Epstein-Glaser Renormalization: Finite Renormalizations, the S-Matrix of Φ^4 Theory and the Action Principle

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Abstract

A formula describing finite renormalizations is derived in the Epstein-Glaser formalism and an calculation of finite counterterms in Φ^4-theory is performed. The Zimmermann identities and the action principle for changes of parameters in the interaction are presented independent of the adiabatic limit. Additionally a comparison with BPHZ renormalization is presented including a derivation of the Hopf algebra structure of renormalization in the Epstein-Glaser approach.

Zusammenfassung

Im Epstein-Glaser Formalismus wird eine Formel für endliche Umrenormierungen hergeleitet. Eine Berechnung endlicher Counterterme wird in der Φ^4 Theorie durchgeführt. Die Zimmermann Identitäten und der Teil des Wirkungsprinzips, der Änderungen von Parametern in der Wechselwirkung beschreibt, werden unabhängig vom adiabatischen Limes hergeleitet. Zusätzlich wird das Epstein-Glaser Verfahren mit der BPHZ Renormierung verglichen, dabei wird die Hopf Algebra Struktur der Renormierung im Epstein-Glaser Zugang hergeleitet.

Contents

Chapter 1

Introduction

Renormalization is an old art of removing divergencies which occur unavoidably in QFT. In course of time many calculational techniques, more or less mathematical, were developed. Usually the more mathematical formulations of renormalization were too abstract for practical purposes, like the Epstein-Glaser approach [EpGl1] of renormalization. Apart from the work of Scharf [Scha] and Stora [PoSt] [Sto] nothing was done in this framework for a long time. Nevertheless a further development of the Epstein-Glaser method is worth-wile because it turned out that this method is best suited for the construction of theories on curved space times [BrFr] [DüFr]. Its advantages are the local character and the formulation in position space.
Another more abstract formulation of renormalization theory is the BPHZ- renormalization. In this framework some fundamental results of the structure of renormalization were achieved, namely the forest formula and the action principle. The latter describes how Green's functions change by a variation of parameters in a theory. In the derivation of the action principle Lowenstein [Low] used the Gell-Mann-Low formula to express Green's functions in terms of free fields. In this context the action principle is a consequence of simple properties of free field insertions into the S-matrix. In the present work we call these properties action principle because they describe the underlying basic structure and can be proved independent of the adiabatic limit. The properties for the Green's functions theirselves then follow in the adiabatic limit because the Gell-Mann-Low formula is valid if this limit can be performed.
Dütsch and Fredenhagen [DüFr2] give another derivation of the action principle. In contrast to our derivation they use insertions into time-ordered products (T-products) of interacting fields. Using the fact that interacting fields are up to a factor insertions in the S-matrix one can transform the two formulations into each other. In their comparison with the usual action principle they only consider variations of the interacting part of the Lagrangian with mass dimension 4. In this case their action principle coincides with the usual one in the adiabatic limit. Breitenlohner and Maison [BrMa] succeeded in formulating the action principle also in dimensional renormalization.
In this work we will give a formulation of finite renormalizations in the Epstein-Glaser approach corresponding to the forest formula. Furthermore we give a formulation of T-products and insertions in the Epstein-Glaser formalism so that the derivation of the part of the action principle concerning changes in the interaction is analogous to that of Lowenstein [Low]. To fill the gap between theoretical formula-

tion and practical calculations we demonstrate how to renormalize the S-matrix in Φ^4 theory up to the third order.

Many calculations concerning renormalization can be found in the book of Zavialov [Zav]. Often they are similar to the results presented here, but they are not formulated in the sense of distributions. In this sense the structure of the renormalization presented in [Zav] corresponds to the formulas given in the theorems (6.1) and (6.2).

In [Pra3] the energy momentum tensor and an operator product expansion of two time ordered fields is discussed in the Epstein-Glaser theory.

A part of the content of this work can be found in [BrPiPr] and [Pin]. Additionally we discuss in this work comparisons with other renormalizations. We hope it is helpful for the reader to get a better understanding of the methods.

This work is divided into seven chapters. After this introduction we briefly repeat the basics of causal perturbation theory in the framework of the Wightman axioms.

In the third chapter we give a mathematical description of the time-ordered product. The inductive construction of Epstein and Glaser [EpGl1] is described. The main result of this section is the description of finite renormalizations by a family of functions Δ_n.

Chapter 4 prepares the calculations of section 5 and contains a short summary of [Scha] [EpGl1] [Fre] [BrFr]. After some microlocal analysis we see that renormalization is nothing else than an extension of distributions on an appropriate space of test functions. We repeat the calculation of the Lorentz invariant form of the extension presented in [BrPiPr] which is used in the calculations and discuss some properties of the W-operator. Finally we compare Epstein-Glaser renormalization with differential and BPHZ renormalization. Recently Kreimer discovered the structure of a Hopf algebra in the usual momentum space renormalization [Kre]. A connection to noncommutative geometry [CoKr1] and the unsolved mathematical Riemann-Hilbert problem [CoKr2] was worked out by Connes and Kreimer. With our results of chapter 5 and the fourth-order terms in Φ^4 theory Gracia-Bondía and Lazzarini [GBLa] have found the same Hopf algebra structure also in Epstein-Glaser renormalization of Φ^4 theory. We give another mathematical more rigorous derivation of this Hopf algebra structure based on the formula for finite renormalizations.

In chapter 5 we show how renormalization of the S-matrix of Φ^4 theory is done up to the third order. In the results we only list terms surviving in the adiabatic limit, but the calculation can be performed without using the existence of this limit. In contrast to other renormalizations in momentum space the complexity does not grow with the number of loops but with the number of vertices in a diagram. Thus the second order calculation is simple because subdivergencies first appear in the third order calculation. To come back to the abstract formulation of the Epstein-Glaser approach we list the normalization conditions for a scalar theory. Some of them are an abstract form of the rules used in the calculations. With the Gell-Mann-Low formula we derive the Dyson-Schwinger equations (DSE) from the normalization condition N4 in Φ^4 theory.

In the next chapter we derive the action principle for changes of parameters in the interaction analogously to [Low]. The first subsection is based on the theorem of perturbative renormalization theory [PoSt]. Using the form of finite renormalizations of section 2 we are able to determine explicitly the counterterms in the Lagrangian

compensating a change of renormalization. Then we define insertions into T-products and show that they have some of the properties of the insertions of [Low]. Relations between insertions of different degrees are described by the Zimmermann identities [Zim] that are formulated in the framework of Epstein-Glaser renormalization. Finally we prove the formulation of the part of the action principle describing changes in the interaction in terms of insertions in the S-matrix independent of the adiabatic limit. The action principle for changes in the parameters of the free Lagrangian turns out to be more complicated, so we postpone it.
These results and the missing part of the action principle allow a derivation of the renormalization group equations (RGE) as in [Low]. But to derive this a better understanding of the role of the mass term and its derivations is needed. The RGE and the DSE are also valid outside perturbation theory. They provide an important tool for nonperturbative constructions and methods, see for example [Zin] and [Sti]. The action principle is used in algebraic renormalization [PiSo] which leads to a systematic renormalization of the standard model of electroweak interaction to all orders of perturbation theory [Kra]. This work is a first step of a translation of these methods into the Epstein-Glaser formulation of renormalization.
We always use the manifold $\mathbb{R}^4$ with Minkowski metric for spacetime. With the work of [Fre], [BrFrKö], [BrFr], [Rad] it should be easy to formulate the arguments on curved space times; often one only has to replace $\mathbb{R}^4$ by an arbitrary Lorentz manifold $\mathcal{M}$.

Chapter 2

Causal Perturbation Theory and Epstein-Glaser Renormalization

A mathematical precise formulation of a QFT was given by Gårding and Wightman [WiGa]. They treat a QFT as a tupel

$$(\mathcal{H}, U, \phi, D, |0\rangle) \tag{2.1}$$

of a separable Hilbert space $\mathcal{H}$, a unitary representation of the restricted Poincaré group $\mathcal{P}_+^\uparrow$, field operators ϕ, a dense subspace D of $\mathcal{H}$ and the vacuum vector $|0\rangle$. The tupel has to fulfill the Wightman axioms [StWi]. Among others they state that the fields ϕ are local operator-valued distributions which are well-defined on the dense domain D of the Hilbert space. D contains the vacuum. In the construction of the S-matrix it is sufficient to regard the dense domain $D_0 \subset D \subset \mathcal{H}$ generated by all vectors which can be constructed by applying a finite set of field operators to the vacuum:

$$D_0 = \{\psi \in \mathcal{H} | \psi \in \text{span}\,(\phi(f_1)\dots\phi(f_n)|0\rangle, n \in \mathbb{N})\}. \tag{2.2}$$

In free theories such a tupel satisfying the Wightman axioms is easily found. Here we only treat the example of a free scalar field. In a scalar theory with particles of mass m the Hilbert space $\mathcal{H}$ is the following Fock space

$$\mathcal{F} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k. \tag{2.3}$$

Every state is represented by a ray in this Fock space which is the direct sum of the Hilbert spaces $\mathcal{H}_0 = \mathbb{C}$ describing the vacuum, and

$$\begin{aligned}\mathcal{H}_i = \{\, \text{f} : \mathbb{R}^3 \times \mathbb{R}^3 \times \dots \times \mathbb{R}^3 \longrightarrow \mathbb{C} \mid \text{f square-integrable with} \\ \text{respect to the measure} \frac{d^3\vec{k}_1 \dots d^3\vec{k}_i}{2\omega(\vec{k}_1)\dots 2\omega(\vec{k}_i)}, \text{f symmetric}\,\},\end{aligned} \tag{2.4}$$

consisting of all i-particle wave functions in the momentum representation.
We use

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}. \tag{2.5}$$

The norm of a Fock space vector

$$\mathcal{F} \ni | f > = \Big(f_0, f_1(\vec{k}_1), \ldots, f_n(\vec{k}_1, \ldots, \vec{k}_n), \ldots \Big) \tag{2.6}$$

is given by

$$\begin{aligned}
(< f \mid f >)^{\frac{1}{2}} &= \left(\sum_{i=0}^{\infty} \|f_i\|^2 \right)^{\frac{1}{2}} \\
&= \left(| f_0 |^2 + \sum_{n=1}^{\infty} \int \frac{d^3\vec{k}_1 \ldots d^3\vec{k}_i}{2\omega(\vec{k}_1) \ldots 2\omega(\vec{k}_i)} \mid f_n(\vec{k}_1 \ldots d\vec{k}_n) \mid^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.7}$$

The norm of a physical state should be finite. Fock space vectors with a finite number of non-zero components automatically fulfill this condition, they are called finite. The set of finite vectors is dense in the Fock space.
We assign to every square-integrable function $g \in \mathcal{H}_1$ the creation operator $a^+(g)$ and the annihilation operator $a(g)$. They are defined by their action on elements of the Fock space: the creation operator $a^+(g)$ transforms a state with $n-1$ particles into a state with n particles:

$$\Big(a^+(g) \, |f > \Big)_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g(\vec{k}_i) f_{n-1}(\vec{k}_1, \ldots, \check{\vec{k}}_i, \ldots, \vec{k}_n). \tag{2.8}$$

The annihilation operator transforms a state with $n+1$ particles into a state with n particles:

$$\Big(a(g) \, |f > \Big)_n = \sqrt{n+1} \int \frac{d^3\vec{k}_{n+1}}{2\omega(\vec{k}_{n+1})} f_{n+1}(\vec{k}_1, \ldots \vec{k}_{n+1}) \overline{g(\vec{k}_{n+1})}, \tag{2.9}$$

particularly the vacuum is annihilated. One can prove that these operators satisfy the following commutation relations:

$$\left[a(g_1), a^+(g_2) \right] = \int \frac{d^3\vec{k}}{2\omega(\vec{k})} \overline{g_1(\vec{k})} g_2(\vec{k}), \tag{2.10}$$

$$\left[a(g_1), a(g_2) \right] = \left[a^+(g_1), a^+(g_2) \right] = 0. \tag{2.11}$$

The field operators are now constructed with the help of the operator-valued distributions $a(\vec{k})$ and $a^+(\vec{k})$. By smearing them with test functions one obtains the annihilation and the creation operators:

$$a(g) = \int \frac{d^3\vec{k}}{2\omega(\vec{k})}\, \overline{g(\vec{k})} a(\vec{k}), \tag{2.12}$$

$$a^+(g) = \int \frac{d^3\vec{k}}{2\omega(\vec{k})}\, a^+(\vec{k}) g(\vec{k}). \tag{2.13}$$

The operator-valued distributions fulfill the following commutation relations:

$$\left[a(\vec{k}), a^+(\vec{k}')\right] = 2\omega(\vec{k})\ \delta(\vec{k}-\vec{k}'), \tag{2.14}$$

$$\left[a(\vec{k}), a(\vec{k}')\right] = \left[a^+(\vec{k}), a^+(\vec{k}')\right] = 0. \tag{2.15}$$

$a(\vec{k})$ is an operator defined on all finite vectors in the Fock space whose wave functions are continuous, but $a^+(\vec{k})$ is only defined as a quadratic form on these wave functions. This construction is the so-called Fock representation. The existence of the vacuum and the relations (2.10), (2.11) are characteristic for it. Every irreducible representation with these properties is unitarily equivalent to the representation constructed above. Although $a^+(\vec{k})$ is not an operator, every bounded operator can be written as a normal product of of $a(\vec{k})$ and $a^+(\vec{k})$. There are many unbounded operators which can be represented in the same way, for example the particle number operator N.
Free real scalar fields are operator-valued distributions satisfying the Klein-Gordon equation in the sense of distributions:

$$\left(\Box + m^2\right)\phi(f) = \phi\left(\left(\Box + m^2\right) f\right) = 0 \qquad \forall\, f \in \mathcal{D}\left(\mathbb{R}^4\right). \tag{2.16}$$

They are constructed with the help of the classical solutions of this equation consisting of the positive frequency solutions

$$u_{\vec{k}}(\vec{x},t) = N_{\vec{k}} e^{-i\left(\omega(\vec{k})t - \vec{k}\vec{x}\right)} \tag{2.17}$$

and the negative frequency solutions

$$u^*_{\vec{k}}(\vec{x},t) = N_{\vec{k}} e^{i\left(\omega(\vec{k})t - \vec{k}\vec{x}\right)} \tag{2.18}$$

with

$$N_{\vec{k}} = \left(2\,(2\pi)^3\,\omega(\vec{k})\right)^{-\frac{1}{2}}, \tag{2.19}$$

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}. \tag{2.20}$$

Combining these functions of the classical solutions with the operator-valued distributions $a(\vec{k})$ and $a^+(\vec{k})$ in the following way,

$$\phi(\vec{x},t) = \int d^3\vec{k}\left(a^+(\vec{k})u^*_{\vec{k}}(\vec{x},t) + a(\vec{k})u_{\vec{k}}(\vec{x},t)\right), \tag{2.21}$$

one obtains free fields which are operator-valued distributions on an appropriate space of test functions (e.g. $\mathcal{S}(\mathbb{R}^4)$ or $\mathcal{D}(\mathbb{R}^4)$) and satisfy the following commutation relations:

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = \left[\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{y},t)\right] = 0, \tag{2.22}$$

$$\left[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)\right] = i\delta(\vec{x}-\vec{y}). \tag{2.23}$$

The field operators $\phi(f)$ are obtained by smearing the field distribution with the test function f:

$$\phi(f) = \int d^4x\ \phi(x)f(x). \tag{2.24}$$

If the Fourier transforms of two different test functions f_1 and f_2 coincide on the hyperboloid $k^2 = k_0^2 - \vec{k}^2 = m^2$, they yield the same field: $\phi(f_1) = \phi(f_2)$.
The aim of a QFT is to describe interacting fields. In this work we only treat the case of Φ^4-interaction. The equation of motion of the fields in Φ^4-theory differs from the Klein-Gordon equation by a nonlinear term:

$$\left(\Box + m^2\right)\Phi + \frac{\lambda}{3!}\Phi^3 = 0. \tag{2.25}$$

One can prove that a classical solution $\phi\left(\vec{x},t\right)$ exists for all times t for reasonable initial conditions $\phi\left(\vec{x},0\right), \dot{\phi}\left(\vec{x},0\right)$.

Describing scattering experiments one assumes that long times before and after the scattering the distance between the particles is so large that they do not interact. In this way it is allowed to describe them in terms of free fields of the Fock representation constructed above:

$$\lim_{t\to-\infty}\left(\Phi_{int}(x) - \Phi^{in}(x)\right) = 0,$$
$$\lim_{t\to\infty}\left(\Phi_{int}(x) - \Phi^{out}(x)\right) = 0,$$

in the sense of matrix elements on a dense domain of $\mathcal{F}$. As two irreducible Fock representations are unitarily equivalent there exists a unitary operator S with

$$\Phi^{in}\left(x\right) = S\ \Phi^{out}\left(x\right)\ S^{-1}. \tag{2.26}$$

This picture of asymptotically free fields is not exact because the self-interaction of the fields is disregarded.
The interacting fields should have all the fundamental physical properties of the free ones like Poincaré invariance or locality, formulated in the Wightman axioms. A construction of the interacting fields and the S-matrix is possible up to now only within perturbation theory.
Not every formulation of perturbation theory takes into account the distributional character of the fields and interaction terms and makes sure that the operators are well defined on D (or in the case of the S-matrix on D_0). Causal perturbation theory has all those properties. It was founded by the work of Stueckelberg [Stu], Bogoliubov and Shirkov [BoSh]. In their formulation every interaction term is accompanied by a test function

$$g \in \mathcal{S}\left(\mathbb{R}^4\right), \qquad\qquad g : \mathbb{R}^4 \longrightarrow [0,1] \tag{2.27}$$

which switches the interaction on and off at different space time points ($\mathcal{S}$ is the Schwartz space of test-functions). g vanishes at infinity and therefore provides for a cut-off for long-range interactions. All quantities of the causal construction are formal power series in this function; especially the S-matrix is constructed with the following ansatz:

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots \int d^4x_n \; S_n\left(x_1, \ldots, x_n\right) g\left(x_1\right) \ldots g\left(x_n\right). \tag{2.28}$$

The coefficients S_n are symmetric operator-valued distributions smeared with the switching function $g(x)$ in such a way that each term in the sum is a well-defined operator on D_0.
Epstein and Glaser [EpGl1] have found that the S_n can be determined by a few physical properties of the S-matrix, namely

1. Translation covariance
2. Lorentz covariance
3. Unitarity
4. Causality

and the renormalization conditions. Actually, for the inductive construction of the S_n proposed by Epstein and Glaser only causality is important. To prove that the resulting $S(g)$ is a well-defined operator on D_0 they apply Wick's theorem and use translation covariance. In [BrFr] it is shown how translation covariance can be substituted by a condition on the wave-front sets of the S_n. This makes the method work on curved space times, too. The other properties are realized by the normalization conditions.
With some calculations, the above properties are transferred to the following properties of the coefficients S_n:

1. Translation covariance: $S_n(x+a) = U(a)\, S_n(x)\, U^{-1}(a)$, where $U(a)$ is a representation of the translation in the Fock space.
2. Lorentz covariance: $S_n(\Lambda x) = U(\Lambda)\, S_n(x) U^{-1}(\Lambda)$, where Λ is an element of the proper Lorentz group and U is a representation of this group.
3. In the following, we will often work with sets of indices. We will denote with $J = \{1, \ldots, n\}$ the full set of indices and with I any subset of J. We use $I^c = J \backslash I$. For $I = \{i_1, \ldots, i_k\}$ we use the notation

$$S_k(x_j | j \in I) := S_k(x_{i_1}, \ldots, x_{i_k}). \tag{2.29}$$

   From the unitarity condition the following equation is obtained:

$$\sum_{\substack{I \subset J \\ |I|=k}} S_k(x_i | i \in I)\, S^+_{n-k}(x_j | j \in J \setminus I) = 0. \tag{2.30}$$

4. Causality of the S-matrix yields the factorization property:

$$S_n\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right) = S_i(x_1, \ldots, x_i)\, S_{n-i}(x_{i+1}, \ldots, x_n) \tag{2.31}$$

   for $\{x_1, \ldots, x_i\} \gtrsim \{x_{i+1}, \ldots, x_n\}$, where $x \gtrsim y$ means that y does not lie in the forward light cone of x.

   This property implies the symmetry and the locality of the S_n:

   $[S_n(x_1, \ldots, x_n), S_m(y_1, \ldots, y_m)] = 0$ if all the x_i are spacelike to the y_j.

The factorization property (2.31) is used to identify the coefficients S_n with time-ordered products. If we further demand $S_1 = \mathcal{L}_{int}$ the higher S_n can be constructed inductively [EpGl1]. This is described in the next section. In this work we only treat Φ^4 theory in explicit calculations, in this case we have $T_1(x) = V(x) = -\frac{\lambda}{4!} : \phi^4(x) :$.
From the S-matrix the interacting fields can be obtained by the following formula of Bogoliubov [BoSh]:

$$\phi_{int\mathcal{L}}(x) = \frac{d}{dh(x)} S^{-1}(\mathcal{L}_{int}) S(\mathcal{L}_{int} + h(x)\phi(x))\Big|_{\lambda=0}. \tag{2.32}$$

They are formal power series in the switching function g like the S-matrix. To get rid of this function at the end of the construction the adiabatic limit $g(x) \to 1$ has to be performed. This limit bears some problems, because all infrared divergencies appear which were avoided in the local formulation. By definition the adiabatic limit exists in the weak sense if all Green's functions exist in the sense of tempered distributions for $g \to 1$. If it can further be shown that for $g \to 1$ the S-matrix is a unitary operator

one says that the adiabatic limit exists in the strong sense. The adiabatic limit is constructed by choosing a sequence of functions g_n with $\lim_{n\to\infty} g_n = 1$, but one has to be careful because sometimes the result depends on the choice of the g_n.
The weak adiabatic limit was proved to exist for massive theories [EpGl1], QED and massless $\lambda : \Phi^{2n}$: theories [BlSe]. The existence of the adiabatic limit in the strong sense has only been proved for massive theories [EpGl2].

Chapter 3

The Time-Ordered Product

Usually T-products are defined as multilinear functions on Wick monomials of quantized fields [BrFr]. In this section we use another definition of T-products, which was first introduced in [Boa]. Let $\mathcal{A}$ be a commutative algebra generated by the so-called classical symbolical fields ϕ_i and their derivatives. They are called symbols because they are not subject to a relation like the Klein-Gordon equation. Therefore the fields and their derivatives are linearly independent. We regard

$$\mathcal{D}\left(\mathbb{R}^4, \mathcal{A}\right) \ni f = \sum_i g_i(x)\phi_i, \tag{3.1}$$

where the sum is a finite sum over elements ϕ_i of the algebra $\mathcal{A}$. An element f is then given by its coefficients $g_i \in \mathcal{C}_0^\infty(\mathbb{R}^4)$.
The time-ordered product (T-product) is a family of maps $T_n, n \in \mathbb{N}$, called T_n-products. They are functions from $\left(\mathcal{D}\left(\mathbb{R}^4, \mathcal{A}\right)\right)^{\otimes n}$ into the operators on $\mathcal{H}$ with the following properties:

1. $T_0 = 1$
   $T_1(f) = \sum_i : \phi_i(g_i) : \qquad \forall \quad f \in \mathcal{D}\left(\mathbb{R}^4, \mathcal{A}\right),$
   where the sum is taken over all generators of $\mathcal{A}$. Each local field is the image of an element $f \in \mathcal{D}\left(\mathbb{R}^4, \mathcal{A}\right)$ under T_1. We define the T-products such that $T_n(g(d\phi)) = T_n((d^t g)\phi)$ is fulfilled for derivatives d up to the second order (^{t} means the transposition of an operator). T_1 is not injective because the Wick products obey the wave equation. For example, in a free scalar theory we obtain
   $T_1\left(g\Box\phi + gm^2\phi\right) =: \phi\left(\left(\Box + m^2\right)g\right) := 0.$

2. Symmetry in the arguments:

$$T_n\left(f_1, \ldots, f_n\right) = T_n\left(f_{\pi_1}, \ldots, f_{\pi_n}\right) \qquad \forall\ \pi \in S_n \qquad \forall\ f_i \in \mathcal{D}\left(\mathbb{R}^4, \mathcal{A}\right), i = 1, \ldots n \tag{3.2}$$

   where S_n is the set of all permutations of n elements.

3. The factorization property:

$$T_n(f_1,\ldots,f_n) = T_i(f_1,\ldots,f_i)\,T_{n-i}(f_{i+1},\ldots,f_n) \tag{3.3}$$

if $(\mathrm{supp} f_1 \cup \ldots \cup \mathrm{supp} f_i) \gtrsim (\mathrm{supp} f_{i+1} \cup \ldots \cup \mathrm{supp} f_n)$ and $f_i \in \mathcal{D}(\mathbb{R}^4, \mathcal{A})\ \forall\, i$.

Remark: The T_n can be defined as multilinear functionals in the arguments f_i, for instance $T_2(f_1+f_2,f_3) = T_2(f_1,f_3)+T_2(f_2,f_3)$. The linearity depends on the mass dimension of the arguments f_i and the definition of the T_i. If all arguments are treated in the renormalization (which defines the $T-$product) as of the same mass dimension the T-products are linear. In this case it is sufficient to know their values for one kind of interaction term f (polarization identity), and we can omit the indices of the f.
We now want to construct higher T_n-products with the help of the factorization identity as expressions of lower T_n-products. This is possible if $\cap_{i\in J}\mathrm{supp} f_i = \emptyset$ with $J=\{1,\ldots,n\}$, in other words if the total diagonal

$$D_n = \{(x_1,\ldots,x_n) \in \mathbb{R}^{4n} | x_1 = x_2 = \ldots x_n\} \tag{3.4}$$

is not in the support of the tensor-product of the f_i. In calculations this is achieved by multiplication of the distribution with a causal partition of unity:

Definition 3.0.1 *A causal partition of unity in $\mathbb{R}^{4n}\setminus D_n$ is a set of $\mathcal{C}^\infty$-functions $p_I^{(n)}\,:\,\mathbb{R}^{4n}\setminus D_n \longrightarrow \mathbb{R}$ with the following properties:*

1. $\mathrm{supp}\ p_I^{(n)} \subset \{(x_1,\ldots,x_n)\in\mathbb{R}^{4n}\setminus D_n \mid x_i \gtrsim x_j\ \forall i\in I, j\in I^c\}$
2. $\sum_{\substack{I\subseteq J\\ I\neq\emptyset}} p_I^{(n)}\Big|_{\mathbb{R}^{4n}\setminus D_n} = 1.$

Since $\mathrm{supp}(f_1\otimes\ldots\otimes f_n)$ is contained in a compact region, $\mathrm{supp}(f_1\otimes\ldots\otimes f_n)\cap C^I$ is contained in a compact region where $C^I = \{(x_1,\ldots,x_n)\subset\mathbb{R}^{4n}|\ x_i\gtrsim x_j\ \forall i\in I, j\in I^c\}$. In this region the part $p_I^{(n)}$ of the partition of unity can be written as a finite sum of factorized terms $p_I^{(n)} = \sum_k p^1_{I,k}(x_1)\cdot\ldots\cdot p^n_{I,k}(x_n)$. Then a T-product is constructed according to the factorization property as follows:

$$T_n^0(f_1,\ldots,f_n) = T_n^0\left(\bigotimes_{k=1}^n f_k\right) = \sum_{\substack{I\subseteq J\\ I\neq\varnothing}}\sum_k T_{|I|}\left(\bigotimes_{j\in I} p^j_{I,k} f^I_j\right) T_{|I^c|}\left(\bigotimes_{l\in I^c} p^l_{I,k} f^I_l\right). \tag{3.5}$$

The last step in this construction of Epstein and Glaser [EpGl1] is the extension of the right-hand side of (3.5), if the support of the tensor product of the f_i contains the total diagonal D_n. We call this shortly the extension of the T-product to the total diagonal. This extension is not unique, so there are several T-products differing from

another by finite renormalizations. In renormalization theory this corresponds to the free choice of the renormalization constants.
Now we can describe the structure of finite renormalizations with the following theorem which arose from discussions with K. Fredenhagen (it can be seen as the precise formulation of a formula given by [BoSh]):

Theorem 3.0.1 *Let $T, \hat{T}$ be two different T-products. Then there are functions $\Delta_n : \mathcal{D}\left(\mathbb{R}^{4n}, \mathcal{A}^n\right) \to \mathcal{D}\left(\mathbb{R}^4, \mathcal{A}\right)$ with $\operatorname{supp} \Delta_n \subset D_n$ and*

$$\hat{T}_n\left(\bigotimes_{j\in J} f_j\right) = \sum_{P\in Part(J)} T_{|P|}\Big[\bigotimes_{O_i\in P} \Delta_{|O_i|}\left(\bigotimes_{j\in O_i} f_j\right)\Big]. \tag{3.6}$$

For the interpretation of the operator-valued distribution Δ_n we go back to equation (3.5). The lower T-products on the right-hand side of (3.5) consist themselves of products of lower T-products which were extended to a subdiagonal of $\mathbb{R}^{4n}$. We call the extension to these subdiagonals the *renormalization of subdivergences* and the extension to the total diagonal in the last step the *renormalization of the superficial divergence*.
Fixing the extensions of all T_n-products to the diagonals defines us a special T-product. The distribution $\tilde{\Delta}_n = T_1(\Delta_n)$ is the difference of $\hat{T}$ and T in the renormalization of the superficial degree of divergence of (3.5) having all subdivergencies renormalized according to $\hat{T}$.
Proof of the theorem: We construct the Δ_n inductively by the following formula:

$$\begin{aligned}\tilde{\Delta}_n &:= \hat{T}_n\left(\bigotimes_{j\in J} f_j\right) - \sum_{\substack{P\in Part(J)\\ |P|>1}} T_{|P|}\Big[\bigotimes_{O_i\in P} \Delta_{|O_i|}\left(\bigotimes_{j\in O_i} f_j\right)\Big] \\ &= T_1\left(\Delta_n\left(\bigotimes_{j\in J} f_j\right)\right).\end{aligned} \tag{3.7}$$

To show that this construction makes sense we prove by induction over n that the support of $\tilde{\Delta}_n$ is contained in D_n. Since T_1 is surjective there is a $\Delta_n\left(\bigotimes_j f_j\right) \in \mathcal{D}(\mathbb{R}^4, \mathcal{A})$ with $T_1(\Delta_n) = \tilde{\Delta}_n$. Equation (3.6) is automatically fulfilled by this construction. Now we show by induction that $\operatorname{supp}(\tilde{\Delta}_n) \subset D_n$.
Beginning of the induction:

- $|n| = 1$: $\quad \tilde{\Delta}_1(f) = T_1(f)$
- $|n| = 2$: $\quad \tilde{\Delta}_2\left(f_1, f_2\right) = \hat{T}_2\left(f_1, f_2\right) - T_2\left(\Delta_1(f_1)\Delta_1(f_2)\right) = \hat{T}_2\left(f_1, f_2\right) - T_2\left(f_1, f_2\right),$
  and we saw in (3.5) that two T_2-products are equal outside the diagonal. So $\operatorname{supp}\tilde{\Delta}_2 \subset D_2$.

Now we assume $\cap_{i=1}^n \operatorname{supp} f_i = \emptyset$ and have to show $\tilde{\Delta}_n\left(f_1 \ldots f_n\right) = 0$. We obtain with (3.5):

$$\hat{T}_n\left(\bigotimes_{j\in J} f_j\right) = \sum_{I\subset J}\sum_k \hat{T}_{|I^c|}\left(\bigotimes_{j\in I^c} p^j_{I,k} f_j\right) \hat{T}_{|I|}\left(\prod_{j\in I} p^j_{I,k} f_j\right) \tag{3.8}$$

With the induction hypothesis ($\mathrm{supp}\tilde{\Delta}_m \subset D_m$ for all $m < n$) we also obtain a factorization of the second term of $\tilde{\Delta}_n$

$$\sum_{\substack{P\in Part(J)\\ |P|>1}} T_{|P|}\Big[\bigotimes_{O_i\in P} \Delta_{|O_i|}\left(\otimes_{j\in O_i} f_j)\right)\Big] =$$
$$= \sum_{I\subset J}\sum_{k}\sum_{\substack{S\in Part(I^c)\\ T\in Part(I)}} T_{|S|}\Big[\bigotimes_{O_i\in S} \Delta_{|O_i|}\left(\otimes_{j\in O_i} p^j_{I,k} f_j\right)\Big] T_{|T|}\Big[\bigotimes_{U_i\in T} \Delta_{|U_i|}\left(\otimes_{j\in U_i} p^j_{I,k} f_j\right)\Big] \tag{3.9}$$

(since the support of the $\Delta_{|O_i|}$ is contained in a set of points belonging to a partition $I \subset J$ with $O_i \subset I$ or $O_i \subset I^c$). Therefore we obtain

$$\tilde{\Delta}_{|J|}\left(\otimes_{j\in J} f_j\right) =$$
$$= \sum_{I\subset J}\sum_{k}\Bigg\{\hat{T}_{|I^c|}\left(\otimes_{j\in I^c} p^j_{I,k} f_j\right)\left(\hat{T}_{|I|}\left(\otimes_{j\in I} p^j_{I,k} f_j\right)\right.$$
$$\left. - \sum_{P\in Part(I)} T_{|P|}\Big[\bigotimes_{O_i\in P} \Delta_{|O_i|}\left(\otimes_{j\in O_i} p^j_{I,k} f_j\right)\Big]\right)$$
$$+ \left(\hat{T}_{|I^c|}\left(\otimes_{j\in I^c} p^j_{I,k} f_j\right) - \sum_{P\in Part(I^c)} T_{|P|}\Big[\bigotimes_{U_i\in P} \Delta_{|U_i|}\left(\otimes_{j\in U_i} p^j_{I,k} f_j\right)\Big]\right)\cdot$$
$$\cdot \sum_{P\in Part(I)} T_{|P|}\Big[\bigotimes_{O_i\in P} \Delta_{|O_i|}\left(\otimes_{j\in O_i} p^j_{I,k} f_j\right)\Big]\Bigg\} = 0$$

because the Δ_n of lower order fulfill equation (3.6) ■.

Chapter 4

Renormalization as an Extension of Distributions

4.1 Motivation

In this section we show that the extension of numerical distributions corresponds to renormalization. We derive some properties of our extension procedure, compare it with the usual way of renormalization in momentum space and apply it in the next section explicitely to the renormalization of the second and third order of the S-matrix in Φ^4 theory, namely:

$$\begin{aligned} S^{(2)}(g) &= \frac{1}{2}\left(\frac{i\lambda}{4!}\right)^2 T_2\left((g\phi^4)^{\otimes 2}\right) &(4.1)\\ S^{(3)}(g) &= -\frac{1}{3!}\left(\frac{i\lambda}{4!}\right)^3 T_3\left((g\phi^4)^{\otimes 3}\right). &(4.2)\end{aligned}$$

By means of Wick's theorem the T_n-products appearing in the expansion of the S-matrix can be transformed into integrals of linear combinations of products of numerical distributions with Wick products. For instance T_2 in $S^{(2)}(g)$ has the form:

$$\begin{aligned} T_2^0\left(g\phi^4, g\phi^4\right) &= \sum_{k=0}^{4} \binom{4}{k}\binom{4}{k}(4-k)!\,\cdot \\ &\quad \cdot \int d^4x_1 \int d^4x_2\; t^0(x_1,x_2) : \phi^k(x_1)\,\phi^k(x_2) : g(x_1)g(x_2). \end{aligned} \tag{4.3}$$

with $\quad t^0(x_1,x_2) \overset{x_1\neq x_2}{=} \left(i\Delta_F(x_1-x_2)\right)^{4-k}.$

From theorem 0 of [EpGl1] we know that the product of a well-defined translation invariant numerical distribution with a Wick product is well-defined on D_0. Therefore we only have to take care of the numerical distributions denoted with t^0 in the following. In renormalization theory for non-coincident points t^0 is a product of Feynman propagators and therefore it is Poincaré invariant in Minkowski space. Problems arise from the fact that Feynman propagators are distributions and that the product of distributions is not always defined. The domain of definition depends on the singularities of the individual factors and are determined by methods of microlocal analysis.

4.2 Some Microlocal Analysis

The content of this subsection can be found in [BrFrKö] [Hör] [Fre]. Let $\mathcal{M} = \mathbb{R}^{4n}$ or $\mathcal{M} \subset \mathbb{R}^{4n}$ be a manifold of dimension $4n$ and $\mathcal{D}(\mathcal{M})$ be all $\mathcal{C}^\infty$-functions on $\mathcal{M}$ with compact support. The singular support of a distribution $u \in \mathcal{D}'(\mathcal{M})$, $sing\ \mathrm{supp}\ u$, is defined as the set of all points in $\mathcal{M}$ without any open neighbourhood on which the restriction of u is a $\mathcal{C}^\infty$ function.
If a distribution v with compact support has no singularities its Fourier transform $\hat{v}$ is asymptotically bounded for large ξ by

$$|\hat{v}(\xi)| \leq C_N (1 + |\xi|)^{-N} \qquad \forall\, N \in \mathbb{N}, \xi \in \mathbb{R}^{4n}, \tag{4.4}$$

where C_N are constants for each N. Any distribution $u \in \mathcal{D}'(\mathcal{M})$ is regular in $Y \subset \mathcal{M}$ if for all functions $f \in \mathcal{C}_0^\infty(Y)$ the Fourier transform of fu is asymptotically bounded for large ξ by

$$|\widehat{fu}(\xi)| \leq C_N (1 + |\xi|)^{-N} \qquad \forall\, N \in \mathbb{N}, \xi \in \mathbb{R}^{4n}, \tag{4.5}$$

where C_N are constants for each N. With $sing\ \mathrm{supp}\ u$ we denote the set of points of $\mathcal{M}$ where u is not regular.
Let u be singular at $x \in X \subset \mathcal{M}$, $x \in sing\ \mathrm{supp}\ u$. $\Sigma_x \subset \mathbb{R}^{4n} \setminus 0$ is the set of all ξ of the cotangent space $T_x^*(X)$ such that (4.5) is not fulfilled for any function $f \in \mathcal{C}_0^\infty(X)$ with $f(x) \neq 0$ in a conic neighbourhood V of ξ. Σ_x is a cone describing the direction of the high frequencies causing the singularities at x. The pair of x and $\xi \in \Sigma_x$ is an element of the wave-front set:

Definition 4.2.1 *If $u \in \mathcal{D}'(\mathcal{M})$, then the closed subset of $\mathcal{M} \times \left(\mathbb{R}^{4n} \setminus \{0\}\right) \subset T^*(\mathcal{M})$ defined by*

$$WF(u) = \left\{(x,\xi) \in \mathcal{M} \times \left(\mathbb{R}^{4n} \setminus \{0\}\right) \mid \xi \in \Sigma_x(u)\right\} \tag{4.6}$$

is called the wave-front set of u. The projection on $\mathcal{M}$ is $sing\ \mathrm{supp}\ u$.

Multiplication with a smooth function $a \in \mathcal{C}^\infty$ and differentiation do not enlarge the wave-front set:

$$WF(au) \subset WF(u), \tag{4.7}$$
$$WF(D^\alpha u) \subset WF(u). \tag{4.8}$$

For the existence of the pointwise product of two distributions u, v at x it is sufficient for them to fulfill the condition

$$(x,0) \notin WF(u) \oplus WF(v) = \{(x, \xi_1 + \xi_2) | (x, \xi_1) \in WF(u), (x, \xi_2) \in WF(v)\}. \quad (4.9)$$

Thus the product at x exists if u or v or both are regular in x. If u and v are singular at x the product exists if the sum of the second components in the wave-front sets of u and v at x cannot be the zero-vector. If the product exists, its wave-front set will fulfill

$$WF(uv) \subset WF(u) \cup WF(v) \cup (WF(u) \oplus WF(v)). \quad (4.10)$$

We now check whether the products of Feynman propagators of scalar fields appearing in (4.3),

$$(i\Delta_F(x_1 - x_2))^n \qquad \text{with } n = 2, 3, 4 \quad (4.11)$$

exist for all $x_1 - x_2$. Since $i\Delta_F(x_1 - x_2)$ is for $x_1 \neq x_2$ a solution of the Klein-Gordon equation, its singular support is contained in the characteristic set of the Klein-Gordon operator, the forward and backward light-cone.
The wave-front set of the Feynman propagator [DuHö] [Jun] [Rad] has the following form:

$$\begin{aligned} WF(\Delta_F) &= \{(x_1, k; x_2, k') \in T^*_{x_1}\mathbb{R}^4 \times T^*_{x_2}\mathbb{R}^4 | (x_1, k) \sim (x_2, -k'), k \in \bar{V}_{\underline{+}} \text{if } x_1 \in J_{\underline{+}}(x_2)\} \\ &\cup \{(x_1, k; x_1, -k), k \neq 0\}. \end{aligned} \quad (4.12)$$

Here $\bar{V}_{\underline{+}}$ is the closed forward respectively backward light cone and $J_{\underline{+}}(x_2)$ are all points in $\mathcal{M}$ in the future of x_2 or its past, respectively, which can be connected with x_2 by a causal curve γ. $(x_1, k) \sim (x_2, m)$ means that x_1 and x_2 can be connected by a light cone with cotangent vectors k and m at x_1 and x_2.
The sum of the second components of the wave-front sets of two propagators can vanish only at $x_1 = x_2$. Therefore the products (4.11) are defined on $(\mathbb{R}^4)^2 \setminus D_2$.
With (4.10) we are also able to investigate higher products of Feynman propagators occuring in higher orders of the S-matrix. For instance we discuss the wave-front set of the numerical distribution

$$t = \underbrace{(i\Delta_F(x_1 - x_2))^2}_{u} \underbrace{(i\Delta_F(x_2 - x_3))^2}_{v} \underbrace{(i\Delta_F(x_1 - x_3))}_{w}. \quad (4.13)$$

which occurs in our second order calculation of the S-matrix in Φ^4 theory.

$$\begin{aligned} WF(uvw) \subset\ & WF(u) \cup WF(v) \cup WF(w) \\ & \cup (WF(u) \oplus WF(v)) \cup (WF(w) \oplus WF(v)) \cup (WF(u) \oplus WF(w)) \\ & \cup (WF(u) \oplus WF(v) \oplus WF(w)). \end{aligned} \quad (4.14)$$

The wave-front sets in the first line of (4.14) yield problems for $x_3 \neq x_2 = x_1$ (in $WF(u)$) and for $x_2 = x_3 \neq x_1$ (in $WF(v)$), the ill-definedness of the product on this subset is treated in the renormalization of subdivergences. The remaining sums of wave-front sets can contain a zero component for $x_1 = x_2 = x_3$. This ill-definedness of the numerical distribution is called the superficial divergence.

4.3 Power Counting of Divergences

In momentum space calculations the ill-definedness of products of Feynman propagators corresponds to UV divergencies of loop integrals. Divergent terms can be found by counting the powers of momenta in the loop integrals (power counting). The superficial degree of divergence of a diagram corresponds in position space to the singular order at the total diagonal of the numerical distribution belonging to this diagram. We first introduce the definition of the scaling degree of a distribution:

Definition 4.3.1 *The scaling degree of a numerical distribution $t \in \mathcal{D}'(\mathbb{R}^{4n})$ at the origin is defined by*

$$sd(t) := \inf \left\{ \omega \left| \lim_{\epsilon \to 0} \epsilon^{\omega} t\left(\epsilon x_1, \ldots, \epsilon x_n\right) = 0 \textit{ in the sense of distributions} \right. \right\}. \qquad (4.15)$$

The singular order is now given by

Definition 4.3.2 *The singular order of a numerical distribution $t \in \mathcal{D}'(\mathbb{R}^{dn})$ at the origin is defined by*

$$sing\ ord\, t = [sd(t)] - nd \qquad (4.16)$$

where d is the space time dimension.

There exist different definitions of the scaling degree in the literature. The definition above is the Steinmann scaling degree [Ste]. The singular order of a distribution with respect to this scaling degree is not larger than the superficial degree of divergence obtained by power counting in momentum space [BrFr]. In [BrFr] there is also given a definition of a scaling degree of a distribution with respect to a submanifold, called microlocal scaling degree. With this microlocal scaling degree we obtain

$$sd\left(\left(i\Delta_F\left(x_1 - x_2\right)\right)^{4-k}\right) \quad = \quad 8 - 2k. \qquad (4.17)$$

At $x_1 = x_2$ the products of Feynman propagators have the singular order

$$sing\ ord\left(\left(i\Delta_F\left(x_1 - x_2\right)\right)^{4-k}\right) = 4 - 2k. \qquad (4.18)$$

For the distribution t in (4.13) we obtain $sd(t) = 10$ at $x_1 = x_2 = x_3$ whereas at $x_1 = x_2 \neq x_3$ and at $x_2 = x_3 \neq x_1$ we have $sd(t) = 4$ and at $x_2 \neq x_1 = x_3$ it holds $sd(t) = 2$.

The singular order of the superficial divergence is 2, the singular orders corresponding to the subdivergencies are 0 and -2.

4.4 Extension of Distributions

In momentum space integrals of diagrams with negative degree of divergence are finite in Euclidean calculations. To make divergent diagrams meaningful as well one has to renormalize them, and this procedure is not unique. Correspondingly we have the following two theorems in position space which are proved in [BrFr] on curved space times. The first one states that the extension of a distribution of negative singular order exists and is unique. The proof can already be found in [Fre]:

Theorem 4.4.1 *If $t^0(x_1,\ldots,x_n)\in\mathcal{D}'\left(\mathbb{R}^{4n}\setminus 0\right)$ has singular order $\delta<0$ at the origin, then a unique $t(x_1,\ldots,x_n)\in\mathcal{D}'\left(\mathbb{R}^{4n}\right)$ exists with the same singular order at 0 and*

$$t^0(\phi)=t(\phi)\qquad\qquad \forall\ \phi\in\mathcal{D}\left(\mathbb{R}^{4n}\setminus 0\right). \tag{4.19}$$

Distributions with zero or positive singular order δ can be extended, but the extension is not unique. We first remark that they are only defined on test functions vanishing sufficiently fast at 0:

$$\infty > sing\ ord\, t^0=\delta\geq 0\qquad\Rightarrow\qquad t^0\in\mathcal{D}'_\delta\left(\mathbb{R}^{4n}\right) \tag{4.20}$$

with

$$\begin{aligned}\mathcal{D}_\delta\left(\mathbb{R}^{4n}\right)\ &=\ \left\{\phi\in\mathcal{D}(\mathbb{R}^{4n})\mid D^\alpha\phi(0)=0\quad\text{for all multi-indices}\right.\\ &\qquad\left.\alpha=(\alpha_1,\ldots,\alpha_n)\ \text{ with } |\alpha|\leq\delta\right\}.\end{aligned} \tag{4.21}$$

Theorem 4.4.2 *For all $t^0(x_1,\ldots,x_n)\in\mathcal{D}'_\delta\left(\mathbb{R}^{4n}\right)$ with $sing\ ord\,t^0=\delta$ at 0, $0\leq\delta<\infty$, there exist numerical distributions $t\in\mathcal{D}'\left(\mathbb{R}^{4n}\right)$ with the same singular order δ at 0 and*

$$t^0(\phi)=t(\phi)\qquad\qquad \forall\ \phi\in\mathcal{D}_\delta\left(\mathbb{R}^{4n}\right). \tag{4.22}$$

In [BrFr] this is proved on curved space times.
To construct extensions of distributions with positive singular order a projection operator on test-functions is used.

Definition 4.4.1 *With*

$$W^{(k)}\left(\delta,w,(x_1,\ldots,x_k)\right):\ \mathcal{D}\left(\mathbb{R}^{4n}\right)\longrightarrow\ \mathcal{D}_\delta\left(\mathbb{R}^{4n}\right)$$

$$\begin{aligned}\phi(x_1,\ldots,x_n)\ \ \mapsto\ \ &\phi(x_1,\ldots,x_n)-\\ &-w(x_1)\cdot\ldots\cdot w(x_k)\sum_{\alpha_i=0\ \forall i>k}^{|\alpha|\leq\delta}\frac{x^\alpha}{\alpha!}D^\alpha\phi(y_1,\ldots,y_k,x_{k+1},\ldots x_n)\Big|_{y_1=\ldots=y_k=0}\end{aligned} \tag{4.23}$$

we define a projection operator on test-functions of the order δ in $x_1 \ldots x_k$ for each function $w \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ fulfilling $w(0) = 1$ and $D^\alpha w(0) = 0$ for all multi-indices α with $0 < |\alpha| \leq \delta$ and $\alpha_i = 0$ for all $i > k$.

$W^{(k)}$ is a modified Taylor subtraction operator in the k variables $x_1, \ldots, x_k$ of the test-functions. Since the function w has compact support the result is a function with compact support, and vanishes up to the order δ at 0. Therefore the singular distribution $t^0(x)$ with singular order δ is defined on all $W^{(n)}(\delta, w, x)\,\phi(x)$ with $x = (x_1, \ldots x_n)$. For $k < n$ the operator $W^{(k)}$ is used in the renormalization of subdivergences.
The following construction of the ("superficial") extension of a numerical distribution t^0, well-defined outside the origin, with $sing\ ord\ t^0 = \delta$ fulfills (4.22):

$$\begin{aligned} < t(x), \phi(x) > &= < t^0(x), W^{(n)}(\delta, w, x))\,\phi(x) > + \\ &+ \sum_{|\alpha| \leq \delta} \frac{(-1)^{|\alpha|} c^\alpha}{\alpha!} < \delta^\alpha(x), \phi(x) > . \end{aligned} \tag{4.24}$$

The free constants c^α express the ambiguity in the extension of the distribution, which remains after the function $w(x)$ in the W-operator is fixed; it holds

$$< t, wx^\alpha >= c^\alpha \qquad \text{for } |\alpha| \leq \delta. \tag{4.25}$$

The form of the c^α can further be restricted by demanding the invariance of the distribution t under symmetry operations, for instance the Lorentz invariance. The most simple choice of the c^α would be $c^\alpha = 0$. But we will see that this choice is incompatible with Lorentz invariance for $|\alpha| > 1$. Another choice of the c^α leads to the form of BPHZ subtraction at momentum q in momentum space. This choice of the c^α for $q \neq 0$ is of course again incompatible with the Lorentz invariance for $|\alpha| > 1$.
In [BrFr] it is shown that every extension of a distribution can be written as (4.24) with the following generalized W-operation:

$$W\phi = \phi - \sum_{\alpha \leq \delta} v_\alpha \partial^\alpha \phi(0) \tag{4.26}$$

with v_α being smooth functions of compact support with $\partial^\alpha v_\beta(0) = \delta^\alpha_\beta$. In the following we will restrict ourselves to the treatment of extensions with W-operators of the form (4.23).

4.5 The Lorentz Invariant Extension in Scalar Theories

It is possible to determine the free constants c^α of an extension, such that the result is a Lorentz invariant distribution. In [BrPiPr] the main idea of the calculation is introduced and an explicit result is obtained for scalar distributions in one variable. A further development of the techniques yields an inductive formula for a Lorentz invariant extension of an arbitrary distribution [Pra2]. In our calculations of the

second and third order of the S-matrix in Φ^4-theory we only need the calculations of [BrPiPr] which are repeated now.
We denote the action of the Lorentz group on $\mathbb{R}^{4n}$ by $x \to D(\Lambda)\,x$. This leads naturally to an action on $\mathcal{D}\left(\mathbb{R}^{4n}\right)$ and $\mathcal{D}'\left(\mathbb{R}^{4n}\right)$, respectively, given by

$$\begin{aligned} &\mathcal{D}\left(\mathbb{R}^{4n}\right) \ni \phi \mapsto \phi_\Lambda, && \phi_\Lambda(x) := \phi\left(D\left(\Lambda^{-1}\right)x\right) \\ &\mathcal{D}'\left(\mathbb{R}^{4n}\right) \ni t \mapsto t_\Lambda, && \langle t_\Lambda, \phi\rangle := \langle t, \phi_{\Lambda^{-1}}\rangle . \end{aligned}$$

According to equation (4.24), the extension has the form

$$\langle t, \phi\rangle = \left\langle t^0, \phi - w \sum_{|\alpha|\leq\omega} \frac{x^\alpha}{\alpha!} D^\alpha\phi(0)\right\rangle + \sum_{|\alpha|\leq\omega} \frac{c^\alpha}{\alpha!} D^\alpha\phi(0). \tag{4.27}$$

After performing a Lorentz transformation we obtain

$$\langle t_\Lambda, \phi\rangle = \langle t, \phi_{\Lambda^{-1}}\rangle = \left\langle t^0, \phi - w_\Lambda \sum_{|\alpha|\leq\omega} \frac{x^\alpha}{\alpha!} D^\alpha\phi(0)\right\rangle + \sum_{|\alpha|\leq\omega} \frac{(D(\Lambda)c)^\alpha}{\alpha!} D^\alpha\phi(0). \tag{4.28}$$

In order to be a Lorentz invariant extension, the difference of (4.27) and (4.28) has to be zero:

$$\sum_{|\alpha|\leq\omega} \frac{D^\alpha\phi(0)}{\alpha!} \left\langle t^0, (w - w_\Lambda)\, x^\alpha\right\rangle = -\sum_{|\alpha|\leq\omega} \left((D(\Lambda)-1)\,c\right)^\alpha \frac{D^\alpha\phi(0)}{\alpha!}. \tag{4.29}$$

So we have to solve

$$\left\langle t^0, (w - w_\Lambda)\, x^\alpha\right\rangle = -\left((D(\Lambda)-1)\,c\right)^\alpha \tag{4.30}$$

for all α, where c^α is a tensor of rank $|\alpha|$ and $D(\Lambda)$ is the corresponding tensor representation of the Lorentz group.
From now on, we restrict ourselves to the case of distributions in one coordinate. In this case only the totally symmetric part of the c^α contributes to (4.27). Using Lorentz indices, (4.30) reads

$$\left\langle t^0, (w - w_\Lambda)\, x^{\mu_1} \dots x^{\mu_n}\right\rangle = -\left(\Lambda^{\mu_1}{}_{\beta_1} \dots \Lambda^{\mu_n}{}_{\beta_n} - \delta^{\mu_1}_{\beta_1} \dots \delta^{\mu_n}_{\beta_n}\right) c^{\beta_1\dots\beta_n} \tag{4.31}$$

where $n = |\alpha|$. Using infinitesimal transformations we can solve these equations for c inductively.
In the case $|\alpha| = 0$, (4.30) is fulfilled for all choices of c since the 1-dimensional representation of the Lorentz group is trivial. For $|\alpha| \geq 1$, the solution is unique up to Lorentz invariant contributions consisting only of symmetrized tensor products of the metric tensor $g^{\mu\nu}$ (which generate Lorentz invariant counterterms like $\Box\delta(x)$).
We use the generators of Lorentz transformations

$$\left(l^{\alpha\beta}\right)^\nu{}_\mu = g^{\beta\nu}\delta^\alpha_\mu - g^{\alpha\nu}\delta^\beta_\mu. \tag{4.32}$$

The representation of a Lorentz transformation on $\mathbb{R}^4$ has the form

$$\Lambda^\nu{}_\mu = \delta^\nu_\mu + \frac{1}{2}\Theta_{\alpha\beta}(l^{\alpha\beta})^\nu{}_\mu + O(\Theta^2) \tag{4.33}$$

with infinitesimal parameters $\Theta_{\alpha\beta}$ satisfying $\Theta_{\alpha\beta} = -\Theta_{\beta\alpha}$. We obtain

$$w(x) - w(\Lambda^{-1}x) = \frac{1}{2}\Theta_{\alpha\beta}(l^{\alpha\beta})^\nu{}_\mu x^\mu \partial_\nu w(x) + O(\Theta^2). \tag{4.34}$$

We use the abbreviations

$$n!! = \begin{cases} 2 \cdot 4 \cdot \ldots \cdot n & \text{for } n \text{ even} \\ 1 \cdot 3 \cdot \ldots \cdot n & \text{for } n \text{ odd} \end{cases} \tag{4.35}$$

and

$$b^{(\alpha_1 \ldots \alpha_n)} = \frac{1}{n!} \sum_{\pi \in S_n} b^{\alpha_{\pi(1)} \ldots \alpha_{\pi(n)}} \tag{4.36}$$

for the total symmetric part of a tensor b. Now we prove by induction over $|\alpha|$ that the symmetric part of c^α for $|\alpha| > 0$ is given by (up to the aforementioned ambiguity for even $|\alpha|$):

$$\begin{aligned} c^{(\alpha_1 \ldots \alpha_n)} = \frac{(n-1)!!}{(n+2)!!} \sum_{s=0}^{\left[\frac{n-1}{2}\right]} \frac{(n-2s)!!}{(n-1-2s)!!} g^{(\alpha_1\alpha_2} \ldots g^{\alpha_{2s-1}\alpha_{2s}} \times \\ \times \left\langle t^0, (x^2)^s x^{\alpha_{2s+1}} \ldots x^{\alpha_{n-1}} \left(x^2 \partial^{\alpha_n)} w - x^{\alpha_n)} x^\beta \partial_\beta w \right) \right\rangle. \end{aligned} \tag{4.37}$$

At the beginning of the induction we determine the c^α for $|\alpha| = 1$ and $|\alpha| = 2$.

1. $|\alpha| = 1$. We obtain

$$\begin{aligned} ((D(\Lambda) - 1)\, c)^\nu &= \frac{1}{2}\Theta_{\alpha\beta}(l^{\alpha\beta})^\nu{}_\mu c^\mu \\ &\overset{(4.34)(4.30)}{=} - \left\langle t^0, \frac{1}{2}\Theta_{\alpha\beta}(l^{\alpha\beta})^\rho{}_\mu x^\mu \partial_\rho w x^\nu \right\rangle, \end{aligned} \tag{4.38}$$

which yields (using the independence of $\Theta_{\alpha\beta}$)

$$(l^{\alpha\beta})^\nu{}_\mu c^\mu = - \left\langle t^0, (l^{\alpha\beta})^\rho{}_\sigma x^\sigma \partial_\rho w x^\nu \right\rangle. \tag{4.39}$$

Inserting the expression (4.32) for the $(l^{\alpha\beta})$ yields

$$g^{\beta\nu} c^\alpha - g^{\nu\alpha} c^\beta = - \left\langle t^0, \left(x^\alpha \partial^\beta w - x^\beta \partial^\alpha w \right) x^\nu \right\rangle. \tag{4.40}$$

Contracting finally with $g_{\nu\beta}$ on both sides yields

$$c^\alpha = \frac{1}{3} \left\langle t^0, x^2 \partial^\alpha w - x^\alpha x^\beta \partial_\beta w \right\rangle. \tag{4.41}$$

2. $|\alpha| = 2$. We obtain from (4.31)

$$\langle t^0, (w - w_\Lambda)\, x^{\alpha_1} x^{\alpha_2} \rangle = -\left(D(\Lambda)^{\alpha_1}{}_{\beta_1} D(\Lambda)^{\alpha_2}{}_{\beta_2} - \delta^{\alpha_1}_{\beta_1} \delta^{\alpha_2}_{\beta_2} \right) c^{\beta_1 \beta_2}. \tag{4.42}$$

With (4.33) and (4.34) we get

$$\left\langle t^0, (l^{\alpha\beta})^\rho{}_\mu x^\mu \partial_\rho w x^{\alpha_1} x^{\alpha_2} \right\rangle = (l^{\alpha\beta})^{\alpha_1}{}_\sigma c^{\sigma\alpha_2} + (l^{\alpha\beta})^{\alpha_2}{}_\sigma c^{\sigma\alpha_1}. \tag{4.43}$$

Inserting the form (4.32) of the $(l^{\alpha\beta})$ and contracting both sides with $g_{\beta\alpha_1}$ yields

$$4c^{\alpha\alpha_2} - g^{\alpha\alpha_2} c^\mu{}_\mu = -\langle t^0, x^\alpha x^{\alpha_2} x^\sigma \partial_\sigma w - x^{\alpha_2} x^2 \partial^\alpha w \rangle. \tag{4.44}$$

As in the case $|\alpha| = 0$ we choose $c^\mu{}_\mu = 0$ and obtain (by setting $\alpha = \alpha_1$):

$$c^{(\alpha_1\alpha_2)} = -\frac{1}{4} \left\langle t^0, x^{\alpha_1} x^{\alpha_2} x^\sigma \partial_\sigma w - x^2 x^{(\alpha_1} \partial^{\alpha_2)} w \right\rangle. \tag{4.45}$$

We now assume that (4.37) holds for all integers smaller than n and describe the induction $|\alpha| = n - 2 \to |\alpha| = n$. With the examples of the beginning of the induction, it is easy to see how the calculation is done for higher $|\alpha|$. For $|\alpha| = n$ we obtain instead of (4.40) the following equation:

$$\sum_{i=1}^n \left(g^{\beta\alpha_i} \delta^\alpha_\mu - g^{\alpha\alpha_i} \delta^\beta_\mu \right) c^{\mu\alpha_1 \ldots \check{\alpha}_i \ldots \alpha_n} = \\ = -\left\langle t^0, \left(x^\alpha \partial^\beta w - x^\beta \partial^\alpha w \right) x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_n} \right\rangle. \tag{4.46}$$

On the left-hand side only the symmetric traceless part of the c^α yields a contribution. Contracting (4.46) with $g_{\beta\alpha_1}$ yields

$$(n+2) c^{\alpha\alpha_2 \ldots\ldots \alpha_n} - \sum_{i \geq 2} g^{\alpha\alpha_i} c_\rho{}^{\rho\alpha_2 \ldots \check{\alpha}_i \ldots \alpha_n} = \\ = -\langle t, \left(x^\alpha x^\rho \partial_\rho w - x^2 \partial^\alpha w \right) x^{\alpha_2} \ldots x^{\alpha_n} \rangle. \tag{4.47}$$

The second term on the left-hand side of (4.47) is determined by the induction hypothesis because it is the solution of the problem for $|\alpha| = n - 2$ for the distribution $x^2 t$. Setting $\alpha = \alpha_1$ and symmetrizing in the indices $\alpha_1 \ldots \alpha_n$, the form (4.37) is obtained. This was the repetition of the calculations published in [BrPiPr].
Choosing the coefficients according to (4.37) the remaining freedom in the renormalization procedure is the choice of the function w and the Lorentz-invariant counterterms. In the following we set all counterterms not depending on w equal to 0, so only the contributions of (4.37) are nonvanishing. In the calculations of the next section we need the coefficients $c^{(\alpha)}$ for $|\alpha| = 0$ and $|\alpha| = 2$, because we want to renormalize in Φ^4 theory only diagrams with 2 and 4 external legs. Therefore we have $sing\, ord\, t \leq 2$. Furthermore we only regard the case of symmetric testfunctions.
In the case $|\alpha| = 0$ all choices of c are Lorentz invariant and we set $c = 0$. For $|\alpha| = 2$ we have

$$c^{(\alpha_1\alpha_2)} = -\frac{1}{4}\left\langle t^0, x^{\alpha_1}x^{\alpha_2}x^\sigma\partial_\sigma w - x^2x^{(\alpha_1}\partial^{\alpha_2)}w\right\rangle. \tag{4.48}$$

In the following we denote with $t^0$the class of all Lorentz invariant extended distributions, in this case we obtain by partial integration

$$c^{(\alpha_1\alpha_2)} = -\frac{1}{4}\left\langle\partial^{(\alpha_2}x^2x^{\alpha_1)}t^0 - \partial_\beta x^{\alpha_1}x^{\alpha_2}x^\beta t^0, w\right\rangle. \tag{4.49}$$

The numerical distributions t^0 are products of Feynman propagators and depend only on x^2:

$$\partial_\sigma t^0 = 2x_\sigma(t^0)' \tag{4.50}$$

Therefore we have

$$\begin{aligned} c^{\alpha_1\alpha_2} &= \left\langle x^{\alpha_1}x^{\alpha_2}t^0, w\right\rangle - \frac{1}{4}\left\langle x^2g^{\alpha_1\alpha_2}t^0, w\right\rangle \\ &= \left\langle t^0, (x^{\alpha_1}x^{\alpha_2} - \frac{1}{4}x^2g^{\alpha_1\alpha_2})w\right\rangle. \end{aligned} \tag{4.51}$$

In the following calculations there contribute only terms which are even in the variables. Since only the support of the function w is important we can assume without loss of generality that the function is even, too. It follows that the Lorentz invariant extension of a distribution of singular order 2 with a symmetric test function has according to (4.24) the form

$$\langle t, \phi\rangle = \left\langle t^0, \phi(x) - w(x)\phi(0) - \frac{w(x)}{8}x^2\Box\phi(0)\right\rangle. \tag{4.52}$$

4.6 Properties of the W-Operator

1. As a projection operator, $W^{(k)}$ fulfills $W^{(k)2} = W^{(k)}$.

   For $n \geq l \geq m$, $\delta \geq \delta'$, the following relation holds:

$$\begin{aligned} W^{(l)}\left(\delta', w', (x_1, \ldots, x_l)\right) W^{(m)}\left(\delta, w, (x_1, \ldots, x_m)\right)\phi\left(x_1, \ldots, x_n\right) = \\ W^{(m)}\left(\delta, w, (x_1, \ldots, x_m)\right)\phi\left(x_1, \ldots, x_n\right). \end{aligned} \tag{4.53}$$

2. The renormalization scale

   Characteristic for every renormalization procedure is the occurence of a mass scale, the renormalization scale. In the extension of the T-products the W-operation depends on a function $w(x)$. Since the argument of this function should be dimensionless it depends implicitly on a mass scale: $w = w(mx)$. The shape of the function w describes the subtraction procedure; in the region of $w = 1$ the subtraction is the full Taylor subtraction, whereas nothing is subtracted outside the support of w.

Varying the mass scale m with fixed w, we change the support region of the function w and regulate in this way the subtraction procedure. In the limit $m \to 0$ we have $w(mx) \equiv 1$ and the Taylor subtraction acts everywhere, whereas in the limit $m \to \infty$ the support of $w(mx)$ shrinks to the origin and we subtract only at this point.

In the following, we continue to write $w(x)$, and only if we need the dependence on the mass scale we will write this function as $w(mx)$.

A variation of the mass scale yields

$$w((m+\delta m)z) - w(mz) = \delta m \frac{\partial}{\partial m} w(mz) = \delta m z^{\mu} \partial_{\mu} w. \tag{4.54}$$

4.7 Comparison with Differential Renormalization

Differential renormalization (DR) [FrJoLa] is another renormalization method in coordinate space. The singular distributions are written as derivatives of less singular distributions which contain a logarithmic mass scale, for instance

$$\left.\frac{1}{x^4}\right|_R \equiv -\frac{1}{4}\Box \frac{\ln(M^2x^2)}{x^2} \tag{4.55}$$

in massless Φ^4 theory. This yields:

$$\int d^4x \, \left(\frac{1}{x^4}\right)_R f(x) = -\frac{1}{4}\int d^4x \frac{\ln M^2x^2}{x^2} \left(\Box f(x)\right), \tag{4.56}$$

In [FrJoLa] this formula was explained by saying that there are hidden counterterms canceling the divergent surface integral. They arise in partial integrations where the distributions are treated as functions.

In [AgCuMu] these techniques are improved by giving a set of rules for the calculations. They tell us how to expand Feynman graphs in a set of basic functions and fix their renormalization so that the Ward identities of the theory are automatically fulfilled at one-loop order for abelian and non-abelian gauge theories. This is called constrained DR (CDR).

In many renormalization prescriptions a Taylor subtraction is used. In this case Zavialov had the idea that the remaining part after the subtraction is the residue term of a Taylor expansion and can be written in the form of an integral operator [Zav]. Applying this to the Taylor subtraction of the W-operation formalism, it is shown in [Pra1] that the W-operation acts on the test functions through differentiation. The integral operator of [Pra1] is more complicated than the one given by [Zav] because the W-operation is not a real Taylor subtraction but modified by the function w. Explicit calculations with this more complicated formula succeeded up to now only in the case where $w(x)$ is assumed to be the step function $\Theta(x)$. Since $\Theta(x)$ is not a test function this is only allowed if the singular support of the distribution does not intersect with the singular support of $\Theta(x)$. In Minkowski space renormalization this calculation is not applicable because the singular support of the Feynman propagator lies on the whole light-cone.

Here we give another comparison of the two methods in the euclidean space which holds for all allowed test functions but becomes unfortunately very complicated in the case of higher singularities. Starting point of our derivation is equation (4.56):

$$
\begin{aligned}
< \frac{1}{(x^2)^2}, \phi(x) >_{Ren} &= -\frac{1}{4}\int d^4x \frac{\ln M^2x^2}{x^2}\left(\partial_\mu\partial^\mu\phi(x)\right) \\
&= \lim_{\epsilon\to 0}\Bigg[-\frac{1}{4}\int_{\mathbb{R}^4\setminus B_\epsilon(0)} d^4x \left(\Box\frac{\ln(M^2x^2)}{x^2}\right)\phi(x) + \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^4\setminus B_\epsilon(0)} d^4x\ \partial^\mu\left(\left(\partial_\mu\frac{\ln(M^2x^2)}{x^2}\right)\phi(x)\right)\Bigg],
\end{aligned}
\tag{4.57}
$$

the surface term of the first partial integration vanishes. The second term on the right-hand side yields the surface integral

$$
\frac{1}{4}\int_{B_\epsilon(0)} df^\mu\partial_\mu\left(\frac{\ln(M^2x^2)}{x^2}\right)\phi(x), \tag{4.58}
$$

where $B_\epsilon(0)$ is a sphere of radius ϵ around the origin and $df^\mu = \epsilon^3\hat{x}^\mu d\hat{x}$ is the outward normal volume element of the 3-sphere S_ϵ.

A Taylor expansion of $\phi(x)$ for small x modified for $x \neq 0$ yields

$$
\phi(x) = w(x)\phi(0) + w(x)x^\mu\partial_\mu\phi(0) + O(x^2) \tag{4.59}
$$

with $w(x) \in \mathcal{C}_0^\infty\left(\mathbb{R}^4\right)$ and $w(0) = 1$.

Inserting (4.59) in the surface integral (4.58), only the first term yields a nonvanishing contribution for $\epsilon \to 0$:

$$
\begin{aligned}
&\frac{1}{4}\int_{B_\epsilon(0)} df^\mu\left(\partial_\mu\frac{\ln(M^2x^2)}{x^2}\right)w(x)\phi(0) = \\
&= \frac{1}{4}\int_{\mathbb{R}^4\setminus B_\epsilon(0)} d^4x\ \partial^\mu\left(\left(\partial_\mu\frac{\ln(M^2x^2)}{x^2}\right)w(x)\phi(0)\right) \\
&= \frac{1}{4}\int_{\mathbb{R}^4\setminus B_\epsilon(0)} d^4x\left(\Box\left(\frac{\ln(M^2x^2)}{x^2}\right)w(x)\phi(0) + \partial_\mu\frac{\ln(M^2x^2)}{x^2}\partial^\mu w(x)\phi(0)\right).
\end{aligned}
\tag{4.60}
$$

Together with the other term of (4.57) it yields

$$
\begin{aligned}
< \frac{1}{(x^2)^2}, \phi(x) >_{Ren} &= \int d^4x \frac{1}{(x^2)^2} \left(\phi(x) - w(x)\phi(0)\right) + \frac{1}{4} \int d^4x \; \partial^\mu w(x) \partial_\mu \frac{\ln(M^2x^2)}{x^2} \phi(0) \\
&= \int d^4x \frac{1}{(x^2)^2} W^{(1)}(0, w, x)\phi(x) + \frac{1}{4} \int d^4x \; \partial^\mu w(x) \partial_\mu \frac{\ln(M^2x^2)}{x^2} \phi(0),
\end{aligned}
\tag{4.61}
$$

which is exactly the extension of $\frac{1}{x^4}$ according to (4.24).
Fixing the function $w(mx)$ and the scales according to $M = m$ in (4.61), the coefficient c corresponding to DR is determined:

- In case of $w(mx) = \Theta\left(1 - m^2x^2\right)$ treated in [Pra1] we obtain:

$$
\begin{aligned}
c &= \frac{1}{(x^2)^2} \int d^4x \left(\partial^\mu \Theta\left(1 - m^2x^2\right)\right) \partial_\mu \frac{\ln(M^2x^2)}{x^2} \\
&= \int d^4x \; m^2 \delta\left(1 - m^2x^2\right) \left(\frac{1}{x^2} - \frac{\ln(M^2x^2)}{x^2}\right) \\
&= \int_0^\infty dr \; 2\pi^2 r^3 m^2 \delta\left(1 - m^2x^2\right) \left(\frac{1}{r^2} - \frac{\ln(M^2r^2)}{r^2}\right) \\
&= 2\pi^2 - 2\pi^2 \ln \frac{M^2}{m^2}.
\end{aligned}
\tag{4.62}
$$

  We see that the renormalization of the basic function $\frac{1}{x^4}$ differ in [FrJoLa] and [Pra1] because in the formula of [Pra1] all the coefficients c are zero.

- With (4.61) we are moreover able to determine the coefficient c for the function

$$
w(x) = \begin{cases} \exp\left(-\frac{1}{1-m^2x^2}\right) & \text{for } |mx| \le 1 \\ 0 & \text{for } |mx| > 1 \end{cases} .
\tag{4.63}
$$

  We obtain

$$
\begin{aligned}
c &= -\frac{1}{4} \int_{|x^2|<m^{-2}} d^4x \; \frac{m^2}{(1 - m^2x^2)^2} \left(\frac{1}{x^2} - \frac{\ln(M^2x^2)}{x^2}\right) \exp\left(-\frac{1}{1 - m^2x^2}\right) \\
&= 2\pi^2 \int_0^1 dy \; \left[\frac{y}{(1 - y^2)^2} - \frac{y}{(1 - y^2)^2} \ln \frac{M^2y^2}{m^2}\right] \exp\left(-\frac{1}{1 - y^2}\right) \\
&= \frac{\pi^2}{e} \left(1 - \ln \frac{M^2}{m^2}\right) + 2\pi^2 \int_0^1 dy \; \frac{y \ln y^2}{(1 - y^2)^2} \exp\left(-\frac{1}{1 - y^2}\right).
\end{aligned}
\tag{4.64}
$$

  The contribution of the second integral has to be determined numerically, with the help of the functions `evalf` and `int` of the computer program package Maple [Map] one obtains

$$c = -0.63017689 - \frac{\pi^2}{e} \ln \frac{M^2}{m^2}.$$

The transformation of renormalized terms with higher singularities becomes more difficult and the meaning of the rules of CDR in the language of the W-operation also remains an open question.

4.8 Comparison with BPHZ-Renormalization

The definition of composite operators and the Zimmermann identities [Zim] were first formulated in BPHZ-renormalization in the version described by Zimmermann [Zim1]. To show the connection between the BPHZ-formulation and our approach we compare in this section these two ways of renormalization.
In BPHZ-renormalization the integrand I_Γ of a divergent momentum space Feynman integral is changed by the so called R-operation, such that only the integrand R_Γ of the finite part remains. If there are more than one divergence in a diagram the forest formula describes the combinatorics of the R-subtraction procedures.
In the first subsection we compare the R-operation with the W-operation. It turns out that the coefficients c^α of the W-operation can be chosen in such a way that the W-operation coincides with the R-operation, in particular one obtains the same renormalized distributions in both ways. In the second subsection we compare the combinatorics of the two ways of renormalization. There are differences because the definition of divergent subdiagrams is different in coordinate and momentum space. We derive the Hopf algebra structure in Epstein-Glaser renormalization. In the last subsection we compare the two methods in an explicit calculation of a diagram. Despite of the different combinatorics the result of the two methods is the same.

4.8.1 Subtraction Procedure

Bogoliubov's R-operation is a Taylor subtraction in momentum space. More precisely, if we have a diagram without subdivergences the R-operation is a subtraction of the first terms of the Taylor series in the external momenta of the diagram. Performing a Fourier transformation the external momenta are conjugate to the differences of vertex coordinates of the diagram. These coordinate differences belong to the arguments of the W-operation in Epstein-Glaser renormalization because the Feynman propagators of a diagram depend only on these differences. If every vertex has an external line the number of momenta in the BPHZ subtraction coincides with the number of arguments of the W-operation. If there are vertices with no external lines in a diagram there will be more arguments in the W-operation than in the Taylor subtraction of the R-operation.
In [Pra1] it is shown that the BPHZ-regularized distribution with subtraction point q corresponds to the distribution renormalized with the W-operation using the function $w(x) = \exp(iqx)$; unfortunately this function is not allowed in the W-operation because it has no compact support. Therefore one has to prove in calculations with

this function in each step that everything is well-defined. The advantage is that the calculation itself becomes more easy with this function. In the case where q is totally spacelike this choice of $w(x)$ is allowed according to [Pra3].
The right choice of the c_α also leads with allowed functions $w(x)$ to a Taylor subtraction at a point q in momentum space. Under the assumption that there exists a point $q \in \mathbb{R}^{4n}$ where the derivative $D^b\hat{T}(q)$ exists in the usual sense of functions for all $|b| \leq \omega$, it is shown in [Scha] that

$$\hat{T}_{BPHZq}(p) := \hat{T}_R(p) - \sum_{|b|\leq\delta} \frac{(p-q)^b}{b!} D^b\hat{T}_R(q) \tag{4.65}$$

(where $\hat{T}_R(p)$ is the distribution in momentum space renormalized with the W-operation with coefficients $c_\alpha = 0$) fulfills the normalization condition

$$D^b\hat{T}_{BPHZq}(q) = 0 \qquad\qquad \forall\ |b| \leq \omega \tag{4.66}$$

and is further uniquely specified by this condition.

4.8.2 Combinatorics

The combinatoric in Epstein-Glaser renormalization is hidden in the structure of finite renormalizations. We know from Theorem (3.0.1):

$$\hat{T}_n\left(\bigotimes\nolimits_{j\in J} f_j\right) = \sum_{P\in Part(J)} T_{|P|}\Big[\bigotimes_{O_i\in P} \Delta_{|O_i|}\left(\bigotimes\nolimits_{j\in O_i} f_j\right)\Big]. \tag{4.67}$$

We now only regard the contribution of T_n to a special diagram γ with n vertices. A subdiagram of γ is defined by the vertices belonging to it. All lines of γ between the vertices of the subdiagram belong to the subdiagram. γ_{O_i} is the subdiagram with vertices V_j with $j \in O_i$. $\gamma \setminus \gamma_{O_i}$ is the reduced diagram obtained from γ by contracting each line belonging to γ_{O_i} to a point. We define $\gamma_{O_i} = \emptyset$ if O_i consists of only one element. We use

$$\gamma_P = \prod_{i=1}^{|P|} \gamma_{O_i} \tag{4.68}$$

With this we obtain from (4.67):

$$\hat{T}_n^\gamma\left(\bigotimes\nolimits_{j\in J} f_j\right) = T_{|P|}^\gamma\left(\bigotimes\nolimits_{j\in J} f_j\right) + \sum_{\substack{P\in Part(J)\\ |P|<n}} T_{|P|}^{\gamma\setminus\gamma_P}\Big[\bigotimes_{O_i\in P} \Delta_{|O_i|}^{\gamma_{O_i}}\left(\bigotimes\nolimits_{j\in O_i} f_j\right)\Big], \tag{4.69}$$

where Δ^γ is defined inductively by

$$T_1\left(\Delta_n^\gamma\left(\bigotimes_{j\in J} f_j\right)\right) = \hat{T}_n^\gamma\left(\bigotimes_{j\in J} f_j\right) - \sum_{\substack{P\in Part(J)\\ |P|>1}} T_{|P|}^{\gamma\setminus\gamma_P}\left[\bigotimes_{O_i\in P}\Delta_{|O_i|}^{\gamma_{O_i}}\left(\bigotimes_{j\in O_i} f_j\right)\right]. \tag{4.70}$$

In the following we omit the lower indices and the arguments of the T-products.
In momentum space, a subdiagram is defined by the lines belonging to it. All vertices which are endpoints of lines of a subdiagram belong to the subdiagram. So in momentum space the set of subdiagrams is larger because not all lines between two vertices of the diagram have to belong to the subdiagram. Proper subdiagrams of nonegative dimension are called renormalization parts.
Zimmermann described in [Zim1] the combinatorics of BPHZ renormalization and gave the following formula for the finite part R_Γ of an integrand I_Γ of a Feynman integral belonging to a diagram Γ:

$$R_\Gamma(K,q) = I_\Gamma(K,q) + \sum_{\gamma_1\cdots\gamma_c} I_{\Gamma\setminus\gamma_1\ldots\gamma_c}(K,q)\prod_{\tau=1}^{c}\left(-t_{q^{\gamma_\tau}}^{d(\gamma_\tau)}\bar{R}_{\gamma_\tau}\right). \tag{4.71}$$

In the above formula I_Γ is the integrand belonging to the diagram Γ, q are the corresponding basic internal momenta (they are also called external momenta) and K can be chosen as loop momenta. t_q^d applied to a function $f(q)$ denotes the Taylor series in the components of the vectors q up to the order d. The sum is over all sets of renormalization parts of Γ which are mutually disjoint and Γ itself. $\bar{R}_\gamma$ is the integrand of Γ with renormalized subdivergencies:

$$\bar{R}_\gamma(K^\gamma,q^\gamma) = I_\gamma(K^\gamma,q^\gamma) + {\sum}'_{\substack{\gamma_1\cdots\gamma_c\\ \gamma_i\cap\gamma_j=\emptyset}} I_{\Gamma\setminus\gamma_1\ldots\gamma_c}(K^\gamma,q^\gamma)\prod_{\tau=1}^{c}\left(-t_{q^{\gamma_\tau}}^{d(\gamma_\tau)}\bar{R}_{\gamma_\tau}\right). \tag{4.72}$$

Here the sum $\sum'$ is over all sets of renormalization parts $\gamma_a \neq \gamma$ of γ which are mutually disjoint and K^γ, q^γ are the loop and external momenta of the diagram γ.
Comparing (4.72) with (4.69) we see that both renormalizations consist of a sum of terms, each belonging to a set of renormalization parts of Γ. The difference between the two formulations is the different definition of subdiagrams and that (4.69) is formulated for finite and (4.69) for infinite renormalizations.
Kreimer [Kre] found a Hopf algebra structure hidden in the combinatorics (4.72) of the singularities and we now show that there can be found the same structure in Epstein Glaser renormalization.
Recently Bondía and Lazzarini discussed the Hopf algebra structure in Epstein Glaser renormalization, but they adapted a formulation of the forest formula of Zimmermann with the following form:

$$R_\Gamma f(\Gamma) = \left[1 + \sum_{\mathcal{F}}\prod_{\gamma\in\mathcal{F}}(-S_\gamma)\right] f(\Gamma), \tag{4.73}$$

where the sum is over all nonempty sets whose elements are proper, divergent (and may be connected) subgraphs of Γ. The problem with this formula is that R_Γ is not an operator defined on the space of distributions. Especially it cannot be adjoint to give an operator on the testfunction $f(\Gamma)$ is smeared with.
In our derivation we first describe the algebra of graphs and then we show how the antipode is connected with the structure of renormalization given in (4.69).
We take the algebra $\mathcal{A}$ of graphs with multiplication and addition as in [CoKr2]:
$\mathcal{A}$ has a basis labelled by all Feynman graphs Γ which are disjoint unions of connected graphs:

$$\Gamma = \bigcup_{j=1}^{n} \Gamma_j. \tag{4.74}$$

The empty graph $\Gamma = \emptyset$ is the unit element of the algebra.
The product in $\mathcal{A}$ is bilinear and given on the basis by the operation of disjoint union:

$$\begin{aligned} m : \mathcal{A} \otimes \mathcal{A} &\rightarrow \mathcal{A} \\ m(\Gamma_1, \Gamma_2) = \Gamma_1 \cdot \Gamma_2 &= \Gamma_1 \cup \Gamma_2. \end{aligned} \tag{4.75}$$

This means that the product of two graphs is a graph consisting of this two disconnected subgraphs.
For the definition of a coproduct we need the notion of a subgraph in Epstein-Glaser renormalization discussed above. In a graph Γ with n vertices labelled with the set of indices $J = \{1, \dots, n\}$ each subset I of J defines a subgraph γ_I consisting of all vertices labelled by I and all lines joining them.
If the set I consists of only one element, the subgraph has only one point and is defined as the empty graph.
Every partition P of J defines a set of subgraphs, more precisely if $P = \{O_1, \dots, O_k\}$ the subgraphs are those belonging to the subsets O_i.
Now we define a coproduct on $\mathcal{A}$:

$$\begin{aligned} c &: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \\ c(\Gamma) &= \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + {\sum}'_{\substack{P \in Part J \\ 1 \neq |P| \neq n}} \gamma_P \otimes \Gamma \setminus \gamma_P \qquad \forall \Gamma \neq \emptyset, \end{aligned} \tag{4.76}$$

where the sum $\sum'$ is over all partitions with subgraphs of γ_{O_i} where γ_{O_i} is the empty graph or has only two or four external lines. $\mathbb{I}$ is the unit element of the algebra and Γ is a diagram with $|J|$ vertices. $\Gamma \setminus \gamma_P$ is obtained from Γ by shrinking all its nonempty subdiagrams to a point. In the special case $\Gamma = \emptyset$ we define

$$c(\emptyset) = \mathbb{I} \otimes \mathbb{I}. \tag{4.77}$$

The proof of the coassociativity is analogous to that of [CoKr2].

A counit is defined by

$$\begin{aligned}\bar{e}(\Gamma) &= 1 \qquad for \qquad \Gamma = \emptyset \\ \bar{e}(\Gamma) &= 0 \qquad for \qquad \Gamma \neq \emptyset\end{aligned} \tag{4.78}$$

The defining equation for the antipode is

$$m(s \otimes id)c(\Gamma) = \bar{e}(\Gamma) \cdot \mathbb{I}. \tag{4.79}$$

From this equation we obtain a recursion formula for the antipode:

$$s(\Gamma) = -\Gamma - \sum_{\substack{P \in Part(J) \\ 1 \neq |P| \neq n}} s(\gamma_P) \cdot \Gamma \setminus \gamma_P \tag{4.80}$$

for $\gamma \neq \mathbb{I}$ and

$$s(\mathbb{I}) = \mathbb{I}. \tag{4.81}$$

This antipode describes the structure of singularities. We now want to show that we can derive the same structure from the formula of finite renormalizations (4.69). We have (omitting the lower indices of the T-products)

$$\hat{T}^\Gamma = T^\Gamma + \Delta^\Gamma + \sum_{\substack{P \in Part(J) \\ 1 < |P| < n}} T^{\Gamma \setminus \gamma_P} \left[\Delta^{\gamma_P} \right]. \tag{4.82}$$

Now we formulate this equation on the level of numerical distributions and denote with t^γ the numerical distribution belonging to the diagram γ and with δ^γ the numerical distribution belonging to the diagram $T_1(\Delta^\gamma)$:

$$\hat{t}^\Gamma = t^\Gamma + \delta^\Gamma + \sum_{\substack{P \in Part(J) \\ 1 < |P| < n}} t^{\Gamma \setminus \gamma_P^{(i)}} \delta^{\gamma_P^{(i)}}, \tag{4.83}$$

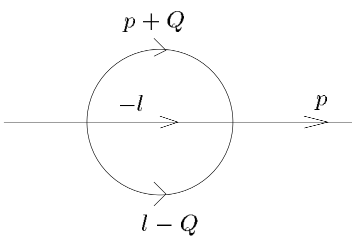
where we sum over the multiindices (i) having one value for each connected component of the O_i belonging to P (compare [CoKr2]). We now assume that t are the regularized but unrenormalized distributions and $\hat{t}$ are the regularized and renormalized distributions. Then $\delta(t)$ describes the renormalization of the superficial divergence. Taking only the parts of (4.83) which are divergent when the regularization is removed and summing over all multiindices (i) which are not explicitly written down in the following, we obtain:

$$\delta^\Gamma = -t^\Gamma - \sum_{\substack{P \in Part(J) \\ 1 < |P| < n}} t^{\Gamma \setminus \gamma_P} \delta^{\gamma_P}, \tag{4.84}$$

which has to be interpreted as the structure of singularities in renormalization. This recursive equation for the singularity of δ^Γ has exactly the same structure as equation (4.80) for the antipode (one has to identify a diagram with the singularity of its numerical distribution).

4.8.3 Example

In this subsection we show that the BPHZ renormalization of the setting sun yields the same result as the Epstein-Glaser renormalization.



yields the same result as the Epstein-Glaser renormalization.
In contrast to Epstein-Glaser renormalization this diagram has three subdivergencies in BPHZ renormalization. We show that the contributions of the renormalization of the subdiagrams vanish, such that the result of both renormalizations is the same. We obtain

$$\begin{aligned}\bar{R} &= i\tilde{\Delta}_F(l-Q)i\tilde{\Delta}_F(l)i\tilde{\Delta}_F(p+Q) - i\tilde{\Delta}_F(l-Q)i\tilde{\Delta}_F(l)i\tilde{\Delta}_F(l) \\ &\quad -i\tilde{\Delta}_F(l)i\tilde{\Delta}_F(l-Q)i\tilde{\Delta}_F(l-Q) - i\tilde{\Delta}_F(p+Q)i\tilde{\Delta}_F(l)i\tilde{\Delta}_F(l)\end{aligned} \tag{4.85}$$

In the following Taylor subtraction in p all contributions not depending on p vanish. We show that the contribution of the last term vanishes, too. We obtain from it the expression

$$-\int d^4Q \int d^4l(i\tilde{\Delta}_F(p+Q)(i\tilde{\Delta}_F(l))^2 + i\tilde{\Delta}_F(Q)(i\tilde{\Delta}_F(l))^2 + \frac{p^2}{2}(\Box i\tilde{\Delta}_F(Q))(i\tilde{\Delta}_F(l))^2) \tag{4.86}$$

The contributions of the renormalization of the subdiagram vanish in the momentum integration, because of

$$\int d^4Q \int d^4l \; i\tilde{\Delta}_F(p+Q)(i\tilde{\Delta}_F(l))^2 = \int d^4Q \int d^4l \; i\tilde{\Delta}_F(Q)(i\tilde{\Delta}_F(l))^2 \tag{4.87}$$

and

$$\int d^4Q \int d^4l \, \frac{p^2}{2} (\Box i\tilde{\Delta}_F(Q))(i\tilde{\Delta}_F(l))^2) = 0. \tag{4.88}$$

There only remains the Taylor subtraction of the Epstein-Glaser renormalization, so the results of the two methods are the same in this example.

Chapter 5

Renormalization of the S-Matrix in Φ^4 Theory

5.1 Introduction

In this section the second and third order of the S-matrix in Φ^4 theory are renormalized using the formalism developed in the previous section (The result of the fourth-order calculation is presented in [GBLa]). Already in second order we will see that the operators $\tilde{\Delta}_n$ of theorem (3.0.1) consist only of linear combinations of $: \partial_\mu \phi \partial^\mu \phi :, : \phi^2 :$ and $: \phi^4 :$. In third order we demonstrate an explicit calculation of a diagram with subdivergencies to make clear how the subtraction works and that the result is indeed independent of the partition of unity used in the inductive construction. The result of the calculations is given in the adiabatic limit, otherwise there would be many additional terms.

Before starting the calculations it is useful to make some remarks. We will have to work with expressions of the typical form

$$\int du \int dx \; t^0(u) \;\; (W\,(\delta, w, u)\, A(x,u) g(x,u))\,, \tag{5.1}$$

where $t^0(u)$ is the numerical distribution with singular order δ, g is a test-function with compact support and A is a Wick product of fields at x and u. With u and x we denote a tupel of coordinates $(u_1, \ldots u_n)$ and $(x_1, \ldots x_n)$ respectively. We always assume that the function $w(u)$ of the W-operation is even in all components of u_i^μ for all i. In the calculations we use the following facts:

1. All test functions are symmetric in the variables, so all odd terms in the Taylor subtraction of W will vanish.

2. To make the extension Lorentz invariant, we use the form of the subtraction in (4.37) and (4.52) respectively. Here again the contributions of the coefficients c^α with $|\alpha|$ odd vanish.

3. For $\delta \geq 1$ there appear terms with derivatives of g in the Taylor subtraction. These terms vanish in the adiabatic limit. Because the limit exists in massive Φ^4-theory in the strong sense, we omit them from the beginning.

4. In [Düt] it is shown that the adiabatic limit of vacuum diagrams exists only for a special choice of the c^α. This choice is Lorentz invariant and the contributions of the vacuum diagrams vanish in the adiabatic limit, too.

5. We assume $D^\alpha w(0) = 0$ for all multiindices α with $0 < |\alpha| < 3$ for all test-functions w used in the renormalization.

We denote with $()_R$ the extension of a numerical distribution and with $()_E$ the extension of a T_m-product to the total diagonal D_m. A numerical distribution without index R stands for the whole class of its lorentz invariant extensions.

5.2 Renormalization of the Second Order

The second order term in the S-matrix has the form

$$\begin{aligned} S^{(2)}(g) &= \frac{1}{2}\left(\frac{-i\lambda}{4!}\right)^2 T_2\left(g\Phi^4, g\Phi^4\right) \\ &= -\frac{\lambda^2}{2(4!)^2}\int d^4x_1 \int d^4x_2 \sum_{k=0}^{4} \begin{pmatrix} 4 \\ k \end{pmatrix} \begin{pmatrix} 4 \\ k \end{pmatrix} (4-k)! \,\cdot \\ &\quad \cdot \left(\left(i\Delta_F\left(x_1 - x_2\right)\right)^{4-k}\right)_R : \phi^k\left(x_1\right)\phi^k\left(x_2\right) : g\left(x_1\right) g\left(x_2\right). \end{aligned} \tag{5.2}$$

With the singular orders (4.18) of the Feynman propagators we obtain nontrivial contributions of the extension from terms with $k = 0, 1, 2$.

1. $k = 0$ yields a vacuum diagram. Its contribution vanishes in the adiabatic limit.

2. For $k = 1$ we have

$$\begin{aligned} S^{(2)}_{(k=1)}(g) &= -\frac{96\lambda^2}{2!(4!)^2}\int d^4x_1 \int d^4x_2 \left(i\Delta_F\left(x_1 - x_2\right)\right)^3_R \cdot \\ &\qquad \cdot : \phi\left(x_1\right)\phi\left(x_2\right) : g\left(x_1\right) g\left(x_2\right) \\ &\overset{(4.52)}{=} -\frac{\lambda^2}{12}\int d^4v \int d^4u \left(i\Delta_F(u)\right)^3 \Big[: \phi(u+v)\phi(v) : g(u+v)g(v) \\ &\qquad - w(u) : \phi^2(v) : g^2(v) - \frac{1}{8}w(u)u^2 : \phi(v)\Box\phi(v) : g^2(v)\Big]. \end{aligned} \tag{5.3}$$

3. For $k = 2$ we obtain

$$S^{(2)}_{(k=2)}(g) = -\frac{\lambda^2}{16}\int d^4u \int d^4v \,(i\Delta_F(u))^2 \Big[: \phi^2(u+v)\phi^2(v) : g(u+v)g(v) - w(u) : \phi^4(v) : g^2(v)\Big]. \tag{5.4}$$

Remark: The individual terms in the square brackets are not defined, only their sum is convergent and well-defined. The result depends on the function $w(x)$. We use this dependence to read off the form of the finite renormalizations $\tilde{\Delta}_2$: to choose another Lorentz invariant T-product with extensions of the form (4.24) we can only change the function w. The difference of two w-functions has no support at 0 and gives a well-defined contribution. Denoting with $\hat{T}$ the renormalization at the scale $m + \delta m$ and with T the renormalization at m we obtain with (4.54):

$$\begin{aligned}\tilde{\Delta}_2 &= \left(S^{(2)}_{\hat{T}} - S^{(2)}_{T}\right) = \\ &= \int d^4v \left[A^{(2)} : \phi^2(v) : + B^{(2)} : \partial_\mu\phi(v)\partial^\mu\phi(v) : + C^{(2)} : \phi^4(v) :\right] g^2(v)\end{aligned} \tag{5.5}$$

with

$$\begin{aligned}A^{(2)} &= \delta m \,\frac{\lambda^2}{12}\int d^4u \,(i\Delta_F(u))^3 \; u^\mu(\partial_\mu w) \\ B^{(2)} &= -\delta m \,\frac{\lambda^2}{96}\int d^4u \,(i\Delta_F(u))^3 \; u^\nu(\partial_\nu w)u^2 \\ C^{(2)} &= \delta m \,\frac{\lambda^2}{16}\int d^4u \,(i\Delta_F(u))^2 \; u^\mu(\partial_\mu w).\end{aligned} \tag{5.6}$$

At this stage we are able to see how the theoretically predicted form of the higher $\tilde{\Delta}_n$ is realized in the calculations. $\tilde{\Delta}_n$ is the difference in the superficial renormalization of diagrams with four and two external legs. Diagrams with four external legs are superficially logarithmically divergent and of the form

$$\int du \int dv \; t^0(v,u)W(0,w,u) : \phi\left(l_1(u,v)\right)\phi\left(l_2(u,v)\right)\phi\left(l_3(u,v)\right)\phi\left(l_4(u,v)\right) : \\ g\left(l_5(u,v)\right)\ldots g\left(l_{n+4}(u,v)\right), \tag{5.7}$$

where $u = (u_1, \ldots, u_{n-1})$ are the difference variables and l_i, $i = 1, \ldots, n+4$ are linear combinations of v and the u_i of the form $l_i(v,u) = v + a_i u_i$ with $a_i \in \mathbb{R}$. Varying the mass scale in W we obtain only the following contribution

$$\int dv \; C^{(n)} : \phi^4(v) : g^n(v). \tag{5.8}$$

Diagrams with two external legs are quadratically divergent and of the form

$$\int du \int dv\, t^0(v,u) W(2,w,u) : \phi\left(l_1(u,v)\right) \phi\left(l_2(u,v)\right) : g\left(l_3(u,v)\right) \ldots g\left(l_{n+2}(u,v)\right). \quad (5.9)$$

Varying the mass scale in the W-operation yields the following contributions:

$$\int dv \left(A^{(n)} : \phi^2(v) : + B^{'(n)} : \phi(v) \Box \phi(v) \right) g^n(v). \quad (5.10)$$

By partial integration we obtain the contribution

$$\int dv\, B^{(n)} : \partial_\mu \phi(v) \partial^\mu \phi(v) : g^n(v), \quad (5.11)$$

because the contributions with derivatives of g vanish in the adiabatic limit. Finite renormalizations to all orders in Φ^4 theory consist only of shifts in the coupling constant, in mass-like terms and kinetic terms of $\mathcal{L}_{int}$ whose coefficients $A^{(n)}$, $B^{(n)}$ and $C^{(n)}$ have to be determined in the calculations.

5.3 Renormalization of the Third Order

The third order term in the S-matrix has the form

$$S_3(g) \;=\; \frac{i\lambda^3}{3!(4!)^3} T_3\left(g(x_1)\phi^4, g(x_2)\phi^4, g(x_3)\phi^4\right) \quad (5.12)$$

and with the structure (3.5) of the higher T-products we obtain

$$\begin{aligned} S_3(g) \;=\; & \frac{i\lambda^3}{3!(4!)^3} \int d^4x_1 \int d^4x_2 \int d^4x_3 \Bigg\{ \sum_{\substack{l\in\{1,2,3\}, m<n \\ m,n \in \{1,2,3\}\setminus l}} \\ & \left(f_l^{(3)} + f_{mn}^{(3)} \right) \Bigg[\sum_{i=0}^{4} \sum_{j_1=0}^{4-i} \sum_{j_2=0}^{min(4-j_1,4-i)} K(i,j_1,j_2) \cdot \\ & \cdot \left(\left(i\Delta_F(x_m - x_n) \right)^i \right)_R \left(i\Delta_F(x_l - x_m) \right)^{j_1} \left(i\Delta_F(x_l - x_n) \right)^{j_2} \cdot \\ & \cdot : \phi^{4-j_1-j_2}(x_l) \phi^{4-i-j_1}(x_m) \phi^{4-i-j_2}(x_n) : g(x_1) g(x_2) g(x_3) \Bigg] \Bigg\}_E, \end{aligned} \quad (5.13)$$

with the factor

$$\begin{aligned} K(i,j_1,j_2) \;=\; & \binom{4}{i} \binom{4}{i} \binom{4}{j_1} \binom{4-j_1}{j_2} \binom{4-i}{j_1} \binom{4-i}{j_2} (i)! j_1! j_2! \\ =\; & \frac{(4!)^3}{((4-j_1-j_2)! j_1! (4-i-j_1)! j_2! (4-i-j_2)! i!}. \end{aligned} \quad (5.14)$$

Reordering the terms in expression (5.13) we arrive at

$$
\begin{aligned}
S_3(g) \quad = \quad & \frac{i\lambda^3}{3!(4!)^3}\int d^4x_1\int d^4x_2\int d^4x_3\sum_{i=0}^{4}\sum_{j_1=0}^{4-i}\sum_{j_2=0}^{min(4-j_1,4-i)} K(i,j_1,j_2)\cdot \\
& \cdot\Big\{(f_{12}^{(3)}+f_3^{(3)})\left((i\Delta_F(x_1-x_2))^i\right)_R(i\Delta_F(x_1-x_3))^{j_1}\,(i\Delta_F(x_3-x_2))^{j_2} \\
& +(f_{13}^{(3)}+f_2^{(3)})\left((i\Delta_F(x_1-x_3))^{j_1}\right)_R(i\Delta_F(x_1-x_2))^{i}\,(i\Delta_F(x_3-x_2))^{j_2} \\
& +(f_{23}^{(3)}+f_1^{(3)})\left((i\Delta_F(x_2-x_3))^{j_2}\right)_R(i\Delta_F(x_1-x_3))^{j_1}\,(i\Delta_F(x_1-x_2))^{i}\Big\}_R\cdot \\
& \cdot :\phi^{4-j_1-j_2}(x_3)\phi^{4-i-j_2}(x_2)\phi^{4-i-j_1}(x_1): g(x_1)g(x_2)g(x_3). \qquad (5.15)
\end{aligned}
$$

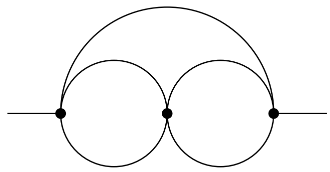
There are two kinds of W-operations in the calculation, that is Taylor subtractions in one and two variables corresponding to the renormalization of superficial divergencies and subdivergences. In the subtraction in one variable u we use the test-function $w(u)$ as before. The test-function of the W-operator in two variables u and v is chosen to be $\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v)$ where $\tilde{w}$ is a test-function in one variable. Since u and v are difference variables the function is symmetric in all coordinates.
We now list the 14 topologically different diagrams occuring in the sum with their values of i, j_1 and j_2. Furthermore the singular orders of the whole diagram $\delta(x_1,x_2,x_3)$ and of the subdiagrams, $\delta(x_i,x_j)$ consisting only of the vertices x_i and x_j are listed. N is the number of all different diagrams in the sum with the same topological structure.

Number	i	j_1	j_2	$\delta(x_1,x_2,x_3)$	$\delta(x_1,x_2)$	$\delta(x_1,x_3)$	$\delta(x_2,x_3)$	N
1	4	0	0	0	4	-4	-4	3
2	3	0	1	0	2	-4	-2	6
3	2	2	0	0	0	0	-4	3
4	2	1	1	0	0	-2	-2	3
5	3	1	1	2	2	-2	-2	3
6	2	2	1	2	0	0	-2	3
7	2	2	2	4	0	0	0	1
8	3	0	0	-2	2	-4	-4	3
9	2	1	0	-2	0	-2	-4	6
10	1	1	1	-2	-2	-2	-2	1
11	2	0	0	-4	0	-4	-4	3
12	1	1	0	-4	-2	-2	-4	3
13	1	0	0	-6	-2	-4	-4	3
14	0	0	0	-8	-4	-4	-4	1

The contributions of the diagrams 1 and 7 vanish in the adiabatic limit.

Diagrams 2, 3 and 4 are superficially logarithmically divergent, they have four external legs and contribute to the renormalization of the coupling constant. Diagram 5 and 6 have two external lines and are therefore superficially quadratically divergent. They yield contributions to the mass and field strength renormalization. All the other diagrams are superficially convergent, they do not depend on $\tilde{w}$ and yield no contribution to $\tilde{\Delta}_3$. Here we only demonstrate the calculation of

Diagram 6:



With $i = 2, j_1 = 2$ and $j_2 = 1$ we obtain from (5.14) $K(2,2,1) = \frac{(4!)^3}{4}$ and inserting this in (5.15) we get the following expression:

$$
\begin{aligned}
S_3^{(6)}(g) \quad = \quad & \frac{i\lambda^3}{4!} \int d^4x_1 \int d^4x_2 \int d^4x_3 \ :\phi(x_2)\phi(x_3): g(x_1)g(x_2)g(x_3) \cdot \\
& \cdot \Big[\left(f_{12}^{(3)} + f_3^{(3)} \right) \left(i\Delta_F(x_1 - x_2)\right)_R^2 \left(i\Delta_F(x_1 - x_3)\right)^2 i\Delta_F(x_2 - x_3) \\
& \quad + \left(f_{13}^{(3)} + f_2^{(3)} \right) \left(i\Delta_F(x_1 - x_2)\right)^2 \left(i\Delta_F(x_1 - x_3)\right)_R^2 i\Delta_F(x_2 - x_3) \\
& \quad + \left(f_{23}^{(3)} + f_1^{(3)} \right) \left(i\Delta_F(x_1 - x_2)\right)^2 \left(i\Delta_F(x_1 - x_3)\right)^2 \left(i\Delta_F(x_2 - x_3)\right)_R \Big]_R
\end{aligned}
$$

At first we have to perform the superficial renormalization, therefore we introduce new variables $u = x_2 - x_1$, $v = x_1 - x_3$ and apply the operator $W^{(2)}(2, \tilde{w}, v, u + v)$ to the Wick-monomial and the test functions. Because of remark 3 made before the calculations the derivatives in the subtraction act only on the Wick-monomial depending only on one variable $u + v$. In this case $W^{(2)}$ has the form of a subtraction operator in one variable given by (4.52).

$$
\begin{aligned}
S_3^{(6)}(g) \;=\; & \frac{i\lambda^3}{4!}\int d^4u\int d^4v\int d^4x_3 \\
& \cdot\Big[\Big(f_{12}^{(3)}+f_3^{(3)}\Big)\,(i\Delta_F(u))_R^2\,(i\Delta_F(v))^2\,i\Delta_F(u+v) && (5.16)\\
& \quad +\Big(f_{13}^{(3)}+f_2^{(3)}\Big)\,(i\Delta_F(u))^2\,(i\Delta_F(v))_R^2\,i\Delta_F(u+v) && (5.17)\\
& \quad +\Big(f_{23}^{(3)}+f_1^{(3)}\Big)\,(i\Delta_F(u))^2\,(i\Delta_F(v))^2\,(i\Delta_F(u+v))_R\Big] && (5.18)\\
& \cdot\Big[:\phi(u+v+x_3)\phi(x_3):g(u+v+x_3)g(v+x_3)g(x_3)\\
& \quad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v):\phi^2(x_3):g^3(x_3)\\
& \quad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v)\frac{(u+v)^2}{8}:\phi(x_3)\Box\phi(x_3):g^3(x_3)\Big].
\end{aligned}
$$

Because of $(i\Delta_F(u+v))_R = i\Delta_F(u+v)$ we now have to renormalize the subdivergencies of the lines (5.16) and (5.17). Using $\left(f_{12}^{(3)}+f_3^{(3)}\right)_{u=0} = 1$ we obtain from line (5.16):

$$
\begin{aligned}
& (i\Delta_F(u))^2\,W^{(1)}(0,w,u)\,\Big(f_{12}^{(3)}+f_3^{(3)}\Big)\,(i\Delta_F(v))^2\,i\Delta_F(u+v)\cdot\\
& \quad\cdot\Big[:\phi(u+v+x_3)\phi(x_3):g(u+v+x_3)g(v+x_3)g(x_3)-\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v):\phi^2(x_3):g^3(x_3)\\
& \qquad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v)\frac{(u+v)^2}{8}:\phi(x_3)\Box\phi(x_3):g^3(x_3)\Big]\\
= \;& (i\Delta_F(u))^2\,(i\Delta_F(v))^2\\
& \Big\{\Big(f_{12}^{(3)}+f_3^{(3)}\Big)\,i\Delta_F(u+v)\Big[:\phi(u+v+x_3)\phi(x_3):g(u+v+x_3)g(v+x_3)g(x_3)\\
& \qquad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v):\phi^2(x_3):g^3(x_3)\\
& \qquad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v)\frac{(u+v)^2}{8}:\phi(x_3)\Box\phi(x_3):g^3(x_3)\Big]\\
& \quad -i\Delta_F(v)\Big[:\phi(v+x_3)\phi(x_3):g^2(v+x_3)g(x_3)-\tilde{w}^2(v):\phi^2(x_3):g^3(x_3)\\
& \qquad -\tilde{w}^2(v)\frac{v^2}{8}:\phi(x_3)\Box\phi(x_3):g^3(x_3)\Big]\Big\}.
\end{aligned}
$$

Interchanging u and v, we obtain from this the renormalization of the subdivergence of line (5.17). Finally we obtain the following result:

$$
\begin{aligned}
S_3^{(6)}(g) \;\; = \;\; & \frac{i\lambda^3}{8}\int d^4u \int d^4v \int d^4x_3 \;\; (i\Delta_F(u))^2\,(i\Delta_F(v))^2 \\
& \Big\{ i\Delta_F(u+v)\Big[:\phi(x_3)\phi(u+v+x_3): g(u+v+x_3)g(v+x_3)g(x_3) \\
& \qquad\qquad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v) :\phi^2(x_3): g^3(x_3) \\
& \qquad\qquad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v)\frac{1}{8}\,(u+v)^2 :\phi(x_3)\Box\phi(x_3): g^3(x_3)\Big] \\
& -i\Delta_F(v)w(u)\Big[:\phi(x_3)\phi(v+x_3): g^2(v+x_3)g(x_3) \\
& \qquad\qquad -\tilde{w}^2(v) :\phi^2(x_3): g^3(x_3) \\
& \qquad\qquad -\tilde{w}^2(v)\frac{v^2}{8} :\phi(x_3)\Box\phi(x_3): g^3(x_3)\Big] \\
& -i\Delta_F(u)w(v)\Big[:\phi(u+x_3)\phi(x_3): g(u+x_3)g^2(x_3) \\
& \qquad\qquad -\tilde{w}^2(u) :\phi^2(x_3): g^3(x_3) \\
& \qquad\qquad -\tilde{w}^2(u)\frac{u^2}{8} :\phi(x_3)\Box\phi(x_3): g^3(x_3)\Big]\Big\}.
\end{aligned}
$$

The renormalized diagrams 1-5 and 7-11 are given in the appendix A.
Now it becomes clear that the independence of the partition of unity proved in [BrFr] is a consequence of the fact that the Taylor subtractions of the subdivergencies act on the partition of unity. In this explicit third-order calculation we can see that not only the T-products as a whole as shown in [BrFr] but also the contributions to the individual diagrams are independent of the partition of unity.

5.4 The Normalization Conditions

We have seen how renormalization works in the Epstein-Glaser formalism in Φ^4-theory. To come back to a more theoretical formulation we present the normalization conditions for Φ^4-theory. Some of them are an abstract formulation of techniques used in the previous calculations. In contrast to the calculations of the previous section the following conditions are independent of the adiabatic limit. The normalization conditions introduced in [DüFr] and extended in [Boa] restrict the ambiguities in the renormalization. We repeat these conditions in the more simple form fitting to Φ^4 theory:

- Condition N1 demands the Lorentz covariance of the T-products; it is described in [BrPiPr] and [Pra2] how to realize this in the extension procedure.

- Condition N2 gives the form of the adjoint of T on $\mathcal{D}$ and makes sure that the S-matrix is unitary:

$$
T_n(f_1,\ldots f_n)^+ = \sum_{P\in Part J}(-1)^{|P|+n}\prod_{p\in P} T_n\left(f_i^+, i\in p\right),
$$

where the sum is over the ordered partitions of J.

- It is shown in [Boa] that condition N3 is equivalent to the Wick expansion of the time-ordered products. The Wick expansion has the form

$$\begin{aligned} T_n\left(g_1\phi^{l_1},\ldots,g_n\phi^{l_n}\right) &= \int dx_1\ldots\int dx_n \sum_{\substack{0\leq k_1,\ldots,k_n\\ k_i<l_i}} <0|T\left(\phi^{k_1}(x_1)\ldots\phi^{k_n}(x_n)\right)|0> \\ &\prod_{i=1}^{n}\frac{l_i!}{k_i!(l_i-k_i)!} :\phi^{l_1-k_1}(x_1)\ldots\phi^{l_n-k_n}(x_n): g(x_1)\ldots g(x_n). \end{aligned} \tag{5.19}$$

Each contribution to the sum in (5.19) corresponds to a diagram with $\frac{1}{2}\sum_i k_i$ internal lines and $\sum_i(l_i-k_i)$ external lines. Condition N3 reads

$$\begin{aligned} &[T_n\left(f_1,\ldots,f_n\right),T_1\left(g\phi\right)] = \\ &= \sum_{k=1}^{n} iT_n\left(f_1,\ldots,\Delta\frac{\partial f_k}{\partial\phi},\ldots,f_n\right) + \sum_{k=1}^{n} iT_n\left(f_1,\ldots,(\partial_\mu\Delta)\frac{\partial f_k}{\partial(\partial_\mu\phi)},\ldots,f_n\right) \end{aligned} \tag{5.20}$$

for all f_i containing only fields and their first derivatives with

$$\Delta(x_k) = \int d^4y\ \Delta_{11}(x_k-y)g(y) \tag{5.21}$$

where $\Delta_{11}(y-x_k)$ is the commutator function $i\Delta_{11}(y-x_k) = [:\phi(y):,:\phi(x_k):]$. The derivative of f is implicitly defined by N3 for $n=1$ for all $f\in\mathcal{D}\left(\mathbb{R}^4,\mathcal{A}\right)$ containing only linear combinations of fields and their first derivatives. We obtain

$$\frac{\partial}{\partial\phi}\left(g_n(y)\phi^n\right) = ng_n(y)\phi^{n-1}, \qquad \frac{\partial}{\partial\phi}\left(g(y)\partial_\mu\phi\right) = 0. \tag{5.22}$$

By demanding

$$T_n\left(f_1,\ldots,f_n,g(z)(\Box+m^2)\phi\right) = T_n\left(f_1,\ldots,f_n,((\Box_z+m^2)g(z))\phi\right) \tag{5.23}$$

we can extend condition N3 to T-products containing one factor $(\Box+m^2)\phi$.

- Condition N4 has the form

$$T_{n+1}\left(f_1,\dots,f_n,-\left(\Box_y+m^2\right)g(y)\phi\right)=i\sum_{k=1}^{n}T_n\left(f_1,\dots,g\frac{\partial f_k}{\partial\phi},\dots,f_n\right) \quad (5.24)$$

where the $f_i \in \mathcal{D}\left(\mathbb{R}^4,\mathcal{A}\right)$ contain only combinations of fields and their first derivatives.

We now show that in Φ^4 theory the Dyson-Schwinger equations are a consequence of N4. With the Gell-Mann-Low formula the Green's functions $G_i\left(x_1,\dots x_n\right)$ have the form

$$\frac{\left\langle 0\left|T\phi(x_1)\dots\phi(x_i)\sum_{n=0}^{\infty}\frac{i^n}{n!}\int dy_1\dots\int dy_n\left(\frac{-\lambda}{4!}\right)^n\phi^4(y_1)\dots\phi^4(y_n)g(y_1)\dots g(y_n)\right|0\right\rangle}{\left\langle 0\left|T\sum_{n=0}^{\infty}\frac{i^n}{n!}\int dy_1\dots\int dy_n\left(\frac{-\lambda}{4!}\right)^n\phi^4(y_1)\dots\phi^4(y_n)g(y_1)\dots g(y_n)\right|0\right\rangle}. \quad (5.25)$$

By this we obtain using N4 and (5.22)

$$\begin{aligned}\int d^4z\ \left(\Box_z+m^2\right)G_i\left(x_1,\dots x_{i-1},z\right)=\\ =\int d^4z\ \left[i\sum_{k=1}^{i-1}\delta(x_k-z)G_{i-2}\left(x_1,\dots\check{x}_k\dots x_{i-1}\right)+\frac{\lambda g(z)}{6}G_{i+2}\left(x_1,\dots x_{i-1},z,z,z\right)\right]\end{aligned} \quad (5.26)$$

where $G_{i+2}\left(x_1,\dots x_{i-1},z,z,z\right)$ denotes the vacuum expectation value of the field product $\phi(x_1)\cdot\dots\phi(x_{i-1})\phi^3(z)$. These are the Dyson-Schwinger equations [Riv].

Chapter 6

Topics concerning the Action Principle

6.1 Main Theorem of Perturbative Renormalization Theory

The procedure of renormalization depends on the renormalization scale and we can say that two T-products T and $\hat{T}$ are renormalizations at different scales m, even though their scale dependence is not obvious in the abstract formulation of section 3. Given two renormalization prescriptions T and $\hat{T}$ belonging to different scales m one can pass from one to the other by a suitable change of the scale dependent quantities in the Lagrangian. This is the content of the main theorem of perturbative renormalization theorywhich is proved in [PoSt] in the framework of causal perturbation theory. Popineau and Stora give a construction of the changes of parameters which is, due to the inductive form, not suited for calculations in higher orders. In this section we derive the explicit form of the changes of parameters in the Lagrangian compensating a change of renormalization $T \to \hat{T}$ which corresponds to the structure of renormalization given in [Zav]. In the following calculations we use the abbreviations

$$\begin{aligned} f &:= g(x)\phi^4 \\ TE_k(f) &:= \sum_{n=k}^{\infty} \frac{(-i)^n}{n!} T_n\left(f^{\bigotimes n}\right). \end{aligned}$$

Then the S-matrix reads, for example,

$$S = TE_0(f) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} T_n\left(f^{\bigotimes n}\right), \qquad \text{and} \qquad \Delta E_k(f) = \sum_{n=k}^{\infty} \frac{(-i)^n}{n!} \Delta_n\left(f^{\bigotimes n}\right). \quad (6.1)$$

The following theorem describes how to absorb a change in the renormalization prescription $\hat{T} \to T$ in a change $f \to f_r$ of the physical parameters in the Lagrangian:

Theorem 6.1.1 *An S-matrix renormalized according to $\hat{T}$ can be expressed by an S-matrix renormalized according to T in the following way:*

$$\hat{T}E_0(f) = TE_0(\Delta E_1(f)) =: TE_0(f_r). \tag{6.2}$$

Remark: Because of $\hat{T}E_0(f) = 1 + T(f) + \hat{T}E_2(f)$ and $\Delta E_1(f) = f + \Delta E_2(f)$ we can derive from (6.2) the following recursion relation:

$$T\Delta E_2(f) = \hat{T}E_2(f) - TE_2(f + \Delta E_2(f)). \tag{6.3}$$

Proof: We prove the theorem by the following calculation:

$$\hat{T}E_0(f) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \hat{T}\left(f^{\otimes n}\right) \overset{(3.6)}{=} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \sum_{P \in Part(J)} T\left(\bigotimes_{O_i \in P} \Delta\left(f^{\otimes |O_i|}\right)\right). \tag{6.4}$$

Here we sum over all partitions of the set $J = \{1, \ldots, n\}$. Now we take a fixed partition P and denote with N_i the number of elements O of P with $|O| = i$.
Then the relation $n = \sum_{i=1}^{k} iN_i$ is true with $k \leq n$. There are

$$\frac{n!}{N_1! \ldots N_k! 1!^{N_1} \ldots k!^{N_k}} \tag{6.5}$$

different partitions P with the same numbers N_i. Therefore we have

$$\hat{T}E_0(\mathbf{V}) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \sum_{\sum iN_i = n} \frac{n!}{N_1! \ldots N_k! 1!^{N_1} \ldots k!^{N_k}} T_n \left[(\Delta(f))^{N_1} \ldots \left(\Delta_k(f^{\bigotimes k})\right)^{N_k}\right] \tag{6.6}$$

and obtain

$$\begin{aligned} \hat{T}E_0(f) &= \lim_{k\to\infty} T\left[\sum_{N_1=0}^{\infty} \frac{(-i)^{N_1}}{N_1!} (\Delta(f))^{N_1} \cdot \sum_{N_2=0}^{\infty} \frac{(-i)^{2N_2}}{N_2! 2!^{N_2}} (\Delta(f \otimes f))^{N_2} \cdot \ldots \right. \\ &\quad \left. \ldots \cdot \sum_{N_k=0}^{\infty} \frac{(-i)^{kN_k}}{N_k! k!^{N_k}} \left(\Delta_k(f^{\bigotimes k}\right)^{N_k}\right] = T\left(E_0\left(\Delta E_1(f)\right)\right) \qquad \blacksquare \end{aligned}$$

In renormalizable theories, $f_r = \Delta E_1(f)$ consists only of linear combinations of the field combinations in the original Lagrangian. In scalar Φ^4 theory $\Delta E_1(f)$ is a linear combination of the monomials $\partial_\mu \Phi \partial^\mu \Phi$, Φ^2 and Φ^4. The coefficients depend on the renormalization conditions and this dependence is described by the Callan-Symanzik equations and the renormalization group equations. The most elegant way to derive them uses the action principle [Low]. In the formalism of Epstein and Glaser the partition of the Lagrangian into a free and an interacting part is important. The free part of the Lagrangian defines the Hilbert space, fields and their masses of the free theory. It is not clear that a contribution of a mass term to the interacting Lagrangian $\Delta E_1(f)$ can be interpreted as a shift in the mass.

6.2 Insertions

To derive the action principle we need the notion of an insertion.

Definition 6.2.1 *An insertion of degree a of an element $g \in \mathcal{D}\left(\mathbb{R}^4, \mathcal{A}\right)$ into a T-product is defined by*

$$I^{(a)}(g,T)_{(n)}\left(\bigotimes_{j\in J} f_j\right)) \;:=\; T_{n+1}\left(\bigotimes_{j\in J} f_j \otimes g^{(a)}\right). \tag{6.7}$$

where $g^{(a)}$ means that the vertex belonging to g is treated in the extension procedure defining T_{n+1} as a vertex of mass dimension a.

Remark: If the degree a of the insertion corresponds to its physical mass dimension M it is called a soft insertion. In the case $a > M$ all diagrams containing the vertex of the insertion are over-subtracted and we have a hard insertion. The relation of two insertions of the same Wick monomial with different degrees is given by the Zimmermann identities derived in the next section.
Since the definition of the T-product depends on the degree of a vertex and this degree is not a linear function the T-products are not a priori linear. With the possibility to prescribe with the index $^{(a)}$ how a vertex is treated in a T-product, we can define the T-products as linear functions.
The behaviour of an insertion into the S-matrix under a change of renormalization is given by the following theorem.

Theorem 6.2.1 *An insertion into the S-matrix renormalized according to $\hat{T}$ can be expressed as an insertion into the S-matrix renormalized according to T in the following way:*

$$\begin{aligned} I^{(a)}\left(g, \hat{T}E_0(f)\right) &= I^{(a)}\left(\Delta(gE_0(f)), TE_0(f_r)\right) \\ &=: I^{(a)}\left(g_r(f), TE_0(f_r)\right). \end{aligned} \tag{6.8}$$

Remark: Analogously to (6.3) we obtain after some calculation the recursion relation $T\Delta(gE_1(f)) = I^{(a)}\left(g, \hat{T}E_1(f)\right) - I^{(a)}\left(g, TE_1(f_r)\right) - I^{(a)}\left(\Delta(gE_1(f)), TE_1(f_r)\right).$

Proof: We know from Theorem (6.1.1) that

$$\hat{T}E_0(h) = TE_0(\Delta E_1(h)). \tag{6.9}$$

Let a be the degree of the insertion g, λ a parameter and $h = f + g\lambda$ such that the vertex $g\lambda$ is treated in the subtraction procedure of the T-products as a vertex of mass dimension a. Then we can interchange differentiation and integration in the following calculation:

$$\begin{aligned} I^{(a)}\left(g,\hat{T}\right)(E_0(f)) &= i\frac{\partial}{\partial\lambda}\hat{T}E_0(h)\Big|_{\lambda=0} \\ &\overset{(6.9)}{=} i\frac{\partial}{\partial\lambda}T\left(E_0\left(\Delta E_1(f+g\lambda)\right)|_{\lambda=0}\right) \\ &= I^{(a)}\left(g\Delta(E_0(f)),T\right)\left(E_0(\Delta E_1(f))\right) \blacksquare \end{aligned} \tag{6.10}$$

The following properties of insertions are interesting:

- Interacting fields have up to a factor S^{-1} the form of an insertion:

$$\begin{aligned} \int d^4x A_{int\mathcal{L}}(x) &= \frac{\delta}{\delta h(x)}S_T(\mathcal{L})^{-1}S_T(\mathcal{L}+hA)|_{h=0} \\ &= S_T(\mathcal{L})^{-1}I\left(A,S_T(\mathcal{L})\right). \end{aligned} \tag{6.11}$$

- Insertions into the S-matrix of monomials f_j occuring in the interaction act as counting operators of vertices of the kind f_j. The proof is analogous to the one given in [Low]: The form of the S-matrix of a theory with interaction vertices $f_1, \ldots f_n$ of mass dimensions $a_1, \ldots a_n$,

$$\begin{aligned} S_T &= TE_0\left(f_1^{(a_1)}+\ldots+f_n^{(a_n)}\right) \\ &= \sum_{c_i=0}^{\infty}\frac{(-i)^{\sum c_i}}{c_1!\ldots c_n!}T_{c_1+\ldots+c_n}\left(\left(f_1^{(a_1)}\right)^{\otimes c_1}\ldots\left(f_n^{(a_n)}\right)^{\otimes c_n}\right), \end{aligned} \tag{6.12}$$

  implies that the contribution of a diagram γ with c_i vertices of the kind f_i to the S-matrix has the form

$$S_{(c_1,\ldots,c_n)} = \frac{(-i)^{\sum c_i}}{c_1!\ldots c_n!}T_{c_1+\ldots+c_n}\left(\left(f_1^{(a_1)}\right)^{\otimes c_1}\ldots\left(f_n^{(a_n)}\right)^{\otimes c_n}\right). \tag{6.13}$$

  An insertion of a vertex f_j of degree a_j into the S-matrix yields

$$\begin{aligned} I^{(a_j)}(f_j,S_T) &\overset{def.}{=} I^{(a_j)}(f_j,T)\left(E_0\left(f_1^{(a_1)}+\ldots+f_n^{(a_n)}\right)\right) \\ &= T\left(\sum_{c_i=0}^{\infty}\frac{c_j(-i)^{\sum c_i}}{c_1!\ldots c_n!}\left(f_1^{(a_1)}\right)^{\otimes c_1}\ldots\left(f_n^{(a_n)}\right)^{\otimes c_n}\right) \\ &= \sum_{c_i=0}^{\infty}c_jS_{(c_1,\ldots,c_n)}. \end{aligned} \tag{6.14}$$

- Zimmermann Identities

  The Zimmermann identities [Zim] are relations of insertions of different degrees. Comparing the two insertions

$$\begin{aligned} I^{(a)}(g,T)\left(\bigotimes\nolimits_{j\in J} f_j\right) &= T\left(\bigotimes\nolimits_{j\in J} f_j \otimes g\right), \\ I^{(b)}(g,T)\left(\bigotimes\nolimits_{j\in J} f_j\right) &= T^*\left(\bigotimes\nolimits_{j\in J} f_j \otimes g\right) \end{aligned} \tag{6.15}$$

  we obtain the operator-valued distribution Δ mediating between the two T-products. Using the properties

$$\begin{aligned} \Delta(f_i) &= f_i, \\ \Delta(g) &= g, \\ \Delta\left(\bigotimes_{i\in J} f_i\right) &= 0 \text{ for } |J| > 1 \end{aligned} \tag{6.16}$$

  we obtain with (3.6) the Zimmermann identities

$$T^*\left(g \otimes \bigotimes_{i\in J} f_i\right) = T\left(g \otimes \bigotimes_{i\in J} f_i\right) + \sum_{\substack{O_k\subseteq J \\ O_k\neq\emptyset}} T\left(\bigotimes_{l\notin O_k} f_l \otimes \Delta\left(\bigotimes_{l\in O_k} f_l \otimes g\right)\right) \tag{6.17}$$

  In case of insertions into the S-matrix this formula simplifies to

$$\begin{aligned} I^{(b)}(g,T)(E_0(f)) &\overset{(6.8)}{=} I^{(a)}(\Delta(gE_0(f)),T)(E_0(f)) \\ &= I^{(a)}(g,T)(E_0(f)) + I^{(a)}(\Delta(gE_1(f)),T)(E_0(f)). \end{aligned} \tag{6.18}$$

- The Action Principle

  The action principle describes the effects of a change of parameters of a theory [Sib]. Lowenstein [Low] has formulated it in terms of insertions into the S-matrix, and we follow his derivation. In the adiabatic limit it can be transformed with the Gell-Mann-Low formula into the usual formulation in Green's functions.

  According to the main theorem (6.1) a change of the renormalization T can be absorbed by finite counterterms shifting the quantities in the Lagrangian. Therefore we can assume without loss of generality that the T-product is fixed. A change of a parameter $p \to p + \delta p$ can cause changes in the free Lagrangian $\mathcal{L}_0 \to \mathcal{L}_0 + \delta\mathcal{L}_0$ and in the interaction part of the Lagrangian:$f \to f + \delta f$. The action principle states that a change of the S-matrix caused by a variation of a parameter p can be expressed by an insertion into the S-matrix. We only regard the case where the interaction part f depends on the parameter p. In this case the T-product is independent of p and we obtain by a trivial differentiation the first part of the action principle:

$$\frac{\partial}{\partial p} T\left(E_0(f)\right) = I^{(4)}\left(\frac{\partial f}{\partial p}, T E_0(f)\right). \tag{6.19}$$

Chapter 7

Summary and Outlook

We have derived the structure of finite renormalizations in the Epstein-Glaser formalism. The form of the S-matrix after a finite renormalization and the Zimmermann identities followed from this result. Furthermore we were able to derive the Hopf algebra structure in the combinatorics which was first discovered by Kreimer [Kre]. In ϕ^4 theory we have derived the Dyson-Schwinger equations as a consequence of N4.
Furthermore we have performed the renormalization of the S-matrix in Φ^4-theory up to third order. The result depends on the test-function w used in the renormalization, and finite counterterms can be read off. Comparing this procedure with other renormalizations in momentum space, we can say that Epstein-Glaser renormalization is better suited for the treatment of diagrams with few vertices and many loops, whereas momentum space renormalization has advantages in the renormalization of diagrams with many vertices and few loops.
Finally we defined insertions into T-products. With an extension of the normalization condition N4 it was possible to show that our insertions have the same properties as the insertions used by Lowenstein to derive the action principle. With this an analogous derivation of the action principle in the Epstein-Glaser formalism independent of the adiabatic limit was possible.
There are still many open questions: the action principle for changes of fields has to be derived and a complete translation of the derivation of the Callan Symanzik equations as in [Low] has to be worked out. One has to check further if there appear problems in the translation of these methods on curved space times.

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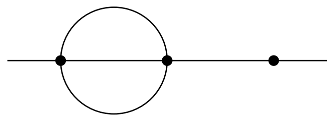
Appendix A

Results of the Third Order Calculations

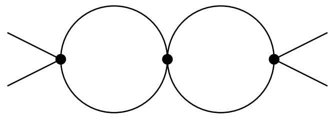
The contributions of the diagrams 1 and 7 vanish in the adiabatic limit.
Diagrams 2, 3 and 4 are superficially logarithmically divergent, they have four external legs and contribute to the renormalization of the coupling constant. Here we list only the results, weightened with N:

Diagram 2:



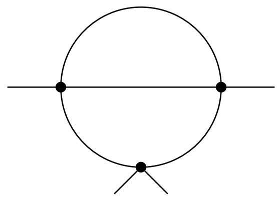
$$\begin{aligned} S_3^{(2)}(g) \quad = \quad & \frac{i\lambda^3}{36} \int d^4u \int d^4v \int d^4x_3 \, (i\Delta_F(u))^3 \, i\Delta_F(v) \\ & \Big\{ : \phi^3(x_3)\phi(u+v+x_3) : g(u+v+x_3)g(v+x_3)g(x_3) \\ & \quad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v) : \phi^4(x_3) : g^3(x_3) \\ & \quad -w(u) : \phi^3(x_3)\phi(v+x_3) : g^2(v+x_3)g(x_3) \\ & \quad +w(u)\tilde{w}^2(v) : \phi^4(x_3) : g^3(x_3) \\ & \quad -\frac{w(u)}{8}u^2 : \phi^3(x_3)\Box\phi(v+x_3) : g^2(v+x_3)g(x_3) \\ & \quad +\frac{w(u)}{8}u^2\tilde{w}(v)\,(\Box\tilde{w}(v)) : \phi^4(x_3) : g^3(x_3) \Big\}. \end{aligned} \tag{A.1}$$

Diagram 3:



$$
\begin{aligned}
S_3^{(3)}(g) \;=\; & \frac{i\lambda^3}{32}\int d^4u \int d^4v \int d^4x_3\, (i\Delta_F(u))^2\,(i\Delta_F(v))^2 \\
& \Big\{ :\phi^2(v+u+x_3)\phi^2(x_3): g(v+u+x_3)g(v+x_3)g(x_3) \\
& \quad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v) :\phi^4(x_3): g^3(x_3) \\
& \quad -w(u) :\phi^2(v+x_3)\phi^2(x_3): g^2(v+x_3)g(x_3) \\
& \quad +w(u)\tilde{w}^2(v) :\phi^4(x_3): g^3(x_3) \\
& \quad -w(v) :\phi^2(x_3+u)\phi^2(x_3): g(x_3+u)g^2(x_3) \\
& \quad +w(v)\tilde{w}^2(u) :\phi^4(x_3): g^3(x_3)\Big\}. \qquad \textbf{(A.2)}
\end{aligned}
$$

Diagram 4:

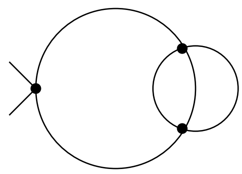


$$
\begin{aligned}
S_3^{(4)}(g) \;=\; & \frac{i\lambda^3}{6}\int d^4u \int d^4v \int d^4x_3 \\
& \Big\{ (i\Delta_F(u))^2\, i\Delta_F(u+v) i\Delta_F(v)\,\cdot \\
& \quad \cdot\Big[:\phi^2(x_3)\phi(v+x_3)\phi(u+v+x_3): g(u+v+x_3)g(v+x_3)g(x_3) \\
& \qquad -\tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v) :\phi^4(x_3): g^3(x_3)\Big] \\
& -(i\Delta_F(u))^2\,(i\Delta_F(v))^2\, w(u)\Big[:\phi^2(x_3)\phi^2(v+x_3): g^2(v+x_3)g(x_3) \\
& \qquad -\tilde{w}^2(v) :\phi^4(x_3): g^3(x_3)\Big]\Big\}. \qquad \textbf{(A.3)}
\end{aligned}
$$

We see that $\tilde{w}$ indeed appears only in combination with the Wick monomial $:\phi^4(x_3):$.

Diagram 5 and 6 have two external lines and are therefore superficially quadratically divergent. They yield contributions to the mass and field strength renormalization:

Diagram 5:



$$
\begin{aligned}
S_3^{(5)}(g) \quad = \quad & \frac{i\lambda^3}{24} \int d^4u \int d^4v \int d^4x_3 \, (i\Delta_F(u))^3 \, i\Delta_F(v) \\
& \Big\{ i\Delta_F(u+v) \Big[: \phi^2(x_3) : g(u+v+x_3) g(v+x_3) g(x_3) \\
& \qquad\qquad - \tilde{w}(u)\tilde{w}(v)\tilde{w}(u+v) : \phi^2(x_3) : g^3(x_3) \Big] \\
& - i\Delta_F(v) w(u) \Big[: \phi^2(x_3) : g^2(v+x_3) g(x_3) \\
& \qquad\qquad - \tilde{w}^2(v) : \phi^2(x_3) : g^3(x_3) \\
& \qquad\qquad - \frac{u^2}{8} \left(\Box \tilde{w}(v)\right) \tilde{w}(v) : \phi^2(x_3) : g^3(x_3) \Big] \\
& - \frac{w(u)}{8} u^2 \left(\Box i\Delta_F(v)\right) \Big[: \phi^2(x_3) : g^2(v+x_3) g(x_3) \\
& \qquad\qquad\qquad - \tilde{w}^2(v) : \phi^2(x_3) : g^3(x_3) \Big] \\
& + \frac{w(u)}{8} u^2 \left(\partial_\mu i\Delta_F(v)\right) \left(\partial^\mu \tilde{w}(v)\right) \tilde{w}(v) : \phi^2(x_3) : g^3(x_3) \Big\}. \qquad \textbf{(A.4)}
\end{aligned}
$$

All the other diagrams are superficially convergent, but some of them contain subdivergences. We obtain
Diagram 8:

$$
\begin{aligned}
S_3^{(8)}(g) \quad = \quad & \frac{i\lambda^3}{288} \int d^4u \int d^4v \int d^4x_3 \; (i\Delta_F(u))^3 \\
& \Big\{ : \phi^4(x_3)\phi(v)\phi(u+v) : g(u+v) g(v) g(x_3) \\
& - w(u) : \phi^4(x_3)\phi^2(v) : g^2(v) g(x_3) \\
& - \frac{w(u)}{8} u^2 : \phi^4(x_3) \left(\Box\phi(v)\right) \phi(v) : g^2(v) g(x_3) \Big\},
\end{aligned}
$$

(A.5)

Diagram 9:

$$
\begin{aligned}
S_3^{(9)}(g) \quad = \quad & \frac{i\lambda^3}{24} \int d^4u \int d^4v \int d^4x_3 \;\; (i\Delta_F(u))^2 \, i\Delta_F(v) \\
& \Big\{ : \phi^3(x_3)\phi^2(u+v+x_3)\phi(v+x_3) : g(u+v+x_3)g(v+x_3)g(x_3) \\
& -w(u) : \phi^3(x_3)\phi^3(v) : g^2(v)g(x_3) \Big\},
\end{aligned}
\tag{A.6}
$$

Diagram 11:

$$
\begin{aligned}
S_3^{(11)}(g) \quad = \quad & \frac{i\lambda^3}{384} \int d^4u \int d^4x_2 \int d^4x_3 \;\; (i\Delta_F(u))^2 \\
& \Big\{ : \phi^4(x_3)\phi^2(x_2)\phi^2(u+x_2) : g(u+x_2)g(x_2)g(x_3) \\
& -w(u) : \phi^4(x_3)\phi^4(x_2) : g^2(x_2)g(x_3) \Big\}.
\end{aligned}
\tag{A.7}
$$

The last three diagrams do not depend on $\tilde{w}$ and yield no contribution to $\tilde{\Delta}_3$.

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