Quantum states on inflationary cosmological models and their Hadamard property.

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Summary

- Cosmological Scenario /
  Geometry of the spacetimes under consideration
- Interplay of the field theory in the bulk and on the horizon.
- Pullback of some states.
- Their microlocal spectral properties.

Bibliography

- C. Dappiaggi, NP, V. Moretti in press on CMP
- C. Dappiaggi, NP, V. Moretti to appear

Models of the Universe

- The universe is described as a curved spacetime (M, g)
- In first approximation: M is $I \times S$
  - I is the interval of the *"cosmological time"*
  - S is a 3d manifold: the *"space"*, it has an high symmetry.
  - **homogeneous** and **isotropic**.
- The metric g is of Freedmann Robertson Walker type

$$g = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\mathbb{S}^2(\theta,\varphi)\right].$$

- recent observation seems to say that
  - $\kappa = 0$.
  - $a(t) = e^{Ht}$, H is the Hubble parameter (very small but not zero).

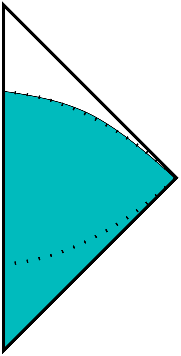
Going back in time: Inflationary scenario

- We assume a **phase of rapid expansion** at the beginning.
- Then just after the big bang:

$$a(t) = e^{H_0 t}\,, \qquad H_0 >> H$$

- More precisely: with $\kappa = 0$, conformally related with Minkowski.

$$\tau(t) = \int_{t_0}^{t} \frac{1}{a(t')} dt'$$



$$g_{FRW} = a^2(\tau)\left[-d\tau^2 + dr^2 + r^2 d\mathbb{S}^2(\theta,\varphi)\right].$$

It is with an interval $I' \subset \mathbb{R}$ and $\tau \to -\infty$

Form of the spacetime models we are considering

- If $a(t) = e^{Ht}$ we have de Sitter spacetime. (or $a(\tau) = -\frac{1}{H\tau}$).
- In order to have an inflationary scenario let's assume

$$a(\tau) = -\frac{1}{H\,\tau} + O\left(\tau^{-2}\right)\ , \qquad \frac{da(\tau)}{d\tau} = \frac{1}{H\,\tau^2} + O\left(\tau^{-3}\right)\ ,$$

$$\frac{d^2 a(\tau)}{d\tau^2} = -\frac{2}{H\,\tau^3} + O\left(\tau^{-4}\right).$$

- For $\tau \to -\infty$ the space time **"looks like"** de Sitter. (Positive cosmological constant), exponential acceleration in the proper time t.

Horizon

- **Cosmological horizon** $(\tau \to -\infty)$.

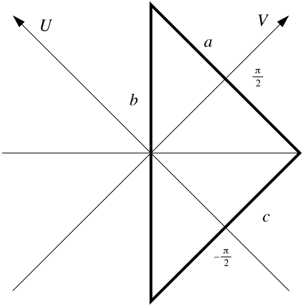
$$U = \tan^{-1}(\tau - r)\ , \qquad V = \tan^{-1}(\tau + r),$$

- Conformall null infinity $\Im^-$ correspond to the horizon *(region c in the figure)*
- Metric on the horizon is degenerate:

$$g|_{\Im^-} = H^{-2}\left(d\mathbb{S}^2(\theta, \varphi)\right)\ ,$$

- Conformal Killing vector ∂_τ tangent to $\Im^-$

$$\mathcal{L}_{\partial_\tau} g = -2\partial_\tau\left(\ln a\right) g,$$



Metric fluctuations

Remark

Homogeneity and Isotropy are over idealization.

- Fluctuations about those spacetimes needs to be taken into account.
- They should be responsible for the formation of the structures we see in the sky (galaxies).
- They should be responsible for the anisotropies in the CMB too.
- It is believed that they are of quantum origin.

A prototype of these fluctuations

After some linearization we end up with *[Bardeen, Mukhanov Feldman Brandenberger]*

$$P\Phi = 0, \qquad P = -\Box + \xi R + m^2$$

It looks like a free quantum field theory on a curved background!

- They are *"born"* on the quantum ground states and soon after this they *"become"* classical.

Problem 1:

What is a ground state for a QFT in a curved spacetime? How can it be chosen?

Problem 2:

Is it possible to assume that the fluctuations become classical?

Problem 1

Problem 1:

What is a ground state for a QFT in a curved spacetime? How can it be chosen?

- There is a preferred time direction, ∂_t.
- **But** the spacetime is not static, (∂_t is not timelike Killing)
- Conformal equivalence with a patch of Minkowksi spacetime.
- **But** the theory is not conformally invariant (If $\xi \neq 1/6$ and $m \neq 0$). We cannot directly take the Minkowski "vacuum".

- In the equation of motion

$$-\frac{\partial^2}{\partial \tau^2}\frac{\Phi}{a(\tau)} + \triangle \frac{\Phi}{a(\tau)} + V(\tau)\frac{\Phi}{a(\tau)} = 0,$$

the "potential" $V(\tau)$ vanishes for $\tau \to -\infty$.

Problem 2

Problem 2:

Is it possible to assume that the fluctuations become classical?

- *Minimal Requirement:* **the variance** of the perturbations needs to be bounded.
- This is guaranteed by the **Hadamard property**, or better by the microlocal spectral condition.
- Unfortunately is not so easy to verify this property in a general spacetime, (at least if the spacetime is not static...)

QFT in the spacetime

- Real solutions of

$$P\Phi = 0 \,, \qquad P = -\Box + \xi R + m^2 \,,$$

generated by compactly supported initial data on Cauchy surf.

- The symplectic structure $(\mathcal{S}(M), \sigma_M)$.

$$\sigma_M(\Phi_1, \Phi_2) = \int_\Sigma d\Sigma \left(\Phi_2 \nabla_n \Phi_1 - \Phi_1 \nabla_n \Phi_2 \right) \,, \quad \forall \Phi_1, \Phi_2 \in \mathcal{S}(M)$$

- The Weyl operators associated to $(\mathcal{S}(M), \sigma_M)$

$$W(\phi_1)W(\phi_2) = e^{i\sigma_M(\phi_1,\phi_2)} W(\phi_1 + \phi_2), \qquad W^\dagger(\phi) = W(-\phi).$$

- They generate the C^*-algebra of local observables.

Analyses of the classical solutions

$\Phi \in \mathcal{S}(M)$ can be decomposed in modes ($\mathbf{k} \in \mathbb{R}^3$, $k = |\mathbf{k}|$,)

$$\Phi(\tau, \vec{x}) = \int_{\mathbb{R}^3} d^3\mathbf{k} \left[\phi_{\mathbf{k}}(\tau, \vec{x}) \widetilde{\Phi}(\mathbf{k}) + \overline{\phi_{\mathbf{k}}(\tau, \vec{x}) \widetilde{\Phi}(\mathbf{k})} \right],$$

with respect to the functions

$$\phi_{\mathbf{k}}(\tau, \vec{x}) = \frac{1}{a(\tau)} \frac{e^{i\mathbf{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}} \chi_{\mathbf{k}}(\tau),$$

$\chi_{\mathbf{k}}(\tau)$, is solution of the differential equation

$$\frac{d^2}{d\tau^2}\chi_{\mathbf{k}} + (V_0(\mathbf{k}, \tau) + V(\tau))\chi_{\mathbf{k}} = 0,$$

$$V_0(\mathbf{k}, \tau) := k^2 + \left(\frac{1}{H\tau}\right)^2 \left[m^2 + 2H^2\left(\xi - \frac{1}{6}\right)\right], \quad V(\tau) = O(1/\tau^3).$$

- With the normalization

$$\frac{d\overline{\chi_{\mathbf{k}}(\tau)}}{d\tau}\chi_{\mathbf{k}}(\tau) - \overline{\chi_{\mathbf{k}}(\tau)}\frac{d\chi_{\mathbf{k}}(\tau)}{d\tau} = i. \quad \forall \tau \in (-\infty, 0)$$

In the case of de Sitter spacetime, $V(\tau) = 0$, and

$$\chi_{\mathbf{k}}(\tau) = \frac{\sqrt{-\pi\tau}}{2}\overline{e^{\frac{i\pi\nu}{2}} H^{(2)}_{\nu}(-k\tau)},$$

with

$$\nu = \sqrt{\frac{9}{4} - \left(\frac{m^2}{H^2} + 12\xi\right)},$$

- where $H^{(2)}_{\nu}$ is the Hankel function of second kind.

Perturbative solutions in the general case

- V perturbation potential over the de Sitter solution $\chi_{\mathbf{k}}$.
- The retarded fundamental solutions $S_{\mathbf{k}}$
- Then the general solutions $\rho_{\mathbf{k}}$.

$$\rho_{\mathbf{k}}(\tau) = \chi_{\mathbf{k}}(\tau)$$

$$+(-1)^n \sum_{n=1}^{+\infty} \int_{-\infty}^{\tau} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n S_{\mathbf{k}}(\tau, t_1) S_{\mathbf{k}}(t_1, t_2) \cdots$$

$$S_{\mathbf{k}}(t_{n-1}, t_n) V(t_1) V(t_2) \cdots V(t_n) \chi_{\mathbf{k}}(t_n),$$

Convergence

if $|Re\nu| < 1/2$ and $V = O(\tau^{-3})$ or
if $|Re\nu| < 3/2$ and $V = O(\tau^{-5})$

Projection of the quantum theory on the Horizon

$\Im^-$ topologically equivalent to $\mathbb{R}\times\mathbb{S}^2$, coordinates (ℓ,θ,φ).

The symplectic space of real wavefunctions $(\mathcal{S}(\Im^-),\sigma)$:

$$\mathcal{S}(\Im^-)=\left\{\psi\in C^\infty(\mathbb{R}\times\mathbb{S}^2)\mid\psi\in L^\infty,\partial_\ell\psi\in L^1,\widehat{\psi}\in L^1,k\widehat{\psi}\in L^\infty\right\},$$

$$\sigma(\psi,\psi')=\int\limits_{\mathbb{R}\times\mathbb{S}^2}\left(\psi\frac{\partial\psi'}{\partial\ell}-\psi'\frac{\partial\psi}{\partial\ell}\right).\quad\forall\psi,\psi'\in\mathcal{S}(\Im^-)$$

A symplectic structure, with data on the null surfaces

$$W_{\Im^-}(\psi)=W^*_{\Im^-}(-\psi),\qquad W_{\Im^-}(\psi)W_{\Im^-}(\psi')=e^{\frac{i}{2}\sigma(\psi,\psi')}W_{\Im^-}(\psi+\psi').$$

Preferred state on the Horizon

- ∂_τ restricted on the Horizon $H^\alpha \partial_\ell$.
- Positive frequencies w.r.t. to ∂_ℓ.
  $\widehat{\psi}(k,\theta,\varphi) = \int_{\mathbb{R}} \frac{e^{ik\ell}}{\sqrt{2\pi}} \psi(\ell,\theta,\varphi) d\ell.$

$$\mu(\psi,\psi') = Re \int_{\mathbb{R}\times\mathbb{S}^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\varphi)}\widehat{\psi}'(k,\theta,\varphi)\, dk\, d\mathbb{S}^2(\theta,\varphi),$$

It defines a pure gaussian state

$$\lambda(W(\psi)) = e^{\frac{\mu(\psi,\psi)}{2}},$$

Projection on the horizon and pull back of the states

$$\gamma : \mathcal{S}(M) \to C^\infty(\Im^-), \qquad \gamma(\Phi) = \Phi|_{\Im^-}$$

Theorem

Φ can be restricted on $\Im^-$, it becomes $\gamma\Phi \in \mathcal{S}(\Im^-)$, preserving the symplectic form

$$\sigma(\gamma\Phi, \gamma\Phi') = H^{-2}\sigma_M(\Phi, \Phi'). \quad \forall \Phi, \Phi' \in \mathcal{S}(M)$$

Theorem

$\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^-)$ *generated by*

$$\imath\left(W_M(\Phi)\right) = W\left(-H^{-1}\gamma(\Phi)\right), \forall \Phi \in \mathcal{S}(M),$$

is an injective $*-$homomorphism: An embedding of algebras.

Pullback of states

Given any state $\omega : \mathcal{W}(\Im^-) \to \mathbb{C}$, it can be pulled back to the algebra $\mathcal{W}_M$ with $\imath^*(\omega)$.

- In particular the preferred state

$$\lambda_M(a) := \lambda(\imath(a)). \quad \forall a \in \mathcal{W}(M)$$

- In the de Sitter spacetime, λ_M is the Bunch-Davies state
- That state is the state considered by cosmologist as the "ground states" for the analyses of perturbation.
- If $\nu \sim 3/2$ we have on Σ_τ

$$\lambda_M(x,y) \sim \int e^{i\mathbf{k}(x-y)} P(k) d\mathbf{k}^3 \,, \qquad P(k) \sim \frac{\alpha}{|\mathbf{k}|^{\sim 3}} + \frac{\beta}{|\mathbf{k}|^{\sim 1}}$$

Hadamard property and Microlocal spectral condition

A two point function of a state is **Hadamard** if

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\ln\sigma_\epsilon(x,y) + W(x,y)$$

Radzikowski has given another equivalent characterization using the **microlocal analyses**

$$WF(\omega) = \left\{((x,k_x),(y,-k_y)) \in (T^*M)^2 \setminus 0 \mid (x,k_x) \sim (y,k_y), k_x \triangleright 0\right\},$$

Hadamard property for these states

Now we tackle the second problem, namely if it is Hadamard.

$$\lambda_M(f,g) = \lim_{\epsilon \longrightarrow 0^+} -\frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{\psi_f(\ell,\theta,\varphi)\psi_g(\ell',\theta,\varphi)}{(\ell - \ell' - i\epsilon)^2} d\ell d\ell' d\mathbb{S}^2(\theta,\varphi),$$

where $\psi_f = \gamma E f$

Theorem

λ_M is a distribution that satisfy the μSC

$$WF(\lambda_M) = \Gamma =$$

$$= \left\{((x,k_x),(y,-k_y)) \in (T^*M)^2 \setminus 0 \mid (x,k_x) \sim (y,k_y), k_x \triangleright 0\right\},$$

hence it is Hadamard

Sketch of the proof. $\supset$

Having

$$\lambda_M(f, Pg) = \lambda_M(Pf, g) = 0, \qquad \lambda_M(f, g) - \lambda_M(g, f) = E(f, g),$$

then the inclusion $\supset$ descends from the *Proposition 6.1 Strohmaier Verch Wollenberg (2002)*.

Sketch of the proof. $\subset$

- The state can be seen as a "composition" of distribution

$$\lambda_M(f,g) = \langle T(Ef)_{\restriction \Im^-}, (Eg)_{\restriction \Im^-} \rangle.$$

- The **restriction** of one entry of E on $\Im^-$ is meaningful

$$WF(E)_{\Im^-} = \emptyset \quad \Longrightarrow \quad \widetilde{E} := E_{\restriction \Im^-} \in \mathcal{D}'(\Im^- \times M)$$

- $WF'(T) \cap WF\left(\widetilde{E} \otimes \widetilde{E}\right)_{\Im^- \times \Im^-} = \emptyset$ we can **multiply** them.
- Consider the distribution $K \in \mathcal{D}'(\Im^- \times \Im^- \times M \times M)$

$$K = (T \otimes I) \cdot \left(\widetilde{E} \otimes \widetilde{E}\right),$$

K is the kernel of the following map

$$\mathcal{K} : C_0^\infty(\Im^- \times \Im^-) \to \mathcal{D}'(M \times M)$$

- We would like to make sense to the following expression, and to control its wave front set

$$\lambda_M(f,g) \sim \text{“}\mathcal{K}(1\otimes 1)(f\otimes g)\text{”}$$

- $\chi(\ell) \in C_0^\infty(\mathbb{R})$ such that $\chi(0) = 1$ and

$$\chi_n(\ell,\theta,\varphi) = \chi\left(\frac{\ell}{n}\right). \quad \forall n \in \mathbb{N}$$

Hence we can define the following sequence

$$\lambda_n = \mathcal{K}(\chi_n(\ell)\chi_n(\ell')) \in \mathcal{D}'(M\times M).$$

We have that

$$WF(\lambda_n) \subset \Gamma =$$

$$= \left\{((x,k_x),(y,-k_y)) \in T^*M^2 \setminus 0 \mid (x,k_x) \sim (y,k_y), k_x \triangleright 0\right\},$$

Theorem

λ_n tends to λ_M in the Hörmander topology $\mathcal{D}'_\Gamma(M \times M)$:

1. *In the topology of $\mathcal{D}'(M \times M)$*

$$\lambda_n \to \lambda_M$$

2. $$\sup_n \sup_{k \in V} |k|^N |\widehat{\lambda_n(\cdot\ \phi)}| < \infty, \qquad N = 1, 2, 3, \ldots$$

   $\phi \in C_0^\infty(M \times M)$, The closed cone $V \cap \Gamma = \emptyset$.

Hence $WF(\lambda_M) \subset \Gamma$

Conclusion and open questions

Summary:

- There is a way of defining a preferred state in some cosmological models.
- It has interesting properties:
  - "Positive frequency" w.r.t. to the conformal time
  - It has a good singular behavior.

Open Questions:

- Stability of these states, in particular how the power spectrum changes in time.
- The role of regulaziation needs to be addressed in the analyses of the power spactrum.
- How to deal with interacting theories?