Algebraic quantum field theory

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Contents

CHAPTER 1

Quantum systems with many degrees of freedom

1. Superposition principle and superselection sectors

In the standard formulation of quantum physics states are described by vectors Φ of some Hilbert space $\mathfrak{H}$. A Hilbert space is a complex vector space with a positive definite scalar product

$$\begin{array}{ccc} \mathfrak{H} \times \mathfrak{H} & \to & \mathbb{C} \\ \Phi, \Psi & \mapsto & \big(\Phi, \Psi\big) \end{array}$$

which is antilinear in the left and linear in the right factor (physicist's convention) and in terms of which a norm

$$\|\Phi\| = \big(\Phi, \Phi\big)^{\frac{1}{2}}$$

on $\mathfrak{H}$ can be introduced. In addition, $\mathfrak{H}$ is assumed to be complete with respect to this norm, i.e. all Cauchy sequences in $\mathfrak{H}$ converge.

Observables correspond to selfadjoint operators A on $\mathfrak{H}$, i.e. $A : \mathfrak{H} \to \mathfrak{H}$ is a linear continuous map which fulfils

$$\big(\Phi, A\Psi\big) = \big(A\Phi, \Psi\big) \text{ for all } \Phi, \Psi \in \mathfrak{H} \ .$$

Every continuous linear map on $\mathfrak{H}$ is bounded on the unit sphere

$$\mathfrak{H}_1 = \{\Phi \in \mathfrak{H}, \|\Phi\| = 1\}$$

Its supremum on $\mathfrak{H}_1$ defines a norm on the algebra of linear continuous operators on $\mathfrak{H}$.

In applications often unbounded (and hence discontinuous) linear operators occur. They are usually defined only on some dense subspace of the Hilbert space (their domain of definition). To include them into the set of observables leads to some mathematical subtleties which we want to avoid. We therefore restrict ourselves, whenever possible, to bounded operators.

The physical interpretation of quantum mechanics relies on an association of probability measures $\mu_{\Phi,A_1,\dots,A_n}$ to all nonzero vectors Φ and pairwise commuting selfadjoint operators $A_i, i = 1, \dots, n$. The probability measure is uniquely determined by its moments

$$\int d\mu_{\Phi,A_1,\dots,A_n} p(a_1, \dots, a_n) = \big(\Phi, p(A_1, \dots, A_n)\Phi\big)\|\Phi\|^{-2}$$

for all polynomials p in n variables. For a region $G \subset \mathbb{R}^n$ the probability that the measurement of $(A_1, \dots, A_n)$ yields some point $(a_1, \dots, a_n) \in G$ is then given by $\mu_{\Phi,A_1,\dots,A_n}(G)$.

The probability distributions associated to a vector Φ depend only on the ray $\{\lambda\Phi, \lambda \in \mathbb{C}\}$. But in a linear superposition

$$\Psi = \alpha\Phi_1 + \beta\Phi_2$$

the probabilities described by Ψ depend on the choice of the vectors Φ_1 and Φ_2 in their respective rays. The possibility of superposition is a crucial property of quantum theory and is responsible for interference effects.

Due to the possibility of interference quantum mechanical states are quite different from states in classical physics, where a state can be labeled by a point of phase space, or, in case of incomplete knowledge, by a probability distribution in phase space. In principle, quantum theory applies also to macroscopical systems and leads there to conclusions which are in sharp contrast to classical physics (and experience from daily life) as may be exemplified by the example of Schrödinger's cat. Even more peculiar are the restrictions on the notion of reality which follow from the violation of Bell's inequalities.

Though quantum mechanical states cannot always be superimposed. Of course, the vectors in Hilbert space can be linearly combined, but it can happen that the relative phase between the vectors cannot be observed. This phenomenon was first observed by Wick, Wightman and Wigner. They considered the superposition of a state of a particle with spin $\frac{1}{2}$ with that of a spin 0 particle,

$$\Psi = \alpha\Phi_{\frac{1}{2}} + \beta\Phi_0 \ .$$

A rotation by 2π changes the state vector of the particle with spin $\frac{1}{2}$ by a factor of -1 and leaves the state vector of the spin 0 particle unchanged. Thus Ψ is transformed to

$$\Psi' = -\alpha\Phi_{\frac{1}{2}} + \beta\Phi_0 \ .$$

But a rotation by 2π has no observable effect, hence for all observables A we find the same expectation values as before. This implies that the matrix elements of any observable A between a spin $\frac{1}{2}$ state and a spin 0 state must vanish. Hence the state corresponding to Ψ is equivalent to the density matrix

$$\varrho = |\alpha|^2 |\Phi_{\frac{1}{2}}\rangle\langle\Phi_{\frac{1}{2}}| + |\beta|^2 |\Phi_0\rangle\langle\Phi_0| \ .$$

The nonexistence of coherent superpositions leads to a decomposition of the state space of quantum physics into so-called superselection sectors. The Hilbert space $\mathfrak{H}$ can be decomposed into a direct sum of mutually orthogonal subspaces

$$\mathfrak{H} = \bigoplus_i \mathfrak{H}_i \ .$$

Every vector $\Phi \in \mathfrak{H}$ has a unique decomposition $\Phi = \sum_i \Phi_i$ with $\Phi_i \in \mathfrak{H}_i$. The phases of the vectors Φ_i have no physical meaning. The

observables are operators which leave the subspaces $\mathfrak{H}_i$ invariant and may be written as diagonal block matrices

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & 0 & \vdots & \vdots \end{pmatrix}$$

where A_i is the restriction of A to $\mathfrak{H}_i$. Only states within the same superselection sector can be superimposed.

2. Algebraic formulation of quantum theory

The algebra of observables $\mathfrak{A}$ consists in the presence of superselection sectors of elements of the form $A = \bigoplus A_i$ with operators A_i on $\mathfrak{H}_i$ and the norm $\|A\| = \sup_i \|A_i\| < \infty$. This algebra has the nontrivial center $Z = \{A \in \mathfrak{A}, [A, B] = 0 \forall B \in \mathfrak{A}\} = \bigoplus_i \lambda_i \mathbf{1}_{\mathfrak{H}_i}$. The subspaces $\mathfrak{H}_i$ are the joint eigenspaces of the elements of the center.

An easy example is the algebra generated by the components of the angular momentum operator. The center of this algebra is generated by the absolute square of the angular momentum. The sectors then consist of states with a prescribed angular momentum quantum number.

In the following we will assume the point of view that a physical system may be characterized by the set of all its observables. In quantum theory this set has the structure of an associative involutive complex algebra ("q-numbers").

In the Born-Heisenberg-Jordan formulation of quantum mechanics, for example, the algebra of observables is the algebra with unity which is generated by position q and momentum p with the canonical commutation relation

$$pq - qp = -i$$

and with the involution

$$p^* = p \ , \ q^* = q \ .$$

Quantum systems with finite dimensional state spaces, which occur in particular in quantum information theory, have as their algebra of observables the algebra of $n \times n$-Matrizen with complex entries and with involution

$$A^*_{ik} = \overline{A_{ki}} \ .$$

In the Hilbert space formulation of quantum mechanics the algebra of observables is the set of bounded linear operators on some Hilbert space $\mathfrak{H}$. Here a linear operator A is called bounded, if its norm

$$\|A\| = \sup_{\Phi \in \mathfrak{H}, \|\Phi\|=1} \|A\Phi\|$$

is finite. A^* is the adjoint operator (often denoted $A^\dagger$). By definition it is that operator which satisfies

$$(\Phi, A^*\Psi) = (A\Phi, \Psi) \ , \ \Phi, \Psi \in \mathfrak{H}$$

where $(\cdot,\cdot)$ denotes the scalar product of Hilbert space.

Starting from the algebra as the fundamental object of the theory, we need a characterization of states which is independent of a possible realization of the observables as operators on a Hilbert space.

Conceptually, a state is a prescription for the preparation of a system. This concept entails in particular that experiments can be reproduced and is therefore equivalent to the ensemble interpretation where the statements of the theory apply to the ensemble of equally prepared systems.

In quantum physics, a state assigns to an observable not a single value but, in general, a probability distribution of measured values. It is convenient to characterize the probability distributions in terms of their moments. But since the n-th moment is the expectation value of the n-th power, it is sufficient to know the expectation values of all elements of the algebra of observables. Therefore, in the algebraic formulation of quantum physics, one identifies states with expectation value functionals. By definition these are all linear functionals ω on the algebra of obserables which are normalized ($\omega(1) = 1$) and positive ($\omega(A^*A) \geq 0$). Under suitable conditions (e.g. $\sup|\omega(A^n)|^{\frac{1}{n}} < \infty$), such a functional ω induces for a selfadjoint element A of the algebra a uniquely determined probability measure $\mu_{\omega,A}$ with the property

$$\int a^n d\mu_{\omega,A}(a) = \omega(A^n) \ , \ n \in \mathbb{N}_0 \ .$$

Also the spectrum of an element A of the algebra can be characterized in purely algebraic terms. It is defined as the set of all complex numbers $\lambda \in \mathbb{C}$ for which $A - \lambda$ possesses no inverse in $\mathfrak{A}$. Let e.g. $A^2 = 1$. Then

$$(A-\lambda)(A-\mu) = A^2 - (\lambda+\mu)A + \lambda\mu = (1+\lambda\mu) - (\lambda+\mu)A \ ,$$

hence for $\lambda^2 \neq 1$

$$(1-\lambda^2)^{-1}(A+\lambda)$$

is an inverse of $A - \lambda$. The spectrum $\sigma(A)$ of A therefore is contained in the set $\{\pm 1\}$.

The elements of the spectrum can be interpreted as the possible measured values of the observable. In the example above, for A selfadjoint, the probability measure induced by a state ω is concentrated at the points ± 1, with the probabilities

$$p(\pm 1) = \frac{1}{2}(1 \pm \omega(A)) \ .$$

More generally, if the spectrum of A is finite and consists of the real numbers $a_1, \ldots, a_n$, then the probability $p(a_k)$ for the the measurement of a_k is

$$p(a_k) = \frac{\omega(\prod_{i\neq k}(A - a_i))}{\prod_{i\neq k}(a_k - a_i)} \ .$$

A nice property of the algebraic formulation is that it applies also to classical physics. There we choose as our algebra the algebra of continuous function on the (1-point compactification) of the phase space. Let f be such a function and let $\lambda \in \mathbb{C}$ be not in the range of f. Then the function

$$f_\lambda(x) = (f(x) - \lambda)^{-1}$$

is continuous, hence an element of the algebra which is the inverse of $f - \lambda$. If, on the other hand, $\lambda = f(x_0)$ for some point x_O in phase space then $((f - \lambda)g)(x_0) = 0$ for all functions g, hence $f - \lambda$ has no inverse. Thus the spectrum of f coincides with its range.

States are the normalized Radon measures which correspond to probability measures via

$$\mu(f) = \int f d\mu \ .$$

In case the algebra is an algebra of Hilbert space operators, every vector Φ of the Hilbert space with $\|\Phi\| = 1$ induces a state (in the sense of the algebraic formulation) by

$$\omega_\Phi(A) = \big(\Phi, A\Phi\big) \ .$$

States can be mixed by applying the corresponding preparation prescriptions with certain statistical weights. Let ω_i be states and $\lambda_i \in \mathbb{R}$, $\lambda_i > 0$, $\sum \lambda_i = 1$. Then the convex combination

$$\omega = \sum \lambda_i \omega_i$$

is again a state.

For states ω_i which are induced by Hilbert space vectors Φ_i, the state ω can be represented by the density matrix

$$\rho = \sum \lambda_i |\Phi_i\rangle\langle\Phi_i| \ .$$

The expectation value is given by

$$\omega(A) = \operatorname{Tr} \rho A \ .$$

Here the trace of a positive operator A is defined by the formula

$$\operatorname{Tr} A = \sum (\Phi_i, A\Phi_i)$$

with an arbitrary orthonormal basis $(\Phi_i)_i$. The trace is invariant under unitary transformations,

$$\operatorname{Tr} UAU^* = \operatorname{Tr} A \ ,$$

but can assume also the value ∞. Density matrices are by definition posititve operators with unit trace. Operators which can be written as finite linear combinations of positive operators with finite trace are called trace class operators. The trace can be extended to a linear functional on all trace class operators. Furthermore, the set of trace

class operators is a two sided ideal $\mathfrak{I}$ in the algebra of bounded operators on a Hilbert space,

$$T \text{ trace class }, \ A \text{ bounded } \implies AT \text{ and } TA \text{ trace class } .$$

Therefore every density matrix induces a state on the algebra of bounded Hilbert space operators.

3. Operator algebras

For the general discussion the most appropriate concept is that of C*-algebras. C*-algebras possess (as the algebra of bounded Hilbert space operators) a norm with the property

$$\|A^*A\| = \|A\|^2$$

(a so-called C*-norm). Furthermore they are (as normed spaces) complete. Actually, one can prove that every C*-algebra is isomorphic to a norm closed algebra of bounded operators on a (not necessarily separable) Hilbert space. But, in general, the representation of a C*-Algebra by Hilbert space operators is highly nonunique. The freedom in the choice of the representation plays an important rôle in the theory of superselection sectors, in the theory in background gravitational fields, in the theory of phase transitions and, more recently, in the theory of nonequlibrium stationary states. It is the fundamental feature of the algebraic approach that those aspects which can be treated on the level of the algebra of observables, can be separated from aspects which are due to the Hilbert space representation.

An example for a concept which is meaningful already on the level of the algebra is the spectrum of an element. The spectrum of an element A of a C*-algebra lies inside the circle with radius $\|A\|$ around the origin and is real for selfadjoint elements.

Among the C*-algebras, an important subclass are the von Neumann algebras (also called W*-algebras). They can be characterized by the property that every monotoneously increasing bounded net of positive elements has a supremum. Here the order relation is defined by

$$A \geq B \iff \exists C \text{ such that } A - B = C^*C \ .$$

Von Neumann algebras are isomorphic to algebras of Hilbert space operators which are closed in the weak operator topology.

4. GNS construction

The algebraic formulation of quantum theory is closely related to the Hilbert space formulation. To describe this connection we need the concept of a representation.

Definition 4.1. A representation of an involutive unital algebra $\mathfrak{A}$ is a unital *-homomorphism π into the algebra of linear operators on a dense subspace $\mathfrak{D}$ of a Hilbert space.

Here a homomorphism is called a *-homomorphism if

$$(\Phi, \pi(A)\Psi) = (\pi(A^*)\Phi, \Psi) \ \forall \ \Phi, \Psi \in \mathfrak{D} \ .$$

We already saw that every unit vector $\Phi \in \mathfrak{D}$ induces a state of the algebra by

$$\omega(A) = \big(\Phi, \pi(A)\Phi\big) \ .$$

Surprisingly, also the converse holds. This is the famous Gelfand-Naimark-Segal (GNS) construction:

THEOREM 4.2. Let ω be a state on the involutive unital algebra $\mathfrak{A}$. Then there exists a representation π of the algebra by linear operators on a dense subspace $\mathfrak{D}$ of some Hilbert space $\mathfrak{H}$ and a unit vector $\Omega \in \mathfrak{D}$, such that

$$\omega(A) = \big(\Omega, \pi(A)\Omega\big)$$

and $\mathfrak{D} = \{\pi(A)\Omega, A \in \mathfrak{A}\}$.

The proof of this important theorem is simple. First one introduces a scalar product on the algebra in terms of the state ω by

$$\big(A, B\big) := \omega(A^*B) \ .$$

Linearity for the right and antilinearity for the left factor are obvious. Hermiticity

$$\big(A^*, B\big) = \overline{\big(B^*, A\big)}$$

follows from positivity of ω by using the representation of A^*B and B^*A as linear combination of positive elements, implied by the equations

$$2(A^*B + B^*A) = (A+B)^*(A+B) - (A-B)^*(A-B) \ ,$$

$$2(A^*B - B^*A) = -i(A+iB)^*(A+iB) + i(A-iB)^*(A-iB) \ .$$

Furthermore, positivity of ω immediately implies that the scalar product is positive semidefinite

$$\big(A, A\big) = \omega(A^*A) \geq 0 \ .$$

We now study the set

$$\mathfrak{N} = \{A \in \mathfrak{A}, \omega(A^*A) = 0\} \ .$$

It is crucial that $\mathfrak{N}$ is a left ideal of $\mathfrak{A}$. Namely, because of the Cauchy-Schwarz inequality $\mathfrak{N}$ is a subspace of $\mathfrak{A}$. Moreover, for $A \in \mathfrak{N}$ and $B \in \mathfrak{A}$ we have, again because of the Cauchy-Schwarz inequality,

$$\omega((BA)^*BA) = \omega(A^*B^*BA) = \big(B^*BA, A\big)$$

$$\leq \sqrt{\big(B^*BA, B^*BA\big)}\sqrt{\big(A, A\big)} = 0 \ ,$$

hence $BA \in \mathfrak{N}$. We now define $\mathfrak{D}$ as the quotient space $\mathfrak{A}/\mathfrak{N}$. Per constructionem the scalar product is positive definite on $\mathfrak{D}$, thus we can complete $\mathfrak{D}$ and obtain a Hilbert space $\mathfrak{H}$. The representation π is induced by left multiplication of the algebra,

$$\pi(A)(B + \mathfrak{N}) := AB + \mathfrak{N} \ .$$

π is well defined, since $\mathfrak{N}$ is a left ideal of $\mathfrak{A}$. Finally, we set

$$\Omega = 1 + \mathfrak{N} \ .$$

It can easily be verified that the conditions of the theorem are satisfied. It is also straightforward to see that the construction is unique up to unitary equivalence. Namely, let $(\pi', \mathfrak{D}', \mathfrak{H}', \Omega')$ be another quadruple satisfying the conditions of the theorem. Then we define an operator $U : \mathfrak{D} \to \mathfrak{D}'$ by

$$U\pi(A)\Omega = \pi'(A)\Omega' \ .$$

U is well defined, since $\pi(A)\Omega = 0$ if and only if $\omega(A^*A) = 0$; but then we have also $\pi'(A)\Omega' = 0$. Furthermore U preserves the scalar product and is invertible and has therefore a unique extension to a unitary operator from $\mathfrak{H}$ to $\mathfrak{H}'$. Finally, the representations π and π' are unitarily equivalent,

$$\pi'(A) = U\pi(A)U^* \ , \ A \in \mathfrak{A} \ .$$

CHAPTER 2

Principles of Local Quantum Physics

1. Subsystems

A subsystem of a physical system can be identified with a subalgebra of the algebra of observables of the full system such that the unit elements of the two algebras coincide. Two subsystems $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of a given system $\mathfrak{A}$ may be called independent if the subalgebra of the full system generated by them is isomorphic to the tensor product algebra $\mathfrak{A}_1 \otimes \mathfrak{A}_2$.

While on the level of algebras of observables a well defined decomposition of a system into subsystems exists in terms of the tensor product, the situation is more involved for states. We first observe that every state of a system can be restricted to a subsystem just by restricting the expectation values to the observables of the subsystem. Conversely, every state on a sub-C*-algebra of a C*-algebra can be extended to a state on the full algebra. This is a consequence of the Hahn-Banach Theorem, but the extension is, in general, highly nonunique. In case the system consists of two independent subsystems, we may asociate to every pair ω_1 and ω_2 of states of the respective subsystems a state $\omega_1 \otimes \omega_2$ of the full system which is defined by

$$\omega_1 \otimes \omega_2(A_1 \otimes A_2) = \omega_1(A_1)\omega_2(A_2) \ .$$

Convex combination of these so-called product states are called separable. It is a crucial property of noncommutative algebras that there are also nonseparable states. These so-called entangled states are responsible for the violation of Bell's inequalities and they are important for quantum information.

2. The principle of locality

In relativistic field theory appropriate subsystems are the algebras $\mathfrak{A}(\mathcal{O})$ of all observables which can be measured within a given spacetime region $\mathcal{O}$. This association of regions with algebras is then to satisfy the condition of **Isotony**,

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2) \ .$$

Often, the local algebras are defined independently of each other, then the condition of isotony is encoded in a family of unit preserving injective homomorphisms $i_{\mathcal{O}_1\mathcal{O}_2} : \mathfrak{A}(\mathcal{O}_2) \to \mathfrak{A}(\mathcal{O}_1)$ for each pair of regions

$\mathcal{O}_2 \subset \mathcal{O}_1$ such that

$$i_{\mathcal{O}_1\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_3} = i_{\mathcal{O}_1\mathcal{O}_3} \text{ if } \mathcal{O}_3 \subset \mathcal{O}_2 \subset \mathcal{O}_1 \ .$$

The regions then play merely the role of elements of a partially ordered index set $\mathcal{K}$.

One may now associate an algebra $\mathfrak{A}(\mathcal{K})$ to the whole system. Abstractly, this is done in the following way: One considers the free algebra generated by all pairs $(A, \mathcal{O})$ with $A \in \mathfrak{A}(\mathcal{O})$ and divides out the relations

(i) The map $i_{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(\mathcal{K}), A \mapsto (A, \mathcal{O})$ is a unit preserving *-homomorphism.

(ii) $i_{\mathcal{O}_1} \circ i_{\mathcal{O}_1\mathcal{O}_2} = i_{\mathcal{O}_2}$ for $\mathcal{O}_2 \subset \mathcal{O}_1$

The standard situation is that the index set is directed, i.e. for each pair $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ there exists an $\mathcal{O}_3 \in \mathcal{K}$ such that $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}_3$. This is true e.g. if $\mathcal{K}$ is the set of relatively compact open subsets of Minkowski space. For a directed index set $\mathcal{K}$, the algebra $\mathfrak{A}(\mathcal{K})$ is just the union of local algebras,

$$\mathfrak{A}(\mathcal{K}) = \bigcup_{\mathcal{O}\in\mathcal{K}} i_{\mathcal{O}}(\mathfrak{A}(\mathcal{O})) \ .$$

This is called the inductive limit of the net of local algebras.

If the local algebras are C*-algebras the inductive limit has a unique C*-norm inherited from the local algebras. One may then complete $\mathfrak{A}(\mathcal{K})$ and obtains the so called C*-inductive limit $\overline{\mathfrak{A}(\mathcal{K})}$ of the net. It is gratifying that no new relations can occur in this process. Namely, let ϕ be a homomorphism of $\overline{\mathfrak{A}(\mathcal{K})}$ into some C*-algebra $\mathfrak{B}$, and let $A \in \overline{\mathfrak{A}(\mathcal{K})}$ with $\phi(A) = 0$. There is a sequence $\mathcal{O}_n \in \mathcal{K}$ and a sequence $A_n \in i_{\mathcal{O}_n}(\mathfrak{A}(\mathcal{O}_n))$ such that $\|A_n - A\| \to 0$ and $\|\phi(A_n)\| \to 0$. But $\|\phi(A_n\| = \inf\{\|A_n - B_n\|, \phi(B_n) = 0, B_n \in i_{\mathcal{O}_n}(\mathfrak{A}(\mathcal{O}_n))\}$. Thus we can find a sequence $B_n \in i_{\mathcal{O}_n}(\mathfrak{A}(\mathcal{O}_n))$ with $B_n \to A$ and $\phi(B_n) = 0$. In particular, if the local algebras are simple (i.e. have no nontrivial ideal), the same holds true for $\overline{\mathfrak{A}(\mathcal{K})}$.

The situation changes if the index set is no longer directed. A simple example is provided by chiral conformal field theories in 2 dimensions.

The wave equation in 2 dimensions may be separated in light cone coordinates $u = t - x$, $v = t + x$ and has as a general solution a sum of a function of u and a function of v. Massless fields in 2 dimensional Minkowski space which depend only on one of the light cone coordinates are called chiral. They can be considered as fields on the real axis. Moreover, because of conformal invariance, the real line may be embedded into the circle S^1. A natural index set $\mathcal{K}$ then is the set of open nondense intervals $I \subset S^1$. $\mathcal{K}$ is not directed, since the union of two such intervals may cover the whole circle.

Let us look at the example of a chiral Majorana field. For each interval $I \in \mathcal{K}$ we form the CAR-algebra over the Hilbert space $L^2(I)$ with

complex conjugation as involution Γ. The algebra $\mathfrak{A}(I)$ of observables associated to I is then defined as the even subalgebra of $\mathrm{CAR}(L^2(I), \Gamma)$. For $I \subset J$, $I, J \in \mathcal{K}$, $L^2(I)$ has a natural embedding into $L^2(J)$. This induces an embedding

$$i_{IJ} : \mathfrak{A}(I) \to \mathfrak{A}(J) \ .$$

These embeddings obviously satisfy the compatibility relation

$$i_{KJ} \circ i_{JI} = i_{KI} \ , \ I \subset J \subset K \ , \ I, J, K \in \mathcal{K} \ .$$

We now investigate the structure of the universal algebra $\mathfrak{A}(\mathcal{K})$.

The local algebras $\mathfrak{A}(I)$ are generated by bilinear expressions in the Majorana fields,

$$b_I(f, g) = 2\chi(f)\chi(g) \ , \ f, g \in L^2(I), \ \Gamma f = f, \Gamma g = g \ .$$

The canonical anticommutation relations for the Majorana field χ,

(i) the map $f \to \chi(f)$ is $\mathbb{C}$-linear

(ii) the anticommutator of smeared Majorana fields is given by

$$\{\chi(f), \chi(g)\} = (\Gamma f, g) \ ,$$

(iii) the involution is induced by

$$\chi(f)^* = \chi(\Gamma f) \ ,$$

lead to the following relations for the bilinears $b_I(f, g)$, $f, g \in L^2(I, \mathbb{R})$:

(i) the map $f, g \to b_I(f, g)$ is $\mathbb{R}$-bilinear,

(ii) we have

$$b_I(f, f) = \|f\|^2 \ ,$$

(iii) for $f, g, h \in L^2(I, \mathbb{R})$ we have

$$b_I(f, g)b_I(g, h) = \|g\|^2 b_I(f, h) \ ,$$

(iv) the involution is induced by

$$b_I(f, g)^* = b_I(g, f) \ .$$

These relations already determine $\mathfrak{A}(I)$. The universal algebra $\mathfrak{A}(\mathcal{K})$ is the algebra generated by the symbols $b_I(f, g)$ with the relations above and the additional relation

$$b_I(f, g) = b_J(f, g) \ , \ I \subset J \ , \ I, J \in \mathcal{K} \ .$$

Let now $I_1, I_2 \in \mathcal{K}$ be two intervals with disjoint closures. Then there are 2 intervals $J_\pm \in \mathcal{K}$ with the properties

$$I_1 \cup I_2 \subset J_\pm \ , \ J_+ \cup J_- = \mathrm{S}^1 \ .$$

Let $f \in L^2(I_1, \mathbb{R})$, $g \in L^2(I_2, \mathbb{R})$ with $\|f\| = \|g\| = 1$ and consider the element

$$Y = b_{J_+}(f, g)b_{J_-}(g, f) \ .$$

Proposition 2.1. (i) Y does neither depend on the choice of f and g nor on the choice of the intervals $I_{1,2}, J_\pm$ within the above restrictions.

(ii) $Y^2 = 1$.
(iii) $Y \in Z(\mathfrak{A}(\mathcal{K}))$.

PROOF. (i) Let $g' \in L^2(I_2, \mathbb{R})$ with $\|g'\| = 1$. Then

$$b_{J_+}(f, g') = b_{J_+}(f, g) b_{I_2}(g, g')$$

and

$$b_{J_-}(g', f) = b_{I_2}(g', g) b_{J_-}(g, f) \ .$$

Since

$$b_{I_2}(g, g') b_{I_2}(g', g) = b_{I_2}(g, g) = 1$$

we find

$$b_{J_+}(f, g') b_{J_-}(g', g) = Y \ ,$$

hence Y is independent of the choice of g. Using

$$b_{J_+}(f,g)+b_{J_-}(g,f) = \frac{1}{2} b_{J_+}(f+g, f+g) - \frac{1}{2} b_{J_+}(f-g, f-g) = (f,g)+(g.f) = 0$$

we conclude that the rôle of f and g may be interchanged. We may now deform the intervals continuously by choosing f and g appropriately. The interchange of f and g amounts to an exchange of J_+ with J_-. Together this proves the claimed independence of Y.

(ii) In particular we have

$$Y = b_{J_-}(f, g) b_{J_+}(g, f) \ ,$$

hence

$$Y^2 = b_{J_+}(f, g) b_{J_-}(g, f) b_{J_-}(f, g) b_{J_+}(g, f) = 1 \ .$$

(iii) Let $I \in \mathcal{K}$ arbitrary, and choose I_1, I_2 such that $\overline{I_1} \cap \overline{I} = \emptyset = \overline{I_2} \cap \overline{I}$. We may choose $J_\pm$ such that $I \subset J_\pm$. Then from the canonical anticommutation relations we conclude that $b_{J_\pm}(f, g)$, $f \in L^2(I_1, \mathbb{R})$, $g \in L^2(I_2, \mathbb{R})$ commutes with all elements of $\mathfrak{A}(I)$. Since $\mathfrak{A}(\mathcal{K})$ is generated by the local algebras $\mathfrak{A}(I)$, Y is in the center of the algebra.

■

We will later see, that the two eigenvalues of Y correspond to the Ramond (+1) and the Neveu Schwartz (-1) sector of the theory. These two possibilities correspond to the two real line bundles over S^1, the trivial one and the Möbius bundle. They are realized by Majorana Fermions with periodic and antiperiodic boundary conditions, respectively.

3. Haag-Kastler axioms

We start from a system of algebras $\mathfrak{A}(\mathcal{O})$, labeled by a family $\mathcal{K}$ of regions of Minkowski space and satisfying the condition of isotony. In order to represent the algebras of observables of a relativistic quantum field theory, the system is supposed to fulfill a number of axioms which go back to Haag and Kastler.

Locality, in the sense of independence of algebras associated to spacelike separated regions, mainly replaced by the a priori weaker condition of spacelike commutativity,

$$\mathcal{O}_1 \text{ spacelike separated from } \mathcal{O}_2 \Longrightarrow [A, B] = 0 \forall A \in \mathfrak{A}(\mathcal{O}_1), B \in \mathfrak{A}(\mathcal{O}_2) \ ,$$

Covariance, as the existence of a family of isomorphisms $\alpha_L^{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(L\mathcal{O})$ with regions $\mathcal{O}$ and Poincaré transformations L such that for $\mathcal{O}_1 \subset \mathcal{O}_2$ the restriction of $\alpha_L^{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\alpha_L^{\mathcal{O}_1}$ and such that

$$\alpha_{L'}^{L\mathcal{O}} \circ \alpha_L^{\mathcal{O}} = \alpha_{L'L}^{\mathcal{O}} \ ,$$

and the **Time slice axiom**, which says that the algebra of a neighbourhood of a Cauchy surface of a given region coincides with the algebra of the full region, thus expressing the consequence of the existence of a hyperbolic equation of motion.

In addition to these axioms one wants to add a stability condition. This condition may be formulated as the existence of a translation covariant representation, i.e. a representation π of $\mathfrak{A}(\mathcal{K})$ on some Hilbert space $\mathfrak{H}$ together with a strongly continuous unitary representation U of the translation group which implements the translation automorphisms,

$$U(a)\pi(A)U(a)^{-1} = \pi(\alpha_a(A) \ , \ a \in \mathbb{R}^4, A \in \mathfrak{A}(\mathcal{K}) \ ,$$

such that the joint spectrum of the generators of U is contained in the closed forward lightcone,

$$\sigma(P) \subset \overline{V_+} \ , \ e^{iaP} = U(a) \ , \ aP = a^\mu P_\mu \ .$$

These axioms are formulated for a theory on Minkowski space. The condition on locality as well as the time slice axiom can easily be generalized to more general spacetimes. A meaningful generalization for the covariance condition has only recently been found, and the condition of stability was replaced by the so-called microlocal spectrum condition.

Surprisingly, a system of algebras satisfying the Haag-Kastler axioms already fixes the theory; in particular it suffices for a determination of scattering cross sections. The argument which will later be made more prezise may be sketched as follows. Let ω be a vacuum state, i.e. a state which is invariant under Poincaré transformations, and whose GNS representation satisfies the spectrum condition. A detector may then be identified with a positive observable A with vanishing vacuum expectation value. In addition, A should be "almost local" in the sense

that it possesses good approximations by local observables (to be specified). Actually, the spectrum condition forbids a vanishing vacuum expectation value for positive local observables by the Reeh-Schlieder theorem.

Let now ω be some state. The asymptotic particle content may be read of from the coincidence arrangement

$$\omega(\alpha_{t,t\mathbf{v}_1}(A_1)\cdots\alpha_{t,\mathbf{v}_n}(A_n))$$

for $t \to \pm\infty$ where the velocities $\mathbf{v}_i$ are pairwise different. Typically, these expectation values factorize for large $|t|$, and the expectation values of single operators $\alpha_{t,t\mathbf{v}}(A)$ behave as the position densities of freely moving single particles moving with velocity $\mathbf{v}$, hence decay with a power t^{-3}. Thus multiplying the expectation values of coincidence arrangements by $|t|^{3n}$ we get a probability distribution of an arrangement of n particles with prescribed velocities. Comparing these results for $t \to \infty$ with those for $t \to -\infty$ we can compute the scattering cross sections.

4. Positive energy representations; Reeh-Schlieder Theorem

Among the huge number of representations of the algebra of observables there is an important subclass, namely the so-called positive-energy representations.

DEFINITION 2.1. Let $\mathfrak{A}$ be a C*-algebra and (α_t) a 1-parameter group of automorphisms. A positive energy representation (π, U) of the pair $(\mathfrak{A}, \alpha)$ is a representation π of $\mathfrak{A}$ in some Hilbert space $\mathfrak{H}_\pi$ and a strongly continuous unitary representation U of $\mathbb{R}$ (as an additive group) in $\mathfrak{H}_\pi$ such that

(i) U implements α, i.e.

$$U(t)\pi(A)U(t)^{-1} = \pi(\alpha_t(A)) \ .$$

(ii) The generator of U,

$$K := \frac{1}{i}\frac{d}{dt}|_{t=0}U(t)$$

is a positive (in general unbounded) selfadjoint operator.

We would like to identify the operator K with the observable "energy". But a priori it is not clear whether K can be approximated by observables in the algebra $\mathfrak{A}$. Fortunately, we have the following theorem:

THEOREM 2.1. Let (π, U) be a positive energy representation of $(\mathfrak{A}, \alpha)$. Then there exists a strongly continuous 1-parameter group of unitaries $V(t) \in \pi(\mathfrak{A})''$, such that (π, V) is a positive energy representation.

This theorem justifies the name positive energy representation, since the spectral projections of the generator H of V are contained in the weak closure of the algebra of observables. H is not unique; one may add to H an arbitrary positive operator from the center of $\pi(\mathfrak{A})''$.

CHAPTER 3

Particle aspects of local quantum physics

1. The concept of a particle

One of the main motivations for quantum field theory is the desire to understand the dynamics of relativistic particles. Contrary to the nonrelativistic situation where particles are a priori given and can neither be destroyed nor generated, in relativistic particle physics particle production and annihilation are generic phenomena. One may describe this situation in terms of the Fock space where the Hilbert space of states is the direct sum of all finite number particle spaces, and where the operators can be composed of annihilation and creation operators. Actually, for the free field such a representation can be given, and for the interacting case one may hope to find a similar situation which can be treated within perturbation theory. The question however is whether the whole particle structure must be imposed as a condition on physically relevant field theories or whether field theories are generically theories of particles.

In the axioms of local quantum physics, no condition on the particle structure was imposed. In this general framework, one may ask the following questions:

- What is a particle? Under which conditions does a particle exist within a given model?
- Provided single particle states exist, do there then also states exist which can be interpreted as multiparticle states? (Because of the possibility of annihilation and creation the notion of a multiparticle state can be only an asymptotic concept for early or late times.)
- Are all states in positive energy representations interpretable as multiparticle states?

According to Wigner, the state space of an elementary particle is the representation space of an irreducible, strongly continuous positive energy representation of the Poincaré group. Positive energy representation means that the joint spectrum of the generators of the translation subgroup is contained in the closure of the forward lightcone. These representations have been classified by Wigner; besides the trivial representation which may be interpreted as the vacuum one finds the following types: representations with positive mass $m > 0$ and, in the restframe of the particle, an irreducible representation s of the

spin group, $s \in \frac{1}{2}\mathbb{N}_0$, representations with mass zero and an irreducible representation of the isotropy group of some lightlike vector where the latter occur in two types: since the isotropy group of a lightlike vector is the euclidean group, one obtains the so-called helicity representations where the translations of the euclidean group are trivially represented and the representation of the rotation subgroup is labeled by the helicity $h \in \frac{1}{2}\mathbb{Z}$, and the representations where the joint spectrum of the generators of translations is a circle with radius $\rho > 0$. The latter representations seem not to occur in nature. This is in agreement with the fact that no quantum field theory is known which contains such particle states.

In the following we assume that particle states are eigenstates of the mass operator. The behaviour under Lorentz transformation does not play an important rôle in the analysis. We assume in particular that the mass of the particle is an isolated point in the spectrum of the mass operator.

Given a positive energy representation of the algebra of observables with a unique and cyclic vacuum vector Ω and a subspace $\mathfrak{H}_1$ corresponding to the isolated eigenvalue $m > 0$ of the mass operator, one may ask whether there exist local observables which create a single particle state out from the vacuum,

$$A\Omega \in \mathfrak{H}_1, \ A \in \mathfrak{A}(\mathcal{O}) \ .$$

This happens in free field theory where the free field itself has this property. Unfortunately, this is the only case; one can show that particles which can be created by local observables cannot interact (Jost-Schroer-Theorem). But according to the Reeh-Schlieder Theorem, every state can be arbirarily well approximated by vectors of the form $A\Omega$ with a local A. We now show that there are so-called almost local observables which indeed can create single particle states out of the vacuum.

Definition 3.1. An operator $A \in \mathfrak{A}$ is called almost local if there exists a sequence $A_R \in \mathfrak{A}(\mathcal{O}_R)$, $\mathcal{O}_R = \{x \in \mathbb{R}^4, |x^0| + |\mathbf{x}| < R\}$ with

$$\lim_{R\to\infty} R^n \|A - A_R\| = 0 \ \forall n \in \mathbb{N}_0 \ .$$

Namely, let $A \in \mathfrak{A}(\mathcal{O})$ with $P_1 A\Omega \neq 0$, where P_1 is the projection onto the single particle space $\mathfrak{H}_1$. Choose a function $f \in \mathcal{D}(\mathbb{R}^4)$ with the property

$$\operatorname{supp} f \cap \sigma(P) \subset \{p \in \mathbb{R}^4, p^2 = m^2, p_0 > 0\} \ .$$

such that $f(P)A\Omega \neq 0$. Then $f(P)A\Omega \in \mathfrak{H}_1$. Using the translation invariance of Ω we get

$$f(P)A\Omega = \int d^4x \hat{f}(x)\alpha_x(A)\Omega$$

The operator $A(\hat{f}) = \int d^4x \hat{f}(x)\alpha_x(A)$ is an almost local operator. We first convince ourselves that the integral is well defined in the weak operator topology, since the matrix elements of $\alpha_x(A)$ are continuous and bounded functions of x due to the assumed continuity properties of the translations, and $\hat{f}$ is an element of the Schwartz space $\mathcal{S}(\mathbb{R}^4)$. We may then look for a sequence $g_R \in \mathcal{D}(\mathcal{O}_R)$ such that g_R converges to $\hat{f}$ in the sense of the topology of $\mathcal{S}(\mathbb{R}^4)$ and such that g_R coincides with $\hat{f}$ within $\mathcal{O}_{\frac{R}{2}}$. We estimate

$$\|A(\hat{f}) - A(g_R)\| \leq \|A\| \int d^4x |\hat{f}(x) - g_R(x)|$$

$$\leq \|A\| \frac{2^n}{R^n} \int d^4x (|x^0| + |\mathbf{x}|)^n |\hat{f}(x) - g_R(x)|$$

which proves the assertion. Moreover, the function $x \mapsto \alpha_x(A(\hat{f}))$ is infinitely differentiable in the norm sense, and all derivatives are almost local.

2. Haag-Ruelle construction of multiparticle states

We start from a positive energy representation with the following properties: The energy momentum spectrum contains an isolated point zero (the vacuum) and an isolated mass shell $\{p \in \mathbb{R}^4, p^2 = m^2, p_0 > 0\}$ with $m > 0$. Moreover, the vacuum vector Ω is unique (up to a phase) and cyclic. From the previous section we know that under these conditions there exist almost local and smooth (with respect to translations) observables which create single particle vectors from the vacuum. Let $\Psi = A\Omega$ be a single particle vector with compact support in momentum space. We now exploit the fact that the same particle state can be generated also at earlier or later times by using the known time evolution of a solution of the Klein-Gordon equation. Namely, let

$$f(t, \mathbf{x}) = (2\pi)^{-3} \int d^3\mathbf{p}\, e^{-i(\omega(\mathbf{p})t - \mathbf{p}\cdot\mathbf{x})} h(\mathbf{p}) \ ,$$

with a test function $h \in \mathcal{D}(\mathbb{R}^3)$ with $h(\mathbf{p}) = 1$ on the momentum support of Ψ. Let

$$A_f(t) = \int d^3\mathbf{x}\, f(t, \mathbf{x}) \alpha_{(t,\mathbf{x})}(A) \ .$$

Then, since $\alpha_x(A)\Omega = e^{iPx} A\Omega$ because of the translation invariance of the vacuum, it follows from the functional calculus for selfadjoint operators that

$$A_f(t)\Psi = \int d^3\mathbf{x}\, f(t, \mathbf{x}) e^{itP_0 - \mathbf{P}\cdot\mathbf{x}} \Psi = h(\mathbf{P}) e^{-i(\omega(\mathbf{P}) - P_0)t} \Psi = \Psi \ .$$

The localization properties of the almost local operators $A_f(t)$ are determined by the corresponding properties of the wave function f. As

one might expect f is concentrated in the region which can be reached from the origin by velocities

$$\mathbf{v} \in V(f) = \{\mathbf{v} = \operatorname{grad}\omega(\mathbf{p}) \text{ for some } \mathbf{p} \in \operatorname{supp} h\} \ .$$

The argument is based on the principle of stationary phase by which oszillatory integrals can be approximately evaluated. More precizely, we have the following theorem:

THEOREM 3.1. Let $\varepsilon > 0$ and $d(\mathbf{v}) = \operatorname{dist}(\mathbf{v}, V(f))$. Then there are constants $c, c', c_n > 0$, $n \in \mathbb{N}_0$ such that

$$|f(t, t\mathbf{v})| < c_n |t|^{-n} d(\mathbf{v})^{-n} \ , \ t \neq 0, d(\mathbf{v}) > \varepsilon$$

$$\int d^3\mathbf{x} \, |f(t, \mathbf{x})| < c + c'|t|^3)$$

Let now $A_1, \ldots, A_n$ be almost local and smooth single particle generators such that the velocity supports of the single particle states $\Psi_i = A_i\Omega$ are compact and pairwise disjoint. Then we choose as described above positive frequency solutions f_i of the Klein Gordon equation such that also the velocity supports $V(f_i)$ are compact and pairwise disjoint and such that

$$A_{i,f_i}(t)\Omega = \Psi_i \ , \ i = 1, \ldots, n \ .$$

Set $A_i(t) = A_{i,f_i}(t)$. These operators have the following properties:

LEMMA 2.1.

$$\|A_i(t)\| \leq c + c'|t|^3$$

$$\|[A_i(t), A_j(t)]\| < c_n |t|^{-n} \ , \ i \neq j \ .$$

We now define the Haag-Ruelle approximants of scattering states by

$$\Psi(t) := A_1(t) \cdots A_n(t)\Omega \ .$$

THEOREM 3.2. (i) $\Psi(t)$ converges for $t \to \infty$ and for $t \to -\infty$.

(ii) The limit depends only on the single particle states $\Psi_i, i = 1, \ldots, n$.

The limit states are therefore multilinear functionals on the single particle space with values in the full Hilbert space. We therefore introduce the notation

$$\lim_{t \to \pm\infty} \Psi(t) = (\Psi_1 \times \cdots \times \Psi_n)_{\text{out,in}} \ .$$

THEOREM 3.3. (i) The scattering states $(\Psi_1 \times \cdots \times \Psi_n)_{\text{out,in}}$ are symmetric under permutations of the single particle states.

(ii) The scalar product of scattering states is determined by the scalar product of the occuring single particle states,

$$\big((\Psi_1 \times \cdots \times \Psi_n)_{\text{out,in}}, (\Phi_1 \times \cdots \times \Phi_m)_{\text{out,in}}\big) = \delta_{nm} \sum_{\sigma \in P_n} \prod_{i=1}^{n} \big(\Psi_i, \Phi_{\sigma(i)}\big) \ .$$

We thus obtain two isometric embeddings $W_{\text{out,in}}$ of the bosonic Fock space built over the single particle space into the Hilbert space of our theory. Originally, we had to restrict ourselves to the dense subspace of states with nonoverlapping velocities, but in view of the fact that the operators $W_{\text{out,in}}$ are continuous we can extend them to the full Fock space. In particular, we obtain also scattering states for overlapping velocities. The S-matrix elements are now defined by

$$\big(\Phi, S\Psi\big) = \big(W_{\text{out}}\Phi, W_{\text{in}}\Psi\big) \ .$$

Here Φ, Ψ are vectors in Fock space.

In general, one has to expect that the operators $W_{\text{out,in}}$ are not unitary and that their ranges might be different. In these cases the S-matrix as defined above is not unitary. A possible reason could be the existence of particles which are charged, in the sense that the single particle space is not contained in the cyclic space generated from the vacuum. One therefore has to extend the analysis to this more general situation. This leads to the study of superselection sectors.

CHAPTER 4

Theory of superselection sectors

1. Localized sectors and their products

In addition to the vacuum representation π_0 of the algebra of observables there might be other positive energy representations which describe charged particles. We already saw that single particle states in the vacuum sector are necessarily bosons because of the spacelike commutativity of observables. Formally, one could obtain fermionic statistics by admitting also spacelike anticommutativity, but while commutativity has a physical meaning as the impossibility of influence over spacelike distances, the requirement of anticommutativity would be completely ad hoc.

Let π be a representation of $\mathfrak{A}$ which has the property that its states coincide with those of the vacuum representation if one restricts oneself to the observables localized in the spacelike complement of an arbitrary double cone $\mathcal{O}$ (DHR condition). The idea behind this condition is that the charge might be localized within $\mathcal{O}$ and is thus invisible for observations in the spacelike complement ("particle behind the moon argument"). While plausible at the first sight, one should be aware of the fact that the argument cannot be applied to the electric charge which can, due to Gauss law, be detected from the electric flux within the spacelike complement of $\mathcal{O}$. One can, however, in the case of massive particles with an isolated mass shell, prove that a slightly weaker condition is satisfied, namely that in the complement of a spacelike cone the states of the charged representation cannot be distinguished from those of the vacuum representation. One may imagine that the reason is that a possible electric flux can be concentrated within the spacelike cone.

In the following we will use the DHR condition. In case of a localization in a spacelike cone esssentially the same analysis can be performed.

The DHR condition says that the representation π becomes quasiequivalent to π_0 after restriction to $\mathfrak{A}(\mathcal{O}')$ for any double cone $\mathcal{O}$. This means that there are sufficiently many partial intertwiners $F \in \mathcal{F}_{\pi\pi_0}$,

$$\mathcal{F}_{\pi\pi_0} = \{F : \mathcal{H}_{\pi_0} \to \mathcal{H}_\pi,\ F\pi_0(A) = \pi(A)F\ \forall\ A \in \mathfrak{A}(\mathcal{O}')\}$$

in the sense that $\mathcal{F}_{\pi\pi_0}\mathcal{H}_{\pi_0}$ is dense in $\mathcal{H}_\pi$.

Let $\mathcal{F}_{\pi\pi_0} = \bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{F}_{\pi\pi_0}$ and $\mathfrak{A}_0 = \bigcup_{\mathcal{O}\in\mathcal{K}} \mathfrak{A}(\mathcal{O})$. Then $\mathcal{F}_{\pi\pi_0}$ becomes a bimodule over $\mathfrak{A}_0$ by setting

$$B \cdot F \cdot A := \pi(B) F \pi_0(A) \ , \ A, B \in \mathfrak{A}_0 \ .$$

Let us consider $\mathcal{F}_{\pi_0\pi_0}$. By definition

$$\mathcal{F}_{\pi\pi_0}(\mathcal{O}) = \pi_0(\mathfrak{A}(\mathcal{O}'))' \ .$$

By locality, $\pi_0(\mathfrak{A}(\mathcal{O})) \subset \pi_0(\mathfrak{A}(\mathcal{O}'))'$. We require now that the two algebras are even equal ("Haag duality"). This condition expresses the maximality of the local algebras in the sense that they cannot be enlarged without violating locality. It is known that Haag duality holds in typical cases. An exception are theories where a global symmetry is spontaneously broken.

Under the condition of Haag duality, one finds the equality $\mathcal{F}_{\pi_0\pi_0} = \pi_0(\mathfrak{A}_0)$. We can use Haag duality to equip $\mathcal{F}_{\pi\pi_0}$ with an $\mathfrak{A}_0$ valued hermitean product,

$$\langle F, G \rangle = \pi_0^{-1}(F^* G)$$

where $F^*G \in \mathcal{F}_{\pi_0\pi_0} = \pi_0(\mathfrak{A}_0)$ and where we assumed that π_0 is injective. The latter assumption is harmless since by the DHR condition π and π_0 have the same kernel.

The representation spaces $\mathcal{H}_\pi$ are by definition $\mathfrak{A}_0$ left modules,

$$A \cdot \Phi := \pi(A)\Phi \ , \ A \in \mathfrak{A}_0, \Phi \in \mathcal{H}_\pi \ .$$

We may obtain this left module as the tensor product of the bimodule $\mathcal{F}_{\pi\pi_0}$ with the vacuum Hilbert space $\mathcal{H}_{\pi_0}$.