Aufgabe 10
Gegeben:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Gesucht sind die Eigenwerte und Eigenvektoren und Eigenprojektoren von A.

$$\det(A - \lambda\mathbb{E}) = -\lambda^3 + 1$$

$$\Rightarrow \lambda^3 = 1$$

Daraus ergeben sich drei Einheitswurzeln:

$$\lambda_1 = 1$$

$$\lambda_2 = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\lambda_3 = e^{i\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$P_1 = \frac{\begin{pmatrix} -e^{i\frac{2\pi}{3}} & 1 & 0 \\ 0 & -e^{i\frac{2\pi}{3}} & 1 \\ 1 & 0 & -e^{i\frac{2\pi}{3}} \end{pmatrix}\begin{pmatrix} -e^{i\frac{4\pi}{3}} & 1 & 0 \\ 0 & -e^{i\frac{4\pi}{3}} & 1 \\ 1 & 0 & -e^{i\frac{4\pi}{3}} \end{pmatrix}}{\left(1 - e^{i\frac{2\pi}{3}}\right)\left(1 - e^{i\frac{4\pi}{3}}\right)}$$

$$= \frac{1}{\left(-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right)\left(-\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$P_2 = \frac{1}{\left(-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right)\left(i\sqrt{3}\right)}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} -\lambda_3 & 1 & 0 \\ 0 & -\lambda_3 & 1 \\ 1 & 0 & -\lambda_3 \end{pmatrix}$$

$$= \frac{1}{3\lambda_3}\begin{pmatrix} \lambda_3 & -1-\lambda_3 & 1 \\ 1 & \lambda_3 & -1-\lambda_3 1 \\ -1-\lambda_3 & 1 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3}\left(\frac{1}{\lambda_3}+1\right) & \frac{1}{3\lambda_3} \\ \frac{1}{3\lambda_3} & \frac{1}{3} & -\frac{1}{3}\left(\frac{1}{\lambda_3}+1\right) \\ -\frac{1}{3}\left(\frac{1}{\lambda_3}+1\right) & \frac{1}{3\lambda_3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{1}{3}(\lambda_2+1) & \frac{\lambda_2}{3} \\ \frac{\lambda_2}{3} & \frac{1}{3} & -\frac{1}{3}(\lambda_2+1) \\ -\frac{1}{3}(\lambda_2+1) & \frac{\lambda_2}{3} & \frac{1}{3} \end{pmatrix}$$

$$P_3 = \frac{1}{3\lambda_2}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} -\lambda_2 & 1 & 0 \\ 0 & -\lambda_2 & 1 \\ 1 & 0 & -\lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{1}{3}\left(\frac{1}{\lambda_2}+1\right) & \frac{1}{3\lambda_2} \\ \frac{1}{3\lambda_2} & \frac{1}{3} & -\frac{1}{3}\left(\frac{1}{\lambda_2}+1\right) \\ -\frac{1}{3}\left(\frac{1}{\lambda_2}+1\right) & \frac{1}{3\lambda_2} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3}\left(\lambda_3+1\right) & \frac{\lambda_3}{3} \\ \frac{\lambda_3}{3} & \frac{1}{3} & -\frac{1}{3}\left(\lambda_3+1\right) \\ -\frac{1}{3}\left(\lambda_3+1\right) & \frac{\lambda_3}{3} & \frac{1}{3} \end{pmatrix}$$

$$(A-\lambda_1\mathbb{E})\overrightarrow{x_1}=0$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 = y_1,$$

$$y_1 = z_1,$$

$$z_1 = z_1,$$

$$\overrightarrow{x_0} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(A-\lambda_2\mathbb{E})\overrightarrow{x_2}=0$$

$$\Rightarrow y_2 = \lambda_2 x_2,$$

$$z_2 = \lambda_2 y_2,$$

$$x_2 = \lambda_2 z_2,$$

$$\overrightarrow{x_2} = \begin{pmatrix} \lambda_2 \\ \lambda_2^2 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_3 \\ 1 \end{pmatrix}$$

$$(A-\lambda_3\mathbb{E})\overrightarrow{x_3}=0$$

$$\begin{pmatrix} -\lambda_3 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 1 & 0 & -\lambda_3 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_3 = \lambda_3 x_3$$

$$z_3 = \lambda_3 y_0$$

$$x_3 = \lambda_3 z_3$$

$$\overrightarrow{x_3} = \begin{pmatrix} \lambda_3 \\ \lambda_3^2 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_3 \\ \lambda_2 \\ 1 \end{pmatrix}$$

Eigenvektoren:

$$\phi_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\phi_2 = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

$$\phi_3 = \begin{pmatrix} \lambda_1 \\ \lambda_3 \\ \lambda_2 \end{pmatrix}$$

Eigenprojektoren

$$P_1 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$P_2 = \frac{1}{3} \begin{pmatrix} 1 & \lambda_2 & \lambda_3 \\ \lambda_3 & 1 & \lambda_2 \\ \lambda_2 & \lambda_3 & 1 \end{pmatrix} = \frac{1}{3} \phi_2 \phi_2^*$$

$$P_3 = \frac{1}{3} \begin{pmatrix} 1 & \lambda_3 & \lambda_2 \\ \lambda_2 & 1 & \lambda_3 \\ \lambda_3 & \lambda_2 & 1 \end{pmatrix}$$