm}\Box = \mathbb{1} \tag{2.29}$$

*and*

$$\mathrm{supp}(E^{\pm} f) \subset J^{\pm}(\mathrm{supp}\, f) \; . \tag{2.30}$$

*The operators $E^{\pm}$ extend to $f$ with $\mathrm{supp}\, f$ compact to the past/future and for such $f, u = E^{\pm} f$ is the unique solution of $\Box u = f$ with $\mathrm{supp}\, u$ compact to the past/future.*

The initial values can equally be characterized as the pair of functions $(\phi, \pi)$ on the Cauchy surface $\Sigma$, where $\pi = n^{\mu}\nabla_{\mu}\phi$, with $n^{\mu}$ the unit normal to $\Sigma$. Then the space of solutions is defined by

$$\mathscr{S} = \{(\phi, \pi) | \phi, \pi \in C^{\infty}_c(\Sigma)\} \; . \tag{2.31}$$

Hence the Klein-Gordon equation gives rise to a well defined, conserved symplectic structure $\sigma : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ on $\mathscr{S}$ by

$$\sigma((\phi_1, \pi_1), (\phi_2, \pi_2)) = \int_{\Sigma} (\pi_1\phi_2 - \pi_2\phi_1) d^3x \; . \tag{2.32}$$

The fundamental solutions $E$ are actually maps from the space of test functions into the space of solutions: $E : \mathcal{D}(\mathcal{M}) \to \mathscr{S}$. Then for all $\phi \in \mathscr{S}$ and all $f \in \mathcal{D}$, we have [Walb]

$$\int \phi(x) f(x) d^4 x = \sigma(Ef, \phi) \ . \tag{2.33}$$

This means, that for each test function $f$, the function $\sigma(Ef, \cdot)$ on $\mathscr{S}$ is equal to the solution smeared out in spacetime. From this we see that the corresponding Heisenberg observable $\widehat{\sigma}(y, \cdot)$ has an alternative interpretation as a smeared out field operator. This provides the field interpretation as required in the Wightman formalism.

# Chapter 3

# The Maxwell Field

In this chapter we introduce the Maxwell field, the main object of this thesis. The Maxwell equations, published in their original form by JAMES CLERK MAXWELL in 1862, have been the germ of more than one physical theory. The Michelson-Morley experiment and the impossibility to prove the existence of the 'aether', led to the formulation of special relativity. The quest for a quantum theory, respecting special relativity and incorporating the field properties of the Maxwell equations, led to quantum electrodynamics as the first quantum field theory. Within modern quantum field theory one is still interested in quantum electrodynamics, since it provides the simplest example of a gauge field theory.

We begin this chapter with the definition of the Maxwell field in the language of differential geometry. The Maxwell equations pose a wave equation, which we solve explicitly in Minkowski space. After that, we review the usual quantization procedure of the Maxwell field in Minkowski space. This is the so called Gupta-Bleuler quantization.

## 3.1 Maxwell's Equations

The electric and the magnetic field treated as 3-dimensional vectors $\mathbf{E}, \mathbf{B}$ in classical electrodynamics can be implemented in the Maxwell field strength tensor $F_{\mu\nu}$ in relativistic electrodynamics. The Maxwell equations are then formulated in terms of this new quantity. Since the Maxwell tensor is a differential form one can use the operations on differential forms to obtain a final beautified version of these equations.

We start with the well known classical Maxwell equations:

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} \tag{3.1}$$

$$\nabla \cdot \mathbf{E} = \rho \tag{3.2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3.3}$$

$$\nabla \cdot \mathbf{B} = 0\,. \tag{3.4}$$

$\mathbf{E}$ and $\mathbf{B}$ are the 3-dimensional electric and magnetic field strength vectors, $\mathbf{j}$ is the electric current density and $\rho$ is the charge density. $\nabla$ denotes the conventional nabla-operator $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$, implying that $\nabla\times$ and $\nabla\cdot$ are the curl and divergence operators respectively. This is a system of eight coupled differential equations. It can be shown that these equations are invariant under Lorentz transformations. The charge density $\rho$ and the current density $\mathbf{j}$ form the *current four-vector* $J^\mu = (J^0, J^1, J^2, J^3) = (\rho, j^1, j^2, j^3)$. We define the *electromagnetic field strength tensor* or *Maxwell tensor*, as the following antisymmetric (0,2) tensor:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} . \tag{3.5}$$

Sometimes one also needs the dual-field strength tensor $*F$, which in Minkowski space is simply

$$(*F)_{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_3 & -E_2 & E_1 & 0 \end{pmatrix} . \tag{3.6}$$

For calculations it is helpful to notice that

$$F_{i0} = -F_{0i} = E_i \tag{3.7}$$

$$F_{ij} = \sum_k \tilde{\epsilon}_{ijk} B_k \ . \tag{3.8}$$

With the help of these new objects we can combine the two inhomogeneous Maxwell equations and the two homogeneous equations into single tensor equations respectively:

$$\partial^\mu F_{\mu\nu} = -J_\nu \tag{3.9}$$

$$\partial_{[\lambda} F_{\mu\nu]} = 0 \ . \tag{3.10}$$

A further simplification is obtained if we make use of the exterior derivative and the star operator

$$\mathrm{d}(*F) = *J \tag{3.11}$$

$$\mathrm{d}F = 0 \ . \tag{3.12}$$

The action of the star-operator on (3.11) gives $\delta F = J$. Mathematically equation (3.12) simply says, that $F_{\mu\nu}$ is a closed two-form on the spacetime manifold $\mathcal{M}$. Assuming contractibility of $\mathcal{M}$, the Poincaré lemma says that $F$ is exact and thus we can write $F$ as the exterior derivative $\mathrm{d}A$ of a one-form $A$ on $\mathcal{M}$:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{3.13}$$

The quantity $A$ is called the *vector potential* or *gauge field*. $F_{\mu\nu}$ can be viewed as the 'four-dimensional rotation' of $A$. The identity $\mathrm{d}F = \mathrm{dd}A = 0$ follows directly from $\mathrm{d}^2 = 0$ and is known as the *Bianchi identity*. From this we see that the theory is invariant under the transformation

$$A \to A + \mathrm{d}\lambda \tag{3.14}$$

for some zero-form (scalar) $\lambda$. This property of the electromagnetic field is called *gauge invariance*. As we are only interested in source free Maxwell fields, we can put the four-current $J$ equal to zero and get:

$$\delta F = 0 \tag{3.15}$$

$$\mathrm{d}F = 0 \; . \tag{3.16}$$

Henceforth we will call these the Maxwell equations and refer to (3.1)-(3.4) as the classical Maxwell equations.

## 3.2 The Homogeneous Wave Equation

In Minkowski space it is possible to construct the solutions of the homogeneous wave equation explicitly. Whereas on generic curved spacetimes we can only give existence and uniqueness theorems. This will be the main result of this work, but let us first recall the construction of the solutions in flat spacetime. We follow the notes of K. FREDENHAGEN [Frec], but similar constructions are found in many books on differential equations or quantum field theory, e.g. [BLT].

First we solve the homogeneous wave equation for an arbitrary scalar field $u(x)$ in Minkowski space $\mathbb{M}$. The wave equation reads

$$\Box u(x) = 0 \; . \tag{3.17}$$

In Minkowski space the d'Alembertian is simply

$$\Box = \frac{\partial^2}{\partial^2 t} - \sum_{i=1}^{3} \frac{\partial^2}{\partial {x^i}^2} \; . \tag{3.18}$$

Since (3.17) is a second order linear partial differential equation, to find the solution for arbitrary initial values $u_0(\mathbf{x}) = u(0, \mathbf{x})$ and $v_0(\mathbf{x}) = \frac{\partial}{\partial t} u(0, \mathbf{x})$, it is sufficient to find the fundamental solution of the equation. The *fundamental solution* of a differential equation $\Box f(x) = 0$ is a distribution $D(x)$ which fulfills

$$\Box D(x) = 0 \; , \qquad D(0, \mathbf{x}) = 0 \; , \qquad \left.\frac{\partial D}{\partial t}\right|_{t=0} = \delta(\mathbf{x}) \; . \tag{3.19}$$

Inserting the separation ansatz:

$$u(t, \mathbf{x}) = f(t) e^{i\mathbf{k}\mathbf{x}} \tag{3.20}$$

into the wave equation yields

$$\ddot{f} + |\mathbf{k}|^2 f = 0 \; . \tag{3.21}$$

The general solution to this differential equation is

$$f(t) = f(0)\cos(|\mathbf{k}|t) + \frac{1}{|\mathbf{k}|}\dot{f}(0)\sin(|\mathbf{k}|t)\ . \tag{3.22}$$

Suppose that the initial values are given as a superposition of monochromatic waves. Then they have a Fourier expansion:

$$u_0(\mathbf{x}) = (2\pi)^{-\frac{3}{2}} \int d^3k\, \widehat{u}_0(\mathbf{k})e^{i\mathbf{kx}} \tag{3.23}$$

$$v_0(\mathbf{x}) = (2\pi)^{-\frac{3}{2}} \int d^3k\, \widehat{v}_0(\mathbf{k})e^{i\mathbf{kx}}\ . \tag{3.24}$$

In this case the solution of the wave equation is itself a superposition of monochromatic waves

$$u(t,\mathbf{x}) = (2\pi)^{-\frac{3}{2}} \int d^3k \left(\widehat{u}_0(\mathbf{k})\cos(|\mathbf{k}|t) + \frac{1}{|\mathbf{k}|}\widehat{v}_0(\mathbf{k})\sin(|\mathbf{k}|t)\right) e^{i\mathbf{kx}}\ . \tag{3.25}$$

Now let $u_0(\mathbf{x}), v_0(\mathbf{x})$ be arbitrary initial conditions. Inserting their Fourier transforms

$$\widehat{u}_0(\mathbf{x}) = (2\pi)^{-\frac{3}{2}} \int d^3k\, u_0(\mathbf{x})e^{-i\mathbf{kx}} \tag{3.26}$$

$$\widehat{v}_0(\mathbf{x}) = (2\pi)^{-\frac{3}{2}} \int d^3k\, v_0(\mathbf{x})e^{-i\mathbf{kx}} \tag{3.27}$$

into (3.25) yields

$$\begin{aligned} u(t,\mathbf{x}) &= (2\pi)^{-3} \int d^3k\, d^3y \left(u_0(\mathbf{y})\cos(|\mathbf{k}|t) + \frac{1}{|\mathbf{k}|}v_0(\mathbf{y})\sin(|\mathbf{k}|t)\right) e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \\ &= \int d^3y \left[\frac{\partial}{\partial t}D(t,\mathbf{x}-\mathbf{y})\, u_0(\mathbf{y}) + D(t,\mathbf{x}-\mathbf{y})v_0(\mathbf{y})\right] \end{aligned} \tag{3.28}$$

with

$$D(t,\mathbf{x}) = (2\pi)^{-3} \int d^3k \frac{\sin(|\mathbf{k}|t)}{|\mathbf{k}|}\, e^{i\mathbf{kx}} \tag{3.29}$$

$$= \frac{1}{4\pi|\mathbf{x}|}\big(\delta(t-|\mathbf{x}|) - \delta(t+|\mathbf{x}|)\big)\ . \tag{3.30}$$

(Note: In the field theory context the distribution $D(x)$ is usually defined with negative sign, but this is not necessary here)

**Application to E and B.** So far we worked with an arbitrary field $u(t,\mathbf{x})$. Taking the curl of equation (3.1), setting $j = 0$ and using (3.3), one obtains the wave equation for the magnetic field $\Box B = 0$. By a similar procedure the wave equation for the electric field, $\Box E = 0$, is obtained.

The solutions to these wave equations are given by inserting the Maxwell equations with vanishing $\rho, \mathbf{j}$ into the solution (3.28)

$$\mathbf{E}(t,\mathbf{x}) = \int d^3y \left[\frac{\partial}{\partial t} D(t,\mathbf{x}-\mathbf{y})\,\mathbf{E}_0(\mathbf{y}) + D(t,\mathbf{x}-\mathbf{y})\frac{\partial}{\partial t}\mathbf{E}_0(\mathbf{y})\right] \tag{3.31}$$

$$\mathbf{B}(t,\mathbf{x}) = \int d^3y \left[\frac{\partial}{\partial t} D(t,\mathbf{x}-\mathbf{y})\,\mathbf{B}_0(\mathbf{y}) + D(t,\mathbf{x}-\mathbf{y})\frac{\partial}{\partial t}\mathbf{B}_0(\mathbf{y})\right] . \tag{3.32}$$

However, the initial values $\mathbf{E}_0(\mathbf{y}), \mathbf{B}_0(\mathbf{y})$ are not free. They have to satisfy the sourceless classical Maxwell equations. If we assume $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ at the time $t = 0$, then $\partial_t \mathbf{E}$ and $\partial_t \mathbf{B}$ are completely fixed and the solution is

$$\mathbf{E}(t,\mathbf{x}) = \int d^3y \left[\frac{\partial}{\partial t} D(t,\mathbf{x}-\mathbf{y})\,\mathbf{E}_0(\mathbf{y}) + D(t,\mathbf{x}-\mathbf{y})\big(\nabla \times \mathbf{B}_0(\mathbf{y})\big)\right] \tag{3.33}$$

$$\mathbf{B}(t,\mathbf{x}) = \int d^3y \left[\frac{\partial}{\partial t} D(t,\mathbf{x}-\mathbf{y})\,\mathbf{B}_0(\mathbf{y}) - D(t,\mathbf{x}-\mathbf{y})\big(\nabla \times \mathbf{E}_0(\mathbf{y})\big)\right] . \tag{3.34}$$

## 3.3 Quantization of the Maxwell field in Minkowski Space

### 3.3.1 General considerations

The quantization of the Maxwell field in flat spacetime leads already to some special problems, that are not existent for a scalar field. First, the theory contains two candidates for the field operators $A_\mu$ and $F_{\mu\nu}$. Second, there seems to be a general incompatibility between Lorentz invariance and the use of a Hilbert space. Additional problems arise when quantizing the field in curved spacetimes. But these problems are not specific to the Maxwell field and have been discussed to some extent in the last chapter, and will be dealt with in detail in the following chapter.

Concerning the first problem, it is well known, that the classical state of a free electromagnetic field in spacetime $\mathcal{M}$ is defined by the field tensor $F_{\mu\nu}$. In the quantized theory the vector potential $A_\mu$ becomes essential, since there is no Lagrangian theory of the electromagnetic field without a vector potential. Due to gauge invariance the quantum characterization has some redundancy. Two vector potentials $A_\mu, A'_\mu$ with the same field strength tensor $F_{\mu\nu}$ define the same physical state, i.e. they are physically equivalent. In the mathematical formulation this leads to gauge equivalence classes. One can say that the configuration $A_\mu$ of the vector potential is a virtual state of the classical electromagnetic field and that the physical field is the equivalence class of the virtual states [BLOT].

The incompatibility of Lorentz covariance and simultaneous use of a Hilbert space, as illustrated in [SW74], leads to a theory with an indefinite metric. This approach, usually named Gupta-Bleuler theory [Gup50, Ble50], is a completely consistent theory of a quantized electromagnetic field in an extended Wightman formalism. It also has been rigorously constructed in a C*-algebra context by H. GRUNDLING [Gru88] and further refined by H.GRUNDLING and F. LLEDÓ [GL00].

### 3.3.2 Quantization of the vector potential

The covariant quantization of the vector potential is treated in all books on quantum field theory. Nevertheless we hope to clarify the mathematical structure of the heuristic Gupta-Bleuler theory by reciting the most important steps. The following treatment is extracted from various sources: [Freb, Roe98, BLOT, Gru88, GL00].

The source free Maxwell equations can be derived from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ . \tag{3.35}$$

The components $A_\mu$ of the gauge field are taken to be independent field coordinates. They satisfy $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The field equations that follow from this are $\partial_\mu F^{\mu\nu} = 0$, or in terms of $A$

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0 \ . \tag{3.36}$$

The origin of the difficulties with the vector potential $A_\mu$ is, that it has four independent components and at this stage is ambiguous. We can conclude, that the quantum field, associated with $A_\mu$, has less than four degrees of freedom. The ambiguities can be resolved by imposing a gauge condition on $A_\mu$. One possibility is to impose the *Lorentz gauge*

$$\partial_\mu A^\mu = 0 \tag{3.37}$$

This reduces the independent components of $A_\mu$ from four to three. We also note that the condition (3.37) is Lorentz covariant. However, even after imposing the Lorentz condition, the vector field is still not unique. If $A_\mu$ satisfies the Lorentz condition, any $A_\mu + \partial_\mu \lambda$ with $\Box\lambda = 0$ will satisfy it too. The conjugated momenta also pose a problem:

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = \begin{cases} 0 & \mu = 0, \\ -F^{0i} = E_i, & \mu = i, \quad i = 1,2,3. \end{cases} \tag{3.38}$$

Since $\pi_0 = 0$ we cannot express $\partial_0 A_0$ as a function of the momenta and spatial derivatives. Therefore a transition to the Hamiltonian formalism is impossible. With the Lagrangian density (3.35) it is impossible to quantize covariantly.

It was E. FERMI who proposed another Lagrangian for the Maxwell equations, with an already fixed gauge:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2, \qquad \lambda \neq 0 \ . \tag{3.39}$$

The new Lagrangian yields the Maxwell equations in case of the Lorentz gauge (3.37) is satisfied. The field equations, which follow from the new Lagrangian are

$$\Box A^\mu + (\lambda - 1)\partial^\mu(\partial_\nu A^\nu) = 0 \ . \tag{3.40}$$

With the new Lagrangian all momenta are nonzero:

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = \begin{cases} -\lambda(\partial_\mu A^\mu) & \mu = 0, \\ -F^{0i} = E_i, & \mu = i, \quad i = 1,2,3. \end{cases} \tag{3.41}$$

The equation (3.40) can be simplified by the choice of special value for $\lambda$. The setting $\lambda = 1$ is known as the *Feynman gauge* and leads to the equation of motion

$$\Box A^\mu = 0 \; . \tag{3.42}$$

This can be interpreted as the wave equations of four massless Klein-Gordon fields. Hence we require for $A_\mu$ and $\pi_\mu$ canonical equal time commutation relations:

$$[A_\mu(0,\mathbf{x}), \pi_\nu(0,\mathbf{y})] = i g_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}) \tag{3.43}$$

$$[A_\mu(0,\mathbf{x}), A_\nu(0,\mathbf{y})] = 0 = [\pi_\mu(0,\mathbf{x}), \pi_\nu(0,\mathbf{y})] \; . \tag{3.44}$$

The dynamics of the Maxwell field in the Feynman gauge can also be derived from the Lagrangian $\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$. The momenta then become $\pi_\mu = -\partial^0 A_\mu$. Using this we obtain the commutation relation between $A_\mu$ and $\dot{A}_\mu$:

$$[A_\mu(0,\mathbf{x}), \dot{A}_\nu(0,\mathbf{y})] = -i g_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}) \; . \tag{3.45}$$

Now compare this equation with the scalar case, where the commutator is

$$[\phi(0,\mathbf{x}), \dot{\phi}(0,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}) \; . \tag{3.46}$$

The fields $A^\mu$ can be treated exactly like a scalar field, but the field $A^0$ has a commutation relation with wrong sign. This is a direct consequence of the covariance of the equations, expressed by the metric tensor. In a rigorous quantum theory $A^\mu$ must be an operator valued distribution in a Hilbert space $\mathcal{H}$. The wrong sign of the commutation relation for $A^0$ has consequences on the structure of the Hilbert space. It leads to a 'Hilbert space' with *indefinite metric*.

Before we investigate the indefinite metric Hilbert space, let us first introduce the Fourier transform of the Field. We express $A_\mu$ as a superposition of plane waves:

$$A_\mu(x) = \frac{1}{(2(2\pi)^3)^{1/2}} \int_{C_+} \frac{d^3k}{k_0} \left( a_\mu(\mathbf{k})\, e^{-ikx} + a^*_\mu(\mathbf{k})\, e^{ikx} \right) \tag{3.47}$$

where $C_+ := \{k \in \mathbb{M} : k_\mu k^\mu = 0, k_0 \geq 0\}$ is the mantle of the positive light cone, $k_0 = |\mathbf{k}|$. The canonical commutation relations calculated from this are

$$[A_\mu(x), A_\nu(x')] = -i g_{\mu\nu} D(x-x') \; , \tag{3.48}$$

with

$$D(x) = -\frac{1}{(2\pi)^3} \int_{C_+} \frac{d^3k}{k_0} e^{i\mathbf{kx}} \sin(k_0 x_0) \; . \tag{3.49}$$

Now we interpret $A^\mu$ as an operator valued distribution in an Hilbert space, i.e. we consider smeared out field operators

$$A(f) = \int A_\mu(x) f^\mu(x) d^4x \; . \tag{3.50}$$

If we insert equation (3.47) into (3.50) we obtain

$$A(F) = \sqrt{\pi} \int_{C_+} \frac{d^3k}{k_0} (a_\mu(k) \widehat{f}^\mu(k) + a^*_\mu(k) \overline{\widehat{f}^\mu(k)}) \tag{3.51}$$

$$= \frac{1}{\sqrt{2}} (a(f) + a^*(f)) \tag{3.52}$$

where $F = (f^0, f^1, f^2, f^3) \in \mathcal{S}^4(\mathbb{R}^4)$ and

$$\widehat{f}_\mu(k) = (2\pi)^{-2} \int d^4x e^{-ikx} f_\mu(x) \in \mathcal{S}^4_r(\mathbb{R}^4) \tag{3.53}$$

with $\mathcal{S}^4_r(\mathbb{R}^4) := \{F \in \mathcal{S}^4_C(\mathbb{R}^4) \mid \overline{F}(k) = F(-k)\}$. The operators $a(f), a^*(f)$ are defined by

$$a(f) := \sqrt{2\pi} \int_{C_+} \frac{d^3k}{k_0} a_\mu(k) \widehat{f}^\mu(k) \tag{3.54}$$

and

$$a^*(f) := \sqrt{2\pi} \int_{C_+} \frac{d^3k}{k_0} a^*_\mu(k) \overline{\widehat{f}^\mu(k)}\ . \tag{3.55}$$

The usual interpretation of this equation is that the $f^\mu$ are wavefunctions in the momentum representation in a Hilbert space $\mathcal{H}^{(1)}$ and the $a(f), a^*(f)$ are creation and annihilation operators for particles. The wavefunctions are required to be square-integrable, i.e. $\int d^3k |f_\mu(k)|^2 < \infty$. This is not in contradiction to our assumption $f^\mu \in \mathcal{S}(\mathbb{R}^4)$, since the square-integrable functions are just a special class of temperate distributions.

The one-particle Hilbert space $\mathcal{H}^{(1)}$ is the direct sum of the spaces

$$\mathcal{H}_\mu = \{f_\mu : \mathbb{R}^3 \to \mathbb{C} \mid \int \frac{d^3k}{2k_0} |f_\mu(k)|^2 < \infty\}, \qquad \mu = 0, \ldots, 3\ . \tag{3.56}$$

This space has a Hilbert topology defined by the scalar product

$$(f, g) = \int_{C_+} \frac{d^3k}{2k_0} \sum_{\mu=0}^{3} \overline{f_\mu(k)} g_\nu(k) \tag{3.57}$$

whereas the natural 'scalar product' $\langle\ ,\ \rangle$ of this space is indefinite, since the first component has negative sign:

$$\langle f, g \rangle = \int_{C_+} \frac{d^3k}{2k_0} \overline{f_\mu(k)} \{-\eta^{\mu\nu}\} g_\nu(k) \tag{3.58}$$

Thus we don't really have a Hilbert space, but a pseudo-Hilbert space, i.e. a space with Hilbert topology that is also endowed with an indefinite scalar product. The Hilbert topology of $\mathcal{H}^{(1)}$ and the scalar product $\langle\ ,\ \rangle$ are *compatible* in the sense that there exists a bounded linear operator $\widehat{O}$ (in our case the negative metric $-\eta$) with bounded inverse $\widehat{O}^{-1}$ in $\mathcal{H}^{(1)}$ that is hermitian with respect to ( , ) and satisfies

$$\langle f, g\rangle = (f, \widehat{O}g) . \tag{3.59}$$

Furthermore the Hilbert topology of a pseudo-Hilbert space is uniquely defined by the condition of compatibility. Hence it must be possible to construct a true Hilbert space of physical states from the given indefinite form.

The statistical interpretation of quantum mechanics implicitly requires the positivity of the scalar product (3.58). The lack of positivity can only mean that we have unphysical states of photons in the theory. The vector potential $A(x)$ as a four vector allows four degrees of freedom (independent polarizations). Two of them turn out to be spurious states.

Now we construct a physical Hilbert space with physical photons and a positive scalar product.

**Proposition 3.1** *Let $k$ and $q$ be four-vectors satisfying $k^2 = 0$ and $k \neq 0$.*

1. *From $kq = 0$ it follows that $q^2 \leq 0$.*
2. *From $kq = 0$ and $q^2 = 0$ it follows $q = ck$ with $c \in \mathbb{R}$.*

In the first step we define a subspace of the Hilbert space $\mathcal{H}^{(1)}$ by

$$\mathcal{H}^{(1)'} = \{f \in \mathcal{H}^{(1)} \mid k_\mu f^\mu(k) = 0\} . \tag{3.60}$$

By the first point of the proposition, the indefinite scalar product is semi-positive, $\langle f, f\rangle \geq 0$, in the space $\mathcal{H}^{(1)'}$ . However this scalar product is still degenerate. We have to factor out the zero-norm part to obtain a positive definite scalar product. Thus in a second step consider the space

$$\mathcal{H}^{(1)''} = \{f \in \mathcal{H}^{(1)} \mid k^\mu f^\nu(k) = k^\nu f^\mu(k)\} \tag{3.61}$$

which is a subset of $\mathcal{H}^{(1)'}$. By the second part of proposition 3.1 the space $\mathcal{H}^{(1)''}$ consists of the states $f \in \mathcal{H}^{(1)'}$ with $\langle f, f\rangle = 0$. The physical Hilbert space is then obtained as the quotient space

(3.62)

where the overline denotes the closure of the space. The full Hilbert space is the direct sum

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)} \tag{3.63}$$

where $\mathcal{H}^{(0)}$ is a one-dimensional Hilbert space and $\mathcal{H}^{(n)}$ is the symmetric tensor product $\mathcal{H}^{(n)} = (\mathcal{H}^{(1)\oplus n})_s$ of $\mathcal{H}^{(1)}$ with itself $n$ times [SW74]. The construction of the physical full Hilbert space is then similar to the one-particle case.

It is standard in quantum field theory to introduce the Fock space $\mathcal{F}_\mathcal{H}$ right from the start and then consider the subspace $\mathcal{F}_{\mathcal{H}'}$ comprising the states $\phi$ that satisfy $\partial^\mu A_\mu^{(+)}\phi = 0$. Then one still has to factor out the zero norm states and consider the quotient space [Roe98].

# Chapter 4

# The Maxwell Field in Curved Spacetime

## 4.1 Quantized Electromagnetic Field on a Manifold

In this section we present results obtained by J. DIMOCK [Dim92]. He carries out the quantization of the electromagnetic field on manifolds, as a generalization of his own results on scalar fields [Dim80] and Dirac fields [Dim82]. A similar approach is followed by E. FURLANI [Fur99]. We want to emphasize, however, that these papers only deal with simply connected spacetimes, since they assume the existence of a globally definable vector potential $A$. This is acceptable as far as one is interested in local effects; It is always possible to choose a local region of spacetime, which is simply connected; Then there exists a uniquely defined vector potential in that region. To work with a vector potential is even necessary, if one wants to introduce an interaction with other fields. It seems to be impossible to formulate a theory of interacting quantum electrodynamics without the use of the vector potential $A$.

### 4.1.1 Existence and uniqueness theorem

Henceforth we denote the electromagnetic gauge field and the field strength tensor on $\mathcal{M}$ by $\mathscr{A}$ and $\mathscr{F}$ respectively. The Maxwell equations on $\mathcal{M}$ then read

$$\mathrm{d}\mathrm{d}\mathscr{A} = 0 \;, \qquad \delta\mathrm{d}\mathscr{A} = 0 \;. \tag{4.1}$$

The first equation is trivial because $\mathrm{d}^2 = 0$ always. So we only have to deal with the second equation.

**Initial data.** Let $\Sigma$ be a compact Cauchy surface on $\mathcal{M}$ and $i : \Sigma \to \mathcal{M}$ the identity map. We define two maps $\rho_0(\mathscr{A}), \rho_1(\mathscr{A})$ in $\Omega^1(\Sigma)$:

$$\begin{aligned} \rho_0(\mathscr{A}) &= i^*(\mathscr{A}) \\ \rho_1(\mathscr{A}) &= (-1) * i^*(*\mathrm{d}\mathscr{A}) \;. \end{aligned} \tag{4.2}$$

The left star-operator in $\rho_1$ refers to the Manifold $(\Sigma,\gamma)$ where $\gamma = i^*g$ is negative definite. For the gauge-field $\mathscr{A}$ we define $A = \rho_0(\mathscr{A})$ and $\Pi = \rho_1(\mathscr{A})$:

$$\begin{aligned} A &= i^*(\mathscr{A}) \\ \Pi &= (-1) * i^*(*\mathrm{d}\mathscr{A}) \, . \end{aligned} \tag{4.3}$$

In the case of a scalar field $\varphi$, $\rho_0(\varphi)$ would be the restriction of $\varphi$ to $\Sigma$ and $\rho_1(\varphi)$ would be the forward normal derivative (see section 2.4.1). Here $\mathrm{d}A = i^*(\mathrm{d}\mathscr{A})$ represents the magnetic field as a 2-form on $\Sigma$, and $\Pi$ represents the electric field as a 1-form on $\Sigma$. $A$ and $\Pi$ are considered to be suitable data for the Cauchy problem.

It is easy to prove that $\delta\Pi = 0$ follows directly from $\delta\mathrm{d}\mathscr{A} = 0$. With (1.43) we have:

$$\begin{aligned} \delta\Pi &= *\mathrm{d}i^*(*\mathrm{d}\mathscr{A}) \\ &= *i^*(\mathrm{d} * \mathrm{d}\mathscr{A}) \\ &= *i^*(* * \mathrm{d} * \mathrm{d}\mathscr{A}) \\ &= *i^*(*\delta\mathrm{d}\mathscr{A}) \\ &= 0 \, . \end{aligned} \tag{4.4}$$

**Proposition 4.1 (Existence)** *For any $A, \Pi \in \Omega^1(\Sigma)$, with $\delta\Pi = 0$, there exists $\mathscr{A} \in \Omega^1(\mathcal{M})$ such that*

$$\delta\mathrm{d}\mathscr{A} = 0, \qquad \rho_0(\mathscr{A}) = A, \qquad \rho_1(\mathscr{A}) = \Pi \, . \tag{4.5}$$

*$A$ has compact support on any other Cauchy surface.*

Two potentials $\mathscr{A}, \mathscr{A}' \in \Omega^1(\mathcal{M})$ are said to be *gauge equivalent*, $\mathscr{A} \sim \mathscr{A}'$, if $\mathscr{A} = \mathscr{A}' + \mathrm{d}\lambda$ for some $\lambda \in \Omega^0(\mathcal{M})$. Gauge equivalence in $\Sigma$ is defined similarly.

**Proposition 4.2 (Uniqueness)**

1. *Let $A, \Pi \in \Omega^1(\Sigma)$ with $\delta\Pi = 0$. If $\mathscr{A}, \mathscr{A}' \in \Omega^1(\mathcal{M})$ are solutions of $\delta\mathrm{d}\mathscr{A} = 0$ with this data then $\mathscr{A} \sim \mathscr{A}'$.*

2. *Let $A, \Pi, A', \Pi' \in \Omega^1(\Sigma)$ with $\delta\Pi = \delta\Pi' = 0$. If $\mathscr{A}, \mathscr{A}'$ are solutions of $\delta\mathrm{d}\mathscr{A} = 0$ with these data then $A \sim A', \Pi \sim \Pi'$ if and only if $\mathscr{A} \sim \mathscr{A}'$.*

This proposition shows the well-posedness of the Cauchy problem for gauge equivalence classes. For an equivalence class $[A]$ in $\Omega^1(\Sigma)$ and $\Pi \in \Omega^1(\Sigma)$ with $\delta\Pi = 0$ there is a unique equivalence class $[\mathscr{A}]$ in $\Omega^1(\mathcal{M})$ such that $\delta\mathrm{d}[A] = 0, \rho_0([A]) = [A]$ and $\rho_1([\mathscr{A}]) = \Pi$.

**Proposition 4.3 (Fundamental Solutions)**

1. *Let $\mathscr{S} \in \Omega(\mathcal{M})$ with $\delta\mathscr{S} = 0$ and $\mathrm{supp}(\mathscr{S})$ compact to the past/future, then $\mathscr{A} = E^\pm\mathscr{S}$ solves $\delta\mathrm{d}\mathscr{A} = \mathscr{S}$.*

2. *If $\mathscr{A} \in \Omega^1(\mathcal{M}), \mathrm{supp}(\mathscr{A})$ is compact to the past/future, and $\delta\mathrm{d}\mathscr{A} = \mathscr{S}$ (so $\delta\mathscr{S} = 0$ and $\mathrm{supp}(\mathscr{S})$ compact to past/future), then $\mathscr{A} \sim E^\pm\mathscr{S}$.*

3. *$\mathscr{A} \in \Omega^1(\mathcal{M})$ satisfies $\delta\mathrm{d}\mathscr{A} = 0$ if and only if $\mathscr{A} \sim E^\pm\mathscr{S}$ for some $\mathscr{S} \in \Omega^1_c(\mathcal{M})$ with $\delta\mathscr{S} = 0$.*

### 4.1.2 Phase space for quantization

**Phase space on the Cauchy surface**

As we have shown in chapter 2 we need a phase space for quantization. The phase space needs to be a symplectic manifold $(\mathscr{P}, \sigma)$. First pick a compact Cauchy surface $\Sigma$ and define a phase space by

$$\mathscr{P}_{0\Sigma} = \{(A, \Pi) \in \Omega^1(\Sigma) \times \Omega^1(\Sigma) : \delta\Pi = 0\} . \tag{4.6}$$

This is simply the set of all points $A, \Pi$ for which the Maxwell equation $\delta \mathrm{d} A = 0$ holds. Furthermore we need a symplectic form on the phase space, which we could define by

$$\begin{aligned}\sigma_{0\Sigma}(A, \Pi; A', \Pi') &= \langle A, \Pi' \rangle - \langle \Pi, A' \rangle \\ &= \int_\Sigma A \wedge *\Pi' - \int_\Sigma \Pi \wedge *A' .\end{aligned} \tag{4.7}$$

This symplectic form is degenerate, since for $\delta\Pi = 0$ we have that $\langle \mathrm{d}\chi, \Pi\rangle = \langle \chi, \delta\Pi\rangle = 0$ even though $\mathrm{d}\chi \neq 0$. As a consequence of this the map $A \to \langle A, \Pi\rangle$ and the symplectic form $\sigma_{0\Sigma}$ are gauge invariant. One can remove the degeneracy by going to equivalence classes.

**Definition 4.4** Let $\mathscr{P}_\Sigma$ denote an equivalence class in $\mathscr{P}_{0\Sigma}$; a point in $\mathscr{P}_\Sigma$ is a pair $([A], \Pi)$ with $\delta\Pi = 0$. We define the symplectic form $\sigma_\Sigma$:

$$\sigma_\Sigma([A], \Pi; [A'], \Pi') = \langle [A], \Pi' \rangle - \langle \Pi, [A'] \rangle . \tag{4.8}$$

**Proposition 4.5** *$\sigma_\Sigma$ is (weakly) non-degenerate on $\mathscr{P}_\Sigma$.*

This proposition means that $(\mathscr{P}_\Sigma, \sigma_\Sigma)$ is a suitable phase space for the Maxwell equations on the Cauchy surface $\Sigma$.

**Dynamical phase space**

It is necessary to reformulate these definitions in order to incorporate time evolution. Let $\mathscr{P}_0$ be the space of all solutions to Maxwell's equations in $\mathcal{M}$:

$$\mathscr{P}_0 = \{\mathscr{A} \in \Omega^1(\mathcal{M}) : \delta \mathrm{d}\mathscr{A} = 0\} . \tag{4.9}$$

On $\mathscr{P}_0 \times \mathscr{P}_0$ we define a new symplectic form $\sigma$ for any Cauchy surface $\Sigma$:

$$\sigma(\mathscr{A}, \mathscr{A}') = \int_\Sigma i^*[\mathscr{A} \wedge *\mathrm{d}\mathscr{A}' - \mathscr{A}' \wedge *\mathrm{d}\mathscr{A}] . \tag{4.10}$$

Let $\rho_\Sigma : \mathscr{P}_0 \to \mathscr{P}_\Sigma$ map a solution to its data on $\Sigma$:

$$\rho_\Sigma(\mathscr{A}) = (\rho_0(\mathscr{A}), \rho_1(\mathscr{A})) = (i^*\mathscr{A}, (-1) * i^* * \mathrm{d}\mathscr{A}) . \tag{4.11}$$

The new form is connected to the old form by $\sigma = \sigma_\Sigma \circ \rho_\Sigma$.

$$\sigma(\mathscr{A}, \mathscr{A}') = \sigma_\Sigma(\rho_\Sigma(\mathscr{A}), \rho_\Sigma(\mathscr{A}')) \ . \tag{4.12}$$

Let $\mathscr{P}$ be the gauge equivalence class in $\mathscr{P}_0$. The gauge invariance of $\rho_\Sigma$ leads to gauge invariance for $\sigma$:

$$\sigma([\mathscr{A}], [\mathscr{A}']) = \sigma_\Sigma(\rho_\Sigma([\mathscr{A}]), \rho_\Sigma([\mathscr{A}'])) \ . \tag{4.13}$$

We can now state that $(\mathscr{P}, \sigma)$ is a symplectic space and $\rho_\Sigma : (\mathscr{P}, \sigma) \to (\mathscr{P}_\Sigma, \sigma_\Sigma)$ is a symplectic isomorphism. The new representation has a crucial advantage over the old one:

**Proposition 4.6** *$\sigma$ defined by* (4.10) *is independent of* $\Sigma$.

Time evolution on $\mathcal{M}$ can be regarded as the symplectic isomorphism $\tau_{\Sigma_1,\Sigma_2} = \rho_{\Sigma_2}\rho_{\Sigma_1}^{-1}$ from $(\mathscr{P}_{\Sigma_1}, \sigma_{\Sigma_1})$ to $(\mathscr{P}_{\Sigma_2}, \sigma_{\Sigma_2})$.

In the representation $(\mathscr{P}, \sigma)$ we consider functions on $\mathscr{P}$ of the form $[\mathscr{A}] \to \sigma([\mathscr{A}], [u])$ for $[u] \in \mathscr{P}$ and find the Poisson bracket

$$\{\sigma([\mathscr{A}], [u]), \sigma([\mathscr{A}], [u'])\} = \sigma([u], [u']) \ . \tag{4.14}$$

Any solution $\mathscr{S} \in \Omega^1_c$ with $\delta\mathscr{S} = 0$ can be expressed in terms of its data on a Cauchy surface:

**Proposition 4.7** *For $[\mathscr{A}] \in \mathscr{P}, \mathscr{S} \in \Omega^1_c(\mathcal{M})$ and $\delta\mathscr{S} = 0$*

$$\langle [\mathscr{A}], \mathscr{S} \rangle = \sigma([\mathscr{A}], [E\mathscr{S}]) \ . \tag{4.15}$$

### 4.1.3 Quantization

To quantize we have to replace functions on $(\mathscr{P}, \sigma)$ by self-adjoint operators on a complex Hilbert space. These operators have to be chosen in such a way that the Poisson bracket becomes $(-i)$ times the commutator. For this we need operators $\widehat{\sigma}([\mathscr{A}], [u])$ on some Hilbert space $\mathcal{H}$ with $[u] \in \mathscr{P}$ satisfying

$$[\widehat{\sigma}([\mathscr{A}], [u]), \widehat{\sigma}([\mathscr{A}], [u'])] = -i\sigma([u], [u']) \tag{4.16}$$

on some dense domain.

The field operators are quantizations of the functions $[\mathscr{A}] \to \langle [\mathscr{A}], \mathscr{S} \rangle$ for $\mathscr{S} \in \Omega^1_c(\mathcal{M}), \delta\mathscr{S} = 0$. The corresponding operator is given by

$$[\mathscr{A}](\mathscr{S}) = \widehat{\sigma}([\mathscr{A}], [E\mathscr{S}]) \ . \tag{4.17}$$

**Proposition 4.8** *For $\mathscr{S} \in \Omega^1_c(\mathcal{M}), \delta\mathscr{S} = 0$.*

1. *$[\mathscr{A}](\mathscr{S})$ satisfies Maxwell's equation $\delta \mathrm{d}[\mathscr{A}] = 0$ in the weak sense that $[\mathscr{A}](\delta\mathrm{d}[\mathscr{S}]) = 0$*

2. $[[\mathscr{A}](\mathscr{S}), [\mathscr{A}](\mathscr{S}')] = i\langle \mathscr{S}, E\mathscr{S}'\rangle$. *In particular if* $\operatorname{supp}\mathscr{S}, \operatorname{supp}\mathscr{S}'$ *are space-like separated the commutator is zero.*

The field strength tensor is defined by $\mathscr{F} = \mathrm{d}[\mathscr{A}]$. The smeared out field operator is then $\mathscr{F}(\omega) = \mathrm{d}[\mathscr{A}](\omega) = [\mathscr{A}](\delta\omega)$ for $\omega \in \Omega^2_c(\mathscr{M})$. The Maxwell equation reads $\delta\mathscr{F}(\theta) = \mathscr{F}(\mathrm{d}\theta) = 0$ for any $\theta \in \Omega^1_c(\mathscr{M})$. One finds the Lichnerowicz commutator [Lic61] for $\mathscr{F}$:

$$[\mathscr{F}(\omega), \mathscr{F}(\omega')] = i\langle \omega, \mathrm{d}E\delta\omega'\rangle \ . \tag{4.18}$$

**Algebraic construction**

As explained before we can pass to an exponential form of the commutation relations introducing

$$W([u]) = \exp\left(i\widehat{\sigma}([\mathscr{A}], [u])\right) \ . \tag{4.19}$$

A representation is now a collection of unitary operators $W([u]), u \in \mathscr{P}$ on some Hilbert space satisfying the Weyl relation

$$W([u])W([u']) = W([u] + [u']) \exp\left(-\frac{1}{2} i\sigma([u], [u'])\right) \tag{4.20}$$

The exponential of the field operator is defined by

$$\exp(i[\mathscr{A}](\mathscr{S})) = W([E\mathscr{S}]) \ . \tag{4.21}$$

Let $\mathfrak{A}$ be the $C^*$-algebra generated by $\exp\left(i[\mathscr{A}](\mathscr{S})\right)$. The algebraic structure of the pair $\left(\exp(i[\mathscr{A}](\mathscr{S})), \mathfrak{A}\right)$ is independent of the representation:

**Proposition 4.9** *If* $W_1, W_2$ *are two representations giving rise to* $(\exp(i[\mathscr{A}](\mathscr{S}))_1, \mathfrak{A}_1)$ *and* $(\exp(i[\mathscr{A}](\mathscr{S}))_2, \mathfrak{A}_2)$*, then there is a* $^*$*-isomorphism* $\alpha : \mathfrak{A}_1 \to \mathfrak{A}_2$ *such that* $\alpha(\exp(i[\mathscr{A}](\mathscr{S}))_1) = \exp(i[\mathscr{A}](\mathscr{S}))_2$.

## 4.2 Cauchy Problem for $F_{\mu\nu}$

In our review we have seen that there is a completely satisfying theory for the electromagnetic vector potential on curved manifolds. But as already pointed out we are not satisfied with this theory, because of the restriction to topologically trivial spacetimes. Schwarzschild-Kruskal spacetime is an important example for a topologically non-trivial spacetime but not the only one. C. MISNER and J. WHEELER describe wormhole solutions to Einstein's equations [MW57]. They interpret charge in terms of source-free electromagnetic fields that are subject to Maxwell's equations for free space but which are trapped in the "worm holes" of a space with multiply connected topology. On the quantum level the influence of these wormholes manifests itself via the presence of spontaneous symmetry breaking and non-Fock representations of the canonical commutation relations and leads to superselection rules for electric and magnetic charges [AS80]. Hence it is important to develop a theory of a directly quantized field strength tensor. As a prerequisite for this, we shall now solve the Cauchy Problem for the field strength tensor.

#### Well posedness

In classical electrodynamics the evolution of a physical system is uniquely determined, if initial values for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are given (section 3.2). In the free field situation these values are only subject to the constraints $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$. We see that two Maxwell equations form the constraints, while the other equations, $\nabla \times \mathbf{B} - \partial_t \mathbf{E} = 0$ and $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$ determine the dynamics. A theory which has uniquely determined solutions, depending on appropriate initial data, possibly subject to constraints, is said to possesses an *initial value formulation*. Two more properties are needed if the theory is to be physical. First, one requires that small changes in the initial data produces only small changes in the solution over any fixed compact region of spacetime. Second, causality must be preserved, that is changes in the initial data in a region $S$ should not produce changes outside the the causal future $J^+(S)$, of this region. A theory which possesses an initial value formulation and satisfies these additional requirements has a *well posed* initial value formulation [Wala].

### 4.2.1 The wave equation in curved spacetime

Let $(\mathcal{M}, g_{\mu\nu})$ be a Lorentz manifold. The *d' Alembertian* $\Box$ on a Lorentz manifold is defined in terms of the operators $\mathrm{d}$ and $\delta$ as

$$\Box = \mathrm{d}\delta + \delta\mathrm{d}. \tag{4.22}$$

Since the Maxwell tensor satisfies $\mathrm{d}\mathscr{F} = 0$ and $\delta\mathscr{F} = 0$ it it also satisfies the *wave equation* on 2-forms

$$\Box\mathscr{F} = 0. \tag{4.23}$$

In a flat spacetime with a Cartesian local coordinate system this reduces to

$$\partial^\lambda \partial_\lambda \mathscr{F}_{\mu\nu} = 0\ . \tag{4.24}$$

#### Explicit d'Alembertian

On curved spacetimes the metric has nontrivial entries and consequently the d'Alembert operator has no longer the simple form (4.24). The wave equation gains extra terms, depending on the order of the $p$-form considered. We give here a rigorous derivation of the d'Alembert operator $\Box$ in local coordinates, acting on an arbitrary $p$-form, on an arbitrary differentiable manifold, following [Lic]. Our result only differs in sign from A. LICHNEROWICZ and is adapted to our notation. Then we apply the solution to 2-forms. The resulting equation determines the Cauchy problem for the Maxwell field tensor in curved spacetimes.

Let us first calculate the expressions for $\mathrm{d}$ and $\delta$ acting on a $p$-form. Note that the partial derivatives used to define the exterior derivative in (1.39), can be replaced with covariant derivatives, since the operator $\mathrm{d}$ is covariant. This can be demonstrated by choosing the Christoffel connection: the contribution of Christoffel symbols, that appear in the covariant derivatives, vanish upon antisymmetrization. So we have

$$(\mathrm{d}A)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\nabla_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}\ . \tag{4.25}$$

Given a $p$-form $A$, using (1.22), we can write:

$$(\mathrm{d}A)_{\mu_1\ldots\mu_{p+1}} = \frac{1}{p!}\delta^{\kappa\nu_1\ldots\nu_p}_{\mu_1\ldots\mu_{p+1}}\nabla_\kappa A_{\nu_1\ldots\nu_p} \ . \tag{4.26}$$

For the coderivative of a $p$-form a very short notation is obtained:

$$(\delta A)_{\mu_2\ldots\mu_p} = \nabla_\lambda A^\lambda{}_{\mu_2\ldots\mu_p} \ . \tag{4.27}$$

**The operator $\delta\mathrm{d}$.** Putting (4.26) and (4.27) together yields

$$(\delta\mathrm{d}A)_{\mu_1\ldots\mu_p} = \frac{1}{p!}\delta^{\kappa\nu_1\ldots\nu_p}_{\lambda\mu_1\ldots\mu_p}\nabla^\lambda\nabla_\kappa A_{\nu_1\ldots\nu_p} \ . \tag{4.28}$$

Now we sorts terms within the sum. The term with $\lambda = \kappa$ is the diagonal term

$$\nabla^\kappa\nabla_\kappa A_{\mu_1\ldots\mu_p} \ . \tag{4.29}$$

If $\lambda$ is equal to one of the $\mu_1$, we obtain

$$-\frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}\nabla^\lambda\nabla_\kappa A_{\lambda\nu_2\ldots\nu_p}. \tag{4.30}$$

Thus the whole expression is

$$(\delta\mathrm{d}A)_{\mu_1\ldots\mu_p} = \nabla^\kappa\nabla_\kappa A_{\mu_1\ldots\mu_p} - \frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}\nabla_\lambda\nabla_\kappa A^\lambda{}_{\nu_2\ldots\nu_p} \ . \tag{4.31}$$

**The operator $\mathrm{d}\delta$.** For this operator we readily find

$$(\mathrm{d}\delta A)_{\mu_1\ldots\mu_p} = \frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}\nabla_\kappa\nabla_\lambda A^\lambda{}_{\nu_2\ldots\nu_p} \ . \tag{4.32}$$

**The operator $\Box$.**

$$\begin{aligned}(\Box A)_{\mu_1\ldots\mu_n} &= \nabla^\kappa\nabla_\kappa A_{\mu_1\ldots\mu_p} \\ &\quad - \frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}\nabla_\lambda\nabla_\kappa A^\lambda{}_{\nu_2\ldots\nu_p} \\ &\quad + \frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}\nabla_\kappa\nabla_\lambda A^\lambda{}_{\nu_2\ldots\nu_p} \\ &= \nabla^\kappa\nabla_\kappa A_{\mu_1\ldots\mu_p} - \frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}[\nabla_\lambda,\nabla_\kappa]A^\lambda{}_{\nu_2\ldots\nu_p} \ .\end{aligned} \tag{4.33}$$

Here we have the commutator of two covariant derivatives. This means now the curvature comes in via (1.13):

$$[\nabla_\lambda,\nabla_\kappa]A^\lambda{}_{\nu_2\ldots\nu_p} = R^\lambda{}_{\rho\lambda\kappa}A^\rho{}_{\nu_2\ldots\nu_p} - \sum_{i=2}^{p} R^\rho{}_{\nu_i\lambda\kappa}A^\lambda{}_{\nu_2\ldots\nu_{i-1}\ \rho\ \nu_{i+1}\ldots\nu_p} \ . \tag{4.34}$$

The Tensor $R^{\lambda}{}_{\rho\kappa\lambda}$ is the Riemann tensor which, by contraction, reduces to the Ricci Tensor: $R^{\lambda}{}_{\rho\lambda\kappa} = R_{\rho\kappa}$. We now have

$$
\begin{aligned}
(\Box A)_{\mu_1\ldots\mu_n} &= \nabla^{\kappa}\nabla_{\kappa}A_{\nu_1\ldots\nu_p} \\
&\quad - \frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}[\nabla_\lambda, \nabla_\kappa]A^{\lambda}{}_{\nu_2\ldots\nu_p} \\
&= \nabla^{\kappa}\nabla_{\kappa}A_{\nu_1\ldots\nu_p} \\
&\quad - \frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}(R^{\lambda}{}_{\rho\lambda\kappa}A^{\rho}{}_{\nu_2\ldots\nu_p} - \sum_{i=2}^{p} R^{\rho}{}_{\nu_i\lambda\kappa}A^{\lambda}{}_{\nu_2\ldots\nu_{i-1}\;\rho\;\nu_{i+1}\ldots\nu_p}) \\
&= \nabla^{\kappa}\nabla_{\kappa}A_{\nu_1\ldots\nu_p} \\
&\quad - (\frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}R_{\rho\kappa}A^{\rho}{}_{\nu_2\ldots\nu_p} + T_{\mu_1\ldots\mu_n})\ ,
\end{aligned}
\tag{4.35}
$$

where we have set

$$
T_{\mu_1\ldots\mu_n} = -\frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}\sum_{i=2}^{p} R^{\rho}{}_{\nu_i\lambda\kappa}A^{\lambda}{}_{\nu_2\ldots\nu_{i-1}\;\rho\;\nu_{i+1}\ldots\nu_p}\ . \tag{4.36}
$$

Let us calculate one of the terms:

$$
\begin{aligned}
(\text{one term of } T_{\mu_1\ldots\mu_n}) &= -\frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_{i-1}\;\nu_i\;\nu_{i+1}\ldots\nu_p}_{\mu_1\ldots\mu_p}R^{\rho}{}_{\nu_i\lambda\kappa}A^{\lambda}{}_{\nu_2\ldots\nu_{i-1}\;\rho\;\nu_{i+1}\ldots\nu_p} \\
&= -\frac{(-1)^{2(i-2)}}{(p-1)!}\delta^{\kappa\nu_i\nu_2\ldots\nu_{i-1}\nu_{i+1}\ldots\nu_p}_{\mu_1\ldots\mu_p}R^{\rho}{}_{\nu_i\lambda\kappa}A^{\lambda}{}_{\rho\nu_2\ldots\nu_{i-1}\nu_{i+1}\ldots\nu_p} \\
&= -\frac{1}{(p-1)!}\delta^{\kappa\sigma\nu_2\ldots\nu_{i-1}\nu_{i+1}\ldots\nu_p}_{\mu_1\ldots\mu_p}R^{\rho}{}_{\sigma\lambda\kappa}A^{\lambda}{}_{\rho\nu_2\ldots\nu_{i-1}\nu_{i+1}\ldots\nu_p} \\
&= -\frac{1}{(p-1)!}\delta^{\kappa\sigma\nu_3\ldots\nu_p}_{\mu_1\ldots\mu_p}R_{\rho\sigma\lambda\kappa}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p}\ .
\end{aligned}
\tag{4.37}
$$

We can do the same with any of the $(p-1)$ terms in the sum, obtaining the same result. Adding all these terms up gives

$$
\begin{aligned}
T_{\mu_1\ldots\mu_n} &= -\frac{p-1}{(p-1)!}\delta^{\kappa\sigma\nu_3\ldots\nu_p}_{\mu_1\ldots\mu_p}R_{\rho\sigma\lambda\kappa}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} \\
&= -\frac{1}{(p-2)!}\delta^{\kappa\sigma\nu_3\ldots\nu_p}_{\mu_1\ldots\mu_p}R_{\rho\sigma\lambda\kappa}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} \\
&= -\frac{1}{(p-2)!}\delta^{\kappa\sigma\nu_3\ldots\nu_p}_{\mu_1\ldots\mu_p}R_{\kappa\lambda\sigma\rho}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p}\ .
\end{aligned}
\tag{4.38}
$$

By the properties of the Riemann tensor we have:

$$\begin{aligned}
R_{\kappa\lambda\sigma\rho}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} &= \frac{1}{2}(R_{\kappa\lambda\sigma\rho}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} + R_{\kappa\lambda\sigma\rho}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p}) \\
&= \frac{1}{2}(R_{\kappa\lambda\sigma\rho}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} - R_{\kappa\lambda\sigma\rho}A^{\rho\lambda}{}_{\nu_3\ldots\nu_p}) \\
&= \frac{1}{2}(R_{\kappa\lambda\sigma\rho}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} - R_{\kappa\rho\sigma\lambda}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p}) \\
&= \frac{1}{2}(R_{\kappa\lambda\sigma\rho} + R_{\kappa\rho\lambda\sigma})A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} \\
&= \frac{1}{2}(-R_{\kappa\sigma\rho\lambda})A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} \\
&= \frac{1}{2}R_{\kappa\sigma\lambda\rho}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} \ .
\end{aligned} \tag{4.39}$$

This yields

$$T_{\mu_1\ldots\mu_n} = -\frac{1}{2(p-2)!}\delta^{\kappa\sigma\nu_3\ldots\nu_p}_{\mu_1\ldots\mu_p}R_{\kappa\sigma\lambda\rho}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} \ . \tag{4.40}$$

And thus the whole expression for $\Box A$ is

$$\begin{aligned}
(\Box A)_{\mu_1\ldots\mu_n} &= \nabla^\kappa\nabla_\kappa A_{\mu_1\ldots\mu_p} \\
&\quad - (\frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}R_{\kappa\rho}A^{\rho}{}_{\nu_2\ldots\nu_p} + T_{\mu_1\ldots\mu_n}) \\
&= \nabla^\kappa\nabla_\kappa A_{\mu_1\ldots\mu_p} \\
&\quad - \frac{1}{(p-1)!}\delta^{\kappa\nu_2\ldots\nu_p}_{\mu_1\ldots\mu_p}R_{\kappa\rho}A^{\rho}{}_{\nu_2\ldots\nu_p} \\
&\quad + \frac{1}{2(p-2)!}\delta^{\kappa\sigma\nu_3\ldots\nu_p}_{\mu_1\ldots\mu_p}R_{\kappa\sigma\lambda\rho}A^{\lambda\rho}{}_{\nu_3\ldots\nu_p} \ .
\end{aligned} \tag{4.41}$$

**The wave equation for the Maxwell field.** We apply (4.41) to the 2-form $\mathscr{F}_{\mu\nu}$ and obtain:

$$\begin{aligned}
(\Box\mathscr{F})_{\mu\nu} &= \nabla^\kappa\nabla_\kappa\mathscr{F}_{\mu\nu} \\
&\quad - \frac{1}{(2-1)!}\delta^{\kappa\nu_2}_{\mu\nu}R_{\kappa\rho}\mathscr{F}^{\rho}{}_{\nu_2} \\
&\quad + \frac{1}{2(2-2)!}\delta^{\kappa\sigma}_{\mu\nu}R_{\kappa\sigma\lambda\rho}\mathscr{F}^{\lambda\rho} \\
&= \nabla^\kappa\nabla_\kappa\mathscr{F}_{\mu\nu} \\
&\quad - (\delta^\kappa_\mu\delta^{\nu_2}_\nu - \delta^\kappa_\nu\delta^{\nu_2}_\mu)R_{\kappa\rho}\mathscr{F}^{\rho}{}_{\nu_2} \\
&\quad + \frac{1}{2}(\delta^\kappa_\mu\delta^\sigma_\nu - \delta^\kappa_\nu\delta^\sigma_\mu)R_{\kappa\sigma\lambda\rho}\mathscr{F}^{\lambda\rho} \ .
\end{aligned} \tag{4.42}$$

And finally

$$(\Box\mathscr{F})_{\mu\nu} = \nabla^\kappa\nabla_\kappa\mathscr{F}_{\mu\nu} - R_{\mu\rho}\mathscr{F}^{\rho}{}_{\nu} - R_{\nu\rho}\mathscr{F}_{\mu}{}^{\rho} + R_{\mu\nu\lambda\rho}\mathscr{F}^{\lambda\rho} \ . \tag{4.43}$$

### 4.2.2 Initial values

Let $(\mathcal{M}, g_{\mu\nu})$ be be a globally hyperbolic manifold. Global hyperbolicity guarantees the existence of a smooth time coordinate $t$ on $\mathcal{M}$. As a consequence $\mathcal{M}$ admits a foliation by a one parameter family of Cauchy surfaces $\{t\} \times \Sigma_t$, that is topologically we have $\mathcal{M} = \mathbb{R} \times \Sigma$ [Wala, Walb]. We denote by $\mathscr{F}$ the field strength as a 2-form on $\mathcal{M}$. Maxwell's equations are $\mathrm{d}\mathscr{F} = 0$ and $\delta\mathscr{F} = 0$. On $\mathcal{M}$ we have the injection map $i : \Sigma \to \mathcal{M}$. We define two maps, the pullback $\rho_0(\mathscr{F})$ and the forward normal $\rho_n(\mathscr{F})$ by

$$\begin{aligned} \rho_{(0)} : \Omega^2(\mathcal{M}) &\to \Omega^2(\Sigma) \\ \mathscr{F} &\mapsto i^*(\mathscr{F}) \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} \rho_{(n)} : \Omega^2(\mathcal{M}) &\to \Omega^1(\Sigma) \\ \mathscr{F} &\mapsto *i^*(*\mathscr{F}) \ . \end{aligned} \tag{4.45}$$

The quantities $B = \rho_{(0)}(\mathscr{F})$ and $E = \rho_{(n)}(\mathscr{F})$ are considered to be the Cauchy data on $\Sigma$ [Fur99]:

$$B = i^*(\mathscr{F}) \tag{4.46}$$

$$E = *i^*(*\mathscr{F}) \ . \tag{4.47}$$

**Initial constraints.** Since $E$ and $B$ should serve as initial data they have to satisfy initial constraints. Indeed we find:

$$\begin{aligned} \delta E &= *\mathrm{d} * *i^*(*\mathscr{F}) \\ &= - * \mathrm{d}i^*(*\mathscr{F}) \\ &= - * i^*(\mathrm{d} * \mathscr{F}) \\ &= - * i^*(* * \mathrm{d} * \mathscr{F}) \\ &= - * i^*(*\delta\mathscr{F}) \\ &= 0 \end{aligned} \tag{4.48}$$

and

$$\begin{aligned} \mathrm{d}B &= \mathrm{d} * i^*(\mathscr{F}) \\ &= i^*(\mathrm{d}\mathscr{F}) \\ &= 0 \ . \end{aligned} \tag{4.49}$$

**The field components on $\Sigma$.** For the formulation of the existence and uniqueness theorem we need expressions of $B$ and $E$ in local coordinates on $\Sigma$. The desired expressions are

$$B_{ij} = \mathscr{F}_{ij} \tag{4.50}$$

$$E^i = N\mathscr{F}^{0i} \ . \tag{4.51}$$

The first is clearly the pullback of $\mathscr{F}$. The second needs a little bit of calculation. Note that the 3+1-splitting gives $N = \sqrt{|g|}/\sqrt{\gamma}$ (1.64). Furthermore on $\Sigma$ we have $\tilde{\epsilon}^{ijk}\tilde{\epsilon}_{ijl} = 2!\delta^k_l$ from (1.31). Using these relations we can calculate $E_l$:

$$\begin{aligned}
E_l &= *i^* * \mathscr{F}_{\mu\nu} \\
&= *i^* \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathscr{F}^{\mu\nu} \\
&= *i^* \frac{1}{2}\sqrt{|g|}\tilde{\epsilon}_{\mu\nu\rho\sigma}\mathscr{F}^{\mu\nu} \\
&= *\frac{1}{2}N\sqrt{\gamma}\tilde{\epsilon}_{\mu\nu jk}\mathscr{F}^{\mu\nu} \\
&= *N\sqrt{\gamma}\tilde{\epsilon}_{0ijk}\mathscr{F}^{0i} \\
&= \frac{1}{2}N\sqrt{\gamma}\epsilon^{jk}{}_l\tilde{\epsilon}_{0ijk}\mathscr{F}^{0i} \\
&= \frac{1}{2}N\frac{\sqrt{\gamma}}{\sqrt{\gamma}}\tilde{\epsilon}^{l'jk}\tilde{\epsilon}_{0ijk}\mathscr{F}^{0i}\gamma_{ll'} \\
&= N\mathscr{F}^{0i}\gamma_{ll'}\delta^{l'}_i \\
&= N\mathscr{F}^{0i}\gamma_{li}\ .
\end{aligned} \tag{4.52}$$

We can easily check if our definition makes sense in Minkowski space. Using the obvious relations $\mathscr{F}_{i0} = \mathscr{F}^{0i}, N = 1$ and noting that $E_i = \gamma_{li}E^l = E^l$ we find $E_i = \mathscr{F}_{i0}$, which is exactly (3.7).

**The normal derivatives on $\Sigma$.** Now we have the values of the field strength components on any Cauchy surface. What we need next are the values of $n^\mu\nabla_\mu\mathscr{F}_{\rho\sigma}$ on the Cauchy surface. We follow Sorkin's treatment [Sor79], but refrain from introducing tensor densities. The idea of the following treatment is to distinguish between time and space coordinates in the Maxwell equations $\mathrm{d}\mathscr{F} = 0$ and $\delta\mathscr{F} = 0$, in order to obtain two initial value constraint and two evolution equations similar to the ones familiar from flat spacetime. But our equations will contain information about the curvature of space.

Consider the sourceless Maxwell equations:

$$\partial_\lambda\mathscr{F}_{\mu\nu} + \partial_\mu\mathscr{F}_{\nu\lambda} + \partial_\nu\mathscr{F}_{\lambda\mu} = 0, \tag{4.53}$$

$$\partial_\mu\mathscr{F}^{\mu\nu} = 0\ . \tag{4.54}$$

Each of these equations gives one initial constraint and one evolution equation. For this one has to separate the time derivatives from the rest. For (4.53) we obtain by this procedure

$$\partial_i\mathscr{F}_{jk} + \partial_j\mathscr{F}_{ki} + \partial_k\mathscr{F}_{ij} = 0 \quad \text{(constraint)} \tag{4.55}$$

$$\partial_0\mathscr{F}_{jk} + \partial_j\mathscr{F}_{k0} + \partial_k\mathscr{F}_{0j} = 0 \quad \text{(evolution)}\ . \tag{4.56}$$

Similarly for (4.54) we have

$$\partial_i\mathscr{F}^{i0} = 0 \quad \text{(constraint)} \tag{4.57}$$

$$\partial_0\mathscr{F}^{0j} + \partial_i\mathscr{F}^{ij} = 0 \quad \text{(evolution)}\ . \tag{4.58}$$

Inserting (4.50) and (4.51) into the constraint equations (4.55) and (4.57) leads to

$$\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} = 0 \tag{4.59}$$

$$\partial_i \frac{E^i}{N} = 0\ . \tag{4.60}$$

With the same insertions the evolution equations (4.56) (4.58) become

$$\partial_0 B_{jk} + \partial_j \mathscr{F}_{k0} - \partial_k \mathscr{F}_{j0} = 0, \tag{4.61}$$

$$\partial_0 \frac{E^j}{N} + \partial_i \mathscr{F}^{ij} = 0. \tag{4.62}$$

Our aim is to describe the evolution of the field completely by objects defined on the Cauchy surface $\Sigma$. Hence, we need to express $\mathscr{F}_{k0}$ and $\mathscr{F}^{ij}$ in terms of $B$ and $E$. First we note that

$$\mathscr{F}^{0m} g_{mk} = -\frac{E^m}{N}\gamma_{mk} = -\frac{E_k}{N} \tag{4.63}$$

and

$$\mathscr{F}^{0m} g_{mk} = g^{00}\mathscr{F}_{0k} + g^{0j}\mathscr{F}_{jk}\ . \tag{4.64}$$

Subtracting (4.63) from (4.64), dividing by $g^{00}$ and reordering yields

$$\frac{1}{g^{00}}\frac{E_k}{N} = -\mathscr{F}_{0k} - \frac{g^{0j}}{g^{00}}\mathscr{F}_{jk} \tag{4.65}$$

$$N E_k = \mathscr{F}_{k0} + N^j B_{jk} \tag{4.66}$$

leading to

$$\mathscr{F}_{k0} = N E_k - N^j B_{jk}\ . \tag{4.67}$$

We have now expressed $\mathscr{F}_{k0}$ by $E$ and $B$. Let us try to do the same with $\mathscr{F}^{jk}$. First decompose $\mathscr{F}_{jk}$:

$$\begin{aligned}
\mathscr{F}_{jk} &= g_{j\mu} g_{k\nu} \mathscr{F}^{\mu\nu} \\
&= g_{jm} g_{kn} \mathscr{F}^{mn} + g_{j0} g_{kn} \mathscr{F}^{0n} + g_{k0} g_{jn} \mathscr{F}^{n0} \\
&= g_{jm} g_{kn} \mathscr{F}^{mn} + g_{j0} g_{kn} \mathscr{F}^{0n} - g_{k0} g_{jn} \mathscr{F}^{0n}\ .
\end{aligned} \tag{4.68}$$

Next use (1.59)(1.65) and (4.51) to obtain

$$\begin{aligned}
\mathscr{F}_{jk} &= \gamma_{jm}\gamma_{kn}\mathscr{F}^{mn} + N_j \gamma_{kn}\frac{E^n}{N} - N_k \gamma_{jn}\frac{E^n}{N} \\
&= \gamma_{jm}\gamma_{kn}\mathscr{F}^{mn} + N_j \frac{E_k}{N} - N_k \frac{E_j}{N}\ .
\end{aligned} \tag{4.69}$$

Using (4.50) and the wedge product this becomes

$$B_{jk} = \gamma_{jm}\gamma_{kn}\mathscr{F}^{mn} + N_j \wedge \frac{E_k}{N} \tag{4.70}$$

or equivalently

$$\mathscr{F}^{jk} = B^{jk} - N^j \wedge \frac{E^k}{N} \; . \tag{4.71}$$

This is the desired equation for $\mathscr{F}^{jk}$. By putting (4.67) and (4.71) into (4.61) and (4.62) we find the interim solution:

$$\frac{\partial}{\partial t} B_{jk} + \partial_j \wedge (NE_k - N^i B_{ik}) = 0 \tag{4.72}$$

$$\frac{\partial}{\partial t}\frac{E^k}{N} + \partial_j (B^{jk} + N^j \wedge \frac{E^k}{N}) = 0 \; . \tag{4.73}$$

It is easily seen that we have here a generalization of the flat spacetime case. If we choose the vector field $t^\mu$ to be a perpendicular unit vector field to $\Sigma$, then $N^i = 0$ and $N = 1$. This leads to

$$\frac{\partial}{\partial t} B_{jk} = -\partial_j \wedge E_k \tag{4.74}$$

$$\frac{\partial}{\partial t} E^k = \partial_j B^{kj} \; . \tag{4.75}$$

These are clearly the ordinary free Maxwell equations (3.1)(3.3). Evaluating (4.72) and (4.73) with the help of (1.62) yields

$$t^\mu \nabla_\mu B_{jk} + \partial_j \wedge (NE_k - N^i B_{ik}) = 0 \tag{4.76}$$

$$t^\mu \nabla_\mu \frac{E^k}{N} + \partial_j (B^{jk} + N^j \wedge \frac{1}{N} E^k) = 0 \; . \tag{4.77}$$

Now setting $t^\mu = Nn^\mu + N^\mu$ we have the final result

$$n^\mu \nabla_\mu B_{jk} = -\frac{1}{N}\partial_j \wedge (NE_k - N^i B_{ik}) - \frac{1}{N} N^l \nabla_l B_{jk} \tag{4.78}$$

$$n^\mu \nabla_\mu \frac{E^k}{N} = -\frac{1}{N}\partial_j (B^{jk} + N^j \wedge \frac{1}{N} E^k) - \frac{1}{N} N^l \nabla_l \frac{E^k}{N} \; . \tag{4.79}$$

**A note on SORKIN's result.** In [Sor79] R. SORKIN works with initial values $B_{ij} = F_{ij}$ and $\mathscr{E}^k = \mathscr{F}^{0k}$, where $F_{\mu\nu}$ is the Maxwell field, $\mathscr{F}^{\mu\nu} = \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$. By interpreting $\partial/\partial t$ as the Lie-derivative $\pounds_t$, setting $\mathbf{N} = (N^1, N^2, N^3)$, $\partial \lrcorner \mathscr{F} = \partial_\mu \mathscr{F}^{\mu\nu}$ and a few more technical assumptions he obtains the index-free notation

$$\pounds_t B + \partial \wedge (NE - \mathbf{N} \lrcorner B) = 0 \tag{4.80}$$

$$\pounds_t \mathscr{E} + \partial \lrcorner (N\mathscr{B} - \mathbf{N} \wedge \mathscr{E}) = 0 \; . \tag{4.81}$$

### 4.2.3 Existence and uniqueness

We are now ready to formulate the existence and uniqueness of the solution to the Cauchy problem, analogous to the existence theorem (4.1). Our proof applies to the electromagnetic field strength tensor $\mathscr{F}$ without employing a gauge field $\mathscr{A}$. This is a more general approach, since our proof holds on multiply-connected spacetimes.

**Proposition 4.10** *For any $B \in \Omega^2(\Sigma), E \in \Omega^1(\Sigma)$, with $\delta E = 0$ and $\mathrm{d}B = 0$, there exists $\mathscr{F} \in \Omega^2(\mathcal{M})$ such that*

$$\mathrm{d}\mathscr{F} = 0, \quad \delta\mathscr{F} = 0, \quad \rho_0(\mathscr{F}) = B, \quad \rho_n(\mathscr{F}) = E\,. \tag{4.82}$$

**Proof:** $\mathrm{d}\mathscr{F} = 0$ and $\delta\mathscr{F} = 0$ gives the wave equation $\Box\mathscr{F} = 0$. In local coordinates $(x^\mu) = (x^0, \mathbf{x})$ in which the Cauchy surface $\Sigma$ is given by $x^0 = 0$ we have the following equations: The field equation:

$$\nabla^\kappa\nabla_\kappa\mathscr{F}_{\mu\nu} - R_{\mu\rho}\mathscr{F}^\rho{}_\nu - R_{\nu\rho}\mathscr{F}_\mu{}^\rho + R_{\mu\nu\lambda\rho}\mathscr{F}^{\lambda\rho} = 0\,. \tag{4.43}$$

The field components on the Cauchy surface:

$$\mathscr{F}_{ij}(0,\mathbf{x}) = B_{ij} \tag{4.50}$$

$$\mathscr{F}^{0i}(0,\mathbf{x}) = \frac{E^i}{N}\,. \tag{4.51}$$

The normal derivatives on the Cauchy surface:

$$n^\mu\nabla_\mu\mathscr{F}_{ij}(0,\mathbf{x}) = -\frac{1}{N}\partial_j \wedge (NE_k - N^j B_{jk}) - \frac{1}{N}N^l\nabla_l B_{jk} \tag{4.78}$$

$$n^\mu\nabla_\mu\mathscr{F}^{0k}(0,\mathbf{x}) = -\frac{1}{N}\partial_j(B^{jk} + N^j \wedge \frac{1}{N}E^k) - \frac{1}{N}N^l\nabla_l\frac{E^k}{N}\,. \tag{4.79}$$

Once this data is given, a unique solution is provided by the following theorem on globally hyperbolic differential equations [Wala], also guaranteeing the well posedness of the initial value formulation:

**Theorem 4.11** *Let $(\mathcal{M}, g_{\mu\nu})$ be a globally hyperbolic spacetime (or a globally hyperbolic region of an arbitrary spacetime) and let $\nabla_\mu$ be any derivative operator. Let $\Sigma$ be a smooth, spacelike Cauchy surface. Consider the system of linear equations for $n$ unknown functions $\phi_1, \ldots, \phi_n$ of the form*

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi_i + \sum_j (A_{ij})^\mu\nabla_\mu\phi_j + \sum_j B_{ij}\phi_j + C_i = 0. \tag{4.83}$$

*(This equation is referred to as a linear, diagonal second order hyperbolic system.) Then this equation has a well posed initial value formulation on $\Sigma$. More precisely, given arbitrary smooth initial data, $(\phi_i, n^\mu\nabla_\mu\phi_i)$ for $i = 1, \ldots, n$ on $\Sigma$ there exists a unique solution of this equation throughout $\mathcal{M}$. Furthermore, the solutions depend continuously on the initial data. Finally, a variation of the initial data outside of a closed subset $S$ of $\Sigma$ does not affect the solution in $D(S)$.*

## 4.3 Outlook

We have shown that the electromagnetic field strength tensor has a well posed initial value formulation on arbitrary globally hyperbolic spacetimes. By this, the existence of retarded and advanced solutions to the inhomogeneous wave equation is guaranteed. Our result is by no means unexpected, and, as we have seen, the components for our proof were available in various publications. But no one, as far as we know, carried out the whole piece of work in one place.

The next step would be now the introduction of commutation relations for the field strength tensor, but this has to remain undone in this thesis. Since the commutator is local object, what we expect is the commutator (4.18) already anticipated by A. LICHNEROWICZ [Lic61] and derived by J. DIMOCK [Dim92].

The algebra constructed in section 4.1.3, is valid locally in any spacetime. It is not so clear, what happens when the global algebra is constructed on multiply-connected spacetimes. The symplectic form used to construct the Weyl algebra might be degenerate under such circumstances and lead to the superselection sectors already constructed by A. ASHTEKAR and A. SEN [AS80] in Schwarzschild-Kruskal spacetime. The expected structure could be similar to a structure which can be found in 2-dimensional conformal chiral field theory, considered as a theory on the circle $S^1$ [Frea].

M. RADZIKOWSKI's characterization of Hadamard states for scalar quantum quantum fields globally hyperbolic spacetime in terms of the wavefront set has not been applied to the Maxwell field so far. The construction of Hadamard states of the Maxwell field on curved spacetime, using methods from microlocal analysis is an open issue. This could in principle be done analogously to the scalar field as in [Jun96]. For the vector potential this is already in preparation by W. JUNKER and F. LLEDÓ [JL].

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**Erklärung**

Ich versichere diese Arbeit unter alleiniger Verwendung der angegebenen Quellen und Hilfsmittel selbständig angefertigt zu haben.

2. Mai 2001 Muharrem Küskü