II. Institut für Theoretische Physik
Universität Hamburg

Cosmological Horizons and Reconstruction of Quantum Field Theory

Claudio Dappiaggi

Hamburg - 07/06/2008

based upon:

- C. D., V. Moretti and N. Pinamonti,
  "Cosmological horizons and reconstruction of quantum field theory"
  arXiv:0712.1770 [gr-qc]

Motivations

- General one: Cosmology is nowadays the main viable source for experimental data related to QFT on curved backgrounds, but... many models, a lot of folk results, few mathematically sound statements.
- Particular one: It was recently shown that it is possible to encode the information of a bulk field theory in terms of a suitable counterpart living on the boundary; this holds both in AdS[a] and in asymptotically flat spacetimes[b].

An idea:

1. What about cosmological spacetimes considering the cosmological horizon as a boundary? Is it feasible?
2. Does it exists also in this scenario a distinguished algebraic state as in the asymp. flat case?

---

[a] K. H. Rehren, Annales Henri Poincare **1** (2000) 607,
M. Dütsch and K. H. Rehren, Annales Henri Poincare **4** (2003) 613.

[b] C. D., V. Moretti and N. Pinamonti: Rev. Math. Phys. **18** (2006), 346

Outline of the talk

1. Looking at the Geometry of the Problem: The distinguished role of the cosmological horizon
2. Looking at the Field Theoretical Side of the Problem: a real scalar QFT on FRW spacetimes and the counterpart on the horizon
3. What Holography teaches us: how to construct a bulk-to-boundary correspondence and the notion of preferred state

Recap. of previous episodes

What is an asymptotically flat spacetime? Why is interesting?

A $4D$ manifold M with a metric g solving Einstein vacuum equations is called asymptotically flat with past timelike infinity at null infinity $\Im^-$, if it exists a second manifold $(\widehat{M}, \widehat{g})$, an embedding $\lambda : M \to \widehat{M}$, a preferred point $i^- \in \widehat{M}$ and a conformal factor $\Omega \geq 0$ such that

1. $\Omega^2 g_{\mu\nu} = \lambda^*(\widehat{g}_{\mu\nu})$ in M,
2. $\lambda(M) = J^+(i^-) \setminus \partial J^+(i^-)$ and $\partial(\lambda(M)) = \Im^- \cup i^-$,
3. $\Omega \in C^\infty(\widehat{M})$ and $\Omega = 0$ on $\Im^- \cup i^-$,
4. $d\Omega \neq 0$ on $\Im^- \cup i^-$ but $\widehat{\nabla}_\mu \widehat{\nabla}_\nu \Omega = -2\widehat{g}_{\mu\nu}$ on i^-,
5. other technical requirements.

N.B. $\Im^-$ plays the role of a preferred codimension one submanifold of a bulk field theory. For a real massless scalar field conformally coupled to scalar curvature, this entails the selection of a preferred bulk Hadamard state etc. etc. etc...

Geometrical Setup

First hypothesis: Cosmological Principle $\Longrightarrow$

$$g_{FRW} = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 dS^2(\theta,\varphi)\right], \quad M \sim I \times X_3$$

where $k = 0, 1, -1$ and $a(t) \in C^\infty(I, R^+)$, being $I \subset \mathbb{R}$.

Important properties:

- Consider a co-moving observer as the integral line $\gamma(t)$ of ∂_t. If $M \setminus J^-(\gamma) \neq \emptyset$, then causal signal departing from each $x \in M \setminus J^-(\gamma)$ never reach $\gamma(t)$. Then we call $\partial J^-(\gamma)$ the (future) **cosmological horizon**
- if one introduces the conformal time $d\tau = \frac{dt}{a(t)}$ and rescales the metric as

$$g_{FRW} = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 dS^2(\theta,\varphi)\right],$$

  then τ ranges in $(\alpha, \beta) \subset \mathbb{R}$. Sufficient condition for the existence of an horizon is $\alpha > -\infty$ and/or $\beta < \infty$.

Second hypothesis: Let us consider a FRW spacetime with $k = 0$ and $M \sim I \times \mathbb{R}^3$.

Third hypothesis: $a(\tau) = \frac{\gamma}{\tau} + O(\frac{1}{\tau^2})$ with $I = (-\infty, 0)$ and $\gamma < 0$ or $I = (0, \infty)$ and $\gamma > 0$.

- If we perform the coordinate change $U = \tan^{-1}(\tau + r)$ and $V = \tan^{-1}(\tau - r)$

$$g_{FRW} = \frac{a^2(U,V)}{\cos^2 U \cos^2 V}\left[-dU dV + \frac{\sin^2(U-V)}{4} dS^2(\theta,\varphi)\right].$$

Theorem: Under the previous assumptions the spacetime (M, g_{FRW}) can be extended to a larger spacetime $(\widehat{M}, \widehat{g})$ which is a conformal completion of the asymptotically flat spacetime at past (or future) null infinity $(M, a^{-2} g_{FRW})$, *i.e.*, "a" plays the role of the conformal factor.

The manifold $M \cup \Im^{\pm}$ enjoys:

1. the vector field ∂_τ is a conformal Killing vector for $\widehat{g}$ in M,
2. the vector ∂_τ becomes tangent to $\Im^{\pm}$ approaching it and coincides with $-\gamma \widehat{\nabla}^b a$,
3. the metric restricted on $\Im^{\pm}$ takes a Bondi-like form $\widehat{g}|_{\Im^{\pm}} = \gamma^2 \left[-2dl da + dS^2(\theta,\varphi)\right]$

Cosmological horizon: general notion

A globally hyperbolic spacetime (M, g) equipped with $\Omega \in C^\infty(M, \mathbb{R}^+)$ and with a future-oriented timelike vector X on M is called an **expanding Universe with cosmological past horizon** if:

1. (M, g) can be isometrically embedded as the interior of a submanifold with boundary $(\widehat{M}, \widehat{g})$ such that $\Im^- = \partial M$ and $\Im^- \cap J^+(M, \widehat{M}) = \emptyset$,
2. Ω can be made smooth on $\widehat{M}$ and $\Omega|_{\Im^-} = 0$, but $d\Omega|_{\Im^-} \neq 0$,
3. X is a conformal Killing field on $\widehat{g}$ in a neighbourhood of $\Im^-$ in M with
$$\mathcal{L}_X(\widehat{g}) = -2X(\ln \Omega)\widehat{g},$$
4. $\Im^- \sim \mathbb{R} \times S^2$ and the metric $\widehat{g}|_{\Im^-}$ takes in a suitable frame the form
$$\widehat{g} = \gamma^2 \left[-2dl d\Omega + dS^2(\theta, \varphi) \right] .$$

N.B.

- $\Im^-$ is a null 3−submanifold and the curves $l \mapsto (l, \theta, \varphi)$ are null $\widehat{g}$−geodesics.
- $(\widehat{M}, \widehat{g})$ is the conformal completion of the asymp. flat spacetime at past infinity $(M, \Omega^{-2} g)$ and $\widehat{g}|_M \equiv g$.

On the role of X

N.B.: An Expanding universe with cosmological horizon is characterised by $(M, g, \Omega, X, \gamma)$.

Question: What tells us X?

1. X is a Killing vector for the metric $\Omega^{-2}g$ in a neighbourhood of $\Im^-$ in M.
2. X extends smoothly to a unique smooth vector field $\widetilde{X}$ on $\Im^-$ which can vanish at most on a closed subset of $\Im^-$ with empty interior.
3. $\widetilde{X}$ has the form $f(\theta, \varphi)\partial_l$ when we represent $\Im^-$ as $\mathbb{R} \times S^2$ and f is smooth and nonnegative.

Consequence: In a FRW universe $f = 1$. Therefore a non constant f is a measure of the failure of (M, g) to be isotropic!

Interplay with bulk isometries

Question: How are isometries of g and of $\widehat{g}$ encoded on the horizon?

Consider an expanding Universe with cosmological horizon and Y a Killing field of (M, g), then

a) Y extends to a smooth vector field of $\widehat{Y}$ on $\widehat{M}$,

b) $\mathcal{L}_{\widehat{Y}}\widehat{g} = 0$ on $M \cup \Im^-$,

c) $\widetilde{Y} = \widehat{Y}|_{\Im^-}$ is uniquely determined by Y and it is tangent to $\Im^-$ iff $\lim_{\Im^-} g(Y, X) = 0$

N.B. Killing vectors of (M, g) are represented on $\Im^-$ faithfully.

Definition: A Killing vector field Y of (M, g) is said to preserve $\Im^-$ iff $g(Y, X) \to 0$ approaching $\Im^-$. A similar statement holds for the local 1-parameter group of isometries generated out of Y.

The group $SG_{\Im^-}$ of isometries of the horizon

What is the group of all isometries preserving the horizon structure?

Definition: The horizon symmetry group $SG_{\Im^-}$ is the set of all diffeomorphisms of $\mathbb{R} \times S^2$ such that, given a Bondi-like frame $(l, z, \bar{z})$

$$z \longrightarrow z' = R(z) \doteq \frac{az+b}{cz+d}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(3)$$

$$l \longrightarrow l' \doteq e^{f(z,\bar{z})} l + g(z, \bar{z}),$$

where $g(z, \bar{z})$ and $f(z, \bar{z})$ lie in $C^\infty(S^2)$.

The composition law between two elements of $SG_{\Im^-}$ is

$$(R, f, g)(R', f', g') = (RR', f' + f \circ R, e^{f \circ R'} g' + g \circ R').$$

The horizon symmetry group has the structure of an **iterated semidirect product**:

$$SG_{\Im^-} = SO(3) \ltimes (C^\infty(S^2) \rtimes C^\infty(S^2)).$$

$SG_{\Im^-}$ "Trivia"

The group of symmetries of the cosmological horizon enjoys the following properties:

- it is an **infinite dimensional nuclear Lie group** as the BMS in asymptotically flat spacetimes
- there is no known theory of representation!!! Mackey's induction techniques cannot be blindly applied!
- each Killing vector Y of (M, g) can be restricted on $\Im^-$ to $\widetilde{Y}$, a generator of the algebra of $SG_{\Im^-}$. Therefore $exp(t\widetilde{Y})$ with $t \in \mathbb{R}$ is a one-parameter subgroup of $SG_{\Im^-}$.

Goal: Construct a $SG_{\Im^-}$ invariant (real scalar) field theory on $\Im^-$!

Field Theory on the Horizon

Prequel: The bulk

N.B. Since (M, g) is globally hyperbolic, Cauchy problems are meaningful.

Proposition: Consider a real scalar field $\phi : M \longrightarrow \mathbb{R}$ on a cosmological spacetime with horizon. If ϕ solves $\left(\Box + \xi R + m^2\right)\phi = 0$ with $\xi \in \mathbb{R}$, $m^2 > 0$ and with compactly supported Cauchy data, then

• $\phi \in C^\infty(M)$

• The set of solutions $S(M)$ of our equation is a symplectic space if endowed with the Cauchy-independent nondegenerate symplectic form:

$$\sigma(\phi_1, \phi_2) \doteq \int\limits_S (\phi_1 \nabla_N \phi_2 - \phi_2 \nabla_N \phi_1)\, d\mu_g^{(S)}$$

• A Weyl C^*-algebra $\mathcal{W}(M)$ can be associated to $(S(M), \sigma)$. This is, up to $*$-isomorphisms, unique and its non vanishing generators $W_M(\phi)$ satisfy:

$$W_M(-\phi) = W_M(\phi)^*, \quad W_M(\phi) W_M(\phi') = e^{\frac{i}{2}\sigma(\phi,\phi')} W_M(\phi + \phi'),$$

Part I: The boundary

What is the space of wavefunctions on the horizon?

Def: The space of real wavefunctions is

$$\mathcal{S}(\Im^-) = \left\{ \psi : \Im^- \to \mathbb{R} \mid \psi \text{ and } \partial_l \psi \in L^2\left(\mathbb{R} \times \mathbb{S}^2, dl dS^2(z, \bar{z})\right)\right\}.$$

N.B.: $\mathcal{S}(\Im^-)$ is a symplectic space if endowed with $\sigma' : \mathcal{S}(\Im^-) \times \mathcal{S}(\Im^-) \to \mathbb{R}$ such that

$$\sigma'(\psi_1, \psi_2) = \int\limits_{\mathbb{R}\times\mathbb{S}^2} \left(\psi_1 \frac{\partial \psi_2}{\partial l} - \psi_2 \frac{\partial \psi_1}{\partial l}\right) dl dS^2(z, \bar{z})),$$

on which the **left action** of $SG_{\Im^-}$ acts as a symplectomorphism, *i.e.*,

- $L(g)\psi(x) \doteq \psi(g^{-1}x) \in SG_{\Im^-}$ iff $\psi(x) \in \mathcal{S}(\Im^-)$ for all $x \in \Im^-$ and for all $g \in SG_{\Im^-}$,
- $\sigma'(L(g)\psi, L(g)\psi') = \sigma'(\psi, \psi')$, for all $g \in SG_{\Im^-}$ and for all $\psi, \psi' \in \mathcal{S}(\Im^-)$

Consequence: We can associate a Weyl C^*-algebra $\mathcal{W}(\Im^-)$ to $(\mathcal{S}(\Im^-), \sigma')$ as well as an $SG_{\Im^-}$-representation α_g:

$$\alpha_g\left(W(\psi)\right) \doteq W(L(g)\psi), \quad \forall W(\psi) \in \mathcal{W}(\Im^-),\ \forall g \in SG_{\Im^-}$$

Part II: The state

We can introduce a distinguished state $\lambda : \mathcal{W}(\Im^-) \to \mathbb{C}$ unambiguously defined as

$$\lambda\left(W(\psi)\right) = e^{-\frac{\mu(\psi,\psi)}{2}}, \quad \forall W(\psi) \in \mathcal{W}(\Im^-)$$

where $\forall \psi, \psi' \in \mathcal{S}(\Im^-)$

$$\mu(\psi, \psi') = \int\limits_{\mathbb{R}\times S^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\varphi)}\widehat{\psi}'(k,\theta,\varphi)dkdS^2(\theta,\varphi),$$

being $\psi(k), \psi'(k)$ the Fourier-Plancherel transform

$$\psi(k) = \int\limits_{\mathbb{R}} dl\, \frac{e^{ikl}}{\sqrt{2\pi}}\psi(l,\theta,\varphi).$$

The state λ enjoys the following (almost straightforward) properties:

- it is quasifree and pure,
- referring to its GNS triple $(\mathcal{H}, \Pi, \Upsilon)$ it is invariant under the left action of the $SG_{\Im^-}$ group.

Furthermore the state λ enjoys the following (much less straightforward) remarkable properties. Let us consider a timelike future directed vector field Y whose projection on the horizon is $\widetilde{Y}$, a generator of the algebra of $SG_{\Im^-}$. Then

- The unitary group $U_t^{\widetilde{Y}}$ which implements $\alpha_{\exp(t\widetilde{Y})}$ $(t \in \mathbb{R})$ leaving fixed the cyclic GNS vector is strongly continuous with nonnegative self-adjoint generator

$$H^{\widetilde{Y}} = -i \left. \frac{dU_t^{\widetilde{Y}}}{dt} \right|_{t=0},$$

- The restriction of $H^{\widetilde{Y}}$ to the 1-particle Hilbert space in the GNS representation of λ has no zero modes iff $\widetilde{Y}$ vanishes on a zero-measure subset of $\Im^-$,

- if $\widetilde{Y} = \partial_l$, then λ is the **unique** quasifree pure state on $\mathcal{W}(\Im^-)$ which is invariant under $\alpha_{\exp(t\partial_l)}$ $(t \in \mathbb{R})$ and the unitary group implementing such representation leaving fixed the cyclic GNS vector is strongly continuous with nonnegative self-adjoint generator,

- Each folium of states on $\mathcal{W}(\Im^-)$ contains at most one pure state which is invariant under $\alpha_{\exp(t\partial_l)}$.

Part III: Bulk to Boundary Interplay

Notice: each element $\phi \in S(M)$ can be extended to a unique smooth solution of the same equation on the whole $\widehat{M}$ and, hence, $\Gamma\phi \doteq \phi|_{\Im^-} \in C^\infty(\Im^-)$.

Hypothesis: Suppose that each element $\phi \in S(M)$

- projects/can be restricted to $\Im^-$ to an element $\Gamma\phi \in \mathcal{S}(\Im^-)$,
- the projection/restriction preserves symplectic forms, *i.e.*, for any $\phi_1, \phi_2 \in S(M)$:

$$\sigma(\phi_1, \phi_2) = \gamma^2 \sigma(\Gamma\phi_1, \Gamma\phi_2),$$

then it exists an isometric $*$-homomorphism $i : \mathcal{W}(M) \to \mathcal{W}(\Im^-)$ unambiguously determined by

$$i(W_M(\phi)) \doteq W(\Gamma\phi) \quad \forall \phi \in \mathcal{W}(M).$$

In other words we see the bulk algebra a sub $*$-algebra of the boundary counterpart.

What's next?!

The injection map between algebras allows to pull-back states!

Big Statement: The distinguished state λ in the boundary identifies a bulk state λ_M as

$$\lambda_M(a) = \lambda(i(a)). \quad \forall a \in \mathcal{W}(M).$$

Furthermore λ_M enjoys some interesting properties:

- it is invariant under the natural action of any bulk isometry Y on the algebra. The one-parameter U_t^Y group implementing such an action leaves fixed the cyclic vector in the GNS representation of λ_M,
- if Y is everywhere timelike and future-directed in M then the 1-parameter group U_t^Y has positive self-adjoint operator,
- the generator has no zero mode in the one-particle subspace if the projection of Y on the horizon vanishes on a zero-measure subset of $\Im^-$.

Epilogue: Testing the (hypotheses of the) correspondence

The last big question: when do the hypotheses hold true?

One can prove the following statements:

Test 1: Consider an expanding Universe with $a(\tau) = \frac{\gamma}{\tau}$ and a Klein-Gordon field with $m^2 + \xi R > 0$. Then, whenever every $m^2 + \xi R > \frac{5}{48} R$, then any $\phi \in S(M)$ can be projected to $\Gamma\phi \in \mathcal{S}(\Im^-)$ and the symplectic form is preserved.

N.B. In this case the state λ_M coincides with the Euclidean Bunch-Davies state.

Test 2: Suppose now that $a(\tau) = \frac{\gamma}{\tau} + O\left(\frac{1}{\tau^2}\right)$ with $\ddot{a}(\tau)$ such that $R = \frac{12}{\gamma^2} + O\left(\frac{1}{\tau}\right)$. Then, if $M = (-\infty, 0) \times \mathbb{R}^3$, $g = g_{FRW}$ and $X = \partial_\tau$, then:

- whenever $m^2 + \xi R > 2$, each $\phi \in S(M)$ extends smoothly to $\Gamma\phi \in \mathcal{S}(\Im^-)$ and the symplectic form is preserved!

The hypotheses hold true in the Friedmann-Robertson-Walker spacetimes we considered at the beginning of the talk.

Conclusions

- Can we test our hypotheses for scalar fields on the other backgrounds we considered?
- Can we prove that the bulk state is Hadamard?
- Can we prove explicitly that the correspondence holds for non scalar-field without claiming "it is a simple extension of the scalar case"? (dedicated to a good friend)
- Can we recast the construction for a scalar field interacting with a non constant potential $V(\phi)$? This could provide useful insights on cosmological theories[a].

[a] See also: C. D., Klaus Fredenhagen & Nicola Pinamonti: Phys. Rev. D. **77** (2008) 104015