Aufgabe 9 : Berechne die Eigenwerte der (3×3)-Matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Zusatz: Berechne die zugehörigen Eigenprojektoren

Beweis.

$$\begin{aligned}
f(\lambda) &= det(A - \lambda\mathbb{E}) \\
&= det\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right) \\
&= det\begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} \\
&= -(\lambda) * (-\lambda) * (-\lambda) + 1 * 1 * 0 + 0 * 1 * 1 - 0 * (-\lambda) - (-\lambda) * 1 * 1 \\
&= -\lambda^3 + \lambda + \lambda \\
&= -\lambda^3 + 2\lambda \\
&= 0 \\
&= \lambda^3 - 2\lambda \\
&= \lambda(\lambda^2 - 2)
\end{aligned}$$

$\lambda_1 = 0, \lambda_1 = \sqrt{2}, \lambda_3 = -\sqrt{2}$

Zusatz:

$$
\begin{aligned}
P_1 &= \frac{(A-\lambda_2\mathbb{E})(A-\lambda_3\mathbb{E})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\\
&= \frac{\left(\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}-\begin{pmatrix}\sqrt{2}&0&0\\0&\sqrt{2}&0\\0&0&\sqrt{2}\end{pmatrix}\right)\left(\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}-\begin{pmatrix}-\sqrt{2}&0&0\\0&-\sqrt{2}&0\\0&0&-\sqrt{2}\end{pmatrix}\right)}{(0-\sqrt{2})(0-(-\sqrt{2})}\\
&= \begin{pmatrix}\frac{1}{2}&0&-\frac{1}{2}\\0&\frac{1}{2}&0\\-\frac{1}{2}&0&\frac{1}{2}\end{pmatrix}\\
P_2 &= \frac{(A-\lambda_1\mathbb{E})(A-\lambda_3\mathbb{E})}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\\
&= \frac{\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\sqrt{2}&1&0\\1&\sqrt{2}&1\\0&1&\sqrt{2}\end{pmatrix}}{\sqrt{2}(\sqrt{2}+\sqrt{2})}\\
&= \begin{pmatrix}\frac{1}{4}&\frac{1}{\sqrt{8}}&\frac{1}{4}\\\frac{1}{\sqrt{8}}&\frac{1}{2}&\frac{1}{\sqrt{8}}\\\frac{1}{4}&\frac{1}{\sqrt{8}}&\frac{1}{4}\end{pmatrix}\\
P_3 &= \frac{(A-\lambda_1\mathbb{E})(A-\lambda_2\mathbb{E})}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\\
&= \frac{\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}-\sqrt{2}&1&0\\1&-\sqrt{2}&1\\0&1&-\sqrt{2}\end{pmatrix}}{-\sqrt{2}(-\sqrt{2}-\sqrt{2})}\\
&= \begin{pmatrix}-\frac{1}{4}&\frac{1}{\sqrt{8}}&-\frac{1}{4}\\\frac{1}{\sqrt{8}}&\frac{1}{2}&\frac{1}{\sqrt{8}}\\\frac{1}{4}&\frac{1}{\sqrt{8}}&\frac{1}{4}\end{pmatrix}\\
&= \begin{pmatrix}-\frac{1}{4}&\frac{1}{\sqrt{8}}&-\frac{1}{4}\\-\frac{1}{\sqrt{8}}&-\frac{1}{2}&-\frac{1}{\sqrt{8}}\\-\frac{1}{4}&\frac{1}{\sqrt{8}}&-\frac{1}{4}\end{pmatrix}
\end{aligned}
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