Algebraic Quantum Field Theory meets Cosmology

Claudio Dappiaggi

II. Institut für Theoretische Physik
Hamburg Universität

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Outline of the Talk

- A top-down approach: looking for a temperature!
- A bottom-up strategy: unveiling the role of the stress-energy tensor!
- A step towards a future project: States of cosmological interest!

Motivations - What we know

The 20th century thought us a few good lessons:

1) Interactions $\longrightarrow$ quantum field theory on **flat spacetime**:

- it works almost perfectly for free and electroweak forces,
- perturbative QFT, renormalization, etc...

2) Gravitational interaction $\longrightarrow$ **General Relativity**.

3) Algebraic approach, $\longrightarrow$ also allows for a rigorous discussion of QFT on curved backgrounds [Brunetti, Dimock, Fredenhagen, Hollands, Kay, Verch, Wald,...]

Natural playground $\longrightarrow$ Cosmology

- unveils the structure and dynamics of the Universe,
- we can fully use QFT on curved background in the algebraic approach.

The Cosmological Principle and FRW - I

We wish to model the geometry of the Universe!

Occam's razor leads to the Cosmological principle, *i.e.*,

- spacetime is homogeneous $\to$ at each instant of time, all space points look the same,
- spacetime is isotropic $\to$ there is at each point an observer who sees an isotropic Universe.

This entails

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right].$$

- the parameter $k = 0, \pm 1$ tells me if spatial section are flat planes, spheres or hyperbolas,
- there is still no dynamical content. This determines $a(t)$ and, to this avail, one needs a good $T_{\mu\nu}$.

The Cosmological Principle and FRW - II

Which $T_{\mu\nu}$? Let start with *classical matter*

We know:

- part of mass-energy in the Universe is ordinary matter (stars, galaxies, clusters),
- their density is so low that they appear like "dust" with density ρ. Hence

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu \quad \zeta^\mu\zeta_\mu = 1$$

- if we also include a contribution from pressure, then

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu + P\left(g_{\mu\nu} + \zeta_\mu\zeta_\nu\right),$$

which is the stress-energy tensor of a **perfect fluid**. This is the the most general choice for $T_{\mu\nu}$ if the matter is classical.

Dynamics of the scale factor

Dynamics is encoded in the Einstein's equations

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.$$

I assume from now on $\Lambda = 0$!

$$G_{tt} = R_{tt} - \frac{R}{2} g_{tt} = 8\pi T_{tt} \longrightarrow 3\frac{\dot{a}^2 + k}{a^2} = 8\pi\rho,$$

$$G_{xx} = R_{xx} - \frac{R}{2} g_{xx} = 8\pi T_{xx} \longrightarrow 3\frac{\ddot{a}}{a^2} = -4\pi(\rho + 3P).$$

Conservation of $T_{\mu\nu}$, *i.e.*, $\nabla^\mu T_{\mu\nu} = 0$ yields

$$\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a} = 0.$$

Notice that the dynamical content boils down to this last equation and to an identity between traces:

$$Tr(G_{\mu\nu}) = -R = 8\pi Tr(T_{\mu\nu}).$$

Dynamics of the scale factor - II

To solve that system we need an **equation of state** $\rho = \gamma P$

We assume from now on $k = 0$, but only for simplicity of the talk!

Eq. of state	scale factor	conservation of $T_{\mu\nu}$
Dust, $P = 0$	$a(t) \propto t^{\frac{2}{3}}$	$\rho a^3(t) = const.$
Radiation, $P = \frac{\rho}{3}$	$a(t) \propto \sqrt{t}$	$\rho a^4(t) = const.$

One should interpret the results, but instead let us look at the assumptions.

To get here we assumed

1. isotropy and homogeneity of the Universe,
2. the behaviour of the stress-energy tensor is classical,
3. pressure and energy density are related by an equation of state.

Are we happy?

Practical and Foundational Problems

The "classical" approach to cosmology is highly unsatisfactory

on a practical ground,

- the model is far too rough in the description of matter,
- it is plagued by many problems, namely
  1. the singularity problem as $a \to 0$, namely $\rho \to \infty$,
  2. the flatness problem,
  3. the homogeneity problem.

and a foundational ground,

- Classical matter cannot account for an explanation of interac. as QFT.

What we would like to know

The quest to solve those problems prompted

- Modern Cosmology $\longrightarrow$ matter is often modelled by a scalar field.
- Bright side
  1. it takes seriously the role of QFT as the natural playground to discuss (quantum) matter and interactions,
  2. it provides a nice exit to most of the problems of standard cosmology,
  3. models of inflation lead to testable consequences, on the temperature of CMB in particular.
- Dark side
  1. still plenty of open problems (dark matter, dark energy...),
  2. it is unclear how to derive these models from "first principles",
  3. unclear concepts in curved backgrounds:(temperature, termal equilibrium)...

On the notion of temperature - I

How to cope with temperature in curved spacetimes?

As a starting point:

- there is a good concept of thermal states ω_β in Minkowski (KMS condition)
- in this case we know how to compute expectation values, *e.g.*, for a free massless scalar field

$$\omega_\beta(:\phi^2:) = \frac{1}{12|\beta|^2} \quad |\beta| = T^{-1},$$

- we can extend it to other observables, *e.g.*,

$$\omega_\beta(\widetilde{\partial}^\mu\widetilde{\partial}^\nu :\phi^2(x):) = -\frac{(4\pi)^2}{4!}B_4\partial^\mu\partial^\nu(\beta^2)^{-1} \doteq \alpha^{\mu\nu}(\beta),$$

$$\widetilde{\partial}^\mu :\phi^2 := \lim_{\zeta\to 0}\partial^\mu_\zeta\left(\phi(x+\zeta)\phi(x-\zeta) - \omega_{vac}(\phi(x+\zeta)\phi(x-\zeta))\mathbb{I}\right).$$

On the notion of temperature - II

On curved backgrounds M, such as FRW, what can we do?

- we have a good notion of normal ordering, *i.e.*, $:\phi^2(x):$ is meaningful,
- we seek states ω_M whose expectation values are "coherent" with the Minkowski ones

$$\omega_M(:\phi^2(x):) = f(x) \doteq \frac{1}{12\beta^2(x)},$$

$$\omega_M(\widetilde{\partial}^\mu \widetilde{\partial}^\nu :\phi^2(x):) = \alpha^{\mu\nu}(\beta(x)),$$

and so on and so forth.

The "mother of all problems"

The idea is enticing but it faces a big problem

• An important observable is the stress-energy tensor $T_{\mu\nu}$, but

• for a massless scalar field in Minkowski $T \doteq Tr(T_{\mu\nu}) = 0$,

$$\omega_\beta(:T:) = 0,$$

• if we look for a spacetime conformally related to Minkowski (as FRW with $k = 0$) and we take

$$\Box_g \phi - \frac{R}{6}\phi = 0, \longrightarrow T = 0,$$

but, for an Hadamard ground state ω_M

$$\omega_M(:T:) = \frac{1}{4\pi^2}\left(\frac{1}{720}(R_{ij}R^{ij} - \frac{R^2}{3} + \Box R)\right).$$

This is the trace anomaly!

The role of the trace anomaly

The natural definition of temperature fails due to the trace anomaly!

Is it an accident or does it play a fundamental role, for example in cosmology?

Best arena where to investigate[1]: semiclassical Einstein's equations!

[1] C.D., Klaus Fredenhagen, Nicola Pinamonti, Phys. Rev. D**77** (2008) 104015

A semiclassical effect - I

Let us look at our framework:

- We fix the background as a FRW spacetime with flat spatial section

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \quad M \equiv \mathbb{R} \times \mathbb{R}^3$$

- we consider for "simplicity of the talk" a scalar field on M

$$\left(\Box_g - \frac{R}{6} - m^2\right)\phi(x) = 0,$$

  which is conformally coupled to scalar curvature.

- we shall seek solutions of $G_{\mu\nu} = 8\pi\langle : T_{\mu\nu} :\rangle_\omega$, $\longrightarrow$ in FRW

$$-R = 8\pi\langle : T :\rangle_\omega.$$

Intermezzo: the quest for an Hadamard state

What is a good choice for ω?

A physically reasonable choice is

1. an ω which is quasi-free (technical condition),
2. an ω which is of Hadamard form,
   - same ultraviolet behaviour as the ground state in Minkowski,
   - only on these states the quantum fluctuations of $T_{\mu\nu}$ are finite.

Hence in a geodesic normal neighbourhood of any point $p \in \mathbb{R} \times \mathbb{R}^3$, the integral kernel of the two-point function is

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y) \ln \frac{\sigma_\epsilon(x,y)}{\lambda} + W(x,y).$$

Intermezzo - II

Let start again from

$$\omega(x,y)=\frac{U(x,y)}{\sigma_\epsilon(x,y)}+V(x,y)\ln\frac{\sigma_\epsilon(x,y)}{\lambda}+W(x,y)$$

One can prove that

- U, V, W are all smooth scalar functions,
- in Minkowski $U=1$ and $V=\frac{m^2}{2\sqrt{m^2(x-y)^2}}J_1(\sqrt{m^2(x-y)^2})$, whereas in curved backgrounds they are a series

$$U(x,y)=\sum_{n=0}^{\infty}u_n(x,y)\sigma^n,\quad V(x,y)=\sum_{n=0}^{\infty}v_n(x,y)\sigma^n,$$

  determined out of recursion relations,
- the singular part, namely U and V, depends only on geometric quantities such as $R, R^2, R_{\mu\nu}R^{\mu\nu}\ldots$
- the choice of a quantum state of Hadamard form lies only in W.

A semiclassical effect - II

Let us *assume* to take an Hadamard state! Then

$$\langle : T : \rangle_\omega = -m^2 \frac{W(x,x)}{8\pi^2} + \frac{v_1(x,x)}{4\pi^2},$$

$$v_1(x,x) = \frac{1}{720}(R_{ij}R^{ij} - \frac{R^2}{3} + \Box R) + \frac{m^4}{8}.$$

Plugging it in the semiclassical Einstein's equations, shaking them a little bit, we end up with

$$-6\left(\dot{H} + 2H^2\right) = -8\pi m^2 \langle : \phi^2 : \rangle_\omega + \frac{1}{\pi}\left(-\frac{1}{30}(\dot{H}H^2 + H^4 + \frac{m^4}{4}\right),$$

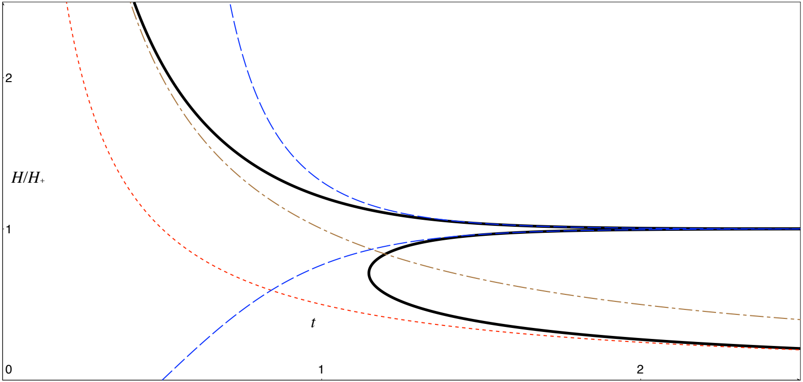
where $H = \frac{\dot{a}(t)}{a(t)}$.

A notion of approximate ground state exists:

$$m^2 \gg R \text{ and } m^2 \gg H^2 \longrightarrow \langle : \phi^2 : \rangle_\omega = \frac{1}{32\pi^2} m^2 + \beta R.$$

A semiclassical effect - III

$$\dot{H} = \frac{-H^4 + H_+^2 H^2}{H^2 - \frac{H_+^2}{4}}, \quad H_+^2 = 360\pi - 2880\pi^2 m^2 \beta.$$



Do Hadamard states really exist?

The bottom-up strategy seems to bear fruit but

- is the result stable if we consider another kind of matter field[2]?
- Are all our assumptions robust enough?

Particularly does an Hadamard state exist on a FRW spacetime?

- Hadamard states are the building block for perturbation theory,
- we ultimately need to tackle interacting models,
- many cosmological predictions of models such as inflation arise out of quantum effects and of perturbation theory.

[2] C.D., Thomas-Paul Hack, Nicola Pinamonti, ArXiv:0904.0612 [math-ph]

Yes, they exist - I

We need a strategy to identify an Hadamard state on a FRW spacetime

1. construct states of low energy and prove they are Hadamard,[3]
2. direct construction of this state: possible, but tricky and time consuming,
3. circumvent the obstacle seeking an alternative approach.

A large class of FRW possesses a distinguished (cosmological) horizon.

Can we use it to implement a bulk-to-boundary correspondence? We know

- it works perfectly in AdS spacetimes via AdS/CFT,
- it fits in the picture of algebraic quantum field theory[4],
- it can be implemented in asymptotically flat spacetimes.

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[3] H. Olbermann, Class. Quant. Grav. **24** (2007) 5011

[4] see also P. L. Ribeiro, arXiv:0712.0401 [math-ph].

Yes, they exist - II

Let us consider a FRW spacetime on $(0,-\infty)\times\mathbb{R}^3$ with[5]

$$ds^2 = a^2(\tau)[-d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)],$$

let us restrict the class of scale factors as:

$$a(\tau) = -\frac{1}{H\tau} + O\left(\tau^{-2}\right) \ ,$$

$$\frac{da(\tau)}{d\tau} = \frac{1}{H\tau^2} + O\left(\tau^{-3}\right) \ , \frac{d^2a(\tau)}{d\tau^2} = -\frac{2}{H\tau^3} + O\left(\tau^{-4}\right).$$

- If $a(\tau) = -\frac{1}{H\tau}$ then $\tau = -e^{-Ht}$, hence cosmological de-Sitter spacetime.
- as $\tau \to -\infty$, the background **"tends to"** de Sitter, Hence we are dealing with an exponential acceleration in the proper time t. This is the the prerequisite of all inflationary models.

[5] C.D., Nicola Pinamonti, V. Moretti Comm. Math. Phys **285** (2009), 1129

Consequences and Properties - I

- There is always a **Cosmological horizon**. Under the coordinate change

$$U = \tan^{-1}(\tau - r) \ , \qquad V = \tan^{-1}(\tau + r),$$

the metric becomes:

$$g_{FRW} = \frac{a^2(U,V)}{\cos^2 U \cos^2 V}\left[-dUdV + \frac{\sin^2(U-V)}{4} dS^2(\theta,\varphi)\right].$$

Theorem:

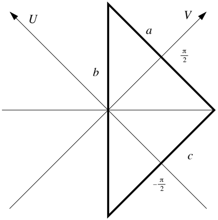
Under the previous assumptions the spacetime (M, g_{FRW}) can be extended to a larger spacetime $(\widehat{M}, \widehat{g})$ which is a conformal completion of the asymptotically flat spacetime at past (or future) null infinity $(M, a^{-2} g_{FRW})$, *i.e.*, "a" plays the role of the conformal factor. The cosmological horizon is

$$\Im^- \doteq \partial J^+(M; \widehat{M}) = \partial M.$$

Consequences and Properties - II

- Conformall null infinity $\Im^-$ corresponds to the horizon *(region c in the figure)* and it is a null degenerate manifold with

$$g|_{\Im^-} = 0 \cdot dl^2 + H^{-2}\left(d\mathbb{S}^2(\theta,\varphi)\right),$$



Furthermore the manifold $M \cup \Im^-$ enjoys:

1. the vector field ∂_τ is a conformal Killing vector for $\widehat{g}$ in M,
2. the vector ∂_τ becomes tangent to $\Im^\pm$ approaching it and coincides with $-H^{-1}\widehat{\nabla}^b a$.

Yes, they exist - III

Let us consider the usual real scalar field

$$P\Phi_f = 0, \qquad P = -\Box + \xi R + m^2 \text{ and } \xi R + m^2 > 0$$

with smooth compactly supported initial datum f on an open set.

- Each solution Φ_f is a smooth function on M, *i.e.*, $\Phi_f \in C^\infty(M)$,
- The set of solutions $S(M)$ of our equation is a symplectic space if endowed with the Cauchy-independent nondegenerate symplectic form:

$$\sigma(\Phi_f, \Phi_g) \doteq \int\limits_\Sigma \left(\Phi_f \nabla_N \Phi_g - \Phi_g \nabla_N \Phi_f\right) d\mu_g^{(\Sigma)},$$

- each $\Phi_f \in S(M)$ can be extended to a unique smooth solution of the same equation on $M \cup \Im^- \longrightarrow \Gamma\Phi_f \doteq \Phi_f|_{\Im^-} \in C^\infty(\Im^-)$.

Yes, they exist - IV

Proposition

For all $\Phi_f \in S(M)$ and $m^2 + \xi R > 0$, then

- $\Gamma\Phi_f \in \left(\mathcal{S}(\Im^-), \sigma_{\Im^-}\right)$, where

$$\mathcal{S}(\Im^-) = \left\{\psi \in C^\infty(\mathbb{R}\times\mathbb{S}^2) \mid \psi \in L^\infty, \partial_l\psi \in L^1, \widehat{\psi} \in L^1, k\widehat{\psi} \in L^\infty\right\},$$

$$\sigma_{\Im^-}(\psi_1, \psi_2) = \int\limits_{\mathbb{R}\times\mathbb{S}^2} (\psi_1\partial_l\psi_2 - \psi_2\partial_l\psi_1)\, d\mu.$$

- $\sigma_{\Im^-}(\Gamma\Phi_f, \Gamma\Phi_g) = H^2\sigma(\Phi_f, \Phi_g)$,

We can introduce a *distinguished state* whose 2-point function is

$$\omega(\psi, \psi') = \int\limits_{\mathbb{R}\times S^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\varphi)}\widehat{\psi}'(k,\theta,\varphi)dkdS^2(\theta,\varphi).$$

Endgame

Consequence

For all $\Phi_f \in S(M)$ and $m^2 + \xi R > 0$, then

- the projection Γ induces a pull-back of any boundary state in the bulk:

$$\omega_M(\Phi_f, \Phi_g) \doteq \omega(\Gamma\Phi_f, \Gamma\Phi_g).$$

Main Result

The state ω_M arising from the distinguished one on $\Im^-$ is[a]

- always of Hadamard form,
- the Bunch-Davies state in de Sitter spacetime,
- a natural distinguished cosmological "ground (vacuum) state" to be used in the study of linear perturbations,
- invariant under the natural action of any bulk isometry.

[a] C.D., N. Pinamonti, V. Moretti: 0812.4033 to appear on J. Math. Phys.

Conclusions

Hurdled Problems

- the original top-down idea $\longrightarrow$ the conformal anomaly as key ingredient,
- the bottom-up strategy brings,
    1. the existence of late time stable solutions for the semiclassical Einstein's equations,
    2. clarification of the origin of the conformal anomaly for Dirac fields
    3. the identification of a distinguished ground state for free scalar field theories on certain FRW spacetimes.

Problems yet to hurdle

- identify a good notion of termal equilibrium at least in FRW spacetimes (using N. Pinamonti, 0806.0803 to appear on CMP),
- create a valuable companion tool of experiment with the aim to rule out the pathological cosmological models,
- understand the role of quantum effects for all kind of fields in phenomena such as dark matter, dark energy, baryogenesis ...