Localization and position operators in Möbius covariant theories.

Nicola Pinamonti

II Institute für Theoretische Physik
Hamburg Universität

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Plan of the talk

- **Localization:** As emerging from symmetry.
- The case of **Möbius covariance**.
- New aspect: **Position operators** arising from a modification of the generators of the group.
- **Example:** Massless KG scalars on 2D Minkowski.

- *R. Brunetti, D. Guido and R. Longo,* Rev. Math. Phys. **14**, 759 (2002).
- *L. Fassarella and B. Schroer,* J. Phys. **A 35**, 9123 (2002).
- *B. Schroer,* Nucl. Phys. **B 499**, 519 (1997).
- np, math-ph/0610070 (2006)

Motivations

- **Causality** is an important concept in relativistic physics.

  "Spatially separated events cannot interact."

- In **QFT** at level of *"second quantization"*. Local observables are charactered by $\mathbb{R}$-linear spaces of local wave-functions.

- It is **not** completely **intrinsic**. It seems to depend on the particular representation of the functions.

- *Brunetti Guido and Longo:* Localization ($\mathbb{R}$-linear spaces) descends from symmetrey group.

- Do observables compatible with this localization exsit?

- We analyze the case of Möbius covariant theories.

Is it a trivial task?

- **Quantum mechanics:** *Example:* Particle on the line. $L^2(\mathbb{R}, dx)$ states, $|\psi(x)|^2$ probability distribution.
- Coordinate: $X : \psi(x) \mapsto x\psi(x)$, self-adjoint operator.
- *Local* states in $[a, b]$ are: $L^2([a, b], dx) \subset L^2(\mathbb{R}, dx)$.
- If $\psi \in L^2([a, b], dx)$ and $\|\psi\| = 1$

$$a \leq (\psi, X\psi) \leq b$$

We say X its compatible with locality.

Relativistic situation

In relativistic theories

- **Example:** Scalar KG field on 2D Minkowski.
- Chose a space-like Hypersurface, then
- *Localization* and *coordinate* can be defined as above. This is called Newton Wigner (NW) localization.

- **Problem**: NW Localization is not preserved by evolution.

  (Classical information cannot travel faster then light?).

  It seems not Physically reasonable.

Quantization scheme and localization

For flat spacetime:

- **First quantization**: *(a la Wigner)*
  - One-particle Hilbert space $\mathcal{H}$.
  - (anti)-unitary representation of the Poincarré group.
- **Second quantization**:
  - Consider the Fock space $\mathfrak{H} := \mathfrak{F}(\mathcal{H})$ built by $\mathcal{H}$ and the vacuum Ω.
  - Weyl operators $W(\psi)$ on $\mathfrak{H}$.

- **Localization**: Operators need to be smeared.

  $\mathcal{O}$ a region of spacetime.
  Consider real local function with support in a region $\mathcal{O}$.
  By means of causal propagator E.

  $$f : \mathcal{O} \to \mathbb{R}, \Longrightarrow \mathcal{K}_{\mathcal{O}} := \{\psi_f \in \mathcal{H} | \psi_f = Ef, D(f) \subset \mathcal{O}\}$$

  $\mathcal{K}_{\mathcal{O}}$ is a $\mathbb{R}$-linear subset of the one-particle Hilbert space $\mathcal{H}$.

- **von Neumann algebras**. $\mathcal{A}(\mathcal{O}) := \{W(\psi) | \psi \in \mathcal{K}_{\mathcal{O}}\}''$

- if $\mathcal{O}$ is a double cone $\mathcal{A}(\mathcal{O})$ is in standard form:
  Ω is cyclic and separating.

Digression Tomita Takesaki modular theory.

- then if $A \in \mathcal{A}$ (standard) exists an operator S from $\mathcal{A}\Omega$ to $\mathcal{A}\Omega$ realizing the star operation

$$S A \Omega = A^* \Omega$$

- Has a polar decomposition $S := J\Delta^{1/2}$
- Δ self-adj. positive. $\Delta^{it}\mathcal{A}\Delta^{-it} = \mathcal{A}$ *(modular transf.)*
- J is an anti-unitary operator. $J\mathcal{A}J = \mathcal{A}'$ *(modular conj.)*
- $\mathcal{A}$ on Ω satisfy the KMS condition w.r. to modular transf.
- For Wedges in Minkowski spacetime, have a geometrical meaning: J is a **Reflection** and Δ^{it} are **Boosts** *(Bisognano Wichmann)*
- Be $\psi = A\Omega$, with $A^* = A$, in the one particle Hilbert state then: $S\psi = \psi$. And also if $\psi \in \mathcal{K}$: $S\psi = \psi$.

New scheme

Revert the point of view:

- Recognize $J_{\mathcal{O}}$ and $\Delta_{\mathcal{O}} = e^{-D_{\mathcal{O}}}$ within the group of symmetry for sufficiently many local sets $\mathcal{O}$.
- Consider $S_{\mathcal{O}} := J_{\mathcal{O}} \Delta_{\mathcal{O}}^{1/2}$.
- Assume $\mathcal{K}_{\mathcal{O}} := \{\psi | S\psi = \psi\}$ as a definition for $\mathbb{R}$-linear subspace of $\mathcal{H}$ of object local in $\mathcal{O}$.

Properties:

P $\mathcal{K}_{\mathcal{O}'} = \mathcal{K}'_{\mathcal{O}}$.

P If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{K}_{\mathcal{O}_1} \subset \mathcal{K}_{\mathcal{O}_2}$ *(Isotony)*

P If $\mathcal{O}_1$ and $\mathcal{O}_2$ spatially separed $\mathcal{K}_{\mathcal{O}_1} \cap \mathcal{K}_{\mathcal{O}_2} = \emptyset$ *(Locality)*

P Local function: dense in $\mathcal{H} := \overline{\mathcal{K}_{\mathcal{O}} + i\mathcal{K}_{\mathcal{O}}}$.

Möbius group: geometric aspects

Conformal transformations of $\mathbb{C}$ where $\mathbb{S}^1$ is fixed.

$$x \to \frac{ax+b}{cx+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R}).$$

$PSL(2,\mathbb{R})$ *transformation* on $\mathbb{PR}$.

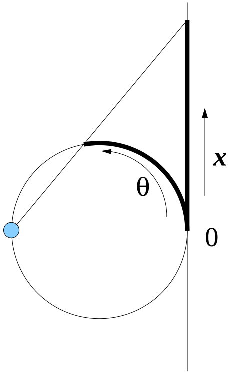
$j : x \to -x$ in $\mathbb{R} \cup \{\infty\}$ *involution*.

Iwasawa decomposition: $g \in PSL(2,\mathbb{R})$

$$g := T(x)\Lambda(y)P(z), \qquad x, y, z \in \mathbb{R},$$

h, d, c: generators

$$[h,d] = h, \qquad [c,d] = -c, \qquad [c,h] = 2d\ .$$



Local sets: $I \subset \mathcal{I}$ proper interval $I = [a, b]$ in $\mathbb{PR}$.

$\forall I$, the decomposition: $g := T_I(x)\Lambda_I(y)P_I(z)$ and a j_I exist.

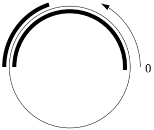
(A) Reflection covariance: j_I maps I to I' and $j_{gI} = g j_I g^{-1}$.

(B) Λ **covariance:** $\Lambda_I(t)$ maps I to I and $\Lambda_{gI}(t) = g\,\Lambda_I(t)\,g^{-1}$.

(C) Positive inclusions:

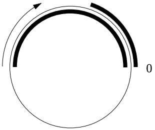
- If $t > 0$, $T_I(t)$ maps I to $I_t \subset I$ and

$$\Lambda_I(b)T_I(t)\Lambda_I(-b) := T_I(e^{2\pi b}t);$$



- If $p < 0$, $P_I(p)$ maps I to $I_p \subset I$ and

$$\Lambda_I(b)P_I(p)\Lambda(-b) := P_I(e^{-2\pi b}p).$$



Lesson: A particular decomposition selects a particular interval.

Properties of $\mathbb{R}$-linear Subspaces

- **Quantum Theory:** $\mathcal{H}$ Hilbert space.
  U_g positive energy (anti)-unitary representation of the Möbius group.

- **Decompositions**: $U_g := T_I(x)\Lambda_I(y)P_I(z)$, and J_I
  - **Generators of $PSL(2,\mathbb{R})$:** Selfadjoint operators H_I, D_I and C_I satisfy:

$$[H_I, D_I] = iH_I, \qquad [C_I, D_I] = -iC_I, \qquad [H_I, C_I] = 2iD_I.$$

  - J_I the corresponding antiunitary transformation.

- **Remark:** A decomposition selects an interval I in an abstract way. Thus intrinsically.

Properties of $\mathbb{R}$-linear Subspaces

Fix a particular decomposition, then

- Modular structure:
  - $\Delta_I := e^{-2\pi D_I}$ *(modular operator)*
  - J_I *(modular conjugation)*

- Real subspaces from modular operators: $S_I := J_I \Delta_I^{1/2}$ and

$$\mathcal{K}_I := \{\psi | S_I \psi = \psi\}$$

- From now on we choose the decomposition for the upper semicircle I_1
  (positive part of $\mathbb{PR}$).
  H, D, C the self adj. generators and J the anti-unitary involution. $\Delta := exp{-}2\pi D$

Digression: POVM

- **Pauli Theorem:** It is not possible to have a selfadjoint operator X, showing CCR with P bounded from below.
- Gen. of rotation $(H + C)/2$ is positive, does not exists a self-adj. operator representing a global coordinate.
- Ordinary **QM**: E energy and T time. Usually this is circumvent enlarging the concept of observable to POVM. (Naimark).
- In KMS states E is not bounded from below, then a selfadjoint T operator exists. (Narnhofer, Thirring)
- We are searching for local coordinates for the interval I: it has to show CCR with the generator of modular transformation.

From positive inclusions: $[H,D]=iH$ $[C,D]=-iC$
Candidates for X showing CCR with D: $-\log H$ and $\log C$

$$\gamma\log(C)-(1-\gamma)\log H+f(D).$$

But we want it being compatible with emerging locality:

$$\text{If } \psi\in\mathcal{K}_{[a,b]\subset I_1}, \qquad \log(a)\,\|\psi\|^2\leq(\psi,X\psi)\leq\log(b)\,\|\psi\|^2.$$

We have the following results

- D is positive on $\psi\in\mathcal{K}_{I_1}$.
  (See also Guido and Longo).
- For every $\psi\in\mathcal{K}_{[a,b]\subset I_1}$, the subsequent inequalities hold

$$a^2(\psi,H\psi)\leq(\psi,C\psi)\leq b^2(\psi,H\psi)\ .$$

Some energy bounds

- If $\psi \in \mathcal{K}_{I_1}$ then $(\psi, D\psi) \geq 0$.

  Proof steps: $J\Delta^{1/2}\psi = \psi$ and $JDJ = -D$.
  $F(\alpha) := (\psi, D\Delta^{\alpha}\psi)$, $F(0) = -F(1)$,
  $\frac{d}{d\alpha}F(\alpha) \leq 0$ if $0 \leq \alpha \leq 1$. Then $F(0) \geq 0$.

- For every $\psi \in \mathcal{K}_{[a,b]\subset I_1}$, the subsequent inequalities hold

$$a^2(\psi, H\psi) \leq (\psi, C\psi) \leq b^2(\psi, H\psi)\ .$$

  Proof steps: $U := e^{-iaH}$, $\psi \in \mathcal{K}_{[a,b]}$ then $\varphi := U\psi \in \mathcal{K}_{I_1}$

$$(\psi, C\psi) = (\varphi, C+2aD+a^2 H\,\varphi) \geq (\varphi, 2aD+a^2 H\,\varphi) \geq (\psi, a^2 H\,\psi)$$

Modular coordinate

Idea: it seems possible to use *"energies"* for measuring positions.
In fact, since log is a monotone function

$$\log(a) \leq (\log\langle C\rangle_\psi - \log\langle H\rangle_\psi)/2 \leq \log(b) \ ,$$

where $\langle C\rangle_\psi = (\psi, C\psi)$.
Eventually we shall see that

$$X = \frac{1}{2}\log(H^{-1/2} C H^{-1/2})$$

NB The domain needs to be fixed properly.

- From H, C, D genearte a representation of $PSL(2,\mathbb{R})$ on $\mathcal{H}$.
  - Decompose $\mathcal{H}$ in irreducible representations $\mathcal{H} = \oplus_i \mathcal{H}_i$.

$$\widetilde{H} := \frac{H^2}{2}, \qquad \widetilde{D} := \frac{D}{2}, \qquad \widetilde{C} := \frac{H^{-1/2} C H^{-1/2}}{2}$$

  - Enjoy $sl(2,\mathbb{R})$ commutation relations.
  - There is a dense set of analytic vectors on every $\mathcal{H}_i$.
  - Generate a positive-energy unitary representation $\widetilde{U}$ of the **covering group** of $SL(2,\mathbb{R})$ on $\mathcal{H}$.
  - For the lowest eigenvalues of rotation gen. we have $\tilde{k} = k/2 + 1/4$
- Let $\psi \in \mathcal{K}_I$ where $I = [a, b] \subset I_1$ then

$$\frac{a^2}{2}\|\psi\|^2 < (\psi, \widetilde{C}\psi) < \frac{b^2}{2}\|\psi\|^2 \,.$$

Position Operator

Since the logarithm is also an operator monotone function, we get

$$X := \frac{1}{2} \log(2\widetilde{C}).$$

- It is self-adjoint on a suitable domain.
- It shows CCR with D:

$$[D, X] := i$$

- It is compatible with emerging locality: $\psi \in \mathcal{K}_{[a,b] \subset I_1}$

$$\log(a) \|\psi\|^2 \leq (\psi, X\psi) \leq \log(b) \|\psi\|^2$$

Massless scalar field on $\mathbb{R}_{1,1}$: coordinate of a Wedge

- 2D Minkowski: $ds^2 = -dt^2 + dx^2$,
- Massless KG equation has two modes, *in*- and *out*-
- One-particle Hilbert space is $L(\mathbb{R}^+, dE) \oplus L(\mathbb{R}^+, dE)$.
- On $L(\mathbb{R}^+, dE)$, the representation of the Möbius group is generated by:

$$H := E, \qquad D = -i\sqrt{E}\frac{d}{dE}\sqrt{E}, \qquad C = -\sqrt{E}\frac{d^2}{dE^2}\sqrt{E},$$

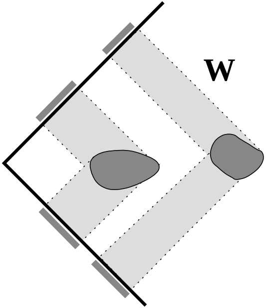
and the anti-unitary involution: the complex conjugation.

If we read them in the following coordinates: $\mathbb{R}_{1,1} := -dv\, du$

The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on wave-function $\partial_v \psi(v)$ reads:

$$U_g \partial_v \psi(v) = \frac{1}{(cv' + d)^2} \partial_{v'} \psi(v') \,, \qquad v' = \frac{dv - b}{a - cv}$$

- Emerging localization is compatible with that of the wedges.
- A Model for Quantum coordinates inside a wedge.
- The scheme, does not work for massive fields: the one particle Hilbert space is only one $L^2(\mathbb{R}^+, dh)$. $(h = p + \sqrt{p^2 + m^2})$
- In this case we get at most an operator measuring a spatial coordinate.



Summary

- Localization can arise from the group properties.
- Also in the case of Möbius covariant theory. *(Positive energy representation)*
- An operator representing a local coordinate arises modifying the energy and the conformal energy
  - CCR with generator of modular transformation.
  - expectation values on local wavefunction compatible with localization.