Semiclassical Einstein equations: A solution with implications in cosmology[2]

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[2] Based on a joint work with C. Dappiaggi and K. Fredenhagen

Summary

- Cosmological Scenario
- Semiclassical Einstein equation
- Stress Tensor regularization
- Solution with massless scalar fields as sources
- Solution with massive fields as sources

Cosmological scenario: geometry

- Physical input: Universe is homogeneous and isotropic. Then FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1+\kappa r^2} + r^2 d\Sigma^2 \right).$$

  $\kappa = 0$ flat, $\kappa = \pm 1$ open or closed.
- recent observation: $a(t) \simeq Ce^{Ht}$, and $\kappa \simeq 0$.

Solutions that have the FRW form.

Cosmological scenario: matter

- We model T_{ab} with a perfect fluid

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b).$$

Homogeneity and isotropy $\Longrightarrow u = \frac{\partial}{\partial t}$, $\rho(t)$ and $P(t)$

- Einstein equations become FRW equations $H = \dot{a}a^{-1}$

$$3H^2 = 8\pi\rho - \frac{3\kappa}{a^2} \tag{1}$$

$$3\dot{H} + 3H^2 = -4\pi\left(\rho + 3P\right) \tag{2}$$

- Type of fluids: $P = w\rho$ and conservation equation
  - Radiation: $w = \frac{1}{3}$, $\rho_R \sim a(t)^{-4}$
  - Dust: $w = 0$, $\rho_M \sim a(t)^{-3}$
  - Cosmological constant: $w = -1$ $\rho_\Lambda = C$

Cosmological scenario: observation

- If we use this to model the present day observation:
  - Radiation is not important.
  - We look for a mixture of ρ_M and ρ_Λ
- To model CMB and Supernovae red-shift observation:

We have a problem

Energy density: $\sim$ 70% *Dark Energy*, $\sim$ 30% *Matter*.
Known matter: only $\sim$ 4%.

- Let's try to see the role of quantum effects.

Gravity: semiclassical approximation

- We would like to have a quantum theory of gravity.
- **Too difficult.**
- At least we would like to have a theory of backreaction.
- We try semiclassically.

$$G_{ab} = 8\pi \langle T_{ab} \rangle .$$

- It should work in some regimes. As in atomic physics: quantum mechanical electron with external classical field.

Range of validity of semiclassical approximation

- A complete satisfactory semiclassical description is impossible. (quantum matter is a source for gravity).
- It should be valid whenever quantum fluctuations are negligible.
- In some model it is unavoidable: Particle creation. (Ex: black holes radiates)
  - Are there quantum effects that can be seen?
  - How is modified the vacuum energy?
  - How can be treated the backreaction effects?
  - Is it a small effect?
  - What implication has the quantum origin of matter on the solutions?

Wald Axioms

In QM T_{ab} are singular objects $\langle T_{ab} \rangle \to \infty$.

We need a renormalization prescription for T_{ab} on CST.

Wald axioms $\Longrightarrow$ meaningful semiclassical approx.
[Wald 77] [Wald 78]

(1.) It must agree with formal results for T_{ab}
(For scalar: $(\Phi, T_{ab}\Psi)$, can be found formally if $(\Phi, \Psi) = 0$).

(2.) T_{ab} should be normal ordering in Minkowski

(3.) Conservation: $\nabla^a \langle T_{ab} \rangle = 0$

(4.) Causality: $\langle T_{ab} \rangle$ at p depends only on $J^-(p)$

(5.) T_{ab} depends on derivatives of the metric up to the second order (or third).

Nice Environment

Problem:

How can we treat matter without fixing the spacetime?

- We can quantize simultaneously and coherently on all spacetime. *[Brunetti Fredenhagen Verch 2003]*.
- Quantum Fields are particular observables that transform suitably under isomorphisms.
- We need another ingredient: reference states on every spacetime.
- Einstein eq.
  - Einstein: consistency criterion.
  - Selects particular elements of the category of local Manifolds and of then of local Algebras.

What we need to search in a quantum theory

Instead of considering FRW equation we use the following.

$$-R = 8\pi T, \qquad \nabla^a T_{ab} = 0$$

- Up to some initial condition this is equivalent to FRW.
- We get
$$3H^2 = 8\pi\rho + 8\pi\frac{C}{a^4} - \frac{3\kappa}{a^2}.$$
C is fixed knowing the whole T_{ab}.
- But it is simpler to perform quantum computation.

Matter: Scalar free field theory

Equation of motion

$$P := -\Box + \xi R + V \,, \qquad P\phi = 0$$

We will be interested in the case $V = m^2$ and $\xi = 1/6$.
Stress Tensor:

$$T_{ab} := \partial_a\phi\partial_b\phi - \frac{1}{6}g_{ab}\left(\partial_c\phi\partial^c\phi + V\phi^2\right) - \xi\nabla_{(a}\partial_{b)}\phi^2$$

$$+\xi\left(R_{ab} - \frac{R}{6}g_{ab}\right)\phi^2 + \left(\xi - \frac{1}{6}\right)g_{ab}\Box\phi^2.$$

Differ by the usual one by terms of the form $\phi P\phi$ *[Moretti 2002]*.

Remarks on T_{ab}

- Trace

$$T = -3\left(\frac{1}{6} - \xi\right)\Box\phi^2 - V\phi^2$$

where we have used $\phi P\phi = 0$ to simplify.

- Conservation equation

$$\nabla_a T^a{}_b = -\frac{1}{2}\phi^2\partial_b V$$

- Classical Ambiguity: $\phi P\phi = 0$

$$T'_{ab} = T_{ab} + Cg_{ab}\left(\phi P\phi + P\phi\phi\right)$$

- T_{ab} can be written by means of balanced derivatives and derivatives of the field ϕ^2

Quantum field theory

- QFT described by $n-$point functions.
- Quasi free states ω described by the two-points function

$$\omega_2(x,y) = \langle \phi(x)\phi(y) \rangle$$

thought as distribution in $\mathcal{D}'(M \times M)$.
- T_{ab} arises as an operation on ω_2 and a coinciding point limit.
- It is not well defined...
- Quasifree states that possess Hadamard property
[Radzikowski 1995] [Brunetti Fredenhagen Köhler 1996]

$$\mathrm{WF}(\omega_2) = \{((x_1,k_1),(x_2,k_2)) \in T^*M^2/\{0\} : -P_\gamma k_2 = k_1 > 0\}$$

- Physically: The fluctuations of the field are always finite on Hadamard states.

Hadamard Two-points function

$$\omega_2 = \frac{1}{8\pi^2}\left(\frac{u}{\sigma_\epsilon} + v \log \sigma_\epsilon + w\right).$$

- u v w are smooth functions,
- u depends only upon the geometry via g_{ab}
- v depends upon g_{ab}, ξ and V
- w characterizes the state.

Some notations: σ half of the square of the geodesic distance

$$v = \sum v_n \sigma^n \qquad\qquad [v](x) = v(x,x)$$

The singular Structure H is fixed and does not depend on the state.

Regularization of the two-points function

Quantum Problem. Regularization with point splitting procedure

$$\langle : \phi(x)\phi(y) : \rangle_\omega := \omega_2(x,y) - H(x,y)$$

It reduces to normal ordering for flat spacetime.
[Hollands Wald, Brunetti Fredenhagen Verch, Moretti]

$$8\pi^2 \langle : \phi P \phi : \rangle_\omega = 6[v_1], \qquad 8\pi^2 \langle : (\nabla_a \phi)(P\phi) : \rangle_\omega = 2\nabla_a [v_1]$$

The conservation equation are satisfied also quantum mechanically

$$\nabla_a \langle : T^a{}_b : \rangle_\omega = -\frac{1}{2} \langle : \phi^2 : \rangle_\omega \partial_b V = -\frac{1}{2}\frac{[w]}{8\pi^2} \partial_b V$$

but unfortunately the trace is different from the classical one.

$$\langle : T : \rangle_\omega := \frac{2[v_1]}{8\pi^2} + \left(-3 \left(\frac{1}{6} - \xi \right) \Box - m^2 \right) \frac{[w]}{8\pi^2}.$$

Some computations.......

$$2[v_1] = \frac{1}{360}\left(C_{ijkl}C^{ijkl} + R_{ij}R^{ij} - \frac{R^2}{3} + \Box R\right) + \frac{1}{4}\left(\frac{1}{6} - \xi\right)^2 R^2 +$$

$$+\frac{m^4}{4} - \frac{1}{2}\left(\frac{1}{6} - \xi\right) m^2 R + \frac{1}{12}\left(\frac{1}{6} - \xi\right)\Box R.$$

Remaining freedom

In the trace $c\Box R$. Wald's fifth axiom does not hold!

- We can add conserved tensors t_{ab} build by curvature only.
- It must behave as T_{ab} under "scale" transformations.
- Some possibilities arises form the variation of

$$t_{ab} = \frac{\delta}{\delta g^{ab}} C \int \sqrt{g} R^2 + D \int \sqrt{g} R_{ab} R^{ab}$$

- ${t_a}^a = \alpha \Box R$
- We use this freedom to cancel the $\Box R$ term from $\langle : T : \rangle$.

Wald's fifth axiom partially holds for $\langle : T'_{ab} : \rangle = \langle : T_{ab} : \rangle - c t_{ab}$

$f(R)$ gravity .....

NB: t_{ab} alone does not guaranty stable solutions.

[Cognola Elizalde Odintsov Zerbini 05, Cognola Zerbini 06]

Equation of the universe

Assuming $\kappa = 0$, we write the equation $-R = 8\pi\langle : T : \rangle$ as follows

$$-6\left(\dot{H} + 2H^2\right) = 8\pi G\left(-3\left(\frac{1}{6} - \xi\right)\Box - m^2\right)\langle : \phi^2 : \rangle_\omega +$$

$$+\frac{G}{\pi}\left(-\frac{1}{30}\left(\dot{H}H^2 + H^4\right) + 9\left(\frac{1}{6} - \xi\right)^2\left(\dot{H}^2 + 4H^2\dot{H} + 4H^4\right)\right)$$

$$+\frac{G}{\pi}\left(\frac{m^4}{4} - 3\left(\frac{1}{6} - \xi\right)m^2\left(\dot{H} + 2H^2\right)\right)$$

If $\xi = 1/6$, namely for the conformal coupling it simplifies a lot:

$$-6\left(\dot{H} + 2H^2\right) = -8\pi G m^2\langle : \phi^2 : \rangle_\omega + \frac{G}{\pi}\left(-\frac{1}{30}\left(\dot{H}H^2 + H^4\right) + \frac{m^4}{4}\right)$$

Conformal invariant theory

If $\xi = \frac{1}{6}$, $m^2 = 0$, the equation does not depend on the state.

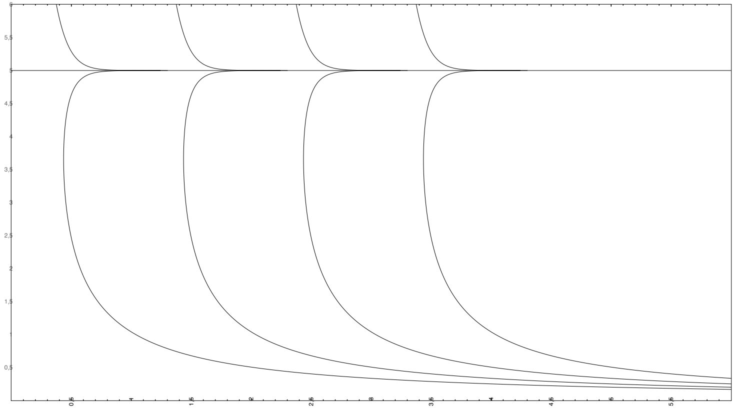
$$\dot{H}\left(H^2 - \frac{H_c^2}{2}\right) = -H^4 + H_c^2 H^2, \qquad H_c^2 = 180\frac{\sqrt{2}\pi}{G}$$

$H = H_c^2$ and $H^2 = 0$ are solutions (de Sitter, Minkowksi).
They are both stable as seen by the general solution

$$Ce^{4t} = e^{1/H} + \left|\frac{H + H_c}{H - H_c}\right|^{1/H_c}$$

- It is as in the Starobinsky model but now with stable de Sitter. *[Starobinsky 80, Vilenkin 85]*
- $H = H_c$ is order of magnitude to big to describe the present expansion velocity of the universe.

For $H_c = 5$. Clearly $H = H_0$ and $H = H_c$ are stable solutions.



Massive model

The state enters through $\langle : \phi^2 : \rangle$. Choose $\langle : \phi^2 : \rangle = A - BR$

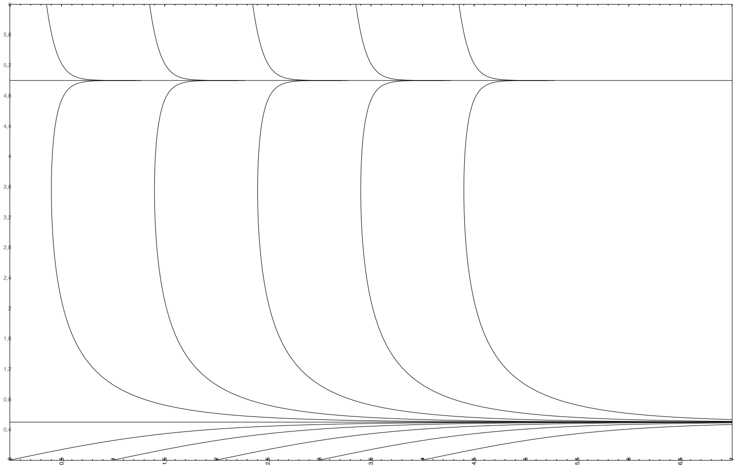
$$\dot{H}\left(H^2 - H_0^2\right) = -H^4 + 2H_0^2 H^2 - M$$

where H_0 and M are two constants with the following values

$$H_0^2 = \frac{180\pi}{G} + B, \qquad M = -\frac{15}{2} m^4 + 240\pi^2 m^2 A$$

If $0 < M < H_0^4$, two fixed stable solutions (de Sitter phases)

$$H_\pm^2 = H_0^2 \pm \sqrt{H_0^4 - M},$$



- H_{-} can be made small by suitable choices of m^2 and A, B
- It models dark energy. (T in general is not 0)
- Smooth exit form rapid expansion in the past. *[Shapiro Sola 02]*

Massive models and adiabatic vacuum $\langle : \phi^2 : \rangle$

On dS Bunch Davies state gives the expected result for $\langle : \phi^2 : \rangle$.

[Parker, Parker and Fulling, Lüders Roberts, Junker Schrohe, Olbermann]

We consider the following conformal transformation

$$\gamma_{ab} = -\frac{g_{ab}}{a(t)^2}, \qquad \tau = \int \frac{dt}{a(t)} \qquad \phi = a^{-1}\widetilde{\phi}$$

$$-\Box_\gamma \widetilde{\phi} + a(\tau)^2 m^2 \widetilde{\phi} = 0$$

Hadamard states remain Hadamard under conformal transformation.

$$-[H] + \frac{[\widetilde{H}]}{a^2} = -\frac{R}{18} - \log a$$

The general solution can be expanded in modes:

$$\widetilde{\phi} = \int d^3k T_k e^{ikx} f(k)$$

Two-points function

The two-points function of the adiabatic vacuum state:

$$\widetilde{\omega}_2(x_1, x_2) = \int d^3k\, \overline{T_k}(\tau_1) T_k(\tau_2) e^{ik(x_1 - x_2)}$$

$\left(\dot{\overline{T_k}} T_k - \overline{T_k} \dot{T}_k\right) = i$ WKB approximation $T_k(\tau) = \frac{e^{i\int_{t_0}^{\tau} \Omega_k d\tau}}{\sqrt{2\Omega_k}}$

Then Ω must satisfy the following equation

$$\Omega_k^2 = k^2 + m^2 a(\tau)^2 + \frac{3}{4}\left(\frac{\Omega_k'}{\Omega_k}\right)^2 - \frac{1}{2}\frac{\Omega_k''}{\Omega_k}$$

Recursively $\Omega_k^{(0)} = k^2 + m^2 a(\tau)^2$ and $\Omega_k^{(n+1)}$ plugging $\Omega_k^{(n)}$ on the right.

$$\langle : \widetilde{\phi}^2 : \rangle \sim \int dk k^2 \left(\frac{1}{2\Omega_k^{(n)}} - \frac{1}{2\Omega_k^{(0)}}\right) + a(\tau)^2 \log(a(\tau)) + a(\tau)^2 C(m^2)$$

Approximation

Transforming back to the original spacetime

$$\langle : \phi^2 : \rangle = C(m^2) - \frac{R}{18} + \frac{1}{a^2} \int dk k^2 \left(\frac{1}{2\Omega_k^{(n)}} - \frac{1}{2\Omega_k^{(0)}} \right)$$

- 0-order O.K.
- 1-order already a problem!

$$\lim_{m \to \infty} \int_0^\infty dk k^2 \left(\frac{1}{2\Omega_k^{(1)}} - \frac{1}{2\Omega_k^{(0)}} \right) = 0$$

- ?

The first approximation with a constant in not so bad. ok!

Summary

- Semiclassical solution of Einstein equation.
- The solutions depend on the quantum states.
- The de Sitter phases could be stable only by modifying the ren. prescription.

Open Questions

- How can we choose a nicer state?
- Fluctuations?
- Inflation?
- Connection with $f(R)$ gravity?
- Origin of R^2 terms in the action? An hint on quantum gravity?

First order approximation

$$\int_0^\infty dk k^2 \left(\frac{1}{2\Omega_k^{(1)}} - \frac{1}{2\Omega_k^{(0)}} \right) =$$

$$a^2 m^2 \int_1^\infty d\omega \sqrt{\omega^2} \left(1 - \frac{1}{\sqrt{1 - \frac{(a^2)''}{4a^4\omega^4 m^2} + \frac{5}{4}\frac{(a')^2}{a^4\omega^6 m^2}}} \right)$$

We use the $\Omega_k^{(n)}(t_0)$ and $\dot{\Omega}_k^{(n)}(t_0)$ as initial condition for $S_k^{(n)}$.

$$\omega_2^{(n)}(x_1, x_2) = \int d^3k \overline{S_k^{(n)}}(\tau_1) S_k^{(n)}(\tau_2) e^{ik(x_1 - x_2)}$$

$\omega^{(n)}$ adiabatic state of order n.
In fact: Be ω_2^H the two-points function of an Hadamard state then

$$\mathrm{WF}^s(\omega_2^{(n)} - \omega_2^H) = \emptyset \quad s < 2(n) + 3/2$$

If n is large enough this condition says that

$$\omega_2^{(n)} - \omega_2^H \in H^s$$

Then, for Sobolev imbedding theorem, if $s \geq 3/2$ ($n \geq 1$) it is in $C^0(\mathcal{O})$.

$$\langle : \widetilde{\phi}^2 : \rangle \sim \lim_{(x,y)\to(x,x)} \left(\omega_2^{(n)} - \widetilde{H}\right)(x, y)$$

x, y are on the same $t = t_0$ surface.

$$\widetilde{H}(x,y) = \int_0^\infty dk \frac{k^2 e^{ik(x-y)}}{\sqrt{k^2 - m^2 a(\tau)^2}} - a(\tau)^2 \log(a(\tau)) - a(\tau)^2 C(m^2)$$

$$C(m^2) = \left(\tfrac{m^2}{4} (\log(m^2) + 2\gamma + 1) \right)$$

$$\langle : \widetilde{\phi}^2 : \rangle \sim \int dk k^2 \left(\frac{1}{2\Omega_k^{(n)}} - \frac{1}{2\Omega_k^{(0)}} \right) + a(\tau)^2 \log(a(\tau)) + a(\tau)^2 C(m^2)$$