Komplexe Zahlen

$z \in \mathbb{C}$:

- $z = x + iy \qquad x = \operatorname{Re} z \in \mathbb{R}, \quad y = \operatorname{Im} z \in \mathbb{R}$

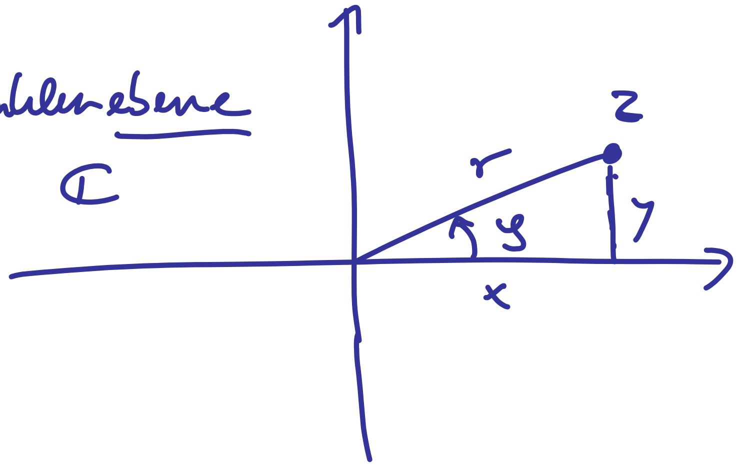
- $z = r\cos\varphi + i r \sin\varphi \qquad$ (Polardarstellung)

  mit $r = \sqrt{x^2+y^2}$

  und $\varphi = \begin{cases} \arccos \dfrac{x}{\sqrt{x^2+y^2}} & \text{für } y \geq 0 \\ 2\pi - \arccos \dfrac{x}{\sqrt{x^2+y^2}} & \text{für } y < 0 \end{cases}$

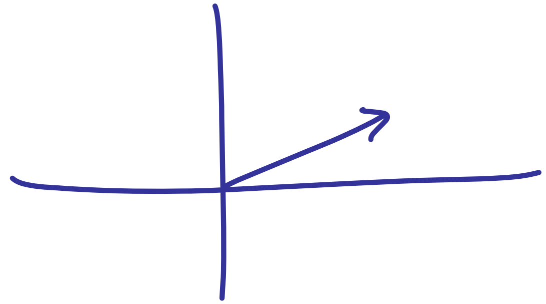
- $z = r e^{i\varphi} \qquad$ (Exponentialdarstellung)

Zahlenebene $\mathbb{C}$



Eulersche Formel:

$$e^{iy} = \cos y + i \sin y \qquad y \in \mathbb{R}$$



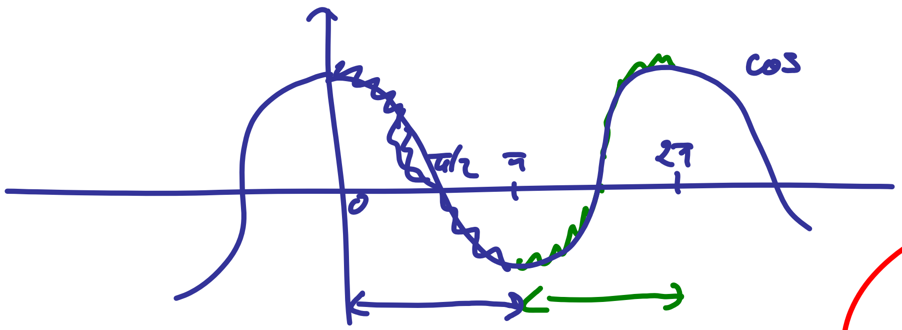
Exponentialdarstellungen:

$$\cos y = \frac{1}{2}\left(e^{iy} + e^{-iy}\right)$$

$$\sin y = \frac{1}{2i}\left(e^{iy} - e^{-iy}\right)$$

Exponentialdarstellung von z ($z = re^{iy}$) wichtig für:

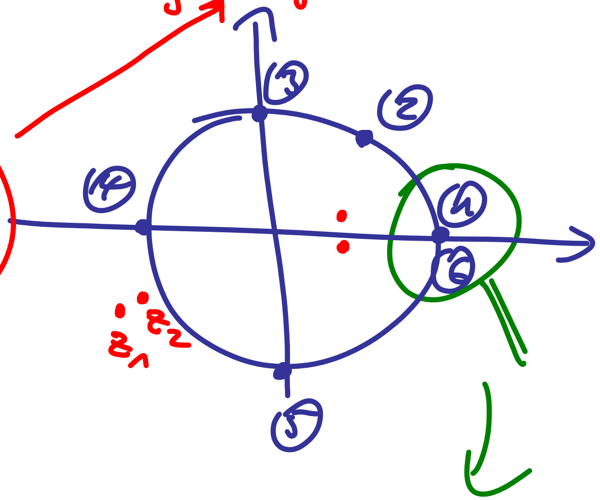
- $z_1 \cdot z_2 = r_1 \cdot r_2 \, e^{i(y_1 + y_2)} = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$
- $z^n = (x + iy)^n = (re^{iy})^n = r^n e^{iny}$
- $\ln z = \ln(re^{iy}) = \ln r + \ln(e^{iy}) = \ln r + iy$
- $\sqrt{z} = \sqrt{x + iy} = \sqrt{re^{iy}} = \sqrt{r}\sqrt{e^{iy}} = \sqrt{r}\, e^{iy/2}$

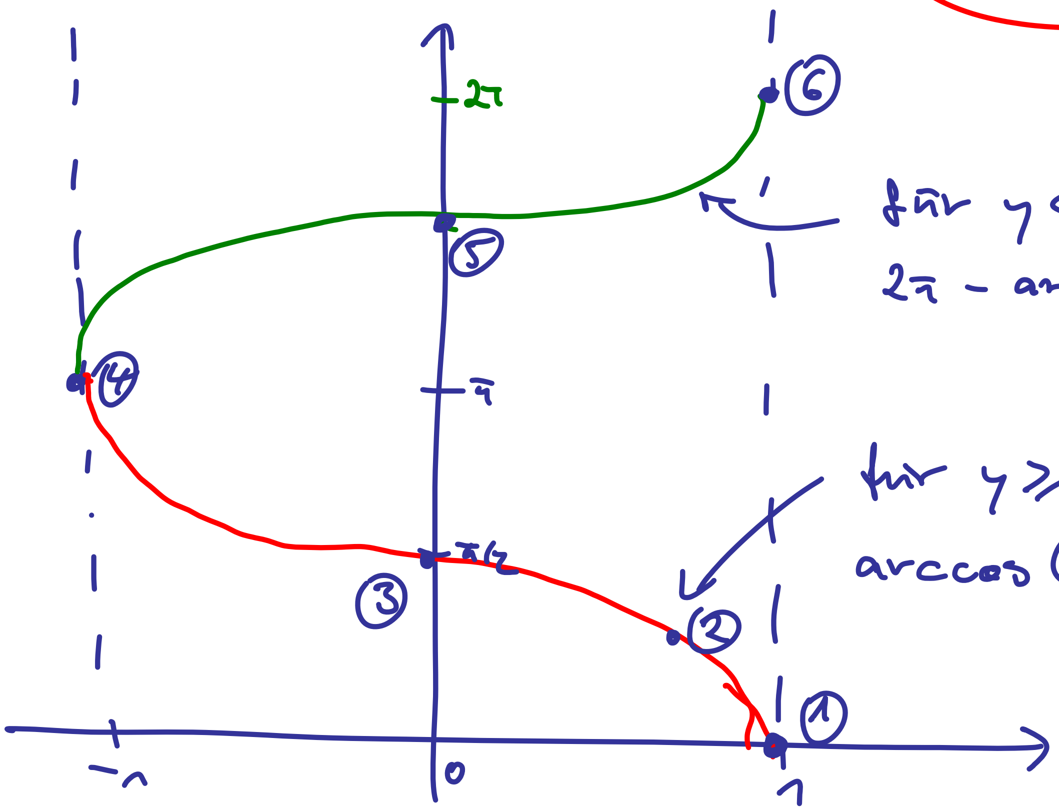


entlang eines Pfades in $\mathbb{C}$ variiert y stetig

$|z_1 - z_2| \ll 1$

$\Rightarrow |y_1 - y_2| \ll 1$



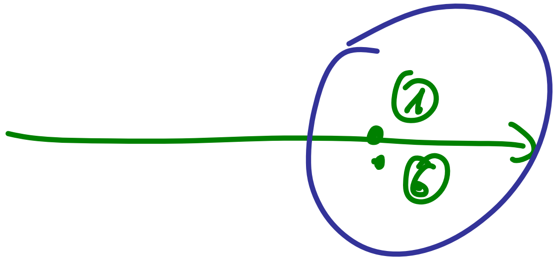


für $y < 0$

$2\pi - \arccos(x/r)$

für $y \geq 0$

$\arccos(x/r)$



falls $y = [0, 2\pi[$

$z \longmapsto \arg z$

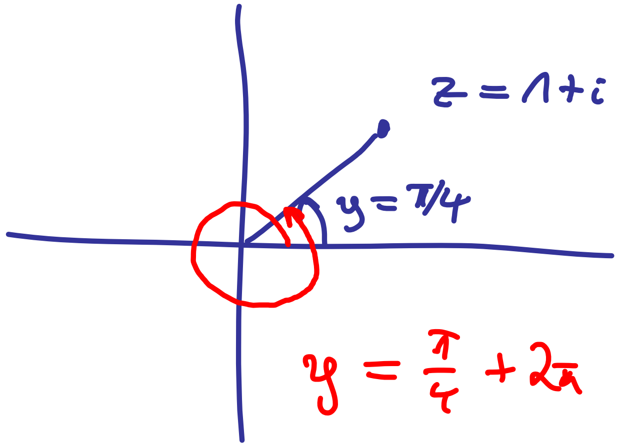
ist eindeutig

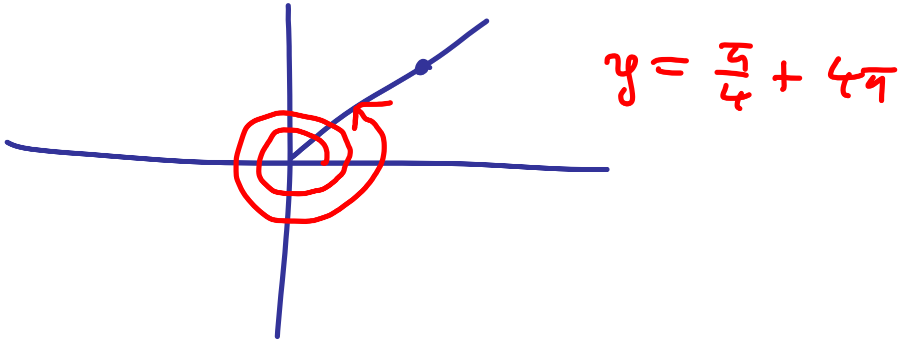
$\varphi \in [0, 2\pi[\quad \Rightarrow \quad z \mapsto \arg z = \varphi$ eindeutig

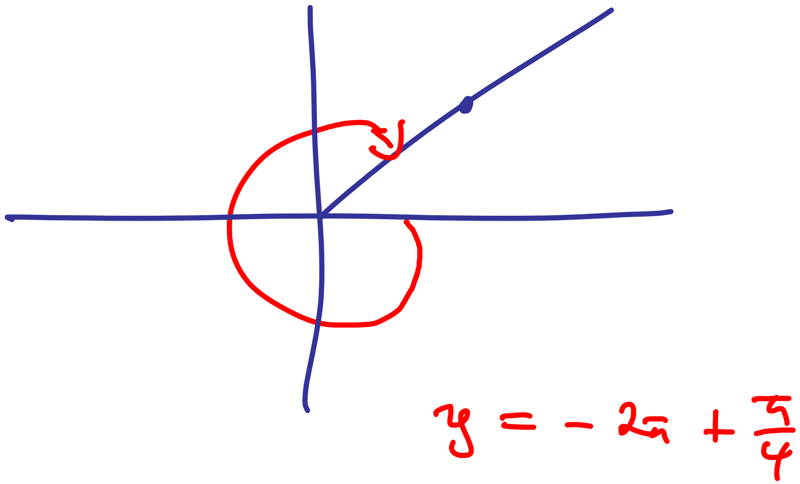
lässt man auch Mehrfachumläufe oder eine Umkehrung des Drehsinns zu, dann ist

$$\varphi \in \mathbb{R}$$

Bsp:



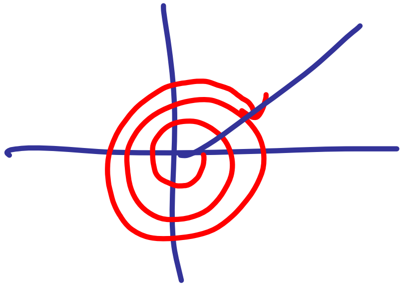




$$[0, 2\pi[\ni \varphi$$

$$\varphi_n = \varphi + n \cdot 2\pi$$

$$n \in \mathbb{Z}$$



$$z = re^{i\varphi} \stackrel{?}{=} re^{i\varphi_n} = re^{i(\varphi + n\cdot 2\pi)} = re^{i\varphi}\cdot \underbrace{e^{in2\pi}}_{\underbrace{\cos(2\pi\cdot n)}_{=1} + i\underbrace{\sin(2\pi\cdot n)}_{=0}} = re^{i\varphi} \quad \checkmark$$

$$\overset{n}{z} \mapsto \arg \overset{n}{z} = \varphi_n = \varphi + n\cdot 2\pi$$

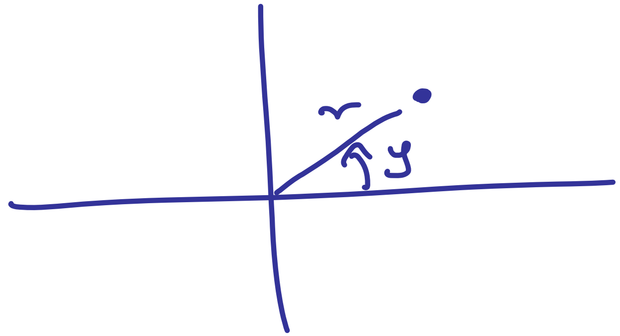
Wozu $\varphi \in \mathbb{R}$?

Einschränkung $\varphi \in [0, 2\pi[$ kann zu Fehlschlüssen verleiten beim Wurzelziehen!

Bsp: Die reellen Lösungen von

$$x^2 = 1$$

sind: $x = 1$, $x = -1$



Frage

Was sind die Lösungen von

$$z^2 = 1$$

in $\mathbb{C}$?

$z = r e^{i\varphi}$ $\quad (\varphi \in [0, 2\pi[\,)$ $\quad \Rightarrow \quad z^2 = r^2 e^{2i\varphi}$

$1 = 1\, e^{i0}$ $\quad (0 \in [0, 2\pi[\,)$

$z = r e^{i\varphi} = 1\, e^{i0} = 1$

$\Uparrow$

$\Rightarrow \quad r^2 e^{2i\varphi} = 1 \cdot e^{i0}$

$\Rightarrow \quad r^2 = 1$ und $2\varphi = 0$ $\quad \Rightarrow \quad r = 1, \; \varphi = 0$

?

Richtig! (basierend auf $y \in \mathbb{R}$)

$$z = re^{iy} \quad (y \in \mathbb{R}) \qquad \Rightarrow \qquad z^2 = r^2 e^{i2y}$$

$$1 = 1 e^{in \cdot 2\pi} \quad (n \in \mathbb{Z})$$

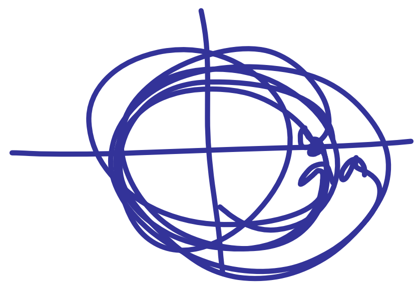
$$z^2 = 1 \quad \Rightarrow \quad r^2 e^{2iy} = 1 \cdot e^{n \cdot 2\pi i}$$

$$\Rightarrow \quad r = 1 \quad \text{und} \quad \underbrace{2iy = n \cdot 2\pi i}_{\Rightarrow \; y = n \cdot \pi}$$

$$z = re^{iy} = e^{iy} = e^{i\pi \cdot n} = \underline{\underline{\pm 1}}$$

$+1$ für n gerade
-1 für n ungerade

$$1 = e^{n \cdot 2\pi i} = \underset{\substack{\parallel \\ 1}}{\cos(n \cdot 2\pi)} + i \underset{\substack{\parallel \\ 0}}{\sin(n \cdot 2\pi)}$$



Bsp! $x^3 = 1$ (in $\mathbb{R}$) $\Rightarrow$ $x = 1$

$z^3 = 1$ (in $\mathbb{C}$)

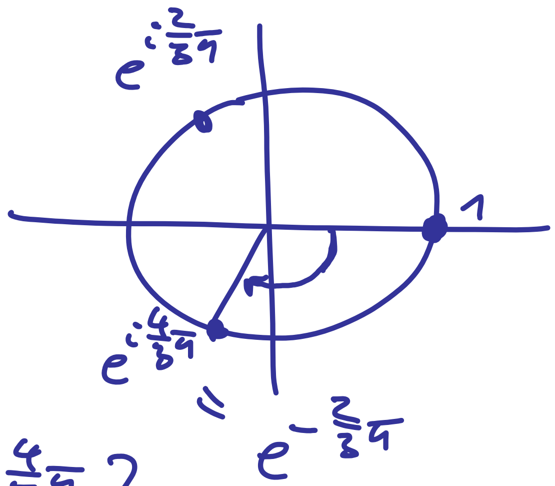
$\varphi \in \mathbb{R}$ $z = r e^{i\varphi} \Rightarrow z^3 = r^3 e^{3i\varphi}$ ($\varphi \in \mathbb{R}$)

$1 = 1\, e^{i\cdot 2\pi n}$ ($n \in \mathbb{Z}$)

$$z^3 = 1 \Rightarrow r^3 e^{3i\varphi} = 1 \cdot e^{i\,2\pi n}$$

$$\Rightarrow r^3 = 1 \text{ und } 3\varphi = 2\pi \cdot n$$

$$\Rightarrow r = 1 \text{ und } \varphi = \tfrac{2}{3}\pi \cdot n$$



$$z = r e^{i\varphi} = e^{i\frac{2}{3}\pi\cdot n} \in \left\{1, e^{i\frac{2}{3}\pi}, e^{i\frac{4}{3}\pi}\right\}$$

$z^3 = 1$ hat in $\mathbb{C}$ genau 3 Lösungen!

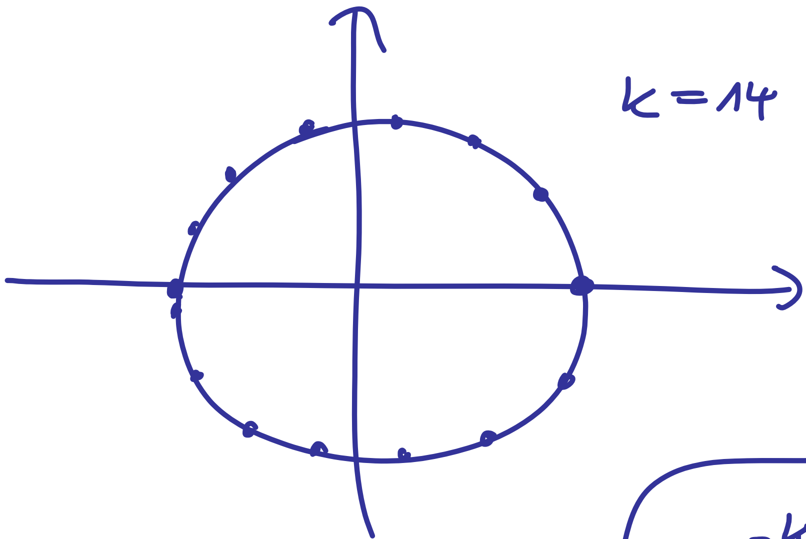
$$z^3 = -1 \Rightarrow z^6 = 1$$

$z^3 = 0$

$\underbrace{z^2 - 2z + 1}_{=(z-1)^2} = 0$

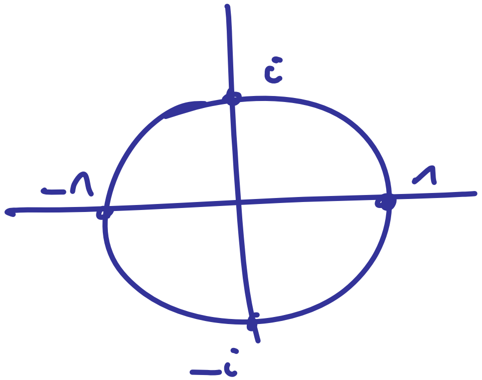
$$z^k = 1 \quad \Rightarrow \quad z = e^{\frac{2\pi i}{k}\cdot n} \qquad (n \in \mathbb{Z})$$

$k = 14$



In $\mathbb{C}$ hat die Gleichung $z^k = 1$ k verschiedene „Einheitswurzeln"

$k = 4$



$$z^k = w = |w|\, e^{i\varphi}$$

$$\Rightarrow \quad z = \sqrt[k]{|w|}\; e^{i\frac{\varphi}{k} + i\frac{2\pi}{k}n} \qquad n \in \mathbb{Z}$$

denn:

$$z^k = \left(\sqrt[k]{|w|}\; e^{i\frac{\varphi}{k} + i\frac{2\pi}{k}n}\right)^k$$

$$= |w|\, e^{\left(i\frac{\varphi}{k} + i\frac{2\pi}{k}n\right)\cdot k}$$

$$= |w|\, e^{i\varphi}\, e^{2\pi i\cdot n} = |w|\, e^{i\varphi} = w$$