i : eine (formale) Lösung der Gleichung $z^2 = -1$

<u>komplexe Zahlen</u> $\mathbb{C} := \{ x + iy \mid x, y \in \mathbb{R} \}$

Rechenregeln: $z_1 + z_2 := (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

es gilt: $\mathbb{R} = \{ x + iy \in \mathbb{C} \mid y = 0 \} = \{ z \in \mathbb{C} \mid \operatorname{Im} z = 0 \} \Rightarrow \mathbb{R} \subset \mathbb{C}$

$\operatorname{Re} z = \operatorname{Re}(x + iy) := x$ Realteil von z

$\operatorname{Im} z = \operatorname{Im}(x + iy) := y$ Imaginärteil von z

$\bar{z} = z^* := x - iy$ die zu z komplex konjugierte Zahl

$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

$z_1 \cdot z_2 = z_2 \cdot z_1$

$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$

$z_1 + z_2 = z_2 + z_1$

$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$1 \cdot z = z \quad 0 \cdot z = 0$

$0 + z = z$

etc.

was ist $\frac{1}{z}$?

$$x_1 y_2 + i y_1 y_2 = (x_1 + iy_1) \cdot y_2 \neq x_1 + i y_1 \cdot y_2$$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\cdot\frac{x-iy}{x-iy} = \frac{x-iy}{x^2-(iy)^2} = \frac{x-iy}{x^2+y^2}$$

$$\frac{1}{x+iy} = \frac{\bar z}{z\cdot\bar z}$$

- $\overline{z_1+z_2} = \bar z_1 + \bar z_2$

dem: $\overline{z_1+z_2} = \overline{x_1+iy_1+x_2+iy_2} = \overline{(x_1+x_2)+i(y_1+y_2)}$

$= x_1+x_2-i(y_1+y_2)$

$\bar z_1+\bar z_2 = \overline{x_1+iy_1} + \overline{x_2+iy_2} = x_1-iy_1+x_2-iy_2$

$= x_1+x_2-i(y_1+y_2)$

- $\overline{z_1\cdot z_2} = \bar z_1\cdot\bar z_2$

dem: $\overline{z_1\cdot z_2} = x_1x_2-y_1y_2-i(x_1y_2+x_2y_1)$

$$\bar{z}_1 \cdot \bar{z}_2 = (x_1 - iy_1)\cdot(x_2 - iy_2) = x_1x_2 - y_1y_2 \underbrace{- iy_1x_2 - ix_1y_2}_{-i(x_1y_2 + x_2y_1)} \quad \checkmark$$

- $\frac{1}{2}(z+\bar{z}) = \frac{1}{2}(x+iy + x-iy) = \frac{1}{2}2x = x = \operatorname{Re} z \Rightarrow$

$$\boxed{\operatorname{Re} z = \frac{1}{2}(z+\bar{z})}$$

- $\boxed{\operatorname{Im} z = \frac{1}{2i}(z-\bar{z})}$

denn: $\frac{1}{2i}(z-\bar{z}) = \frac{1}{2i}(x+iy-(x-iy))$

$= \frac{1}{2i}(x+iy - x + iy)$

$= \frac{1}{2i}2iy = y = \operatorname{Im} z$

- $z = \bar{z} \Rightarrow z \in \mathbb{R}$

$\Rightarrow x+iy = x-iy$

$\Rightarrow iy = -iy$

$\Rightarrow y = -y$

$\Rightarrow y = 0$

$\Rightarrow z = x+iy = x \in \mathbb{R} \quad \checkmark$

- $z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2$
- $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$ $\qquad \frac{1}{i} = \frac{\bar{i}}{i\bar{i}} = \frac{-i}{i(-i)} = \frac{-i}{-i^2} = \frac{-i}{1} = -i$
- $\left(\frac{1}{z_1 \cdot z_2 + z_3} + z_4\right)^*$

$$= \frac{1}{z_1^* \cdot z_2^* + z_3^*} + z_4^*$$

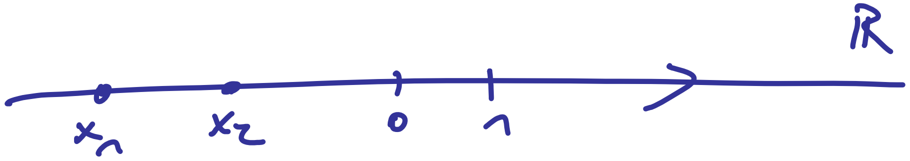
$$\overline{\left(\frac{1}{z}\right)} = \overline{\left(\frac{\bar{z}}{z \cdot \bar{z}}\right)} = \overline{\left(\frac{\bar{z}}{\alpha}\right)} = \frac{1}{\alpha}\bar{\bar{z}} = \frac{z}{\alpha} = \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}$$

$\alpha = z\bar{z} \in \mathbb{R}$

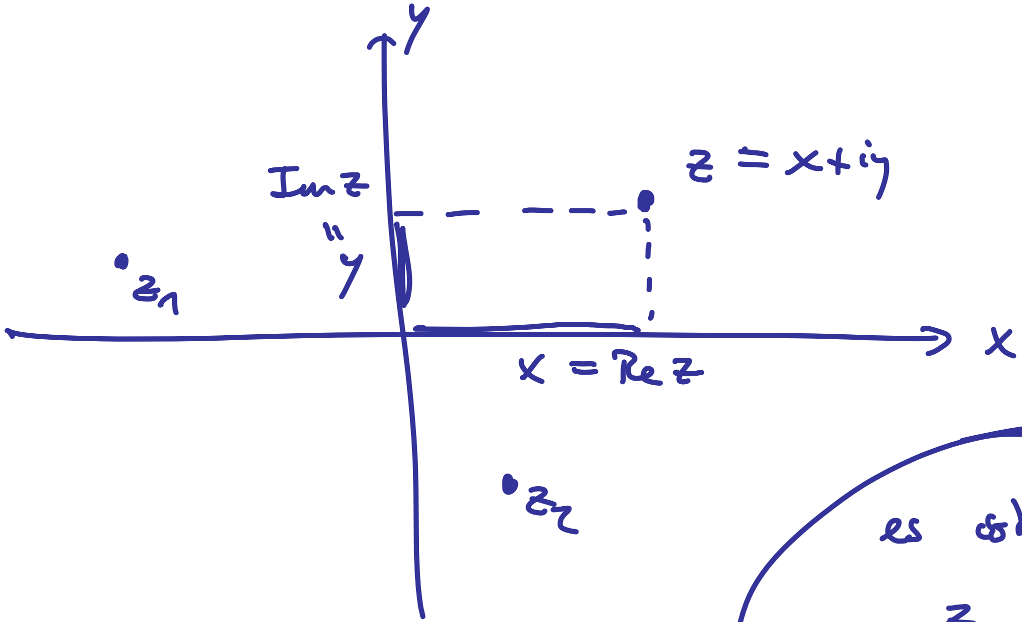
$$\bar{\bar{z}} = \overline{\overline{x+iy}} = \overline{x-iy} = x+iy = z$$

$\mathbb{R}$ kann dargestellt werden durch einen Zahlenstrahl

$x_1 < x_2$

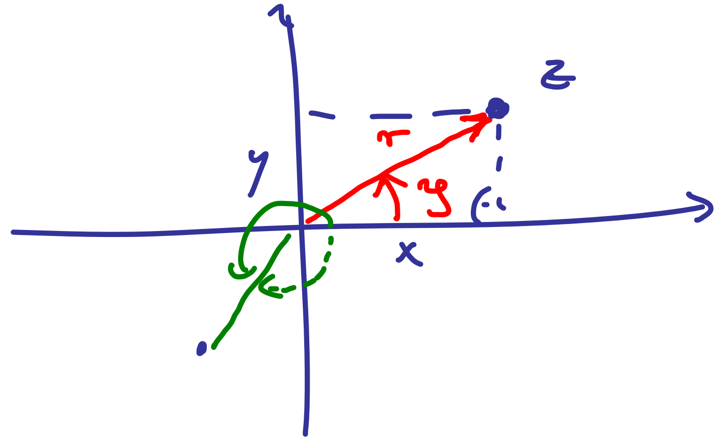


$\mathbb{C} = \{x+iy \mid x,y \in \mathbb{R}\}$ kann dargestellt werden durch eine 2-dim. Zahlenebene $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$



Komplexe (Gaußsche) Zahlenebene

es ist sinnlos, $z_1 > z_2$ ($z_1 < z_2$) zu schreiben



Polardarstellung einer komplexen Zahl: r, φ

Definitionen

Betrag einer komplexen Zahl

$$|z| := r$$

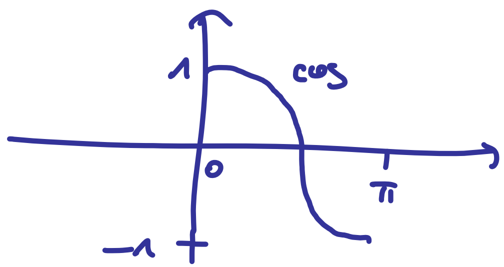
Argument einer komplexen Zahl

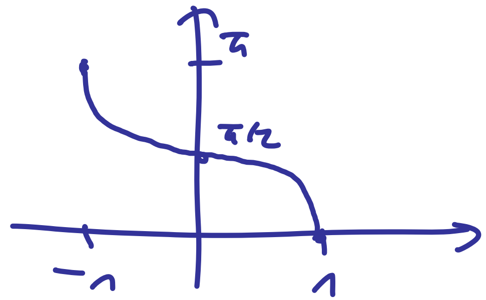
$$\arg z := \varphi \in [0, 2\pi[$$

(1)
$$x = r \cdot \cos \varphi$$
$$y = r \cdot \sin \varphi$$

(2)
$$r = \sqrt{x^2 + y^2}$$

$$\varphi = \arccos \frac{x}{\sqrt{x^2+y^2}} \quad (y \geq 0)$$





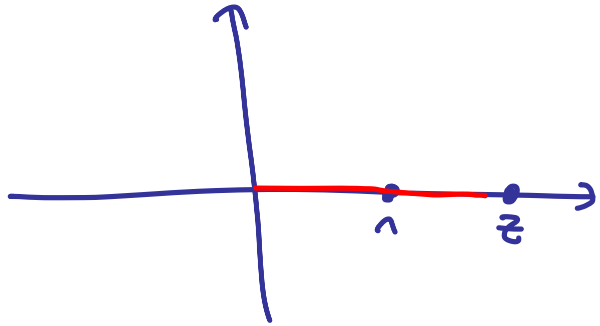
$$\varphi = 2\pi - \arccos \frac{x}{r} \quad (y < 0)$$

Bsp: $z = 2$

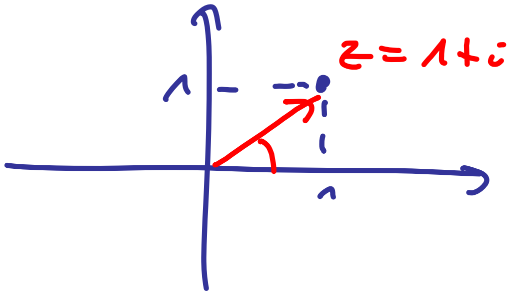
$\operatorname{Re} z = 2 \quad \operatorname{Im} z = 0$

$r = \sqrt{2^2 + 0^2} = 2$

$\varphi = \arccos \frac{x}{\sqrt{x^2+y^2}} = \arccos \frac{2}{2} = \arccos 1 = 0$



Bsp: $z = 1 + i$

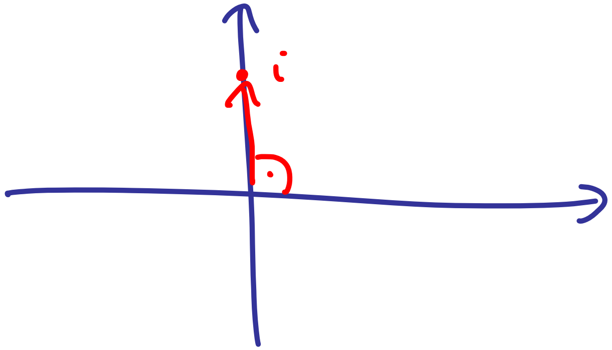


$\operatorname{Re} z = 1 \,, \quad \operatorname{Im} z = 1$

$r = \sqrt{x^2+y^2} = \sqrt{z\overline{z}} = \sqrt{2}$

$\varphi = \arccos \frac{1}{\sqrt{1+1}} = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$

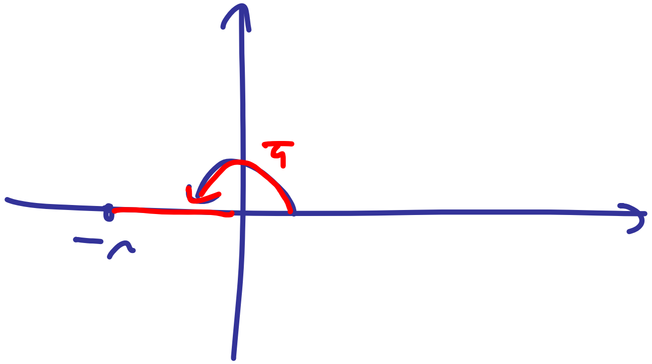
Bsp: $z = i$



$r = 1$

$\varphi = \pi/2$

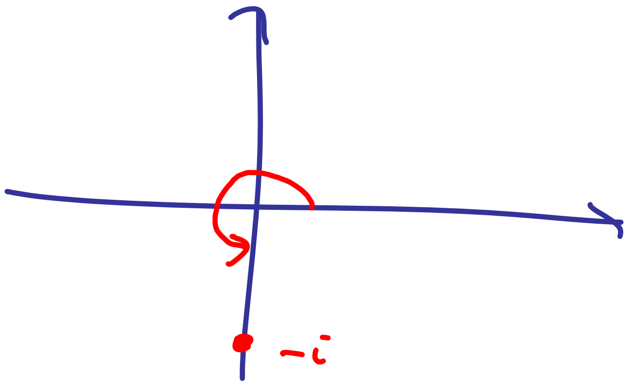
Bsp: $z = -1$



$r = 1$

$\varphi = \pi$

Bsp: $z = -i$



$r = 1$

$$\varphi = \frac{3\pi}{2} = 2\pi - \arccos(0)$$

$$= 2\pi - \frac{\pi}{2}$$

$$= \frac{3\pi}{2}$$

Exponentialform einer komplexen Zahl

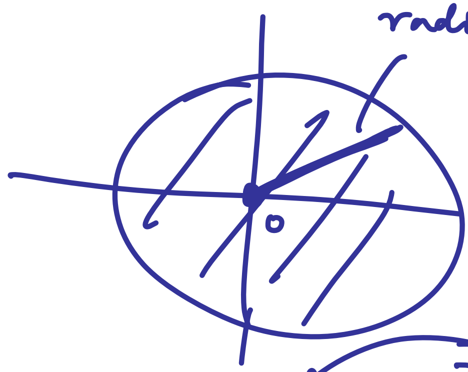
für $x \in \mathbb{R}$ gilt

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

Konvergenz-radius R



für e^z ist $R = \infty$

Def.:

$$e^{iy} = 1 + \frac{1}{1!}iy + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \dots$$

$$= 1 + i\frac{1}{1!}y - \frac{1}{2!}y^2 - i\frac{1}{3!}y^3 + \frac{1}{4!}y^4 + i\frac{1}{5!}y^5 + \dots$$

$$= 1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \dots + i\left(\frac{1}{1!}y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \dots\right)$$

$$= \cos y + i \sin y$$

Eulersche Formel

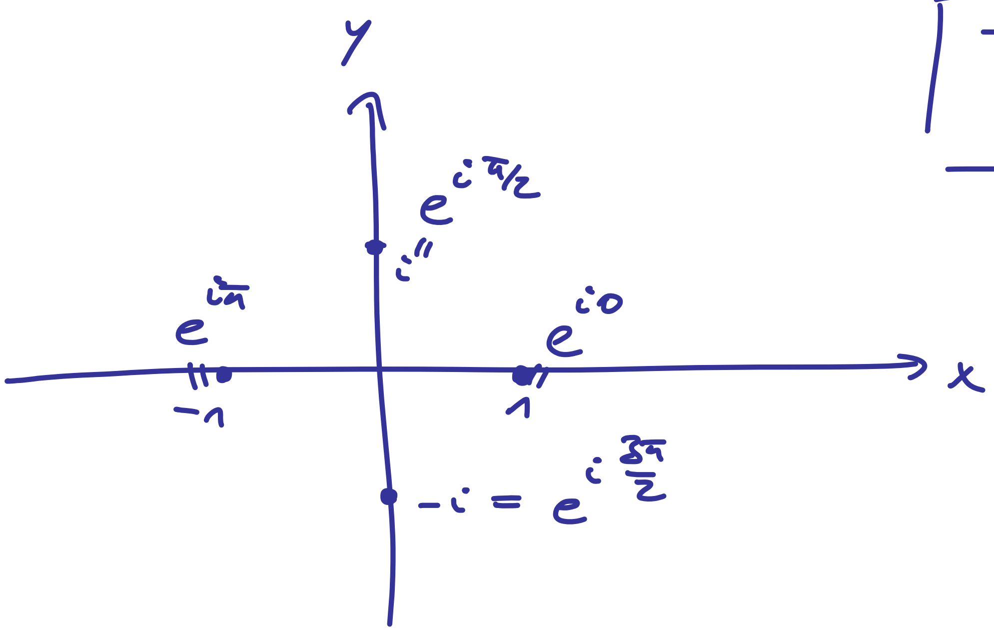
$z \in \mathbb{C}$ $\qquad z = x + iy = r(\cos\varphi + i \sin\varphi) = r e^{i\varphi} = r e^{i \arg z}$

geometrische Darlegung

Eulersche Formel

$\bar{z} = x - iy = r(\cos\varphi - i\sin\varphi) = r e^{-i\varphi}$

$-\sin\varphi = \sin(-\varphi)$
$\cos\varphi = \cos(-\varphi)$



$-1 = e^{i\pi}$

$$\frac{1}{2}\left(e^{iy}+e^{-iy}\right) = \frac{1}{2}\left(\cos y + i\sin y + \cos(-y) + i\sin(-y)\right)$$

$$= \frac{1}{2}\left(\cos y + \cancel{i\sin y} + \cos y - \cancel{i\sin y}\right)$$

$$= \cos y$$

$$\boxed{\cos y = \frac{1}{2}\left(e^{iy}+e^{-iy}\right)}$$

$$\boxed{\sin y = \frac{1}{2i}\left(e^{iy}-e^{-iy}\right)}$$

$$\frac{1}{2i}\left(e^{iy}-e^{-iy}\right)$$

$$\shortparallel$$

$$\frac{1}{2i}\left(\cancel{\cos y} + i\sin y - \cancel{\cos y} + i\sin y\right)$$

$$\shortparallel$$

$$\sin y$$

$$\cos y \cdot \sin y = \frac{1}{2}\left(e^{iy}+e^{-iy}\right)\cdot\frac{1}{2i}\left(e^{iy}-e^{-iy}\right)$$

$$= \frac{1}{2\cdot 2i}\left(e^{2iy}-e^{-2iy}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2i}\left(e^{i(2y)}-e^{-i(2y)}\right)\right) = \frac{1}{2}\sin(2y)$$

$$z_1 = r_1 e^{i\varphi_1} \qquad z_2 = r_2 e^{i\varphi_2}$$

$$z_1 \cdot z_2 = r_1 r_2 e^{i\varphi_1} e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1+\varphi_2)} = |z_1||z_2| e^{i(\varphi_1+\varphi_2)}$$

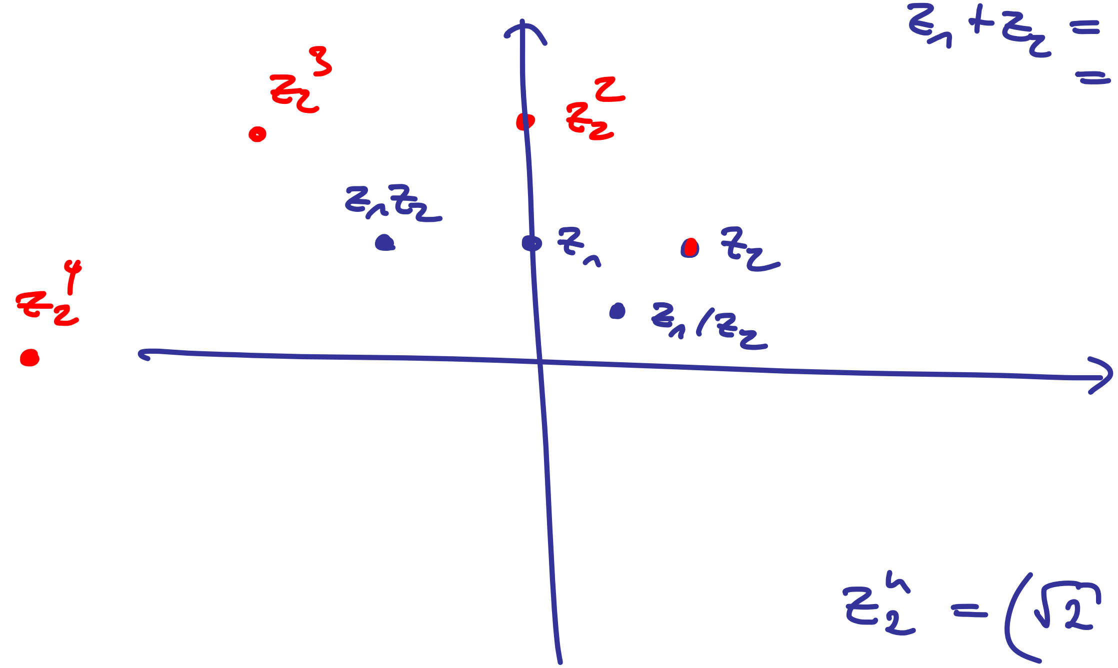
$$z_1/z_2 = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1-\varphi_2)}$$

$$z^n = (r e^{i\varphi})^n = r^n (e^{i\varphi})^n = |z|^n e^{in\varphi}$$

$$z_1 = i$$
$$z_2 = 1+i$$

$$z_1 + z_2 = 1+i+i$$
$$= 1+2i$$

$$z_1 \cdot z_2 = e^{i\frac{\pi}{2}} \cdot \sqrt{2}\, e^{i\frac{\pi}{4}}$$
$$= \sqrt{2}\, e^{i\frac{3}{4}\pi}$$



$$\frac{z_1}{z_2} = \frac{e^{i\pi/2}}{\sqrt{2}\, e^{i\pi/4}} = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}}$$

$$z_2^n = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^n$$
$$= \sqrt{2}^n e^{in\frac{\pi}{4}}$$