On the role of asymptotic structures in quantum field theory over curved backgrounds

Leipzig - 24th LQP meeting

Claudio Dappiaggi

June 23, 2009

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Guideline to the notes

In the early nineties it appeared on the arXiv a preprint, written by G. 't Hooft [33], in which a revolutionary idea was advocated, namely it is possible to encode the information of a physical system living in a black hole spacetime into suitable degrees of freedom of the event horizon. This idea, which soon went under the name of *holographic principle*, is clearly rather heuristic and certainly non constructive since it dows not provide any clue on how to concretely implement. Nonetheless its tantalizing consequences lead soon to formulate an even stronger generalization to a generic curved background in which the role of the event horizon was played by a suitable codimension one submanifold of the considered spacetime.

Soon afterwards, dozens of application, both at a classical and at a quantum level, appeared and a recollection of all the achieved results would require most probably a whole book only for the references. We shall instead be interested on a rather notable niche of these applications, namely [40, 41], which displyed a successful application of the holographic principle in the context of anti-de-Sitter spacetimes and with the language of the algebraic formulation of quantum field theory. Hence, one of the great merit of these papers was the trading of the heuristic ideas of the so-called AdS/CFT correspondence [1] to a mathematically precise correspondence between a quantum field theory in the bulk of an AdS spacetime and a second conformal field theory living on the boundary.

Under the guidance of this application, further attempts to implement 't Hooft paradigm in other classes of physical systems were devised and the aim of these lectures will be to discuss the scenario of asymptotically flat spacetimes [3, 39], which are solutions of Einstein's equation with vanishing cosmological constant having Minkowski as their prototype following the road paved in [16, 18, 20]. To this avail, the role of the distinguished codimension one submanifold on which to encode the bulk datas, will be played by the so-called null infinity and, more generally, by the conformal boundary; hence the causal and conformal structures of the class of manifolds we are interested are central to the implementation of the holographic principle, which is also known as bulk-to-boundary correspondence. For this reason these notes will foucs, in the first chapter, in a thorough discussion of the construction of such a boundary. Since there are several approaches in the literature where the construction of null infinity is proposed, we tried to condense them as much as possible in a unique body with the main goals of proposing a uniform notation and of concentrating in a single place all the informations and the calculations which are needed in order to actively work in this specific topic. Hence we often indulged in providing even the simpler steps needed to prove the various statements trying to minimize the need of browsing through dozens of books, all with their own notation.

The second part focuses instead on the more genuine field theory aspects. Here we also indulged in providing many details on the classical side of the problem, since it is also a realm where one could easily wander in the middle of an endless literature, although we shall often refer to [6] which represents a relatively recent and complete reference. The quantum aspects of our construction are also discussed, but most of the theorems and of the propositions are here only stated since we would simply copy the proofs of [18] and of [37, 38]; hence we refer an interested reader to these cited papers for more informations.

Chapter 1

Asymptotically Flat Spacetimes

1.1 The example of Minkowski spacetime

In the previous section we advocated the need to understand how to discuss the role and the behaviour of physical quantities which peel-off at infinity in a large class of solutions of Einstein's equations whose behaviour does not differ right at the infinity from that of Minkowski spacetime.

Since the flat space is the prototype of all the constructions and the analysis, we shall perform afterwards, it is highly desirable to start looking, first, at this very particular case, as a guiding tool for the generalisations we shall introduce later on. To this avail, we shall present roughly simple computations which slavishly follow those of many text books, [43] in particular. Nonetheless we shall try and, at the same time, we invite a potential reader to focus the attention not just on the results, but actually on the key hypothesis and operations leading to them. The idea will be to really clarify which are the building blocks of the operations we shall perform and, in a second time, to try to transport them in a more general curved scenario.

Therefore let us now focus on Minkowski spacetime $(\mathbb{R}^4, \eta_{\mu\nu})$ and, as said, we shall be interested in the behaviour of radiating fields at infinity, that means we shall consider only massless fields, the prototype being the electromagnetic one describing a photon, or, simply, a scalar field, *i.e.*,

$$\begin{cases} \Phi : \mathbb{R}^4 \to \mathbb{R} \\ \Box_\eta \Phi = 0, \end{cases}$$

where $\Box_\eta$ is the d'Alembert operator constructed with respect to the flat metric. Our goal is to study what happens to Φ at infinity and the paradigm, we shall abide to, is that "one needs to bring infinite to finite" or, in my opinion more appropriately, "we need a natural way to multiply 0 to infinity to get a finite quantity".

On a practical ground we know that, if we associate a Cauchy problem to the above partial differential equation (PDE), say with smooth compactly supported initial data, the solutions propagate along the so-called characteristic strips of the d'Alembert wave operators, namely the light cones. This is tantamount to assert that the Huygens principle is fulfilled and it transports to a level of PDE the particle physics picture that photons, and, more generally, massless objects travel along the light directions. Therefore, if we refer to the standard spherical coordinates of Minkowski spacetime (t, r, θ, φ), it is natural to introduce the so-called *advanced and retarded light coordinates*

$$u = t - r \text{ (retarded)} \qquad v = t + r \text{ (advanced)},$$

so that the flat metrics, parameterised by (u, v, θ, φ), reads:

$$ds^2 = -dudv + \left(\frac{v-u}{2}\right)^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

We wish to extract the information of Φ as $v \to \infty$ (or similarly as $u \to -\infty$) since the energy, which is carried by a massless field at infinity, along a null ray, is contained in the "$\frac{1}{v}$" coefficient. The first possible solution to this problem is to compactify the real axis, namely mapping $\mathbb{R}\cup\infty$ to the circle $\mathbb{S}^1$ or, equivalently, to a closed, and hence compact, interval $I \subset \mathbb{R}$. The most natural analytical realization of this idea is to introduce the map $f \equiv tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ such that, in terms of coordinates,

$$\begin{cases} V = tan^{-1}(v) \\ U = tan^{-1}(u) \end{cases} . \tag{1.1}$$

Hence we shall here proceed compactifying contemporary both light directions. Since $dv = \cos^{-2}(V)dV$ and $du = \cos^{-2}(U)dU$, (1.1) is tantamount to map $\mathbb{R}^4$ to $\widetilde{M} = \left((-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{S}^2\right)$ endowed with the metric

$$ds^2 = \frac{-dUdV}{\cos^2 U \cos^2 V} + \frac{\sin(V-U)^2}{4\cos^2 U \cos^2 V}\left[d\theta^2 + \sin^2 d\varphi^2\right].$$

One can infer per direct inspection that the result falls short of our original goal, since, although the points at infinity would amount to adding the missing $\pm\frac{\pi}{2}$-points in the above domain, the metric becomes there singular, since the cosine vanishes exactly whenever either U or V are equal to $\pm\frac{\pi}{2}$. One could argue that the problem arises due to a not so careful choice of the compactification function f, but, although we shall not prove it, one can convince himself that the problem is much more general and cannot be solved just by a fine-tuning of the mentioned map.

Nonetheless the drawback can be removed if, besides compactifying, we rescale the flat metric by a judicious choice of a smooth and positive function Ω^2, called *conformal factor*, *i.e.*, in the case under analysis,

$$\begin{cases} \widetilde{ds}^2 = \widetilde{g}_{\mu\nu}dx^\mu dx^\nu \doteq \Omega^2 ds^2 = -dUdV + \sin^2(V-U)d\mathbb{S}^2(\theta,\varphi) \\ \Omega^2 \doteq 4\cos^2 U \cos^2 V \in C^\infty(\mathbb{R}^4, \mathbb{R}^+), \end{cases} ,$$

where $\mathbb{R}^+$ are the strictly positive real numbers, whereas $d\mathbb{S}^2(\theta,\varphi)$ will represent, here and henceforth, the standard metric on the 2-sphere, namely $d\theta^2 + \sin^2\theta d\varphi^2$. It is manifest that the new metric is no more singular whenever either U or V are equal to $\pm\frac{\pi}{2}$.

From a mathematical point of view, the operation which is defined by the map f together with the rescaling Ω^2 is called a **conformal transformation** of $(\mathbb{R}^4, \eta_{\mu\nu})$ in $(\widetilde{M}, \widetilde{g})$, whereas, from a physical point of view, we performed a set of operations which alter the background, though they leave unchanged the causal structure of the spacetime. This concept will be made clear in the next section and we shall skip it for now, though we invite the reader to keep in mind since we shall soon come back to it.

Conversely, let us focus on what we have constructed and, to this avail, let us introduce the coordinates $T = V + U$ and $R = V - U$. The metric then reads:

$$\widetilde{ds}^2 = -dT^2 + dR^2 + \sin^2 R\, d\mathbb{S}^2(\theta,\varphi),$$

where $T \in (-\pi,\pi)$ whereas $R \in (0,\pi)$ since we have to recall that, originally, the coordinate r was strictly positive. More precisely $2r = \tan\left(\frac{T+R}{2}\right) - \tan\left(\frac{T-R}{2}\right) > 0$, which, per monotonicity of the tan-function, yields $R > 0$. The new form for $\widetilde{g}$ is rather remarkable because it is well known in the physical literature as the metric of the so-called *Einstein static Universe* $\mathcal{E}$, namely a manifold, static solution of Einstein's equations with a cosmological constant $\Lambda > 0$, topologically $\mathbb{R} \times \mathbb{S}^3$ and endowed with the metric $\widetilde{ds}^2$. Hence the coordinates (T, R, θ, φ) we are using do not cover the full $\mathcal{E}$ and we can, therefore, claim that the conformal transformation, above discussed, is nothing but a mapping of Minkowski spacetime into an open bounded subset $\mathcal{O} \subset \mathcal{E}$. The net advantage of this procedure is that we can close this open set and we call $\partial\mathcal{O}$ the **conformal boundary** of Minkowski spacetime. If one traces back all the performed operations, one can immediately see that $\partial\mathcal{O}$ has the natural interpretation of the image in the static Einstein Universe of the points at infinity in $(\mathbb{R}^4, \eta_{\mu\nu})$. Furthermore, if we suppress the angular coordinates, we can easily draw a diagram representing $\mathcal{O}$ (see figure 1.1), also known as "Penrose diagram" of Minkowski spacetime, and this helps us to recognise that the conformal boundary is actually composed of 5 different distinct "pieces", namely $\partial\mathcal{O} = i^+ \cup \Im^+ \cup i_0 \cup \Im^- \cup i^-$:

- $\Im^+$ and $\Im^-$ which are respectively called *future and past null infinity*. The former is the segment $T + R = \pi$ in the region $R, T > 0$ and it is tantamount to consider the locus $v = +\infty$ in Minkowski spacetime. The latter is defined as $-T + R = \pi$ in the region $R > 0$ and $T < 0$, thus corresponding to $u = -\infty$ in $(\mathbb{R}^4, \eta_{\mu\nu})$. Both have the topology of $I \times \mathbb{S}^2$, where I is an open interval of $\mathbb{R}$,

- i_0, also known as *spatial infinity* because it is the sphere in $\mathcal{E}$ defined as $T = 0$ and $R = \pi$, or, in other words, the locus $r = \infty$ in Minkowski spacetime,

- $i^\pm$ or *future and past time infinity*, namely the spheres $T = \pm\pi$ and $R = 0$. If one goes back to the standard (t, r, θ, φ) coordinates of Minkowski, it means that we are considering $t = \pm\infty$.

It is imperative to notice that, although $\partial\mathcal{O}$ is certainly a boundary, the noun "piece" was not used by chance, because, according to the characterisations just outlined, only $\Im^+$ and $\Im^-$ are codimension 1 subsets of $\mathcal{E}$ whereas $i^\pm$ as well as i_0 are simply 2-spheres and hence they are not proper boundaries.

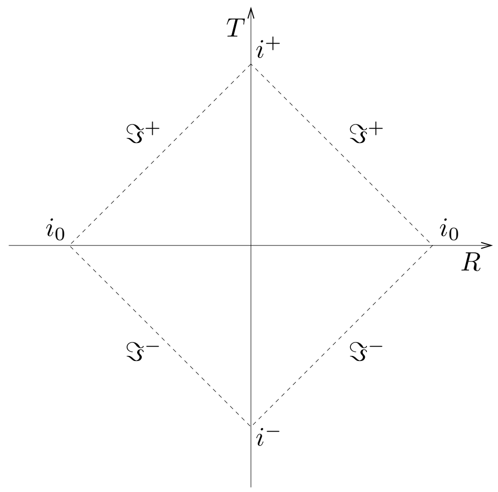

Figure 1.1: We here represent the image of Minkowski spacetime in the Einstein's static Universe. The angular coordinates are supressed and, hence, each point in the plane represents a 2-sphere of radius $\sin^2 R$.

If one gives a closer look to the procedure employed in the above discussion, we can recognise that we have substantially used three main ingredients to tame the infinities of Minkowski spacetime:

1. a compactification map to be applied to the directions along which we want to probe the behaviour of fields at infinity,

2. a conformal transformation of the metric to remove the divergences arising due to the compactification procedure,

3. the identification of an auxiliary (unphysical) spacetime in order to interpret the outcome of the previous two steps as the realization of the physical background as an open subset $\mathcal{O}$ of a larger manifold, $\widetilde{M}$. The boundary $\partial\mathcal{O} \subset \widetilde{M}$ (conformal boundary) is thus interpreted as the image of "infinity" in $\widetilde{M}$.

Question: is it possible to generalise the above procedures to an arbitrary curved background, or, in other words, when does a manifold look like Minkowski spacetime at infinity along null directions?

It is important to stress that, in the formulation of the question, we required explicitly how to mimic the behaviour of a flat spacetime along light-like directions although the compactification process, above depicted, is well-defined and "reasonably regular" also at both $i^{\pm}$ and i_0. Therefore, one could a priori, try to generalise the construction to curved backgrounds seeking to include in the above question also timelike and spacelike directions, leading to well-defined notion of both future and past timelike infinity as well as of spacelike infinity. This is certainly a possibility, though one cannot go that far in pursuing it; even if we shall not give a mathematical proof, one can realize that the only spacetimes fulfilling these regularity conditions at $\Im^{\pm}$, $i^{\pm}$ and i_0 are Minkowski and a few other curved backgrounds which differ from the flat one only in compact regions. Therefore, the best available option will be to generalise the conformal compactification procedure looking for a regular notion either of i^+, of i^- or of i_0. As we shall advocate, the first two cases are the most interesting from a physical point of view, but they have a further "draw-back", namely they lead only to a well-defined notion of null-infinity respectively only in the future and in the past. Conversely the latter scenario, which is the one investigated in [43], leads to a more traditional generalisation of the compactification of Minkowski spacetime which provides a regular construction of both $\Im^+$ and $\Im^-$. In the next few sections we shall first discuss such definition and later that with i^+ as distinguished point; we want also to stress that, from a mere geometrical perspective, one can, in principle, only focus on the behaviour along future or null light-like directions, completely forgetting about $i^{\pm}$ and i_0. This point of view is certainly the less restrictive one (hence leading to the potentially larger class of "asymptotically flat spacetimes") and it has been actively pursued in [30], though, alas, it is so general that it is not always suited to discuss field theoretical interesting scenarios.

1.2 Conformal Transformations

At the end of the previous section, we have argued that, in order to compactify Minkowski spacetime, only few ingredients were needed and they should be used as the building block of the generalisation to curved backgrounds. To this avail, one can see that the first and the third step have a natural and clear counterpart whenever the metric is not the flat one, while the second appears to be a little more tricky since geometric quantities are non-trivially affected by a conformal transformation. One can readily infer it if one notices that Minkowski spacetime, *i.e.*, a manifold with vanishing scalar curvature, has been mapped into an open subset of Einstein's static universe where $\widetilde{R} \neq 0$. Hence we shall devote this section to understand how the mentioned conformal transformations affect the underlying Lorentzian geometry and, more importantly, the related geometric quantities.

To start with, we recall that a four-dimensional Lorentzian spacetime $(M, g_{\mu\nu})$ together with a conformal factor $\Omega \in C^\infty(M, \mathbb{R}^+)$ gives rise to a conformally transformed spacetime $(M, \widetilde{g}_{\mu\nu})$ where

$$\widetilde{g}_{\mu\nu} \doteq \Omega^2 g_{\mu\nu}, \quad \widetilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}.$$

In the preceding section we argued that one of the advantage of conformal transformations is that it does not change the causal structure of the underlying background. Actually the connection between these two concepts is even deeper and we shall make it precise with the following statements:

Lemma 1.2.1. *Two conformally related manifold $(M, g_{\mu\nu})$ and $(M, \widetilde{g}_{\mu\nu})$ have identical causal structure.*

Proof. In order to prove the lemma, one just need to show that, whenever a vector $v^\mu \in T_xM$ for any $x \in M$ is either spacelike, lightlike or timelike, so it is with respect to $\widetilde{g}_{\mu\nu}$ and vice-versa. To this avail, one just notices:

$$g_{\mu\nu} v^\mu v^\nu \leq 0 \implies \widetilde{g}_{\mu\nu} v^\mu v^\nu = \Omega^2 g_{\mu\nu} v^\mu v^\nu \leq 0,$$

since Ω^2 is a strictly positive function. A same conclusion can be reached with the reversed inequality and, since the conformal factor is non-vanishing, everything holds true even if one starts from $\widetilde{g}_{\mu\nu}$ instead of $g_{\mu\nu}$. $\square$

At the same time it holds:

Proposition 1.2.1. *Two spacetimes $(M, g_{\mu\nu})$ and $(M, \widetilde{g}_{\mu\nu})$ with the same causal structure are conformally related.*

Proof. Let $p \in M$ and let $\{t^\mu, x_1^\mu, x_2^\mu, x_3^\mu\}$ be an orthonormal basis for $T_pM \sim \mathbb{R}^4$, that is $g_{\mu\nu}t^\mu t^\nu = -1$ and $g_{\mu\nu}x_i^\mu x_j^\nu = \delta_{ij}$ with $i, j = 1, .., 3$. Let us then construct the light-like vectors $v^\mu = t^\mu + x_i^\mu$ and $u^\mu = t^\mu - x_i^\mu$ for any but fixed i. Accordingly, per hypothesis, also $\widetilde{g}_{\mu\nu}v_\mu v_\nu = \widetilde{g}_{\mu\nu}u_\mu u_\nu = 0$ and this entails by linear algebra manipulations that t^μ and x^μ are orthogonal also with respect to $\widetilde{g}_{\mu\nu}$. Furthermore

$$\frac{\widetilde{g}_{\mu\nu}t^\mu t^\nu}{\widetilde{g}_{\mu\nu}x_i^\mu x_i^\nu} = \frac{\widetilde{g}_{\mu\nu}(u^\mu + v^\mu)(u^\nu + v^\nu)}{\widetilde{g}_{\mu\nu}(u^\mu - v^\mu)(u^\nu - v^\nu)} = \frac{\widetilde{g}_{\mu\nu}(u^\mu v^\nu + u^\nu v^\mu)}{-\widetilde{g}_{\mu\nu}(u^\mu v^\nu + u^\nu v^\mu)} = -1, \quad \forall i = 1, 2, 3 \tag{1.2}$$

where in the before last equality we just used the light-like nature of both u^μ and v^μ. In the same way, we can construct other light-like vectors as $\xi^\mu = t^\mu + \frac{1}{\sqrt{2}}\left(x_i^\mu + x_j^\mu\right)$ with $i \neq j$; if we repeat the same steps as above we are, however, lead to $g_{\mu\nu}x_i^\mu x_j^\nu = 0$, which entails that $\{t^\mu, x_1^\mu, x_2^\mu, x_3^\mu\}$ is an orthogonal basis for $\widetilde{g}_{\mu\nu}$. If we take now into account (1.2), we can draw that $\widetilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$ with $\lambda \in \mathbb{R} \setminus \{0\}$. If we repeat the same procedure for all points $p \in M$, we get the wanted thesis. □

Although the causal structure of two conformally related spacetimes appear to be identical, that does not hold true for all other geometric quantities and we shall now try to understand their relations. As a starting point we consider

•*the covariant derivatives* ∇ and $\widetilde{\nabla}$ which are constructed to be metric compatible, *i.e.*, $\nabla_\rho g_{\mu\nu} = 0$ and $\widetilde{\nabla}_\rho \widetilde{g}_{\mu\nu} = 0$. This yields:

$$\begin{cases} \widetilde{\nabla}_\rho g_{\mu\nu} = \widetilde{\nabla}_\rho \Omega^{-2}\widetilde{g}_{\mu\nu} = -2\Omega^{-3}(\widetilde{\nabla}_\rho\Omega)\widetilde{g}_{\mu\nu} \\ \nabla_\rho \widetilde{g}_{\mu\nu} = \nabla_\rho\Omega^2 \widetilde{g}_{\mu\nu} = 2\Omega\left(\nabla_\rho\Omega\right)g_{\mu\nu} \end{cases}. \tag{1.3}$$

Furthermore, since covariant derivatives must agree on scalars (they coincide simply with the partial derivative which is insensible to the metric), we have to study their action on vectors. In this case, according to standard argument of differential geometry (see chapter 3 of [43]), there must exist a tensor field $C^\rho_{\mu\nu}$ such that

$$\nabla_\mu\omega_\nu = \widetilde{\nabla}_\mu\omega_\nu - \widetilde{C}^\rho_{\mu\nu}\omega_\rho, \quad \forall\omega \in T_pM$$

where metric compatibility yields that

$$\begin{aligned} \widetilde{C}^\rho_{\mu\nu} &= \frac{1}{2}g^{\rho\delta}\left[\widetilde{\nabla}_\mu g_{\nu\delta} + \widetilde{\nabla}_\nu g_{\mu\delta} - \widetilde{\nabla}_\delta g_{\mu\nu}\right] = \frac{\Omega^2}{2}\widetilde{g}^{\rho\delta}\left[-2\Omega^{-3}\left(\widetilde{\nabla}_\mu\Omega g_{\nu\delta} + \widetilde{\nabla}_\nu\Omega g_{\mu\delta} - \widetilde{\nabla}_\delta\Omega g_{\mu\nu}\right)\right] = \\ &= -\frac{1}{\Omega}\left[\delta^\rho_\nu\widetilde{\nabla}_\mu\Omega + \delta^\rho_\mu\widetilde{\nabla}_\nu\Omega - \widetilde{\nabla}^\rho\Omega\widetilde{g}_{\mu\nu}\right] = -2\delta^\rho_{(\mu}\widetilde{\nabla}_{\nu)}\ln\Omega + \widetilde{\nabla}^\rho(\ln\Omega)\widetilde{g}_{\mu\nu}. \end{aligned} \tag{1.4}$$

A similar computation leads us to the similarly useful formula relating

$$\begin{aligned} \widetilde{\nabla}_\mu\omega_\nu &= \nabla_\mu\omega_\nu - C^\rho_{\mu\nu}\omega_\rho, \quad \forall\omega \in T_pM \\ C^\rho_{\mu\nu} &= 2\delta^\rho_{(\mu}\nabla_{\nu)}\ln\Omega - \nabla^\rho(\ln\Omega)g_{\mu\nu}. \end{aligned} \tag{1.5}$$

Notice that a little trick to compute formulas like the last one, without much of an effort, is to send Ω into Ω^{-1} and to add/remove the tilde from the involved geometrical quantities. It goes by itself that this is just an easy path to be used *cum grano salis*.

The usefulness of a formula such as (1.4) becomes manifest thanks to the following lemma which strengthens the relation between causal structures and conformal transformations.

Lemma 1.2.2. *Null geodesics are conformally invariant.*

Proof. Let us take any geodesics $\gamma : I \subset \mathbb{R} \to M$ whose tangent vector, say v^ν, must thus satisfy at any point the defining identity for the covector v_ν:

$$v^\mu\nabla_\mu v_\nu = 0,$$

which, in terms of the covariant derivative $\widetilde{\nabla}$, becomes, out of (1.4),

$$v^\mu \widetilde{\nabla}_\mu v_\nu = v^\mu \nabla_\mu v_\nu + v^\mu \widetilde{C}^\rho_{\mu\nu} v_\rho = v^\mu \left(-2\delta^\rho_{(\mu} \widetilde{\nabla}_{\nu)} \ln \Omega + \widetilde{\nabla}^\rho (\ln \Omega) \widetilde{g}_{\mu\nu} \right) v_\rho = -v^\rho v_\rho \widetilde{\nabla}_\nu (\ln \Omega).$$

To conclude we just need to notice, that according to lemma 1.2.1, the light-like nature of v^μ is the same for conformally related metrics. We can conclude that, under our hypothesis,

$$v^\mu \widetilde{\nabla}_\mu v_\nu = 0,$$

hence v^μ is a geodesic even with respect to $\widetilde{g}_{\mu\nu}$. $\square$

From a physical point of view, the interesting geometrical objects are certainly the Ricci tensor and the scalar curvature since they are, on the one hand, the basic data in the Einstein's equations, while, on the other hand, they enter the fray almost every time we study a classical field on a curved background. Therefore, if we want to use conformal transformations in these physical scenarios, we have to understand how they behave whenever we rescale the metric and, to this end, the starting point is the *Riemann tensor*. Always, bearing in mind, that quantities with the tilde-symbol refer to the metric $\widetilde{g}_{\mu\nu}$, we recall

$$R^\delta_{\mu\nu\rho} v_\delta = [\nabla_\mu, \nabla_\nu] v_\rho, \qquad \widetilde{R}^\delta_{\mu\nu\rho} v_\delta = \left[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu\right] v_\rho.$$

If we expand the commutator and we take into account (1.5), after a few algebraic steps we get to

$$\begin{aligned}
\widetilde{R}^\delta_{\mu\nu\rho} = R^\delta_{\mu\nu\rho} - 2\nabla_{[\mu} C^\delta_{\nu]\rho} + 2C^\eta_{\rho[\mu} C^\delta_{\nu]\eta} = R^\delta_{\mu\nu\rho} - 2\nabla_{[\mu} \left(2\delta^\delta_{(\nu} \nabla_{\rho)} \ln \Omega - \nabla^\delta (\ln \Omega) g_{\nu\rho} \right) + \\
+2 \left(2\delta^\eta_{(\rho} \nabla_{[\mu)} \ln \Omega - \nabla^\eta (\ln \Omega) g_{\rho[\mu} \right) \left(2\delta^\delta_{(\nu]} \nabla_{\eta)} \ln \Omega - \nabla^\delta (\ln \Omega) g_{\nu]\eta} \right) = \\
= R^\delta_{\mu\nu\rho} + 2\delta^\delta_{[\mu} \nabla_{\nu]} \nabla_\rho (\ln \Omega) - 2g_{\rho[\mu} \nabla_{\nu]} \nabla^\delta (\ln \Omega) + 2\nabla_{[\mu} (\ln \Omega) \delta^\delta_{\nu]} \nabla_\rho (\ln \Omega) + \\
-2\nabla_{[\mu} (\ln \Omega) g_{\nu]\rho} \nabla^\delta (\ln \Omega) - 2g_{\rho[\mu} \delta^\delta_{\nu]} \nabla^\eta (\ln \Omega) \nabla_\eta (\ln \Omega). \qquad (1.6)
\end{aligned}$$

The formula simplifies quite a lot if we contract the second and the fourth index (also recall $g_{\mu\nu}\delta^{\mu\nu} = 4$) since

$$\begin{aligned}
\widetilde{R}_{\mu\rho} = \widetilde{R}^\nu_{\mu\nu\rho} = R_{\mu\rho} + \nabla_\mu \nabla_\rho (\ln \Omega) - 4\nabla_\mu \nabla_\rho (\ln \Omega) - g_{\mu\rho} \Box (\ln \Omega) + g_{\rho\nu} \nabla_\mu \nabla^\nu (\ln \Omega) + 2\nabla_\mu (\ln \Omega) \nabla_\rho (\ln \Omega) - \\
\delta^\nu_\mu \nabla_\nu (\ln \Omega) \nabla_\rho (\ln \Omega) - \nabla_\mu (\ln \Omega) \nabla_\rho (\ln \Omega) + g_{\mu\rho} \nabla_\nu (\ln \Omega) \nabla^\nu (\ln \Omega) - 3g_{\mu\rho} \nabla_\eta (\ln \Omega) \nabla^\eta (\ln \Omega) = \\
R_{\mu\rho} - 2\nabla_\mu \nabla_\rho (\ln \Omega) - g_{\mu\rho} \Box (\ln \Omega) + 2\nabla_\mu (\ln \Omega) \nabla_\rho (\ln \Omega) - 2g_{\mu\rho} \nabla_\eta (\ln \Omega) \nabla^\eta (\ln \Omega), \qquad (1.7)
\end{aligned}$$

and, equivalently, since it will be later useful,

$$R_{\mu\nu} = \widetilde{R}_{\mu\nu} + 2\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu (\ln \Omega) + \widetilde{g}_{\mu\nu} \widetilde{\Box} (\ln \Omega) + 2\widetilde{\nabla}_\mu (\ln \Omega) \widetilde{\nabla}_\nu (\ln \Omega) - 2\widetilde{g}_{\mu\nu} \widetilde{\nabla}_\eta (\ln \Omega) \widetilde{\nabla}^\eta (\ln \Omega). \qquad (1.8)$$

As a last step we can construct the relation between the scalar curvatures contracting the free indices with the metric $\widetilde{g}^{\mu\rho} = \Omega^{-2} g^{\mu\rho}$, *i.e.*,

$$\begin{aligned}
\widetilde{R} \doteq \widetilde{g}^{\mu\rho} \widetilde{R}_{\mu\rho} = \Omega^{-2} \left[R - 2\Box (\ln \Omega) - 4\Box (\ln \Omega) + 2\nabla^\mu (\ln \Omega) \nabla_\mu (\ln \Omega) - 8\nabla^\mu (\ln \Omega) \nabla_\mu (\ln \Omega) \right] = \\
= \Omega^{-2} \left[R - 6\Box (\ln \Omega) + 6\nabla^\mu (\ln \Omega) \nabla_\mu (\ln \Omega) \right]. \qquad (1.9)
\end{aligned}$$

Remark 1.2.1. A byproduct of these lengthy calculations is the possibility to construct a *conformal invariant tensor* by a suitable combination of the above quantities, namely the so-called "Weyl tensor" which is nothing but the trace free part of the Riemann tensor:

$$C_{\mu\nu\rho\delta} = R_{\mu\nu\rho\delta} - g_{\mu[\rho} R_{\delta]\nu} + g_{\nu[\rho} R_{\delta]\mu} + \frac{R}{3} g_{\mu[\rho} g_{\delta]\nu} = \widetilde{C}_{\mu\nu\rho\delta}.$$

1.3 General Definition

Since we know have a full control of the behaviour under conformal transformations of the geometric data involved in the realization of a curved background, we are ready to make precise the heuristic idea of a spacetime which looks like Minkowski at infinity along all null directions. The definition here given is also present in [43] and was first introduced by Ashtekar and Xanthopoulos in [31]. Nonetheless, since the reader might not be familiar with the nomenclature of causal structures, let us recall two important notions:

- We call *causal future* (or past) of a point p of a Lorentzian manifold M, the set $J^{\pm}(p, M)$ of all points $q \in M$ such that it exists a curve $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$ whose tangent vector at each point is future directed and either timelike or lightlike. Accordingly the causal future or past of a subset $S \subset M$ is nothing but $\bigcup_{p\in S} J^{\pm}(p, M)$. The symbol M is usually omitted unless potential confusions could arise and, in this case, it will be restored.
- A spacetime is called *strongly causal*, if for all $p \in M$ and for all open neighbourhood $\mathcal{O}_p$ of p, there exists a second neighbourhood $\mathcal{V} \subset \mathcal{O}$ such that no causal curve intersects $\mathcal{V}$ more than once. Notice that this entails, not only that closed timelike curves are forbidden, but also that the initial and final point of any timelike curve cannot get arbitrary close even though they are not coinciding; that yields that a small perturbation of the metric cannot "accidentally" let a closed timelike curve pop up.

Definition 1.3.1. *A four dimensional vacuum spacetime $(M, g_{\mu\nu})$ (a.k.a., physical spacetime), i.e., a solution of Einstein's vacuum field equations, is called* **asymptotically flat at null and spatial infinity** *if there exist*

a) $(\widetilde{M}, \widetilde{g}_{\mu\nu})$ (a.k.a., unphysical spacetime) with $g_{\mu\nu} \in C^\infty(M)$ and $\widetilde{g}_{\mu\nu} \in C^\infty(\widetilde{M}\backslash\{i_0\})$, i_0 being a 2-sphere embedded in $\widetilde{M}$,

b) a conformal isometry from M to $\widetilde{M}$, that is a map $\psi : M \to \psi[M] \subset \widetilde{M}$, $\psi[M]$ being an open subset of $\widetilde{M}$, and a function $\Omega \in C^\infty(\psi[M], \mathbb{R}^+)$ fulfilling $\widetilde{g}_{\mu\nu}|_{\psi[M]} = \Omega^2(\psi^* g)_{\mu\nu}$

such that the following five conditions hold true:

1. *there exists a 2-sphere $i_0 \in \widetilde{M}$ such that $\overline{J^+(i_0)} \cup \overline{J^-(i_0)} = \widetilde{M} \setminus \psi[M]$,*
2. *there exists $\mathcal{O}$, open neighbourhood in $\widetilde{M}$ of $\partial\psi[M]$, such that $(\mathcal{O}, \widetilde{g}_{\mu\nu})$ is strongly causal,*
3. *the function Ω can be extended (not necessarily in a unique way) to a function on the whole $\widetilde{M}$ which is smooth except at most at i_0 where it is twice differentiable,*
4. *Ω must vanish on $\partial J^{\pm}(i_0) \setminus \{i_0\}$, whereas $d\Omega \neq 0$ therein, "d" being the external derivative. Furthermore, on i_0, it holds that $\Omega(i_0) = 0$ and*
$$\lim_{i_0} \widetilde{\nabla}_\mu \Omega = 0, \quad \lim_{i_0} \widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \Omega = 2\widetilde{g}_{\mu\nu}(i_0), \tag{1.10}$$
5. *the map of null directions at i_0 into the space of integral curves of $n^\mu \doteq \widetilde{\nabla}^\mu \Omega$ on $\partial J^{\pm}(i_0) \setminus \{i_0\}$ is a diffeomorphism and, furthermore, for any choice of a function $\omega \in C^\infty(\widetilde{M} \setminus \{i_0\})$ such that $\omega > 0$ on $(\psi[M] \cup J^{\pm}(i_0)) \setminus \{i_0\}$ and $\widetilde{\nabla}_\mu(\omega^4 n^\mu) = 0$ on $\partial J^{\pm}(i_0) \setminus \{i_0\}$, the vector field $\omega^{-1} n^\mu$ is complete on $\partial J^{\pm}(i_0) \setminus \{i_0\}$.*

It is absolutely fair to admit that such a definition is monstrous and certainly hardly marketable unless we show that it is indeed the natural generalisation of the construction of section 1.1 and that each of the above hypothesis lead to important consequences on the structure of the spacetime. Hence, while *a*) and *b*) are a mathematical statement of the conformal compactification process, the requests 1-5 are more subtle to read.

Condition 1 and 2: these are dictating the information on the causal structure of the boundary (and hence of infinity) of the physical spacetime, seen as an open subset of the unphysical one. Particularly it is worth noticing that

- condition 1 sets that i_0 is spatially separated from all points in $\psi[M]$ and hence it is natural to address to it as *spatial infinity*
- condition 1 and 2 grants us that, from a causal perspective, no pathological situation can arise in a neighbourhood of $\partial\psi[M]$, which, thus, can be genuinely called the *conformal boundary* and it is here constituted of three different sets, namely $\partial\psi[M] = \Im^+ \cup i_0 \cup \Im^-$, where $\Im^\pm \doteq \partial J^\pm(i_0) \setminus \{i_0\}$.

Condition 3: this hypothesis apparently only tells us that the conformal factor must be a well-behaved function on the whole unphysical spacetime, but, actually, it also guarantees us that the "Penrose compactification process" is highly non-unique. To wit, if $(\widetilde{M}, \widetilde{g}_{\mu\nu})$ is an unphysical spacetime associated to $(M, g_{\mu\nu})$ for a certain pair (ψ, Ω), then also $(\widetilde{M}, \omega^2\widetilde{g}_{\mu\nu})$ is another good unphysical spacetime with conformal factor $\omega\Omega$ provided that $\omega \in C^\infty(\widetilde{M}, \mathbb{R}^+)$ except at most i_0 where it must be of class C^2 and $\omega(i_0) = 1$. This last condition arises out of (1.10), which entails that, if $\omega\Omega$ is a good conformal factor, then

$$2\widetilde{g}_{\mu\nu}(i_0) = \widetilde{\nabla}_\mu\widetilde{\nabla}_\nu(\omega\Omega) = \widetilde{\nabla}_\mu\widetilde{\nabla}_\nu(\omega)\Omega + 2\omega\widetilde{\nabla}_{(\mu}\widetilde{\nabla}_{\nu)}\Omega + \omega\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\Omega = 2\omega\widetilde{g}_{\mu\nu}(i_0),$$

where in the last equality we used the hypothesis that both Ω and $\nabla_\nu\Omega$ are supposed to vanish at spatial infinity. Henceforth we shall refer to the arbitrariness in fixing ω as a **gauge freedom** for the conformal factor.

Condition 4: this is far the richest of all hypothesis and it dissipates all the doubts that a spacetime fulfilling the above definition really "looks-like" Minkowski at infinity along null directions. Let us see why! The condition on the vanishing of Ω on $\Im^\pm$ tells us that, in order to bring the loci, which are at infinity in the physical spacetime, at a finite distance with respect to any bulk point, an infinite amount of stretch has occurred. In other words, the infinite distances arising in the physical background have been all multiplied by zero, in order to get a well-defined quantity.

The requirement $d\Omega \neq 0$ is even subtler since it tells us that the metric gets flat in a neighbourhood of $\Im^\pm$. Let us show it only for future null infinity, the other case being absolutely identical. We can start from $(M, g_{\mu\nu})$ which we know it is a vacuum spacetime. Hence, out of (1.8), we get

$$R_{\mu\nu} = 0 \Longrightarrow 0 = \Omega R_{\mu\nu} = \Omega\widetilde{R}_{\mu\nu} + 2\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\Omega + \widetilde{g}_{\mu\nu}\widetilde{\Box}\Omega - 3\widetilde{g}_{\mu\nu}\frac{\widetilde{\nabla}^\rho\Omega\widetilde{\nabla}_\rho\Omega}{\Omega}. \tag{1.11}$$

Since both the unphysical metric and the Ricci tensor are smooth, we can perform the limit to $\Im^\pm$ of both sides of the last equality and this entails that, since the first two terms in the right-hand side, must be smooth so must also the third term which is potentially the most pernicious since the denominator vanishes at null infinity. Nonetheless the whole ratio must be a finite quantity, which is only possible if $\widetilde{\nabla}^\rho\Omega\widetilde{\nabla}_\rho\Omega = 0$, namely $n^\mu \doteq \widetilde{\nabla}^\mu\Omega$ is a light-like vector. Furthermore, since, per hypothesis, $\widetilde{\nabla}^\mu\Omega \neq 0$ and since $\Im^+$ is the locus where $\Omega = 0$, it turns out that n^μ must be orthogonal to future null infinity. We can now exploit the gauge freedom in the choice of Ω, namely we are free to select ω which yields $\widetilde{g}'^{\mu\nu} = \omega^2\widetilde{g}^{\mu\nu}$ and $\Omega' = \omega\Omega$. Hence it holds

$$\frac{\widetilde{g}'^{\mu\nu}\widetilde{\nabla}'_\mu\Omega\widetilde{\nabla}'_\nu\Omega}{\Omega} = \omega^{-3}\widetilde{g}^{\mu\nu}\left[\Omega\widetilde{\nabla}_\mu\omega\widetilde{\nabla}_\nu\omega + \omega^2\frac{\widetilde{\nabla}_\mu\Omega\widetilde{\nabla}_\mu\Omega}{\Omega} + 2\omega\widetilde{\nabla}_\mu\Omega\widetilde{\nabla}_\nu\omega\right],$$

where the prime-symbol indicates geometric quantities calculated with respect to $\widetilde{g}'^{\mu\nu}$, whereas the right-hand side of the equality has been calculated out of (1.4) and (1.5) in order to relate $\widetilde{\nabla}'_\mu$ to $\widetilde{\nabla}_\mu$. We can now fine-tune the choice of the gauge factor in such a way that the left-hand side of the above identity vanishes on $\Im^+$. Since Ω is identically zero on null infinity while $\omega \neq 0$, the last assertion is tantamount to impose the following partial differential equation for the gauge factor:

$$\frac{\widetilde{\nabla}^\mu \Omega \widetilde{\nabla}_\mu \Omega}{\Omega} = -2n^\mu \widetilde{\nabla}_\mu (\ln \omega). \tag{1.12}$$

Notice briefly that the ln-function guarantees that any solution of (1.12) yields a strictly positive ω.

That said, let us suppose to have gauge-transformed the conformal factor is such a way that the above analysed term vanishes on null infinity. We can go back to (1.11) and concentrate on the right-hand side of the last equality, which thus reads $\lim\limits_{\Im^+} 2\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \Omega + \widetilde{g}_{\mu\nu} \widetilde{\Box}\Omega = 0$. If we contract with $\widetilde{g}^{\mu\nu}$, we get

$$\begin{cases} \lim\limits_{\Im^+} \widetilde{\Box}\Omega = 0 \\ \lim\limits_{\Im^+} \widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \Omega = 0 \end{cases},$$

which, in turn, yields that

$$\lim_{\Im^+} n^\mu \widetilde{\nabla}_\mu n^\nu = \lim_{\Im^+} \widetilde{\nabla}^\mu \Omega \widetilde{\nabla}_\mu \widetilde{\nabla}^\nu \Omega = 0. \tag{1.13}$$

In other words, since n^μ is also a light-like vector, its integral lines n^μ are *null geodesics* of $\Im^+$. Furthermore it is important to notice that, as a byproduct, this result entails the existence of a remainder of the gauge freedom; as a matter of fact (1.12) fixes only $n^\mu \widetilde{\nabla}_\mu \ln \omega$, that is ω along the null direction fixed by $\widetilde{\nabla}^\mu \Omega$. Since $\Im^+ \doteq \partial J^+(i_0) \setminus \{i_0\}$, we can still assign the gauge factor on any 2-surface $\mathcal{S}$ which intersects only once the integral lines of n^μ. We can now use a bit of information from condition 5), namely we know that the set of such integral lines is diffeomorphic to the map on null directions leaving i_0, which, in turn, is a 2-sphere. That means that the mentioned $\mathcal{S}$ must be diffeomorphic to $\mathbb{S}^2$ and, thus, future null infinity is **topologically equivalent** to $\mathbb{R} \times \mathbb{S}^2$.

To summarise, we have uncovered that $\Im^+$ identifies a light direction, which, together with the information that the light-like vector n^μ is orthogonal to null infinity, entails that $\widetilde{g}_{\mu\nu}$ induces a Riemannian metric on $\mathcal{S}$. Heuristically speaking, one can see per direct inspection that, under the hypothesis that the underlying metric in of Lorentzian signature, once two orthogonal light-like vectors are identified, they can only arise out of a suitable linear combination of a timelike and a spacelike vector; this only leaves open the possibility that the metric on $\mathcal{S}$ has a $(+,+)$ signature. We can now use a standard result in the theory of Riemann surfaces [23] which grants us that, up to a diffeomorphism, every metric on $\mathcal{S}$ must be conformal to that of the unit sphere, that is, introducing the standard (θ, φ)-coordinates on $\mathbb{S}^2$, $(\mathcal{S}, \widetilde{h}_{\mu\nu}) \equiv (\mathbb{S}^2, f^2 h_{\mu\nu})$ where $h_{\mu\nu} dx^\mu dx^\nu = d\theta^2 + \sin^2 \theta d\varphi^2$. If we now recall that we have still the freedom to fix the gauge factor on each $\mathcal{S}$, we can select it in order to counterbalance f and, hence, we can simply consider on $\mathcal{S}$ the metric of the unit sphere.

According to our previous discussion we can introduce a set of coordinates in an open neighbourhood of $\Im^+$ as $(\Omega, u, \theta, \varphi)$ where u is the affine parameter along the integral curves of n^μ, here normalised as $n^\mu \widetilde{\nabla}_\mu u = 1$. This is the so called **Bondi frame** and we can express the metric on $\Im^+$ with these coordinates; if we notice that both ∂_u and ∂_Ω are light-like, *i.e.*, $\widetilde{g}_{uu} = \widetilde{g}_{\Omega\Omega} = 0$, and that the above normalisation condition for u yields $g_{u\Omega} = 1$, we are left with the *Bondi metric*

$$\widetilde{ds}^2|_{\Im^+} = 2d\Omega du + d\theta^2 + \sin^2 d\varphi^2.$$

We are now interested in understanding the form of the metric in the physical spacetime which leads to the Bondi one on null infinity. To this avail let us notice that the vanishing on $\Im^+$ of $\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \Omega$ entails that $\widetilde{g}_{uu}$, $\widetilde{g}_{u\theta}$ and $\widetilde{g}_{u\varphi}$ are $O(\Omega^2)$ as $\Omega \to 0$. Hence

$$ds^2 = \Omega^{-2} \widetilde{ds}^2 + O(1)[du^2 + dud\theta + dud\varphi] + O(\Omega^1)[\text{all others}],$$

which, under the transformation $v = 2\Omega^{-1}$, gives

$$ds^2 = -2dvdu + \frac{v^2}{4}[d\theta^2 + \sin^2\theta d\varphi^2] + O(1)[du^2 + dud\theta + dud\varphi] +$$
$$+ O(v)[d\theta^2 + d\theta d\varphi + d\varphi^2] + O(v^{-1})[dudv + dvd\theta + dvd\varphi] + O(v^{-3})dv^2.$$

In order to interpret this metric from a physical and geometrical point of view, let us first perform a coordinate transformation of the form $v \mapsto v + f(u, \theta, \varphi)$ where the function f is chosen is such a way to get rid of the $O(1)[du^2 + dud\theta + dud\varphi]$, even though, in turn, one acquires a $O(1)[dvd\theta + dvd\varphi]$ term. As a last step, let us introduce the so-called *asymptotically Minkowskian coordinates*

$$t = \frac{u+v}{2}, \quad x = \frac{v-u}{2}\sin\theta\cos\varphi, \quad y = \frac{v-u}{2}\sin\theta\sin\varphi \quad z = \frac{v-u}{2}\cos\theta.$$

After a few algebraic steps, one can see that the physical metric reads

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + O(v^{-1})...,$$

that is the metric assumes up to terms of order v^{-1}, v being the light coordinate, the Minkowski form as $v \to \infty$. Hence this entails that the given definition of asymptotically flat spacetime really mimics the behaviour of a manifold which looks like Minkowski at infinity.

Condition 5: as we already commented, the first part of this condition entails that $\Im^+$ can be read as a *complete* light cone with spatial infinity as its tip; thus the topology is that of $\mathbb{R} \times \mathbb{S}^2$. The second part instead sets a rather strange condition on ω which is actually equivalent to impose (1.12) since

$$0 = \widetilde{\nabla}_\mu(\omega^4 n^\mu) = 4\omega^3\widetilde{\nabla}_\mu(\omega)n^\mu + \omega^4\widetilde{\nabla}_\mu n^\mu,$$

which can be rearranged as

$$n^\mu\widetilde{\nabla}_\mu(\ln\omega) = -\frac{1}{4}\widetilde{\nabla}^\mu\widetilde{\nabla}_\mu\Omega = -\frac{1}{2}\frac{\widetilde{\nabla}^\mu\Omega\widetilde{\nabla}_\mu\Omega}{\Omega},$$

where in the last equality we used that $g_{\mu\nu}$ fulfils per hypothesis Einstein's equations, hence $R_{\mu\nu} = 0$; therefore, if contract (1.8) with $\widetilde{g}^{\mu\nu}$ and we multiply both sides by Ω, we end up with the used identity.

1.3.1 Outlook

In the previous discussion, we tried to somehow convince a potential reader that the employed definition of asymptotically flat spacetime is somehow consistent with our heuristic expectations. At the same time, a closer inspection of the employed procedure reveals that the rich structure of the conformal boundary can be to a certain extent maintained relaxing many of the given hypothesis. From a mathematical point of view, this is certainly a desirable option, but, as we already commented, it does not naturally lead to a physically meaningful conclusion. Let us nonetheless remark a few key point where we could change the taken assumptions:

- the request that $(M, g_{\mu\nu})$ solves the vacuum Einstein's equation in the whole bulk is certainly rather strong and, actually, in our previous calculations we only employed the vanishing of the Ricci tensor in a neighbourhood of null infinity and, thus we could restrict the hypothesis to this milder case. From a physical perspective, this is not meaningful until we consider systems with no matter content, but it provides interesting possibilities whenever the right-hand side of Einstein's equations is different from 0. In this case we could still retain the definition of asymptotically flat spacetime, though, in order to assure that, in a neighbourhood of $\Im^+$, the vacuum Einstein's equations are fulfilled, we should require that $\lim_{\Im^+}\Omega^{-2}T_{\mu\nu}$ is a smooth function.

- the condition on the distinguished role of i_0 is rather delicate. From the perspective of general relativity, it is rather natural to consider it as a point which lies in the compactified unphysical spacetime because it plays a distinguished role whenever one wishes to discuss the ADM formulation of Einstein's theory which leads to notable classical quantities such as the ADM mass which have a natural counterpart on $\Im^+$ (Bondi mass etc...).

  At the same time, from a mere geometric point of view, it is rather manifest that, in most of the consequences of the definition of asymptotically flat spacetime, i_0 plays only a marginal role and, as a matter of fact, this is reflected on the fact that $\Im^+$ and $\Im^-$ could be treated as separate entities (though connected through i_0). Hence one could replace i_0 with either i^+ or i^- as distinguished points (or, of course, also both at the same time), though this choice would allow only to consider spacetimes which are respectively asymptotically flat at future or at past null infinity. It is remarkable that, from a physical point of view, this is a rather appealing and often necessary point of view whenever one deals with classical or quantum fields as we shall briefly discuss in the next sections. In these cases the definition of asymptotic flatness can be almost slavishly adapted from the one we gave above and we shall present it later on in more details.

  In principle one could take even a more radical stance and, for example, try to define a spacetime with either future or past null infinity without reference to any distinguished point. This attitude is followed to a certain extent in [30]. An alternative option, which was recently proposed in [28], calls for using two different conformal factors, one at future null infinity and one at past null infinity, in order to enlarge the class of spacetimes admitting both $\Im^+$ and $\Im^-$.

It is fair to admit that the main problem of all the possible definitions is that, being based on causal structures, they are certainly non-constructive, hence it is difficult to model a non-trivial spacetime which is asymptotically flat. It is possible to prove that a large class of them exists but we shall not indulge in such analysis here; conversely we propose two exercises which are rather helpful in understanding what it is really going on and in not taking too seriously the nomenclature:

Exercise 1: show if Schwarzschild spacetime is asymptotically flat with respect to definition 1.3.1,

Exercise 2: let $ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]$ be the metric of a Friedmann-Robertson-Walker spacetime M with flat spatial section, *i.e.*, an homogeneous and isotropic solution of Einstein's equation. The coordinate $t \in I \subset \mathbb{R}$ while (x, y, z) are coordinates on $\mathbb{R}^3$. Show under which conditions on the function $a(t) \in C^\infty(I, \mathbb{R}^+)$, M together with ds^2 is the image in an "unphysical spacetime" of an asymptotically flat spacetime at past (or future) null infinity $(M', a^{-2}ds^2)$. Notice that, in this case, $\Im^+$ is also called *cosmological horizon*.

Regardless of all the potential subtleties of the definition of asymptotically flat spacetime, we can realize that the structure of $\Im^+$ (as well as that of $\Im^-$ whenever present) can be fully characterised by few data:

1. its topology $\Im^+ \sim \mathbb{R} \times \mathbb{S}^2$ and its differentiable structure which allows us to see null infinity as an intrinsic manifold, $\underline{\Im^+}$, completely detached and independent from $\widetilde{M}$. The mutual relation between $\Im^+$ and $\underline{\Im}^+$ is then encoded in the choice of a smooth diffeomorphism $\psi : \underline{\Im}^+ \to \Im^+ \subset \widetilde{M}$. Hence all quantities arising under restriction to $\Im^+$ can be either pulled-back or pushed-forward to $\underline{\Im^+}$ under the map ψ giving them a more abstract meaning. Only in the case of scalars the pull-back the action of ψ is trivial and therefore we shall not distinguish between scalars on $\Im^+$ and on $\underline{\Im}^+$.
2. the vector n^μ or, more properly, $\underline{n}^\mu = (\psi^* n)^\mu$,
3. the metric $\widetilde{g}_{\mu\nu}$ or, more properly, $\underline{g}_{\mu\nu} = (\psi^*\widetilde{g})_{\mu\nu}$.

Hence, from now on, we can write down relation between quantities directly on $\underline{\Im}^+$ following the standard rules of the pull-back. Therefore

$$\underline{n}_\mu = \psi^*(\widetilde{\nabla}_\mu \Omega) = \widetilde{\nabla}_\mu(\Omega \circ \psi) = 0, \quad \psi^*(\widetilde{g}_{\mu\nu} n^\mu) = (\psi^*\widetilde{g})_{\mu\nu}(\psi^* n)^\mu = \underline{g}_{\mu\nu}\underline{n}^\mu = \underline{n}_\nu = 0, \tag{1.14}$$

where the second chain of identities is a reshuffling of the first one, though it makes manifest that $\underline{g}_{\mu\nu}$ is a non-invertible metric (null structure of $\Im^+$ traded to $\underline{\Im}^+$). According to our previous discussions, the signature of $\underline{g}_{\mu\nu}$ must be of form $(0,+,+)$ and, hence, n^μ is, up to rescaling, the only vector which can be annihilated by the intrinsic metric on $\underline{\Im}^+$. Further informations can be derived out of the other notable geometric quantities, namely $\widetilde{R}_{\mu\nu}$ and $\widetilde{R}$; instead of combining them as usual in the Einstein tensor, we shall consider the somehow unusual tensor $\widetilde{S}_{\mu\nu} = \widetilde{R}_{\mu\nu} - \frac{\widetilde{R}}{6}\widetilde{g}_{\mu\nu}$. If we use the conformal transformation rules (1.7) and (1.9), we get:

$$\widetilde{S}_{\mu\nu} = R_{\mu\nu} - 2\left(\frac{\nabla_\mu\nabla_\nu\Omega}{\Omega} - \frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2}\right) - g_{\mu\nu}\left(\frac{\Box\Omega}{\Omega} - \frac{\nabla^\rho\Omega\nabla_\rho\Omega}{\Omega^2}\right) + \\ +2\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2} - 2g_{\mu\nu}\frac{\nabla^\rho\Omega\nabla_\rho\Omega}{\Omega^2} - \frac{g_{\mu\nu}}{6}\left(R - 6\frac{\Box\Omega}{\Omega}\right).$$

If we multiply both sides by Ω we get the identity

$$\Omega\widetilde{S}_{\mu\nu} + \mathcal{L}_n\widetilde{g}_{\mu\nu} - \widetilde{g}_{\mu\nu}\left(\frac{\widetilde{\nabla}_\mu\Omega\widetilde{\nabla}_\nu\Omega}{\Omega}\right) = \Omega^{-1}S_{\mu\nu} = 0,$$

where we employed the defining relation $\mathcal{L}_n\widetilde{g}_{\mu\nu} = 2\widetilde{\nabla}_\mu n_\nu$ and the previous information on both the behaviour of Ω and of its derivatives in order to conclude that the right hand side must vanish. If we now pull-back this identity to $\underline{\Im}^+$ via ψ^*, we obtain the surprising equation

$$\mathcal{L}_{\underline{n}}\underline{g}_{\mu\nu} - f\underline{g}_{\mu\nu} = 0, \tag{1.15}$$

where $f \doteq \psi^*\left(\frac{\widetilde{\nabla}_\mu\Omega\widetilde{\nabla}_\nu\Omega}{\Omega}\right)$. Here we also used the fact that $\psi^*\Omega = \Omega\circ\psi = 0$ and, thus, $\psi^*(\Omega\widetilde{S}_{\mu\nu}) = \psi^*(\Omega)(\psi^*\widetilde{S})_{\mu\nu} = 0$. We have thus shown that $\underline{n}$ and $\underline{g}_{\mu\nu}$ satisfy the **conformal Killing equation**, *i.e.*, $\underline{n}$ is strictly intertwined with a conformal isometry as one can infer from the following definition.

Definition 1.3.2. *A* conformal isometry *of spacetime $(M, g_{\mu\nu})$ is a map $\upsilon : M \to M$ such that there exists $\Omega \in C^\infty(M, \mathbb{R}^+)$ fulfilling $(\upsilon^*g)_{\mu\nu} = \Omega^2 g_{\mu\nu}$. The generator ξ of a 1-parameter subgroup υ_t of the group of conformal isometries is a conformal Killing isometry which satisfies*

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)} = \alpha g_{\mu\nu}.$$

Eventually let us give a look at the gauge factor and at its behaviour on $\underline{\Im}^+$. Since it is a scalar smooth function, $\underline{\omega} \doteq \psi^*\omega = \omega\circ\psi \equiv \omega|_{\Im^+}$ and, thus, out of the properties of the pull-back, its action on the intrinsic quantities over $\underline{\Im}^+$ is the same as on those over $\Im^+$, *i.e.*,

$$\underline{g}'_{\mu\nu} = \underline{\omega}^2\underline{g}_{\mu\nu} = \psi^*(\omega^2\widetilde{g}_{\mu\nu}) = \psi^*(\widetilde{g}'_{\mu\nu}), \quad \underline{n}'^\mu = \underline{\omega}^{-1}\underline{n}^\mu = \psi^*(\omega^{-1}n^\mu) = \psi^*(n'^\mu).$$

Hence the geometry of $\underline{\Im}^+$ behaves under gauge transformations exactly as that of $\Im^+$, a fact which is fully encoded in the ω-invariant tensor

$$\underline{\Gamma}^{\mu\nu}{}_{\rho\delta} \doteq \underline{g}_{\rho\delta}\underline{n}^\mu\underline{n}^\nu. \tag{1.16}$$

It is interesting to notice that $\underline{\Gamma}^{\mu\nu}{}_{\rho\delta}$ actually contains as much information as $(\underline{g}_{\mu\nu}, \underline{n}^\mu)$; hence it is possible to use it as a starting point and, then, decompose it in a pair $(\underline{g}_{\mu\nu}, \underline{n}^\mu)$, an operation which we shall not describe in detail, though an interested reader can find it in section 4 of [30].

The above apparently pointless and dreamlike discussion on the intrinsic structure of null infinity is actually extremely relevant from a quantum field theory perspective because it is tantamount to the following proposition:

Proposition 1.3.1. *The geometry of $\Im^+$ is fully encoded in the equivalence classes $[\underline{\Im}^+, \underline{n}^\mu, \underline{g}_{\mu\nu}]$ where two such triples (a.k.a., kinematical configurations), say $(\underline{\Im}^+, \underline{n}^\mu, \underline{g}_{\mu\nu})$ and $(\underline{\Im}'^+, \underline{n}'^\mu, \underline{g}'_{\mu\nu})$, are equivalent if and only if there exists a gauge factor $\underline{\omega} \in C^\infty(\underline{\Im}^+, \mathbb{R}^+)$ such that*

$$(\underline{\Im}'^+, \underline{n}'^\mu, \underline{g}'_{\mu\nu}) = (\underline{\Im}^+, \underline{\omega}^{-1}\underline{n}^\mu, \underline{\omega}^2\underline{g}_{\mu\nu}).$$

The structure of $\Im^+$ is then

- **Universal***: for any two asymptotically flat spacetimes $(M, g_{\mu\nu})$ and $(M', g'_{\mu\nu})$ and for any two associated triples $[\underline{\Im}^+, \underline{n}^\mu, \underline{g}_{\mu\nu}]$ and $[\underline{\Im}'^+, \underline{n}'^\mu, \underline{g}'_{\mu\nu}]$, there always exists $\Phi \in Diff(\underline{\Im}^+, \underline{\Im}'^+)$ such that*

$$(\Phi_*\underline{n})^\mu = \underline{n}'^\mu, \quad (\Phi_*\underline{g})_{\mu\nu} = \underline{g}'_{\mu\nu}.$$

Proof. Only the demonstration of universality lacks, but this is a byproduct of the already established existence on $\underline{\Im}^+$ of a coordinate system (u, θ, φ) - induced from the Bondi frame - such that the metric reads $ds^2|_{\underline{\Im}^+} = -0 \cdot du^2 + d\theta^2 + \sin^2\theta d\varphi^2$. If one introduces the counterpart (u', θ', φ') on $\underline{\Im}'^+$, we can set Φ as the application sending a point of coordinates (u, θ, φ), in other one of coordinates (u', θ', φ') such that $u = u'$, $\theta = \theta'$ and $\varphi = \varphi'$. That also yields the wanted relation at level of metrics; at the same time, in the first of these frames, the vector $n^\mu \equiv \frac{\partial}{\partial u}$ and thus also the statement at level of vectors holds automatically. Furthermore, since Φ is nothing but the counterpart of the identity between the two introduced coordinate systems, it is per construction a diffeomorphism. □

From a more physical perspective,let us observe the following: the structure of $\underline{\Im}^+$ is

- **Intrinsic**: there in no physical mean to distinguish between two gauge equivalent triples.

From a physical point of view these notions seem somehow of dubious importance, but a closer look to them actually unveils the strategic importance of future or of past null infinity. Most notably the universality property of $\Im^\pm$ means that, whatever asymptotically flat spacetime we consider, the geometry of its conformal boundary is the same. Therefore, as we briefly commented in the introduction and as we shall better see in the next chapter, this is an ideal scenario if one wants to encode the data of a field theory in the bulk of an asymptotically flat spacetime into those of a second one living on null infinity. Instead of dwelling into a case by case analysis, the universality of $\Im^\pm$ suggests that such second theory should actually be just the same for a given field theory constructed over different bulk spacetimes. The information of the chosen physical manifold must thus be encoded in the way the bulk data are projected into the boundary ones, the latter instead being constructed once and for all. Strong of these heuristic considerations, we have a good indication that our initial intentions were correct and, thus, we are following the right path (though, remember that the road to hell is paved with good intentions).

1.4 Asymptotic symmetries and the BMS group

In order to make precise both the idea that the conformal boundary encodes all the information of the bulk and that it is possible to construct a field theory which lives intrinsically on null infinity, we have to know, in the first case, how to recover bulk isometries from data on $\Im^\pm$, while, in the second, we need to know the structure of the diffeomorphism group of $\Im^\pm$, since invariance under such a group is a prerequisite of a genuine field theory. The lucky aspect of these two problems is that they can be contemporary solved, hence we can kill two birds with one stone and this will be the aim of the section.

Definition 1.4.1. *A symmetry of null infinity $\underline{\Im}^+$ is a diffeomorphism Φ which leaves invariant the intrinsic geometry, hence it maps a kinematical configuration $[\underline{\Im}^+, \underline{n}^\mu, \underline{g}_{\mu\nu}]$ into a gauge equivalent one, that is, there exists $\omega \in C^\infty(\underline{\Im}^+, \mathbb{R}^+)$*

$$(\psi_*\underline{n})^\mu = \omega^{-1}\underline{n}^\mu \quad and \quad (\psi_*\underline{g})_{\mu\nu} = \omega^2\underline{g}_{\mu\nu}.$$

Equivalently, if we start from (1.16),

$$(\psi_*\underline{\Gamma})^{\mu\nu}{}_{\rho\delta} = \underline{\Gamma}^{\mu\nu}{}_{\rho\delta}.$$

The usefulness of the last condition is manifest if we consider any 1-parameter subgroup Φ_t ($t \in \mathbb{R}$) of the diffeomorphisms of $\underline{\Im}^+$ which is generated under exponential map by a vector field ζ^μ. Invariance of $\underline{\Gamma}^{\mu\nu}{}_{\rho\delta}$ under Φ_t translates at an infinitesimal level to

$$\mathcal{L}_\zeta \underline{\Gamma}^{\mu\nu}{}_{\rho\delta} = 0.$$

If we take two of such generators, say ζ_1 and ζ_2 it holds that

$$\mathcal{L}_{\zeta_1}\mathcal{L}_{\zeta_2}\underline{\Gamma}^{\mu\nu}{}_{\rho\delta} = \mathcal{L}_{\zeta_2}\mathcal{L}_{\zeta_1}\underline{\Gamma}^{\mu\nu}{}_{\rho\delta} + \mathcal{L}_{[\zeta_1,\zeta_2]}\underline{\Gamma}^{\mu\nu}{}_{\rho\delta} = 0 \Longrightarrow \mathcal{L}_{[\zeta_1,\zeta_2]}\underline{\Gamma}^{\mu\nu}{}_{\rho\delta} = 0,$$

where $[,]$ is the Jacobi-Lie bracket. The last implication guarantees that the set of the vector fields which are infinitesimal generators of symmetries for $\underline{\Im}^+$ forms a (potentially infinite dimensional) **Lie algebra** $\mathcal{B}$.

We can now draw the first important conclusion of this section:

Theorem 1.4.1. *Let $(M, g_{\mu\nu})$ be an asymptotically flat spacetime with $\left[\underline{\Im}^+, \underline{n}^\mu, \underline{g}_{\mu\nu}\right]$ as a kinematical configuration. Then, any Killing vector field ζ^μ of $g_{\mu\nu}$ can be extended to a smooth vector field $\widetilde{\zeta}^\mu$ on $(\widetilde{M}, \widetilde{g}_{\mu\nu})$ such that $\underline{\zeta}^\mu$ is an infinitesimal symmetry.*

Proof. We adopt notation and conventions of the previous sections. Let thus $\psi[M] \subset \widetilde{M}$ and let us call as $\widetilde{\zeta}^\mu$ the push-forward $(\psi_*\zeta)^\mu$. Since ψ is a local diffeomorphism, we know that $\mathcal{L}_{\widetilde{\zeta}} g_{\mu\nu} = 0$ which, in turn, yields

$$\mathcal{L}_{\widetilde{\zeta}}\widetilde{g}_{\mu\nu} = 2\Omega(\mathcal{L}_{\widetilde{\zeta}}\Omega)g_{\mu\nu} + \Omega^2\mathcal{L}_{\widetilde{\zeta}}g_{\mu\nu} = f\widetilde{g}_{\mu\nu},$$

where $f \doteq 2\Omega^{-1}\mathcal{L}_{\widetilde{\zeta}}\Omega$. Hence $\widetilde{\zeta}$ is a conformal Killing vector field for the unphysical metric $\underline{\text{in } \psi[M]}$. We can now invoke propositions 3.1 and 3.2 in [21] which guarantees that $\widetilde{\zeta}$ admits a smooth extension to $\widetilde{M}$, but this entails in turn the smoothness of f. Hence $\Omega f = 2\widetilde{\zeta}^\mu\widetilde{\nabla}_\mu\Omega$ must vanish on $\Im^+$, or, in other words, ζ^μ is perpendicular to n_μ and, thus, it is tangent to $\Im^+$. Furthermore

$$\mathcal{L}_{\widetilde{\zeta}}n^\mu = \mathcal{L}_{\widetilde{\zeta}}(\widetilde{g}^{\mu\nu}n_\nu) = \left(\mathcal{L}_{\widetilde{\zeta}}\widetilde{g}\right)^{\mu\nu} n_\nu + \widetilde{g}^{\mu\nu}\mathcal{L}_{\widetilde{\zeta}}n_\nu = -fn^\mu + \Omega\widetilde{\nabla}^\mu f,$$

which on $\Im^+$ amounts to $\mathcal{L}_{\widetilde{\zeta}}n^\mu = -fn^\mu$. To conclude, if we put together all these informations, we get that

$$\mathcal{L}_{\widetilde{\zeta}}\left(\widetilde{g}_{\rho\delta}\widetilde{n}^\mu\widetilde{n}^\nu\right) = 0.$$

If we pull-back via the map $\psi : \underline{\Im}^+ \to \Im^+$, we end up with

$$\psi^*\left[\mathcal{L}_{\widetilde{\zeta}}\left(\widetilde{g}_{\rho\delta}\widetilde{n}^\mu\widetilde{n}^\nu\right)\right] = \mathcal{L}_{\underline{\zeta}}\left(\underline{g}_{\rho\delta}\underline{n}^\mu\underline{n}^\nu\right) = \mathcal{L}_{\underline{\zeta}}\underline{\Gamma}^{\mu\nu}{}_{\rho\delta} = 0,$$

which concludes the demonstration. □

The outcome of the previous theorem is that every bulk isometry is a boundary symmetry, but it is not difficult to be convinced that the converse cannot be true. Let $\underline{\zeta}$ be a symmetry of $\underline{\Im}^+$; then, per definition

$$\mathcal{L}_{\underline{\zeta}}\underline{\Gamma}^{\mu\nu}{}_{\rho\delta} = \left(\mathcal{L}_{\underline{\zeta}}\underline{g}\right)_{\rho\delta}\underline{n}^\mu\underline{n}^\nu + 2\underline{g}_{\rho\delta}\left(\mathcal{L}_{\underline{\zeta}}\underline{n}\right)^{(\mu}\underline{n}^{\nu)} = 0. \tag{1.17}$$

If we contract the last equality with any tensor $\beta^{\rho\delta}$ and, separately, with any $\alpha_{\mu\nu}$, we get the **defining relation** for asymptotic symmetries:

$$\left(\mathcal{L}_{\underline{\zeta}}\underline{g}\right)_{\mu\nu} = 2k\underline{g}_{\mu\nu}, \quad \text{and} \quad \left(\mathcal{L}_{\underline{\zeta}}\underline{n}\right)^\mu = -k\underline{n}^\mu, \tag{1.18}$$

where the coefficients on the right hand side turn out to differ only for a factor 2 out of compatibility with (1.17).

In the construction of the asymptotic symmetries of an asymptotically flat spacetime, the leit motiv is the one according to which, since the metric at infinity does look like the flat one, the set of symmetries at infinity should not differ from the Poincaré ones. The next lemma puts a nail in the coffin of such idea.

Lemma 1.4.1. *All the vector fields of the form* $\widetilde{\zeta}^\mu = \alpha(\theta,\varphi)n^\mu$ *with* $\alpha \in C^\infty(\mathbb{S}^2)$ *are asymptotic symmetries fulfilling* (1.18) *with* $k = 0$*; they are called* **supertranslations**.

Proof. We know from (1.15) that $\mathcal{L}_{\underline{n}}\underline{g}_{\mu\nu} = f\underline{g}_{\mu\nu}$ which, in turn, implies $\mathcal{L}_{\alpha\underline{n}}\underline{g}_{\mu\nu} = \alpha f\underline{g}_{\mu\nu}$. At the same time, if we look at $\Im^+$ instead of $\underline{\Im}^+$, we get

$$\mathcal{L}_{\alpha n}n^\mu = n^\nu\widetilde{\nabla}_\nu(\alpha n^\mu) = n^\nu\widetilde{\nabla}_\nu(\alpha)n^\mu,$$

since n^μ gives rise to complete null geodesics. In a Bondi frame $n^\nu\widetilde{\nabla}_\nu(\alpha)n^\mu$ is tantamount to consider $\frac{\partial\alpha(\theta,\varphi)}{\partial u}\frac{\partial}{\partial u} = 0$. If we pull-back to $\underline{\Im}^+$, the last result means $\mathcal{L}_{\alpha\underline{n}}\underline{n}^\mu = 0$. Therefore the only possibility is that $f = 0$. To prove it, one just needs to compute

$$\mathcal{L}_{\underline{\zeta}}\underline{g}_{\mu\nu} = \alpha\mathcal{L}_{\underline{n}}\underline{g}_{\mu\nu} + 2\underline{n}^\rho\underline{g}_{\rho(\mu}\underline{\nabla}_{\nu)}\alpha = 0,$$

since the first term is 0 per construction of the intrinsic structure of $\underline{\Im}^+$ while the second vanished since α depends only on angular coordinates. □

One could notice that the last argument in the proven lemma could be reversed to demonstrate that a vector field proportional to $\underline{n}^\mu$ is an asymptotic symmetry if and only if the proportionality factor depends only on the coordinates transversal to the affine parameter of the integral curves constructed out of $\underline{n}^\mu$. It is also clear that the set of these generators is infinite-dimensional since they are labelled by scalar functions on the 2-sphere which are an infinite dimensional space. Therefore the chance to retrieve the Poincaré group as the asymptotic symmetry group are almost nill, the only left option being the absence of a global exponential map from the algebra of symmetries to a group. In this case, it might be, that one faces a situation similar to the conformal algebra in two dimensions where only a sub-algebra can be exponentiated to a finite-dimensional Lie group, which, in our case, should be the Poincaré group. We shall see (without proving it) that this cannot be the case.

An important remark is in due course.

Remark 1.4.1. It is interesting to note the following two properties of the set $\mathfrak{I}$ of supertranslations:

- The set is closed under liner combination, since, for any $c, c' \in \mathbb{R}$ and $\zeta, \zeta' \in \mathfrak{I}$,

$$\mathcal{L}_{c\underline{\zeta}+c'\underline{\zeta}}(\underline{g}_{\mu\nu}) = c\mathcal{L}_{\zeta}\underline{g}_{\mu\nu} + c'\mathcal{L}_{\zeta'}\underline{g}_{\mu\nu} = 0.$$

- the commutator of two elements of $\mathfrak{I}$, $\underline{\zeta}^\mu = \alpha(\theta,\varphi)\underline{n}^\mu$ and $\underline{\zeta}'^\mu = \alpha'(\theta,\varphi)\underline{n}^\mu$ always vanishes. This can be seen out of a notable property of Lie derivatives (see for example appendix C in [43]):

$$[\underline{\zeta}, \underline{\zeta}']^\mu = \mathcal{L}_{\underline{\zeta}}\underline{\zeta}'^\mu = \alpha'\mathcal{L}_{\underline{\zeta}}\underline{n}^\mu + \mathcal{L}_{\underline{\zeta}}(\alpha')\underline{n}^\mu = \alpha\underline{n}^\nu\underline{\nabla}_\nu\alpha'\underline{n}^\mu = 0,$$

  where we have first used the information that $\underline{\zeta}^\mu$ is an asymptotic symmetry and then the fact that, in (the pull-back on $\underline{\Im}^+$ of) the Bondi frame, $\underline{n}^\nu\underline{\nabla}_\nu$ coincides with the derivative along the u-variable, while α depends only on angular coordinates.

In other words these two properties entail that $\mathfrak{I}$ is an **infinite dimensional Abelian** subalgebra of the Lie algebra of asymptotic symmetries. Since each of its element is labelled by a smooth function on $\mathbb{S}^2$ which, is in turn an Abelian algebra under addition, it turns out that $\mathcal{I}$ is endomorphic to $C^\infty(\mathbb{S}^2)$. Furthermore, let us suppose to take any $\underline{\zeta}^\mu \in \mathcal{B}$ and $\underline{\zeta}'^\mu = \alpha' \underline{n}^\mu$; it holds:

$$[\underline{\zeta}, \underline{\zeta}']^\mu = \mathcal{L}_{\underline{\zeta}} \underline{\zeta}'^\mu = \mathcal{L}_{\underline{\zeta}}(\alpha')\underline{n}^\mu + \alpha' \mathcal{L}_{\underline{\zeta}} \underline{n}^\mu = [\mathcal{L}_{\underline{\zeta}}(\alpha') - \alpha' k]\underline{n}^\mu \in \mathcal{I},$$

where, in the last equality, we used (1.18) together with the fact that the Lie derivative of a scalar function is still a scalar. This last identity thus entails that $\mathcal{I}$ is **an ideal of** $\mathcal{B}$.

Since the algebra of supertranslations is an ideal, it is meaningful to construct the quotient $\frac{\mathcal{B}}{\mathcal{I}}$ and our aim is to understand its structure. The result is not so surprising and it is encoded in the following theorem:

Theorem 1.4.2. *The quotient $\frac{\mathcal{B}}{\mathcal{I}}$ is the Lie algebra of the conformal Killing fields of $\mathbb{S}^2$ and, thus, it is endomorphic to $\mathfrak{sl}(2, \mathbb{C})$, the algebra of complex 2×2 traceless matrices. Therefore the set of asymptotic symmetries forms the so-called* $\mathfrak{bms}$ **algebra** *which is the (internal) semi-direct product*

$$\mathfrak{bms} \equiv \mathfrak{sl}(2, \mathbb{C}) \oplus_\pi C^\infty(\mathbb{S}^2),$$

where $\pi : \mathfrak{sl}(2, \mathbb{C}) \to End(C^\infty(\mathbb{S}^2))$ is realized as the map $\pi(\mathfrak{a})\alpha(\theta, \varphi) \doteq \mathcal{L}_{\underline{\zeta}_\mathfrak{a}}(\alpha \underline{n}^\mu)$, where $\underline{\zeta}_\mathfrak{a}$ is the asymptotic symmetry on $\underline{\Im}^+$ associated to $\mathfrak{a}$.

Proof. Let us take any representative $\underline{\zeta}^\mu$ of an equivalence class in $\frac{\mathcal{B}}{\mathcal{I}}$. Since it is intrinsically defined on $\underline{\Im}^+$, it can be read as the pull-back of a vector tangent to $\Im^+$ and, thus, it must hold $\underline{\zeta}_\mu \underline{n}^\mu = 0$. In other words $\underline{\zeta}_\mu = \underline{g}_{\mu\nu} \underline{\zeta}^\mu$ is orthogonal to $\underline{n}^\mu$. Furthermore the derivative operator acts as

$$\underline{\nabla}_\mu \underline{\zeta}_\nu = \underline{\nabla}_{[\mu} \underline{\zeta}_{\nu]} + \underline{\nabla}_{(\mu} \underline{\zeta}_{\nu)} + \underline{\nabla}_{[\mu} \underline{\zeta}_{\nu]} + \frac{1}{2} \mathcal{L}_{\underline{\zeta}} g_{\mu\nu} = \underline{\nabla}_{[\mu} \underline{\zeta}_{\nu]} + k \underline{g}_{\mu\nu},$$

where, in the last identity, (1.18) has been used and the antisymmetric part can be interpreted as the external derivative. At the same time

$$\mathcal{L}_{\underline{\zeta}} \underline{n}_\mu = \mathcal{L}_{\underline{\zeta}}(\underline{g}_{\mu\nu})\underline{n}^\nu + \underline{g}_{\mu\nu} \mathcal{L}_{\underline{\zeta}}(\underline{n}^\nu) = k \underline{n}_\mu = 0,$$

where we employed first (1.18) and, at the very end, (1.14). To summarise, every element of $\frac{\mathcal{B}}{\mathcal{I}}$ necessarily satisfies

$$a) \;\; \underline{\zeta}_\mu \underline{n}^\mu = 0 \qquad b) \;\; \underline{\nabla}_{(\mu} \underline{\zeta}_{\nu)} = k \underline{g}_{\mu\nu} \qquad c) \;\; \mathcal{L}_{\underline{\zeta}} \underline{n}_\mu = 0. \tag{1.19}$$

We shall now show that (1.19) is also a sufficient condition for a vector field in $\underline{\Im}^+$ to lie in $\frac{\mathcal{B}}{\mathcal{I}}$. Let us thus start from *a)*: this requirement grants us that, since $\underline{n}^\mu$ is the only vector on $\underline{\Im}^+$ which is annihilated by $\underline{g}^{\mu\nu}$, then $\underline{\zeta}_\mu = \underline{g}_{\mu\nu}(\underline{\zeta}^\nu + f \underline{n}^\nu)$. At the same time condition *b)* guarantees that $\mathcal{L}_{\underline{\zeta}} \underline{g}_{\mu\nu} = 2k \underline{g}_{\mu\nu}$ which is one of the prerequisite of (1.18) to consider $\underline{\zeta}^\mu$ an asymptotic isometry. We are thus left with *c)*; since $\underline{\zeta}_\mu$ is defined from the counterpart with upper indices up to a term such as $f \underline{n}^\mu$ and since we are ultimately interested in an equivalence class with respect to $\mathcal{I}$, we can instead set to vanish

$$0 = \mathcal{L}_{\underline{\zeta} + f \underline{n}}(\underline{n})_\mu = \mathcal{L}_{\underline{\zeta}} \underline{n}_\mu + f \underline{n}^\nu \underline{\nabla}_\nu \underline{n}_\mu = \mathcal{L}_{\underline{\zeta}} \underline{n}_\mu + \underline{n}^\nu \underline{\nabla}_\nu (f \underline{n}_\mu) - \underline{n}^\nu \underline{\nabla}_\nu (f) \underline{n}_\mu.$$

Once we raise the μ-index, the last identity yields that $\mathcal{L}_{\underline{\zeta}} \underline{n}^\mu = -k \underline{n}^\mu$ if and only if $\underline{n}^\nu \underline{\nabla}_\nu (f) = k$. If we recall that $\underline{\Im}^+ \sim \mathbb{R} \times \mathbb{S}^2$ and if we interpret the last equality in a Bondi frame, we are facing simply a partial differential equation in the u-variable, whose solution, for given smooth initial data on $\mathbb{S}^2$ certainly exists. Thus we have concluded that the assignment of a vector $\underline{\zeta}^\mu$ fulfilling (1.19) is tantamount to assign an element of $\frac{\mathcal{B}}{\mathcal{I}}$. Let us now concentrate once more on *a)* and *c)* in (1.19). Since standard properties of the Lie derivative entail that $\mathcal{L}_{\underline{\zeta}} \underline{n}_\mu = -\mathcal{L}_{\underline{n}_\mu} \underline{\zeta} = 0$, the two mentioned conditions can also be interpreted as follows: let $\iota : S^2 \hookrightarrow \underline{\Im}^+$ be an embedding of a 2-sphere in the intrinsic representation of null infinity, then there exists a vector field $\underline{\xi}_\mu$ on the 2-sphere such that $\underline{\zeta}_\mu = (\iota_* \underline{\xi})_\mu$. Therefore the remaining condition *b)* can be read as $\iota^*(\underline{D}_{(\mu}\underline{\xi}_{\nu)}) = k(\iota^*\underline{g})_{\mu\nu}$, namely $\underline{\zeta}^\mu$ gives rise via pull-back to a conformal Killing vector field on $\mathbb{S}^2$. Hence the quotient $\frac{\mathcal{B}}{\mathcal{J}}$ must be endomorphic to the algebra of conformal Killing fields on the 2-sphere which is known to be $\mathfrak{sl}(2,\mathbb{C})$ (see for example [23]). This result, together with lemma 1.4.1 and remark 1.4.1, also implies the semidirect structure of the full BMS algebra. □

Our analysis of the asymptotic symmetries cannot be called complete at this stage since the notion of $\mathfrak{bms}$ algebra is certainly interesting, but not physically relevant if one is interested in studying global issues of $\underline{\Im}^+$. As a matter of fact, algebras are local objects which yield information on global quantities as soon as one can associate them a group via exponential map. While, at a level of finite dimensional Lie groups, this is a totally understood problem, as soon as we cope with infinite dimensional groups, we run into troubles because a general theory is somehow lacking and we have to deal with a case by case analysis. Luckily enough, since, in this case, there is an infinite dimensional Abelian subalgebra, and since $\Im^+$ is generated by *complete* null geodesics, we can construct a group of asymptotic symmetries along the lines of the following theorem, which we shall not prove:

Theorem 1.4.3. *There exists a global exponential map* $\exp : \mathfrak{bms} \to BMS$*, the image of which is an infinite dimensional (nuclear Lie) group, homomorphic to the regular semidirect product*

$$BMS \equiv SL(2,\mathbb{C}) \ltimes_T C^\infty(\mathbb{S}^2),$$

such that $\forall(\Lambda,\alpha),(\Lambda',\alpha') \in BMS$ *it holds*

$$(\Lambda,\alpha)(\Lambda',\alpha') = (\Lambda\Lambda', \alpha + T(\Lambda)\alpha').$$

Here T *is a regular* $SL(2,\mathbb{C})$ *representation on* $C^\infty(\mathbb{S}^2)$ *whose form can be explicitly given in a Bondi frame* $(u,z,\bar{z})$ *where* $z = \cot\frac{\theta}{2}e^{i\varphi}$*, namely, for any* $(\Lambda,\alpha) \in BMS$*:*

$$\begin{cases} u \longrightarrow u' = K_\Lambda(z,\bar{z})[u+\alpha(z,\bar{z})] \\ z \longrightarrow z' = \Lambda z = \frac{az+b}{cz+d}, \quad \Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; ad-bc=1 \quad \text{and equally for } \bar{z} \\ K_\Lambda(z,\bar{z}) = \frac{1+|z|^2}{|az+b|^2+|cz+d|^2} \end{cases}. \tag{1.20}$$

Here the action of Λ *on the* $(z,\bar{z})$*-coordinates is intended with respect to its standard representation as a complex* 2×2 *matrix of unit determinant.*

A potential reader should note that BMS stands for Bondi-Metzner-Sachs group, the first authors to recognise its existence [12, 42] and it is the asymptotic symmetry group of all possible asymptotically flat spacetimes. Notice that it has been constructed independently from spatial or future timelike infinity, the only key geometrical condition being the completeness of the null geodesic on $\Im^+$ in order to have a well-defined exponential map. We could analyse the rich properties of this group, but this would require a lecture series on its own; an interested reader could find a lot of informations on the topic in [18, 35, 36]. Nonetheless it is interesting at least to dispel the potential doubts which might arise from the physical idea that the Poincaré group should be the natural asymptotic symmetry group of an asymptotically flat spacetime. Actually the BMS contains not only "the Poincaré" group, but a number of infinite non equivalent copies. In order to understand it, let us consider a generic element $\alpha(z,\bar{z}) \in C^\infty(\mathbb{S}^2)$; barring some subtleties [20], one can always expand it in spherical harmonics as $\alpha(z,\bar{z}) = \sum\limits_{l=0}^{\infty}\sum\limits_{m=-l}^{l}\alpha_{lm}Y_{lm}(z,\bar{z})$. If we consider the subset of elements whose expansion ends at a fixed value of l, these form an Abelian closed subgroup of $C^\infty(\mathbb{S}^2)$ and, particularly, if the sum stops at $l=1$, then such a subgroup is homomorphic to T^4, the 4-dimensional group of translations. A further peculiarity of the set of first 4 spherical harmonics is that, out of a direct computation, it can be shown to be invariant under the $SL(2,\mathbb{C})$-action in (1.20). Hence the set $\mathcal{P} \doteq SL(2,\mathbb{C}) \ltimes_T \{Y_{lm}(z,\bar{z})\}_{l=0}^{1}$ is a closed subgroup of the BMS which is homomorphic to the Poincaré group. The problem is that if we consider any element $g \doteq (0, Y_{l'm'}(z,\bar{z}) \in BMS$ with $l' > 1$

and arbitrary m', then the set $g^{-1}\mathcal{P}g$ is still a subgroup of the BMS homomorphic to the Poincaré group, though distinguished from the original one $\mathcal{P}$. Since there are infinite spherical harmonics with $l' > 1$, one can find infinite different "Poincaré subgroups" of the BMS, none of which is preferred. At the same time this discussion clarifies the origin of the name *supertranslations* since it is an Abelian group which contains the translations as a subgroup. An interesting pair of exercises which can be solved with the data discussed in this chapter is the following:

Exercise 3: Show that the set $\{Y_{lm}(z,\bar{z})\}_{l=0}^{1}$ is the only normal subgroup of the BMS.

Exercise 4: Show that, if the bulk spacetime admits a Killing vector field, then its smooth projection on $\Im^+$ induces on $\underline{\Im}^+$ a generator of the algebra of $\mathcal{P}$, *i.e.*, $\mathfrak{P} = \mathfrak{sl}(2,\mathbb{C}) \oplus_\pi \{Y_{lm}(z,\bar{z})\}_{l=0}^{1}$.

To conclude the section, we would like to stress that the construction of asymptotically flat spacetimes, as discussed here, is based on the critical assumption that the underlying manifold is *four-dimensional*, since, if one plan to work with higher dimension the compactification procedure runs into troubles in odd dimension while, in even, even if it goes through, it leads to a much less rich geometric structure at null infinity in terms of asymptotic symmetries [32]. Paradoxically, the three-dimensional case is the most peculiar since, as studied in [5], it is possible to perform a sort of compactification process analogous to the four-dimensional scenario, but, in this case, it turns out that the asymptotic symmetry group is the semidirect product of two infinite dimensional groups and, thus, it is apparently too complicated to be an effective tool in dealing with physically interesting scenarios.

Chapter 2

Quantum Field Theory on asymptotically flat spacetimes

The goal of this section is to convince the reader that the efforts undertaken to understand the previous chapter were not a waste of time since it possible to use the powerful means of the structure of future (or past) null infinity in order to undercover important and previously never observed properties of certain quantum field theories over a curved background.

In these notes the prototype of fields we shall use is the scalar one, though there is no obstruction to consider other matter constituents endowed with a non trivial spin structure such as the Dirac field or the photon field. We anticipate to the reader that one should always consider only scenarios in which the mass vanishes since, as we shall comment in the conclusions, this term would lead to an apparently pathological situation unless one works in Minkowski where the problem can be somehow circumvented [19]. Nonetheless it is fair to admit that, in the flat case, our approach is as much powerful as redundant since the behaviour both at a classical and at a quantum level of, for example, a field ϕ satisfying the d'Alembert wave equation, is certainly very well understood even without being troubled with the notion of null infinity.

On the opposite, in a general four dimensional asymptotically flat spacetime $(M, g_{\mu\nu})$ the situation turns out to be much more complicated even in the simplest scenario of a scalar field. At a classical level, one faces problems such as that of the well-posedness of the initial value problem and of the choice of the natural counterpart for the d'Alembert wave equation (an arbitrary coupling to scalar curvature arises), that can be dealt without using any notion of conformal transformation. Conversely, at a quantum level, the situation is rather more complicated. The very absence of a huge symmetry group, such as the Poincaré one, deprives us of a powerful tool to study quantum phenomena and an important notion such as that of vacuum state becomes much more complicated and full of subtleties. Therefore the aim of this chapter will be to show that a novel use of the structure of null-infinity, closely mimicking the so-called holographic principle [33], leads to a natural unique way to identify in every asymptotically flat spacetime a distinguished algebraic quasifree state for a suitable bulk scalar field theory, which turns out to fulfil all the mathematical properties which are required to call it a "ground state". To say it in idiomatic words, we shall kill infinite birds with one stone.

2.1 Classical field theoretical aspects

The analysis of the quantum issue, prospected above, cannot be dealt with if we have not a full control of the classical theory. We shall thus dwell into this topic, since, as a byproduct, it will also enlighten the reason why we shall need to slightly amend the standard definition of asymptotic flatness as discussed above and in [43]. One should nonetheless bear in mind that all the key geometrical concepts, we have discussed, are left totally unchanged by such an amendment.

As a starting point, as we already pointed out, the first problem, to address in a classical field theory on a curved background, is the absence of any guarantee that a Cauchy problem can be set up, since there is a priori no good notion of initial surface on which to assign initial data. In order word, we need the certainty that we consider only spacetimes where there is a natural counterpart of the constant time three-dimensional hypersurface of Minkowski spacetime.

Hypothesis: we shall henceforth consider only four dimensional time and space orientable spacetimes $(M, g_{\mu\nu})$ which are *globally hyperbolic*, namely they posses a Cauchy surface Σ, that is a three dimensional surface embedded in M such that

a) the domain of dependence $D(\Sigma) \doteq D^+(\Sigma) \cup D^-(\Sigma) = M$, $D^\pm(\Sigma)$ being the so-called future and past domain of dependence of Σ, *i.e.* the set of all points $p \in M$ such that every past (or future) causal inextensible curve which passes through p intersects Σ,

b) the hypersurface Σ is a closed achronal set, namely there exist no $p, q \in \Sigma$ such that q lies in the chronological future of p, *i.e.* the set of all points in M which can be reached from p with a continuous future directed timelike curve.

Recent results [9, 10, 11] entail that this hypothesis yields that Σ is a smooth submanifold of M which, in turn, can be seen as $M \equiv \Sigma \times \mathbb{R}$. With reference to the construction of an asymptotically flat spacetime and adopting the relative nomenclature, we shall assume that *both* M and $\widetilde{M}$ are globally hyperbolic. This is not a sharp request because, in principle, it suffices that there exists a subspace $\widetilde{V} \subset \widetilde{M}$ which is globally hyperbolic and it contains $\psi[M]$ but, in order to simplify the scenario and the notations we are dealing with, we consider the first stronger condition. The above mentioned papers allow to draw more interesting conclusions on the structure of globally hyperbolic spacetimes, but we shall not dwell into these details.

At the same time, even under the assumption of global hyperbolicty of the underlying spacetime, there is also a further potential arbitrariness, namely there exists no unique "natural equation of motion" for a scalar field as in Minkowski spacetime. In this case Poincaré invariance entails that we have only the freedom to fix the value of m^2 as well as the sign of the energy associated to the scalar field. Afterwards the equation of motion must forcefully be either the Klein-Gordon or the d'Alembert one (this procedure goes under the name of Wigner's construction [7]). In a curved scenario, we can only generalise the results from Minkowski spacetime employing an Occam's razor point of view together with the request that, whenever we set the metric $g_{\mu\nu} = \eta_{\mu\nu}$, we recover the same behaviour as in the flat spacetime. That said, the building block of this chapter will be the following Cauchy problem

$$\begin{cases} \phi : M \to \mathbb{R} \\ (\Box_g + \xi R)\phi = 0, \quad \xi \in \mathbb{R} \\ (\phi|_\Sigma, \partial_n \phi|_\Sigma) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma) \end{cases}, \tag{2.1}$$

where n is the unit vector orthogonal to Σ.

Since we are ultimately interested in understanding how to intertwine a classical field theory in the bulk of an asymptotically flat spacetime with a counterpart on $\Im^+$, we must first analyse the behaviour of (2.1) under a conformal isometry which is the building block of definition 1.3.1. We recall that such transformation is made of two operations, a diffeomorphism ψ and a conformal rescaling Ω; while the first does not really affect the equation of motion of a scalar field, the second drastically alters both $\Box_g$ and R. Let us try to make explicit how; since $\Box$ and $\widetilde{\Box}$ act on a scalar we can use (1.4) to write it as

$$g^{\mu\nu}\nabla_\mu \partial_\nu = \Omega^{-2}\widetilde{g}^{\mu\nu}\left(\widetilde{\nabla}_\mu \widetilde{\partial}_\nu + \widetilde{C}^\rho_{\mu\nu}\widetilde{\partial}_\rho\right) =$$

$$\Omega^{-2}\left[\widetilde{\Box} + 2\widetilde{g}^{\mu\nu}\left(\delta^\rho_{(\mu}\widetilde{\nabla}_{\nu)}(\ln\Omega) + \frac{1}{2}\widetilde{g}_{\mu\nu}\widetilde{\nabla}^\rho(\ln\Omega)\right)\widetilde{\partial}_\rho\right] = \Omega^{-2}\left(\widetilde{\Box} - \widetilde{\nabla}^\rho(\ln\Omega)\widetilde{\partial}_\rho\right). \tag{2.2}$$

If we combine this last formula with (1.8), it is clear that the field ϕ should satisfy a complicated PDE in $(\widetilde{M}, \widetilde{g}_{\mu\nu})$, whose solutions could hardly propagate to $\Im^+$ due to the presence of many terms proportional to $\ln\Omega$ which are divergent on null infinity. At the same time, if we think at the peeling behaviour of a radiating field in Minkowski spacetime, the real information which is carried at infinity is in the coefficient of the term proportion to v^{-1}, where v is one light coordinate. Therefore, in order to extract such datum, we would simply multiply the field by v, an operation, which, in curved background, would see its counterpart in the multiplication by a suitable exponent of the conformal factor Ω. To establish this heuristic idea in mathematical terms, let us start from (2.1) which, introducing $\widetilde{\phi} \doteq \Omega^s\phi$ where $s \in \mathbb{R}$ becomes

$$(\Box_g + \xi R)\phi = \Omega^{-2}\left[\Omega^{-s}\widetilde{\Box}\widetilde{\phi} + s(s+1)\Omega^{-s-2}\widetilde{\nabla}^\rho\Omega\widetilde{\nabla}_\rho\Omega\widetilde{\phi} - 2s\Omega^{-s-1}\widetilde{\nabla}^\rho(\Omega)\widetilde{\partial}_\rho\widetilde{\phi} - s\Omega^{-s-1}\Box\Omega\widetilde{\phi}\right] +$$
$$+2\Omega^{-3}\widetilde{\nabla}^\rho\Omega\left(\Omega^{-s}\widetilde{\partial}_\rho\widetilde{\phi} - s\Omega^{-s-1}\widetilde{\nabla}_\rho\Omega\right)\widetilde{\phi}) + \xi\Omega^{-s-2}\left[\widetilde{R} - 6\widetilde{\Box}(\ln\Omega) - 6\widetilde{\nabla}^\rho(\ln\Omega)\widetilde{\nabla}_\rho(\ln\Omega)\right]\widetilde{\phi} =$$
$$\Omega^{-s-2}(\widetilde{\Box}\widetilde{\phi} + \xi\widetilde{R}\widetilde{\phi}) + (-2s+2)\Omega^{-s-3}\widetilde{\partial}_\rho(\widetilde{\phi})\widetilde{\nabla}^\rho(\Omega) + (6\xi - s)\Omega^{-s-3}\widetilde{\Box}\Omega + \Omega^{-s-4}\widetilde{\phi}\left[s(s-1)\widetilde{\nabla}_\rho\Omega\widetilde{\nabla}^\rho\Omega\right].$$

This apparently looks even worse than before, but, if we set $\xi = -\frac{1}{6}$ and $s = 1$, we get

$$0 = \left(\Box - \frac{R}{6}\right)\phi = \Omega^{-1}\left(\widetilde{\Box} - \frac{\widetilde{R}}{6}\right)\widetilde{\phi} = 0.$$

In other words, we proved the following lemma:

Lemma 2.1.1. *For any $\phi : M \to \mathbb{R}$ solving $= \left(\Box - \frac{R}{6}\right)\phi$ then $\widetilde{\phi} \doteq \Omega^{-1}\phi$ solves the same kind of equation in $(\psi[M], \widetilde{g}_{\mu\nu}) \subset (\widetilde{M}, \widetilde{g}_{\mu\nu})$. This equation will be called* **d'Alembert wave equation conformally coupled to scalar curvature**.

Remark 2.1.1. This statement cannot be slavishly applied at a level of Cauchy problems unless one shows that $\psi[M]$ is a globally hyperbolic sub-spacetime of $\widetilde{M}$ and $\psi(\Sigma)$ is a Cauchy surface for $\psi[M]$. In this case the initial data would simply be

$$(\widetilde{\phi}|_{\psi(\Sigma)}, \partial_n\widetilde{\phi}|_{\psi(\Sigma)}) = \left(\Omega^{-1}\phi|_{\psi(\Sigma)}, -\frac{\partial_{\widetilde{n}}\Omega}{\Omega}\phi|_{\psi(\Sigma)} + \Omega^{-1}\partial_{\widetilde{n}}\phi|_{\psi(\Sigma)}\right),$$

where we used lemma 1.2.1 according to which causal structures are preserved under conformal mapping; hence $\widetilde{n}^\mu \doteq \psi_*(n)^\mu$ is orthogonal to $\psi(\Sigma)$.

In order to solve the potential problem arising from the consideration of this last remark we need a better understanding on the properties of the space of solutions of a Cauchy problem for a scalar field in a globally hyperbolic spacetime. We shall here give only the main proposition while suggesting an interested reader to consult [6] for the proofs and for further informations.

Proposition 2.1.1. *Let us consider a second order hyperbolic partial differential operator P of metric principal type on a globally hyperbolic spacetime $(M, g_{\mu\nu})$, i.e., in a local trivialisation of M*

$$P = -g^{\mu\nu}(x)\partial_\mu\partial_\nu + A^\mu(x)\partial_\mu + B(x), \tag{2.3}$$

where both $A^\mu(x)$ and $B(x)$ are smooth functions. Then one can prove that, for any Cauchy surface $\Sigma \hookrightarrow M$ with normal vector n, the Cauchy problem

$$\begin{cases} P\phi = 0 \\ (\phi|_\Sigma, \partial_n\phi|_\Sigma) = (\phi_0, \phi_1) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma) \end{cases},$$

admits a unique smooth solution such that

$$supp(\phi) \subseteq J^+(supp(\phi_0)) \cup J^+(supp(\phi_1)) \cup J^-(supp(\phi_0)) \cup J^-(supp(\phi_0)).$$

One can per direct inspection realize that the operator $\Box - \frac{R}{6}$, which appears in lemma 2.1.1, falls in the class considered in the above proposition and we can thus apply its results to give an even stronger positive answer to the question raised in remark (2.1.1), namely

Proposition 2.1.2. *Let us consider a four dimensional globally hyperbolic asymptotically flat spacetime* $(M, g_{\mu\nu})$ *whose conformal completion is a globally hyperbolic spacetime* $(\widetilde{M}, \widetilde{g}_{\mu\nu})$. *If* $\phi : M \to \mathbb{R}$ *is a solution of* $\left(\Box - \frac{R}{6}\right)\phi = 0$ *with compactly supported initial data on* Σ, *then* $\widetilde{\phi} \doteq \Omega^{-1}\phi$ *can be extended to a unique solution of* $\left(\widetilde{\Box} - \frac{\widetilde{R}}{6}\right)\widetilde{\phi} = 0$ *in the whole* $(\widetilde{M}, \widetilde{g}_{\mu\nu})$.

Proof. Let $S \hookrightarrow M$ be a Cauchy surface of $(M, g_{\mu\nu})$. Then lemma 1.2.1 entails that $\psi[S]$ is also a Cauchy surface for $\psi[M]$ since causal structures are preserved and, therefore it is meaningful to refer to the support of the Cauchy data on $\psi[S]$, which we shall call $K_{\psi[S]}$. Let $\Sigma \hookrightarrow \widetilde{M}$ be instead a Cauchy surface of $(\widetilde{M}, \widetilde{g}_{\mu\nu})$; since $\widetilde{M} \equiv \Sigma \times \mathbb{R}$, we can choose the considered distinguished copy of Σ to lie in the past of $K_{\psi[S]}$ (see figure 2.1.2). Since this last set is compact and the class of open sets $I^-(p, \widetilde{M}) \cap I^+(q, \widetilde{M})$ with $p, q \in \psi[M]$ is a topological basis for $\psi[M]$ (see [8]), then there exists a finite sequence of points $\{p_i\}_{i=1}^{N<\infty}$ in the future of $K_{\psi[S]}$ such that the following chain of inclusions hold

$$K_{\psi[S]} \subset \bigcup_{i=1}^{N} I^-(p_i, \psi[M]) \subset \bigcup_{i=1}^{N} J^-(p_i, \psi[M]) \subset \bigcup_{i=1}^{N} J^-(p_i, \widetilde{M}).$$

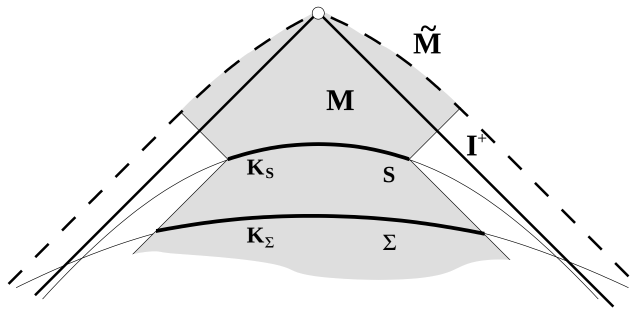

Figure 2.1: This is a graphical representation of both the physical and the unphysical spacetime with their respective Cauchy surfaces. The shaded part represents the region where the initial data supported on K_S propagates.

At the same time this yields that $\bigcup_{i=1}^{N} J^-(p_i, \psi[M]) \cap D^+(\Sigma)$ is a compact set and therefore $K_\Sigma \doteq \bigcup_{i=1}^{N} J^-(p_i, \psi[M]) \cap \Sigma$ is compact since it is a closed subset of a compact set. We can now use proposition 2.1.1 to claim that *supp*$(\phi) \subseteq J^-(K_{\psi[S]}, \psi[M]) \cup J^+(K_{\psi[S]}, \psi[M])$. Furthermore since ϕ is smooth as well as Ω which is also non vanishing in $\psi[M]$, we can evaluate $\Omega^{-1}\phi$ on the compact $K_\Sigma \subset \Sigma$ to construct initial data for a Cauchy problem associated to $\widetilde{\Box} - \frac{\widetilde{R}}{6}$ on the whole $\widetilde{M}$. Still thanks to proposition 2.1.1, we know that the solution of such Cauchy problem is unique and yields a smooth function which, per construction, coincides with $\Omega^{-1}\psi$ on $\psi[M]$. $\Box$

The outcome of this last proposition is extremely important from a physical point of view since it guarantees us that each solution of a d'Alembert wave equation conformally coupled to scalar curvature and with compactly supported initial data in the physical spacetime corresponds to a unique counterpart in the unphysical spacetime. The smoothness of the latter, moreover, entails that such a function can be restricted on null infinity by simple evaluation, hence we projected the bulk data to the boundary. From a classical point of view, the reconstruction of bulk information from the boundary is a little bit involved since neither $\Im^+$ nor $\Im^-$ are Cauchy surfaces, actually they are three dimensional null hypersurfaces; nonetheless they can be used as good initial data for the so-called *Goursat problem* (or characteristic problem). This is a rather special initial value problem in which only one initial datum is given to construct the solution of the relevant PDE ($\widetilde{\Box} - \frac{\widetilde{R}}{6}$ in our case), though uniqueness and smoothness are not globally guaranteed. Nonetheless, in the scenario we are interested, the Goursat problem suffices to construct a unique and smooth solution in $\psi[M] \subset \widetilde{M}$ which coincides, up to the factor Ω^{-1}, to a solution of the PDE we are interested in ($\Box - \frac{R}{6}$ in this particular case) in M. More information of the characteristic problem can be found in [15, 24].

Therefore we can safely claim that we have understood all the properties of a single classical scalar field theory on a globally hyperbolic and asymptotically flat spacetime in order to try to use null infinity as a locus where to encode the bulk information. What we are lacking is actually a last bit of information on the structure of the space of solutions of the Cauchy problem with compactly supported initial data for the d'Alembert wave equation conformally coupled to scalar curvature. Such datum will play a pivotal role in the analysis of the quantum aspects of the theory.

Definition 2.1.1. *A (possibly infinite dimensional) vector space V is called* **symplectic** *if it is endowed with a symplectic form, i.e., a bilinear map $\sigma : V \times V \to \mathbb{R}$ such that*

$$\sigma(a,b) = -\sigma(b,a) \quad \forall a, b \in V.$$

Furthermore σ induces a map $\sigma^\circ : V \to V^*$ such that $\sigma^\circ(a) \doteq \sigma(a,\cdot)$ for all $a \in V$; if σ° is injective then we say that σ is **weakly non degenerate** *whereas, if $\sigma(a,b) = 0$ for all $b \in V$ entails that $a = 0$, then we call σ* **strongly non degenerate**.

While in a finite dimensional scenario, the two possible definitions of non-degenerateness are actually equivalent, they play a rather important role in quantum field theory, where the latter plays a somehow distinguished role. A curious exercise which somehow points in this direction is the following:

Exercise 5: Let $\mathcal{B}$ be an infinite dimensional Banach space and let us call the cotangent bundle of $\mathcal{B}$, the space $T^*\mathcal{B} = \mathcal{B} \times \mathcal{B}^*$, $\mathcal{B}^*$ being the dual space. Then let $\sigma : T^*\mathcal{B} \times T^*\mathcal{B}$ be the bilinear form such that

$$\sigma\left((\alpha_1, \beta_1)(\alpha_2, \beta_2)\right) \doteq \beta_2(\alpha_1) - \beta_1(\alpha_2).$$

Prove that σ is a weakly non degenerate symplectic form which becomes strongly non degenerate if and only if $\mathcal{B} = \mathcal{B}^{**}$, *i.e.*, the Banach space is reflexive. Prove also the, if $\mathcal{B}$ is an Hilbert space, then σ is forcefully strongly non degenerate.

To conclude the section let us state an important result.

Proposition 2.1.3. *The set of solutions $\mathcal{S}(M)$ of $\left(\Box - \frac{R}{6}\right)\phi = 0$ with compactly supported initial data on a globally hyperbolic spacetime $(M, g_{\mu\nu})$ is a symplectic space if endowed with the following strongly non degenerate symplectic form:*

$$\sigma(\phi_1, \phi_2) = \int\limits_\Sigma \left(\phi_1 \nabla_n \phi_2 - \phi_2 \nabla_n \phi_1\right) d\mu(\Sigma), \quad \forall \phi_1, \phi_2 \in \mathcal{S}(M) \tag{2.4}$$

where ∇_n is the derivative along the normal direction to the Cauchy surface and $d\mu(\Sigma)$ is metric induced measure on Σ. Furthermore the symplectic form turns out to be independent from the choice of Σ.

Proof. Let us prove independence of (2.4) from the choice of the Cauchy surface. This turns out to be a byproduct of the divergence theorem since if we take the current $J_\mu = \phi_1 \nabla_\mu \phi_2 - \phi_1 \nabla_\mu \phi_2$, the following integral vanishes due to conservation of J_μ:

$$\int\limits_V \nabla^\mu J_\mu \sqrt{|g|} d^4x = \int\limits_{\partial V} J_\mu n^\mu \sqrt{|h|} d^3x = \int\limits_V \left(\phi_1 \Box \phi_2 - (\Box \phi_1)\phi_2\right) \sqrt{|g|} d^4x = 0,$$

where V is an open set with boundaries constructed as follows: it is the region in $M \equiv \Sigma \times \mathbb{R}$ which is delimited by the set $I \times K$ where $I = (t_0, t_1)$ and K in a compact sufficiently large to contain the support of both ϕ_1 and ϕ_2 when restricted on the Cauchy surfaces at $t = t_0$ and at $t = t_1$. Furthermore in the above formula we used the Stokes theorem in the second equality and $\sqrt{|h|}d^3x$ is the metric induced measure on ∂V. Hence, per construction of V, the above formula boils down to

$$0 = \int\limits_{\Sigma_{t_0}} J_\mu n^\mu d\mu(\Sigma) - \int\limits_{\Sigma_{t_1}} J_\mu n^\mu d\mu(\Sigma),$$

which entails independence of (2.4) from the choice of Σ. We need only to show that σ is a well-defined strongly non degenerate symplectic form. Antisymmetry arises per construction while non degenerateness can be proved as follows: suppose that $\sigma(\phi_1, \phi_2) = 0$ for all possible choices of ϕ_2. Let us then fix the latter in such a way that it vanishes on the Cauchy surface but $\nabla_n \phi_2 = \phi_1$. Then (2.4) becomes

$$\sigma(\phi_1, \phi_2) = \int\limits_{\Sigma} \phi_1^2 d\mu(\Sigma).$$

Since ϕ_1 is compactly supported, the right-hand side coincides with the L^2-norm of ϕ_1 on Σ. Hence $\sigma(\phi_1, \phi_2) = ||\phi_1||^2_{L^2(\Sigma)} = 0$ which can vanish if and only if ϕ_1 on Σ is 0. An identical argument leads to conclude that also $\nabla_n \phi_1 = 0$ on Σ, which entails that ϕ identically vanishes both for independence of the used argument from the Cauchy surface and for linearity of the d'Alembert wave equation conformally coupled to scalar curvature (*i.e.*, the only solution with vanishing initial data is the function identically 0). $\square$

2.1.1 Intermezzo

In the previous discussion all the elements of $\mathcal{S}(M)$ have been characterised as solutions of a Cauchy problem in which suitable smooth initial data of compact support have been assigned on a Cauchy surface. In the discussion of classical issues, but more importantly at a quantum level, it is rather useful to obtain the space $\mathcal{S}(M)$ out of a different, albeit related, concept, namely that of advanced and retarded operator. Their definition is as follows:

Definition 2.1.2. *Let $(M, g_{\mu\nu})$ be a four dimensional time-oriented globally hyperbolic connected space-time. A map $G_+ : C_0^\infty(M) \to C^\infty(M)$ is called an* **advanced (Green) operator** *with respect to an operator P as in* (2.3) *if it satisfies the following conditions:*

- $P \circ G_+ = id : C_0^\infty(M) \to C_0^\infty(M)$
- $G_+ \circ P|_{C_0^\infty(M)} = id : C_0^\infty(M) \to C_0^\infty(M)$
- *$supp(G_+ f) \subseteq J^+(supp f)$ for any $f \in C_0^\infty(M)$.*

Analogously one can introduce the retarded (Green) function G_- which fulfils the same hypothesis except the third which becomes: $supp(G_- f) \subseteq J^-(supp f)$ for any $f \in C_0^\infty(M)$. The difference $E \doteq G_+ - G_-$ is called **causal propagator**.

The most notable consequence of this definition can be summarised in the following theorem whose proof can be found in [6]:

Theorem 2.1.1. *There exists a unique advanced and retarded operator for P.*

Therefore we are free to consider, in the scenarios we are interested in, the causal propagator $E : C_0^\infty(M) \to C_0^\infty(M)$ which thus satisfies:

$$\begin{cases} PE(f) = 0 & \forall f \in C_0^\infty(M) \\ \phi_f \doteq E(f) & \text{is such that} \quad supp(\phi_f) \subset J^+(supp(f)) \cup J^-(supp(f)) \end{cases} . \tag{2.5}$$

In other words, the causal propagator takes a compactly supported function on the whole spacetime M and generates a solution of the equation of motion, we are interested in. At this stage it is natural to wonder which is the relation between the set of solutions generated out of E and $\mathcal{S}(M)$ which is a byproduct of a Cauchy problem. The next proposition unveils the dilemma:

Proposition 2.1.4. *Every solution of a Cauchy problem for P as in (2.3) with compactly supported smooth initial data on a Cauchy surface is in one to one correspondence with ϕ_f where f has to be interpreted as a representative of the equivalence class $\frac{C_0^\infty(M)}{\mathcal{J}}$, i.e., for all $f, f' \in C_0^\infty(M)$ we say that $f \sim f'$ if and only if there exists $g \in C_0^\infty(M)$ such that $f - f' = Pg$.*

Proof. Let us start with $\Longleftarrow$ which is easier. Hence we take any $f \in [f] \in \frac{C_0^\infty(M)}{\mathcal{J}}$ and we construct $\phi_f = E(f)$. This solves $P\phi_f = 0$ and, since M is globally hyperbolic, we can split the underlying background as $\Sigma \times \mathbb{R}$. In order to conclude, we just need to notice that, since $supp(f)$ is compact, we can fix $t \in \mathbb{R}$ so that $K = [J^+(supp(f)) \cup J^-(supp(f))] \cap \Sigma$ is compact; therefore ϕ_f, as well as its derivative along the normal direction to Σ, are smooth and compactly supported. These can be used as initial data for a Cauchy problem of 2.3 and the uniqueness of the solution entails that it must coincide with ϕ_f.

Let us now look at $\Longrightarrow$. We consider smooth compactly supported initial data on a Cauchy surface Σ and we choose any function $f \in C_0^\infty(M)$ which coincides with these data whenever restricted to Σ. It is immediate to see that this must exist, but, certainly it is not a priori unique. Nonetheless the field $\phi_f = E(f)$ solves $P\phi_f = 0$ and it coincides with the solution of the Cauchy problem constructed with the chosen initial data. Let us now suppose that we take a second function, say $f' \in C_0^\infty(M)$ which coincides with the initial data if restricted to Σ. Uniqueness of the solution of Cauchy problems for (2.3) entails that $E(f') = E(f) = \phi_f$. Hence $0 = E(f - f') = (G^+ - G^-)(f - f')$ yields that $G^+(h) = G^-(h)$ where $h \doteq f - f'$. If we apply P we get our of definition 2.1.2 that $PG^+(h) = h = PG^-(h)$ while the support properties of both the advanced and retarded operator yields that $g \doteq G^+(h) = G^-(h)$ is supported in $J^+(supp(h)) \cap J^-(supp(h))$ which is compact, h being compact on its own. Since P is a properly supported operator $g \in C_0^\infty(M)$, hence we have shown that it must exist a smooth compactly supported function g such that $f - f' = P(g)$. □

Remark 2.1.2. From a physical point of view, an important aspect of the last proposition arises from the fact that it provides a clear characterisation of the support properties of the fields we are interested in. Most notably, in the spirit of finding a way to encode their information on null infinity, one can see that a potential disastrous problem might arise with respect to the definition of asymptotically flat spacetime we discussed. If we use the conformal diagram of Minkowski spacetime as a guiding principle (see figure 1.1), it is clear that the solutions of the Cauchy problems, we are interested in, are supported in a region which, at infinity, discards spatial infinity, but actually tends to future (or past) timelike infinity. Unfortunately this point is not a priori included in the compactification process we have employed and this entails, in a physical language, that $\Im^+$ cannot grasp the full information of the bulk field theory because part of it escapes through "a hole" in our background. To avoid this potential problem we can use, as anticipated, a slightly different notion of asymptotic flat spacetime which replaces i_0 with i^+ as distinguished point. It is important to notice that all the considerations on the geometric and asymptotic structures are left unchanged. This notion was first introduced by Friedrich in [25, 26, 27] and used in a bulk-to-boundary context by Moretti in [37]; for sake of completeness we here give the full definition.

Definition 2.1.3. *A four dimensional vacuum spacetime $(M, g_{\mu\nu})$ (a.k.a physical spacetime), i.e., a solution of Einstein's vacuum field equations, is called* **asymptotically flat at future null and time infinity** *if there exists* *a)* $(\widetilde{M},\widetilde{g}_{\mu\nu})$ *(a.k.a unphysical spacetime) with* $g_{\mu\nu}\in C^\infty(M)$ *and* $\widetilde{g}_{\mu\nu}\in C^\infty(\widetilde{M}\backslash\{i^+\})$, i^+ *being a 2-sphere embedded in* $\widetilde{M}$,

b) a conformal isometry from M *to* $\widetilde{M}$, *that is a map* $\psi: M\to\psi[M]\subset\widetilde{M}$, $\psi[M]$ *being an open subset of* $\widetilde{M}$, *and a function* $\Omega\in C^\infty(\psi[M],\mathbb{R}^+)$ *fulfilling* $\widetilde{g}_{\mu\nu}|_{\psi[M]}=\Omega^2(\psi^*g)_{\mu\nu}$

such that the following five conditions hold true:

1. *there exists a 2-sphere* $i^+\in\widetilde{M}$ *such that* $\overline{J^-(i^+,\widetilde{M})}$ *is closed and* $\psi[M]=J^-(i^+)\setminus\partial J^-(i^+,\widetilde{M})$. *Moreover* $\partial\psi[M]=\Im^+\cup i^+$ *where* $\Im^+\doteq\partial J^-(i^+,\widetilde{M})\setminus\{i^+\}$ *is* **future null infinity**,

2. *there exists* $\mathcal{O}$, *open neighbourhood in* $\widetilde{M}$ *of* $\partial\psi[M]$ *such that* $(\mathcal{O},\widetilde{g}_{\mu\nu})$ *is strongly causal,*

3. *the function* Ω *can be extended (not necessarily in a unique way) to a function on the whole* $\widetilde{M}$ *which is smooth except at most at* i^+ *where it is twice differentiable,*

4. Ω *must vanish on* $\partial J^-(i^+)$, *whereas* $d\Omega\neq 0$ *on* $\Im^+$, *"d" being the external derivative. Furthermore, on* i^+, *it also holds that*

$$\lim_{i^+}\widetilde{\nabla}_\mu\Omega=0,\quad \lim_{i^+}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\Omega=2\widetilde{g}_{\mu\nu}(i^+),\tag{2.6}$$

5. *the map of null directions at* i^+ *into the space of integral curves of* $n^\mu\doteq\widetilde{\nabla}^\mu\Omega$ *on* $\partial J^\pm(i^+)\setminus\{i^+\}$ *is a diffeomorphism and, furthermore, for any choice of a function* $\omega\in C^\infty(\widetilde{M}\setminus\{i^+\})$ *such that* $\omega>0$ *on* $(\psi[M]\cup J^-(i^+))\setminus\{i^+\}$ *and* $\widetilde{\nabla}_\mu(\omega^4n^\mu)=0$ *on* $\Im^+$, *the vector field* $\omega^{-1}n^\mu$ *is complete on* $\Im^+$.

Henceforth, whenever we mention an asymptotically flat spacetime, it *must satisfy* this last definition.

2.2 Field Theory at Null Infinity

We are now ready to address the main problem, namely how to encode the bulk data in boundary ones. We have already briefly argued that, from a classical point of view, null infinity could be used as an initial surface of a characteristic problem, but this is certainly not enough to conclude anything at a quantum level. To this avail we need to go one step further, namely we need to suitably embed the bulk data in those of a second field theory which intrinsically lives on $\Im^+$, or more appropriately, in view of the discussions of the previous chapter on an equivalence class $[\underline{\Im}^+,\underline{g}_{\mu\nu},\underline{n}^\mu]$. The aim of this section will be to describe a procedure which ultimately leads to the construction of the field theory at null infinity. We thus invite a potential reader to keep more attention on the logical steps we shall follow more than on the mathematical details which are exquisitely typical of the considered system and they are not present in different, albeit related, contexts. Let us start recalling that the outcome of the previous section and of proposition 2.1.2 in particular is the following lemma:

Lemma 2.2.1. *Each element* $\phi\in\mathcal{S}(M)$ *can be projected to a smooth function* Ψ *on* $\Im^+\subset(\widetilde{M},(\omega\Omega)^2\widetilde{g}_{\mu\nu})$ *for all possible choices of the gauge factor* ω *and*

$$\Psi\doteq\widetilde{\phi}|_{\Im^+}\quad\text{and}\quad \widetilde{\phi}\equiv(\omega\Omega)^{-1}\phi|_{\psi[M]}.$$

One should bear in mind that, since we are dealing only with scalar function, the evaluation of the function on null infinity is tantamount to its pull-back on $\underline{\Im}^+$. This two point of views are for scalar fields identical and thus we shall from now on refer mostly to $\Im^+$.

The philosophy of our next steps will be simply to use Ψ as the building block to construct a field theory on $\Im^+$, but, to this avail, we need to understand first of all how to implement diffeomorphism invariance on null infinity since this is a prerequisite for any physically meaningful field theory. Since, as we stressed before, we are actually interested on the behaviour of Ψ on the triples $[\underline{\Im}^+, \underline{g}_{\mu\nu}, \underline{n}^\mu]$, diffeomorphism invariance is tantamount to require covariance under the action of the BMS group. The reason is the following: a fixed bulk asymptotically flat spacetime induces a set of possible boundary triples which are related one with each other by a gauge transformation. At the same time, as we explained in the previous chapter, a generic diffeomorphism of $\underline{\Im}^+$ maps a triple into a gauge equivalent one if and only if it is an element of the BMS group. Therefore, if we would take the whole $Diff(\underline{\Im}^+, \underline{\Im}^+)$, we would implicitly admit transformations mapping the geometry of null infinity into that of a different bulk spacetime. Since we keep the latter fixed, we must consider only BMS transformations to avoid a potentially pathological situation.

That said, we shall try to implement a BMS action on the functions Ψ and this operation is best performed in a Bondi frame (u, θ, φ), or, more properly, after a stereographic projection from $\mathbb{S}^2 \to \mathbb{C}$, in the frame $(u, z, \bar{z})$. Furthermore, if we recall that the generator $\underline{\zeta}$ of a 1-parameter subgroup γ_t $(t \in \mathbb{R})$ of the BMS can be seen as the restriction to null infinity of a bulk vector field ζ which fulfils $\lim\limits_{\Im^+} \Omega^2(\mathcal{L}_\zeta g)_{\mu\nu} = 0$, we are lead to the following definition:

Definition 2.2.1. *Let us take any but fixed gauge factor ω and a representative $(\underline{\Im}^+, \underline{g}_{\mu\nu}, \underline{n}^\mu)$ of the equivalence class of triples characterising future null infinity of a chosen asymptotically flat spacetime. Then, for any $g = (\Lambda, \alpha) \in BMS$, it acts on smooth functions over $\Im^+$ out of the representation $A_g : C^\infty(\Im^+, \mathbb{C}) \to C^\infty(\Im^+, \mathbb{C})$ which, in a Bondi frame, reads*

$$[A_g \Psi](u', z', \bar{z}') = K_\Lambda^{-1}(z, \bar{z}) \Psi(u, z, \bar{z}), \tag{2.7}$$

where $(u', z, \bar{z}')$ as well as the function K_Λ are defined in (1.20).

One should verify the well-posedness of this definition with respect to the above geometric properties of the BMS, but we shall not dwell into these details which are fully accounted in [18]. In order to carry on our analysis, we must bear in mind that, since $\Im^+$ is three dimensional null hypersurface, the functions Ψ are not bound to satisfy any equation of motion and, therefore, there is no dynamical criterion which allows us to select a natural symplectic space as in the case of the bulk theory. Nonetheless, as we already pointed out, this is a key ingredient if we want both to set up a genuine quantum field theory at null infinity and to discuss its physical implication; it is hence mandatory to introduce a counterpart of $\mathcal{S}(M)$ on $\Im^+$. This is one of the trickiest aspects of the implementation of the bulk to boundary formalism since the absence of an a priori guiding principle forces us to make an educated guess which is only a posteriori justified. In order to better understand the meaning of these remarks, let us give a closer look to the case we are discussing. Let us thus introduce the set:

$$\mathcal{S}(\Im^+) = \left\{ \psi \in C^\infty(\Im^+) \;\middle|\; \psi \text{ and } \partial_u \psi \in L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge d\mathbb{S}^2(z, \bar{z}))\,, \right\}, \tag{2.8}$$

where $d\mathbb{S}^2(z, \bar{z}) = \frac{2dz \wedge d\bar{z}}{i(1+|z|^2)}$ is still the standard measure on the unit 2-sphere.

The first evidence that such a choice is, if not the unique one, at least natural and well-conceived lies in the following theorem which grants us that $\mathcal{S}(\Im^+)$ can be made a symplectic space on which the BMS-representation A acts in a natural way:

Theorem 2.2.1. *The set $\mathcal{S}(\Im^+)$ endowed with the bilinear form $\sigma_{\Im^+} : \mathcal{S}(\Im^+) \times \mathcal{S}(\Im^+) \to \mathbb{R}$:*

$$\sigma_{\Im^+}(\Psi, \Psi') = \int\limits_{\mathbb{R} \times \mathbb{S}^2} (\Psi \partial_u \Psi' - \Psi' \partial_u \Psi)\, du \wedge d\mathbb{S}^2(z, \bar{z}), \quad \forall \psi, \psi' \in \mathcal{S}(\Im^+) \tag{2.9}$$

is a strongly nondegenerate symplectic space on which the BMS representation A acts as a symplectomorphism.

Proof. The well-posedness of the symplectic form as well as its non-degenerateness can be proven following almost slavishly the argument which lead to the same conclusion for σ in proposition 2.1.3 and, thus, we shall not repeat it. Let us instead focus on the representation A and on (2.7) in particular; per direct inspection, one can realize that, beside a change of coordinate, a smooth function over $\Im^+$ is rescaled by K_Λ^{-1}, which we can infer, out of (1.20), to be a smooth function on $\mathbb{S}^2$, hence it is bounded and compactly supported. This entails that, if both ψ and its derivative along the u-coordinate are square integrable, so must be their image under the action of A. In other words $A_g : \mathcal{S}(\Im^+) \to \mathcal{S}(\Im^+)$ for all $g \in BMS$.

We must only prove that the symplectic form is preserved under action of A, or equivalently of A^{-1} since it is a representation of the BMS group. Hence for any but fixed $g \in BMS$ and for any $\Psi, \Psi' \in \mathcal{S}(\Im^+)$, it holds:

$$\sigma_{\Im^+}(A_g^{-1}\Psi, A_g^{-1}\Psi') =$$

$$\int\limits_{\mathbb{R}\times\mathbb{S}^2} \left[(A_g^{-1}\Psi)(u', z, \bar{z}')\partial_{u'}(A_g^{-1}\Psi')(u', z', \bar{z}') - (A_g^{-1}\Psi')(u', z', \bar{z}')\partial_{u'}(A_g^{-1}\Psi)(u', z', \bar{z}')\right] du' \wedge d\mathbb{S}^2(z', \bar{z}') =$$

$$\int\limits_{\mathbb{R}\times\mathbb{S}^2} K_\Lambda^2(z, \bar{z}) \left[\Psi(u, z, \bar{z})\frac{\partial u}{\partial u'}\partial_u\Psi'(u, z, \bar{z}) - \Psi'(u, z, \bar{z})\frac{\partial u}{\partial u'}\partial_u\Psi(u, z, \bar{z})\right] \frac{du'}{du} du \wedge d\mathbb{S}^2(z', \bar{z}') = \sigma_{\Im^+}(\Psi, \Psi'),$$

where we employed in the second equality (2.7), while, in the last one, we used that

$$d\mathbb{S}^2(z', \bar{z}') = K_\Lambda^{-2}(z, \bar{z}) d\mathbb{S}^2(z, \bar{z}),$$

which concludes the proof. $\square$

Although we have now the certitude that we have identified a well-defined symplectic space with respect to the boundary data, this does not suffice since we have also to be sure that it is a natural counterpart of $(\mathcal{S}(M), \sigma)$. This proposition translates this request in a mathematical language:

Proposition 2.2.1. *Let* $\Gamma : \mathcal{S}(M) \to C^\infty(\Im^+)$ *be the so-called* **bulk to boundary projection map** *which associates to any* $\phi \in \mathcal{S}(M)$

$$\Gamma(\phi) \doteq \widetilde{\phi}|_{\Im^+},$$

where $\tilde{\phi}$ *is the solution of* $(\widetilde{\Box} - \frac{\tilde{R}}{6})\tilde{\phi} = 0$ *which coincides with* $\Omega^{-1}\phi$ *in* $\psi[M]$. *Then it holds that*

a) $\Gamma(\mathcal{S}(M)) \subset \mathcal{S}(\Im^+)$,

b) for all $\phi, \phi' \in \mathcal{S}(M)$ *then* $\sigma(\phi, \phi') = \sigma_{\Im^+}(\Gamma\phi, \Gamma\phi')$.

Proof. We already know thanks to lemma 2.2.1 that $\Gamma\mathcal{S}(M) \subset C^\infty(\Im^+)$, while the remaining part of the proof of *a)* relies on two properties of $\mathcal{S}(M)$ proven in [38], namely

- for all $\phi \in \mathcal{S}(M)$, there exists $Q_\phi \in \mathbb{R}$ such that $supp(\Gamma\phi) \subset [Q_\phi, +\infty) \times \mathbb{S}^2$,
- for all $\phi \in \mathcal{S}(M)$, there exist $C_p, M_p > 0$ ($p = 0, 1$) and $u_0 \in \mathbb{R}$ such that for all $u > u_0$ and for all $(z, \bar{z}) \in \mathbb{S}^2$
$$|\partial_u^p(\Gamma\phi)| \leq \frac{M_p}{|C_p u - 1|}.$$

These two conditions imply that the support of $\Gamma\phi$ on null infinity cannot tend to $-\infty$ in the u-variable and that the peeling behaviour as $u \to \infty$ is sufficiently fast to guarantee that the projected function together with its derivative along u are square integrable in $\mathbb{R} \times \mathbb{S}^2$, which, in other words, implies *a)*. To prove *b)* we can write in the unphysical spacetime

$$\sigma(\phi_1, \phi_2) = \int\limits_{\Sigma} \left(\widetilde{\phi}_1 \nabla_n \widetilde{\phi}_2 - \widetilde{\phi}_2 \nabla_n \widetilde{\phi}_1\right) d\mu(\Sigma),$$

where $\widetilde{\phi}_i = \Omega^{-1}\phi_i$ for $i = 1, 2$. Here Σ is a Cauchy surface in the unphysical spacetime and $d\mu(\Sigma)$ the metric induced measure. This last formula can be rewritten as the integral of a 3-form, *i.e.*,

$$\int\limits_{\Sigma} \chi_{\phi_1,\phi_2} = \int\limits_{\Sigma} -\frac{1}{6}\widetilde{g}^{\mu\nu}(\psi_1\partial_\nu\psi_2 - \psi_2\partial_\nu\psi_1)\epsilon_{\mu\rho\delta\eta}\sqrt{|\widetilde{g}|}dx^\rho \wedge dx^\delta \wedge dx^\eta.$$

We can now employ the divergence theorem in a sufficiently large region in $\widetilde{M}$ which includes as its boundary both a part of the Cauchy surface containing the initial data for ϕ_1 and ϕ_2 and a part of null infinity containing the support of their projections; this yields

$$\int\limits_{\Sigma} \chi_{\phi_1,\phi_2} = \int\limits_{\Im^+} \chi_{\phi_1,\phi_2},$$

where the right-hand side, written in a Bondi frame, coincides with $\sigma_{\Im^+}(\Gamma\phi_1, \Gamma\phi_2)$. $\square$

2.3 The bulk and boundary Weyl algebra

We are now ready to use the machinery of the previous section to discuss the quantum properties of the bulk theory in term of the boundary counterpart. Particularly we shall show that the boundary field theory admits a preferred BMS invariant algebraic state which can be pulled-back to the bulk where it defines a distinguished state fulfilling many notables properties in between we should mention in advance,

- invariance under any bulk isometry,
- coincidence with the Poincaré vacuum in Minkowski spacetime,
- the so-called Hadamard property.

Therefore we shall safely claim that we have identified a “natural” ground state for a scalar massless field theory conformally coupled to scalar curvature in any asymptotically flat and globally hyperbolic spacetime.

The procedure we shall employ to get to this result is somehow straightforward and it could in principle be applied to any field theory on any spacetime with a boundary under the hypothesis that a suitable projection map, such as Γ, which preserves the relevant symplectic forms, has been identified. The first step is to introduce a suitable notion of algebra of observables and this justifies a posteriori our perseverance in describing the symplectic properties of the space of functions we use in both the bulk and the boundary.

Definition 2.3.1. *For any symplectic space $(\mathcal{S}, \sigma')$ where σ' is strongly nondegenerate, we call* **Weyl algebra** *(of observables) $\mathcal{W}(\mathcal{S})$ the vector space which is generated by the abstract elements $W(f)$ for all $f \in \mathcal{S}$ and which fulfils the following defining relations for all $f, f' \in \mathcal{S}$:*

$$a)\ W^*(f) = W(-f), \qquad b)\ W(f)W(f) = e^{\frac{i}{2}\sigma'(f,f')}W(f+f'). \tag{2.10}$$

Let us now recall [6, 13] that an algebra $\mathcal{A}$ on the complex numbers endowed with an involution $* : \mathcal{A} \to \mathcal{A}$ is called a **Banach** $*$-algebra if endowed with a norm $||,|| : \mathcal{A} \to \mathbb{R}$ such that

1. $||W^*|| = ||W||$ for all $W \in \mathcal{A}$,
2. it holds completeness with respect to the uniform topology

$$\mathcal{U}_\epsilon(W) = \{W' \in \mathcal{A} \mid ||W' - W|| < \epsilon\}.$$

Furthermore if $||W^*W|| = ||W||^2$ we call it a **C*-algebra**.

Bearing in mind this definition, we can state the following lemma whose proof is given in lemma 5.2.8 of [14]:

Lemma 2.3.1. *The algebra* $\mathcal{W}(\mathcal{S})$ *which fulfils* (2.10) *is, up to* $*$*-isomorphisms a unique* C^**-algebra.*

Remark 2.3.1. The outcome of this lemma is that we can associate to a massless scalar field theory conformally coupled to scalar curvature on an asymptotically flat and globally hyperbolic spacetime two Weyl C*-algebras:

a) to $(\mathcal{S}(M), \sigma)$ we associate $\mathcal{W}(M)$,

b) to $(\mathcal{S}(\Im^+), \sigma_{\Im^+})$ we associate $\mathcal{W}(\Im^+)$.

Let us notice that a formal interpretation of the elements $W(\phi) \in \mathcal{W}(M)$ is $e^{i\sigma(\phi,\Phi)}$ where $\sigma(\phi, \Phi)$ is the field operator symplectically smeared with smooth field equations ϕ with smooth compactly supported initial data. To strengthen our point of view, it is instructive to construct $\mathcal{W}(M)$ in an alternative way: let us pick the previously introduced causal propagator $E \doteq G^+ - G^- : C_0^\infty(M) \to C^\infty(M) \cap \mathcal{S}(M)$ which is actually surjective on $\mathcal{S}(M)$. We can employ it to construct for an $f \in C_0^\infty(M)$:

$$\phi(f) = \int\limits_{\Sigma\times\mathbb{R}} \phi f \sqrt{|g} d^4x = \int\limits_{\Sigma\times\mathbb{R}} \phi PG^+(f)\sqrt{|g|}d^4x =$$

$$\int\limits_{t_1}^{t_2}\int\limits_{\Sigma} \phi PG^+ f\sqrt{|g|}d^4x \quad | \ supp(f) \subset [t_1, t_2] \times \Sigma \ ,$$

where $P = \Box - \frac{R}{6}$ and G^+ is the advanced Green operator. If we choose t_1 such that $G^-(f) = 0$, then $G^+(f) = E(f)$ and thus

$$\phi(f) = \int\limits_{\Sigma_{t_1}} \left[\phi\frac{\partial(G^+f)}{\partial t} - G^+(f)\frac{\partial\phi}{\partial t}\right] d\mu(\Sigma) = \Omega(Ef, \phi).$$

Hence we can simply define the set of elements $V(f)$ for all $f \in C_0^\infty(M)$; out of this we can identify the Weyl algebra elements as the following equivalence classes $W(Ef) = [V(f)]$ where two elements $V(f)$ and $V(f')$ are in relation if $f - f' = Ph$ for a function $h \in C_0^\infty(M)$. The improvement of this approach arises from the fact that the support properties of the causal propagator, together with (2.10) and the identity

$$\int\limits_{\Sigma\times\mathbb{R}} (Ef)f'\sqrt{|g|}d^4x = \sigma(Ef, Ef') \quad \forall f, f' \in C_0^\infty(M)$$

yields the **locality property**:

$$[W(\phi_f), W(\phi'_{f'})] = 0, \qquad \text{if } \ supp(\phi_f) \cap supp(\phi'_{f'}) = \emptyset.$$

A similar locality property can be stated also for $\mathcal{W}(\Im^+)$.

2.4 Unique BMS algebraic state and its distinguished bulk companion

We shall now focus on the boundary Weyl algebra $\mathcal{W}(\Im^+)$ and we shall show that, in this case, it is possible to construct a BMS-invariant Fock representation of the vacuum. We shall proceed in logical sequential steps:

Step 1. Let us fix any $\Psi \in \mathcal{S}(\Im^+)$ and let us recall that $\Im^+ \equiv \mathbb{R}\times\mathbb{S}^2$. Hence one can perform a Fourier transform along the u-direction (actually a *Fourier-Plancherel transform*) as:

$$\begin{cases} \psi_+(u,z,\bar{z}) = \int\limits_{\mathbb{R}^+} dE \frac{e^{-iEu}}{\sqrt{4\pi E}} \widehat{\psi}_+(E,z,\bar{z}) \\ \widehat{\psi}_+(E,z,\bar{z}) = \sqrt{2E} \int\limits_{\mathbb{R}^+} du \frac{e^{iEu}}{\sqrt{2\pi}} \psi(u,z,\bar{z}), \quad E \geq 0 \end{cases},$$

where ψ_+ identifies the so-called *positive frequency* part of ψ and, thus, $\psi = \psi_+ + \overline{\psi}_+$.

Step 2. We call $\mathcal{S}(\Im^+)^{\mathbb{C}}_+ = span_{\mathbb{C}}\{\Psi_+ \mid \Psi \in \mathcal{S}(\Im^+)\}$, where only linear combinations are taken into account. It descends the following proposition:

Proposition 2.4.1. *The symplectic form $\sigma_{\Im^+}$ is well-defined also on $\mathcal{S}(\Im^+)^{\mathbb{C}}_+$ and it yields a unique Hermitian product*

$$\langle\Psi_{1+},\Psi_{2+}\rangle \doteq i\sigma_{\Im^+}(\overline{\Psi_{1+}},\Psi_{2+}) = \int\limits_{\mathbb{R}^+} \overline{\widehat{\Psi}_{1+}(E,z,\bar{z})}\widehat{\Psi}_{2+}(E,z,\bar{z})dE\wedge d\mathbb{S}^2(z,\bar{z}), \tag{2.11}$$

for all possible choices of $\Psi_1,\Psi_2 \in \mathcal{S}(\Im^+)$. The Hilbert space $\mathcal{H} = \overline{\left(\mathcal{S}(\Im^+)^{\mathbb{C}}_+, \langle,\rangle\right)}$ is isomorphic to $L^2(\mathbb{R}^+\times\mathbb{S}^2, dEd\mathbb{S}^2(z,\bar{z})$ while the range of the map $K : \mathcal{S}(\Im^+) \to \mathcal{S}(\Im^+)^{\mathbb{C}}_+$ such that $K(\Psi) = \Psi_+$ is dense in $\mathcal{H}$.

Proof. Since the Fourier-Plancherel transform maps L^2-functions in L^2-ones, $\sigma_{\Im^+}$ is well-defined also on $\mathcal{S}(\Im^+)^{\mathbb{C}}_+$. The right hand side of the second equality in (2.11) can be directly computed out of (2.9) as:

$$i\sigma_{\Im^+}(\overline{\Psi_{1+}},\Psi_{2+}) = i\int\limits_{\mathbb{R}\times\mathbb{S}^2} \left[\overline{\Psi}_{1+}\partial_u\psi_{2+} - \overline{\Psi}_{2+}\partial_u\Psi_{1+}\right] du\wedge d\mathbb{S}^2(z,\bar{z}) =$$

$$i\int\limits_{\mathbb{R}\times\mathbb{S}^2} du\wedge d\mathbb{S}^2(z,\bar{z}) \int\limits_{\mathbb{R}^+\times\mathbb{R}^+} dE_1\wedge dE_2 \frac{e^{-iE_1u}}{\sqrt{4\pi E_1}}\frac{e^{-iE_1u}}{\sqrt{4\pi E_1}}\left[-i(E_1+E_2)\overline{\widehat{\Psi}_{1+}}(E_1,z,\bar{z})\widehat{\Psi}_{2+}(E_2,z,\bar{z})\right] =$$

$$\int\limits_{\mathbb{S}^2\times\mathbb{R}^+\times\mathbb{R}^+} d\mathbb{S}^2(z,\bar{z})\wedge dE_1\wedge dE_2\frac{\delta(E_1-E_2)}{2\sqrt{E_1E_2}}\left[(E_1+E_2)\overline{\widehat{\Psi}_{1+}}(E_1,z,\bar{z})\widehat{\Psi}_{2+}(E_2,z,\bar{z})\right] =$$

$$\int\limits_{\mathbb{S}^2\times\mathbb{R}^+} \overline{\widehat{\Psi}_{1+}}(E,z,\bar{z})\widehat{\Psi}_{2+}(E,z,\bar{z})\; d\mathbb{S}^2(z,\bar{z})\wedge dE.$$

To prove the last part of the proposition, one notices that the Fourier transform defining $\widehat{\psi}_+$ is a bounded isometric injective map and, thus, thanks to the above chain of identities there exists $U : \mathcal{H} \to \mathcal{H}' \subset L^2(\mathbb{R}_+\times\mathbb{S}^2, dEd\mathbb{S}^2(z,\bar{z}))$, where $\mathcal{H}'$ is a closed subspace. In order to prove that it is also an isomorphism, let us take any function $g \in C^\infty_0(\mathbb{R}\times\mathbb{S}^2,\mathbb{C})$ and let us write

$$\Psi(u,z,\bar{z}) = \int\limits_{\mathbb{R}^+\times\mathbb{S}^2} e^{-iEu}g(u,z,\bar{z})\frac{dE}{\sqrt{4\pi E}}dE\wedge d\mathbb{S}^2(z,\bar{z}) \; + \; c.c.$$

This is a rapidly decreasing function, thus also square integrable; this also entails that there must exist $\Psi_+ \in \mathcal{S}(\Im^+)$ such that $g = \widehat{\Psi}_+ = U\Psi_+$. Since $C^\infty_0(\mathbb{R}\times\mathbb{S}^2)$ is dense in the space of square integrable functions, the operator U must be an isomorphism and also $K : \mathcal{S}(\Im^+) \to \mathcal{H}$ is dense since $U^{-1}\left(C^\infty_0(\mathbb{R}\times\mathbb{S}^2)\right) \subset K\left(\mathcal{S}(\Im^+)\right)$, where $U^{-1}\left(C^\infty_0(\mathbb{R}\times\mathbb{S}^2)\right)$ is dense in $U^{-1}\left(L^2(\mathbb{R}\times\mathbb{S}^2)\right) = \mathcal{H}$. □

Step 3. We interpret $\mathcal{H}$ as the **one particle space** out of which we construct per "tensorialization" the bosonic Fock space

$$\mathcal{F}_+(\mathcal{H}) = \mathbb{C}\bigoplus_{n=1}^{\infty}\mathcal{H}^{\otimes_s n},$$

which is built over the the vacuum $\Upsilon \in \mathcal{H}$ normalised to 1. The word "built" means that we can read the field operator Λ symplectically smeared with $\Psi \in \mathcal{S}(\Im^+)$ as

$$\sigma_{\Im^+}(\Theta, \Psi) = ia(\Psi_+) - ia^\dagger(\Psi_+) \doteq \Theta(\Psi),$$

where $a(\Psi_+)$ and $a^\dagger(\Psi_+)$ are respectively the annihilation and creation operator for $\psi_+ \in \mathcal{H}$. The common invariant domain of all the involved operators in the dense linear manifold which is the span of all the vectors with finite numbers of particles. Furthermore $\Theta(\Psi)$ is essentially self-adjoint on this set and it satisfies the **canonical commutation relations**

$$[\sigma_{\Im^+}(\Theta, \Psi), \sigma_{\Im^+}(\Theta', \Psi')] = -i\sigma(\Psi, \Psi').$$

Hence the operators $e^{i\Theta(\psi)}$ can be interpreted as unitary operators acting on the bosonic Fock space and thus they can be read as the image of a unitary reducible representation $\Pi : \mathcal{W}(\Im^+) \to \mathcal{BL}(\mathcal{F}_+(\mathcal{H}))$. All these data can be collected in the following summarising lemma:

Lemma 2.4.1. *The data* $(\Pi, \mathcal{F}_+(\mathcal{H}), \Upsilon)$ *are the* **GNS triple** *of the algebraic pure state* $\lambda_{\Im^+} : \mathcal{W}(\Im^+) \to \mathbb{C}$ *such that*

$$\lambda(W(\Psi)) \doteq e^{-\frac{\langle \Psi_+, \Psi_+ \rangle}{2}}, \tag{2.12}$$

where $\langle , \rangle$ *is the Hermitian product* (2.11).

Step 4. We need at last to understand how the asymptotic symmetry group can play a role in the construction of the state we have just found. Overall this can be summarised in two proposition, the first of which guarantees that the BMS group can be unitarily represented on the one particle space in an unambiguous way. The proof is as straightforward as tedious and an interested reader can find it in [18].

Proposition 2.4.2. *There exists a unique unitary representation of the BMS group* $T : BMS \to \mathcal{BL}(\mathcal{H})$ *such that*

a) $T_g \Pi(W(\Psi)T_g^\dagger = \Pi(W(A\Psi)$ *for any* $g \in BMS$ *and any* $W(\Psi) \in \mathcal{W}(\Im^+)$. *Here* A *is the action* (2.7),

b) $T_g \Upsilon = \Upsilon$ *for all* $g \in BMS$,

c) T *is an irreducible representation on* $\mathcal{H}$ *and* $T_g \Psi_+ = A_{g^{-1}} \Psi_+$, *that is*

$$(T_g \widehat{\Psi}_+)(E, z, \bar{z}) = \frac{e^{iEK_\Lambda(\Lambda^{-1}z, \Lambda^{-1}\bar{z})\alpha(\Lambda^{-1}z, \Lambda^{-1}\bar{z})}}{\sqrt{K_\Lambda(\Lambda^{-1}z, \Lambda^{-1}\bar{z})}} \widehat{\Psi}_+ \left(EK_\Lambda(\Lambda^{-1}z, \Lambda^{-1}\bar{z})\right).$$

While this last proposition guarantees that our construction is compatible with BMS invariance, the group played a somehow passive role in the definition of the state. The coming theorem shows that it can also play an active role in the realization of the structure of the state and this leads to one of the first notable and unexpected results, namely (2.12) satisfies a *uniqueness* property. Hence, as a byproduct, we can safely claim that $\lambda^+_\Im$ is indeed a distinguished algebraic state for the boundary quantum field theory. The proof is unfortunately so technical that the explanation of the employed methods would require another course on their own; hence an interested reader should refer to [37] where all the details are explained with great care.

Theorem 2.4.1. *Let us consider the one-parameter subgroup of the BMS whose elements are* $g_t = (\mathbb{I}, \sqrt{4\pi} t Y_{00}(z, \bar{z}))$ *with* $t \in \mathbb{R}$ *and with* $\mathbb{I}$ *the identity element of* $SL(2, \mathbb{C})$. *These induce rigid translations along the real axis, i.e., in a Bondi frame* $(u, z, \bar{z}) \to (u + t, z, \bar{z})$. *Then* $\lambda_{\Im^+}$ *is the* **unique** *pure quasi-free algebraic state on* $\mathcal{W}(\Im^+)$ *which is:*

- *invariant under* A_{g_t} *where* A *is given in* (2.7),
- *such that the unitary group representation implementing* A_{g_t}, *while leaving fixed* Υ, *is strongly continuous with non-negative generator.*

2.4.1 Back to the bulk

We have constructed a distinguished state for the boundary theory, but this falls one step short from our ultimate goal of retrieving the information of the bulk spacetime. Nonetheless we have now all the ingredients we need. As a matter of fact, if we recollect together proposition 2.2.1 and lemma 2.4.1, we end up with the following fact:

- The projection map Γ induces a $*$-embedding $\Gamma_{\mathcal{W}} : \mathcal{W}(M) \to \mathcal{W}(\Im^+)$ such that
$$\Gamma_{\mathcal{W}}[W(\phi)] = W(\Gamma\phi), \quad \forall W(\phi) \in \mathcal{S}(M)$$
which, in turn, induces a pull-back of any boundary state $\lambda' : \mathcal{W}(\Im^+) \to \mathbb{C}$ to a bulk one as
$$\lambda_M = \Gamma^*_{\mathcal{W}}\lambda' \quad \text{such that} \quad \lambda_M[W(\phi)] \doteq \lambda'[W(\Gamma\phi)]\ \forall W(\phi) \in \mathcal{W}(M).$$
Particularly this yields that we can identify a **distinguished bulk state** starting from (2.12):
$$\widetilde{\lambda} \doteq \Gamma^*_{\mathcal{W}}\lambda_{\Im^+}. \tag{2.13}$$

The claim that $\widetilde{\lambda}$ is distinguished is certainly premature because we have not yet studied the properties of the state in order to assert that it has physically desirable characteristics. On the one hand we would like to urge the reader to realize that the most interesting aspects of the mechanism employed up to now is its universality, not in the sense of asymptotically flat spacetimes, rather from the perspective of a bulk to boundary correspondence. All the steps we have described can be addressed in all the frameworks where one can identify a distinguished notion of codimension 1 submanifold and, thus, the overall idea we used could be adapted in principle in many other cases, an example of which are the Friedmann-Robertson-Walker spacetime [22]. At the same time it is up to now unclear if the properties of the state, which is identified in the bulk, should be analysed case by case or if similar conclusions can be drawn from general considerations. At present we do not have a definitive answer and, hence, we can only address this issue specifically for $\widetilde{\lambda}$; since we wanted to underline only those aspects of our implementation of the holographic machinery which we can expect to be applicable in different contexts, we shall not indulge in the proof of all the characteristics of (2.13). We shall only cite them and an interested reader can find a more detailed discussion in [18, 37, 38].

Theorem 2.4.2. *The state $\widetilde{\lambda} : \mathcal{W}(M) \to \mathbb{C}$ enjoys the following properties:*

1. *it coincides with the Minkowski vacuum if the bulk is $(\mathbb{R}^4, \eta_{\mu\nu})$,*
2. *it is invariant under the action β on $\mathcal{W}(M)$ of the (identity component of the) Lie group G of* isometries *of any asymptotically flat and globally hyperbolic spacetime:*
$$\widetilde{\lambda}[W(\phi)] = \widetilde{\lambda}[\beta_g W(\phi)] = \widetilde{\lambda}[W(\phi \circ g^{-1})], \quad \forall W(\phi) \in \mathcal{W}(M) \text{ and } \forall g \in G,$$
3. *if $\xi \in \mathfrak{g}$ is an element of the Lie algebra of G which is causal and future directed, the 1-parameter group, which implements ξ and leaves fixed Υ in the GNS representation of $\widetilde{\lambda}$ (and hence of $\lambda_{\Im^+}$), is strongly continuous and its self-adjoint generator H^ξ has neither 0 modes nor non-negative spectrum; hence it provides a natural notion of energy,*
4. *the two-point function of $\widetilde{\lambda}$*
$$\widetilde{\lambda}_2(f_1, f_2) = \lim_{\epsilon \to 0^+} -\frac{1}{\pi} \int\limits_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{\Psi_{f_1}(u, z, \bar{z})\Psi_{f_1}(u', z, \bar{z})}{(u - u' - i\epsilon)^2} du \wedge du' \wedge d\mathbb{S}^2(z, \bar{z}) \quad \forall f_1, f_2 \in C_0^\infty(M)$$
is an element of $\mathcal{D}'(M \times M)$ of **Hadamard form**. *Here $\Psi_{f_i} \doteq \Gamma(E(f_i))$ where $i = 1, 2$ and $E(f_i) \in \mathcal{S}(M)$ with f_i as initial datum.*

Hence $\widetilde{\lambda}$ plays the role of natural ground state for a scalar field theory with vanishing mass and conformal coupling to scalar curvature in <u>any</u> asymptotically flat spacetime which is such both at future (or past) null infinity and at future (or past) timelike infinity. Hence it is the natural building block to discuss any quantum phenomenon in the bulk spacetime.

Chapter 3

Outcome and Perspectives

The aim of these lecture notes was to show how it is possible to exploit the powerful means of the conformal and causal structure of a large class of solutions of Einstein's equations is order to unveil important informations on the nature and on the structure of a scalar field theory living on this class of spacetimes. The paradigm we followed called for a rigorous application of the so-called holographic principle according to which it is possible to encode the information of a field theory living in the bulk of a fixed spacetime into the data of a second one constructed on a suitable codimension one submanifold. In the scenario, we considered, the latter was nothing other than the conformal boundary which is most suited to play this role thanks to its property of universality. The outcome of our construction has been the identification of a distinguished algebraic quantum state for the bulk field theory which displays the notable properties of uniqueness and of the Hadamard form; therefore it is a natural candidate to be addressed as a ground state, *i.e.*, it is the counterpart in a generic asymptotically flat and globally hyperbolic spacetime of the Minkowski vacuum.

Despite these hopefully nice and interesting results, it is absolutely fair to admit that the structure of the bulk-to-boundary correspondence is far from being completely understood. Several problems are left open and we would like to briefly comment on them; from a classical perspective, a rather old difficulty of the construction of an asymptotically flat spacetime is its strong intertwining with null directions which makes it rather unsuited to be used in presence of a massive field. There are several different ways to look at this problem, but, the one, which is probably the fastest, is to consider the bulk Klein-Gordon equation $(\Box - \frac{R}{6} - m^2)\phi = 0$. Under a conformal transformation and a rescaling of the field as $\widetilde{\phi} = \Omega^{-1}\phi$, it becomes $(\widetilde{\Box} - \frac{\widetilde{R}}{6} - m^2\Omega^{-1})\widetilde{\phi} = 0$; hence the term $m^2\Omega^{-1}$ acts as a potential which diverges right at $\Omega = 0$, or, in other words, at null infinity. A general statement on the apparent impossibility to project massive fields to $\Im^+$ has been proved in [29], but, recently, in [19], it has been shown that it is possible to circumvent the problem in Minkowski spacetime using tools proper of harmonic analysis. Nonetheless the generalization to a curved background of this result is still an open problem.

Also at a quantum level, there are still interesting problems to be tackled. The first one concerns the structure of $\widetilde{\lambda}$ which, although it comes from a pure algebraic state for the boundary theory, might even not be pure. A solution of this dilemma would have a rather interesting application at a physical level because it might shed a new light on an old open problem, namely the definition of an elementary particle on a curved background. This concept, which is strictly intertwined with Poincaré invariance in Minkowski spacetime, has no natural counterpart in a curved world since a counterpart of the Minkowski vacuum is lacking. In the framework we have studied, the bulk-to-boundary correspondence might overcome this problem [17], but, to this avail, the purity of $\widetilde{\lambda}$ is an important prerequisite.

A further important aspect, which has been, up to now, discarded is the implementation of the bulk-to-boundary correspondence at a level of interacting field theories. This problem is certainly of primary relevance to extract even further unexpected physical informations from the techniques we employed.

Eventually a last question, which deserves an answer, although it is probably a rather hard problem, concerns a better "understanding" of the techniques we employed. To the date the bulk-to-boundary correspondence, as explained in these notes, has been applied to asymptotically flat spacetimes, cosmological Friedmann-Robertson-Walker backgrounds with flat spatial section and the Schwarzschild vacuum solution of Einstein's equation. In all the scenarios, there is a common trait in the steps we followed, but it would be desirable to characterize a priori the class of spacetimes and of field theories in which the approach we followed is deemed to be successful, hence leading to the identification of distinguished bulk state of Hadamard form. This is most probably the most difficult of the open questions we have outlined.

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