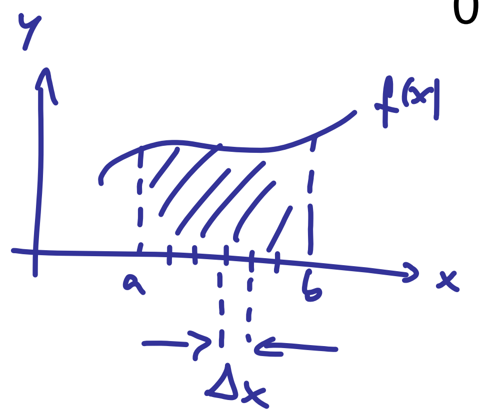
3 Integralrechnung, 3.1 Grundlagen



- (bestimmtes) Integral: $\int_a^b f(x)\,dx = \int_a^b dx\, f(x)$

$$\overset{\text{Def.}}{:=} \lim_{N\to\infty} \sum_{k=1}^{N} f(a+k\cdot\Delta x)\cdot \Delta x \qquad \left(\Delta x = \frac{b-a}{N}\right)$$

- f heißt integrierbar, falls der Limes existiert
- es gilt für $f: D \to \mathbb{R}$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ falls $F: D \to \mathbb{R}$ eine Stammfunktion von f ist

- F ist eine St. fkt. zu f, falls $F'(x) = f(x)$
- ist F eine St. fkt., so ist $F(x) + c$ ebenfalls eine St. fkt. ($c \in \mathbb{R}$ beliebig)
- Hauptsatz der D.+I.-R.:

$$\frac{d}{dx}\int_a^x f(y)\,dy = f(x) \qquad F(x) = \int_a^x f(y)\,dy$$ ist eine St. fkt. (a beliebig)

3.2 Integrationstechniken

Partielle Integration: Umkehrung der Produktregel der D.-R.

$$(u(x)\cdot v(x))' = u'(x)\cdot v(x) + u(x)\,v'(x)$$

$$\Rightarrow \int_a^b (\underbrace{u(x)v(x)})'\,dx = \int_a^b \left(u'(x)v(x) + u(x)v'(x) \right) dx$$

$u(x)\,v(x)$ ist St.fkt von $(u(x)v(x))'$

$$\Rightarrow u(b)\,v(b) - u(a)\,v(a) =: u(x)v(x)\Big|_a^b = \int_a^b u'(x)v(x)\,dx + \int_a^b u(x)\,v'(x)\,dx$$

$$\Rightarrow \boxed{\int_a^b u'(x)\,v(x)\,dx = u(x)\,v(x)\Big|_a^b - \int_a^b u(x)\,v'(x)\,dx}$$

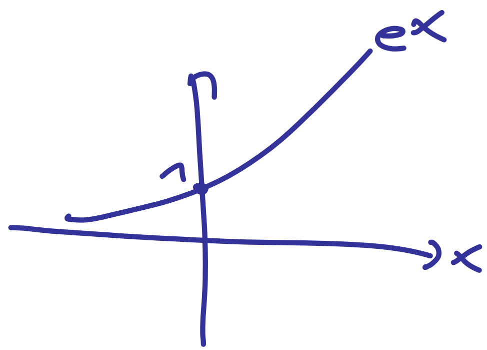
partielle Integration

Bsp: $\int_{-\infty}^{0} dx\, x\, e^x = \int_{-\infty}^{0} x\, e^x\, dx = \int_{-\infty}^{0} \underset{u'(x)}{e^x}\, \underset{v(x)}{x}\, dx$

$u(x) = e^x$
$v'(x) = 1$

$$= e^x \cdot x \Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^x \cdot 1\, dx$$

$$= \lim_{a \to -\infty} \left(e^x \cdot x \Big|_a^0 - \int_a^0 e^x\, dx \right)$$



$$= \underbrace{e^0 \cdot 0}_{0} - 0 - \underbrace{e^x \Big|_{-\infty}^{0}}_{1-0} = -1$$

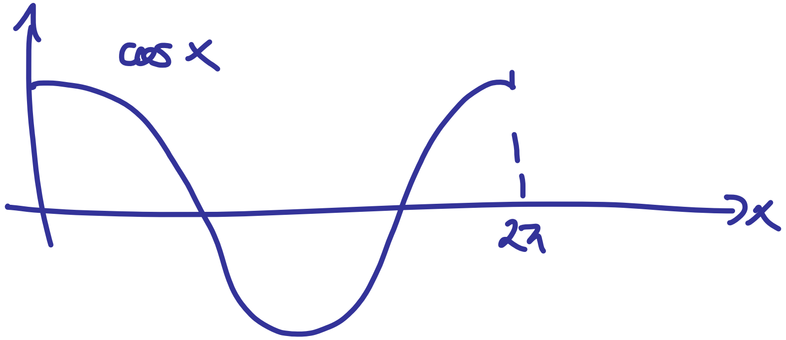
Bsp.: $\int_0^{2\pi} \cos^2 x \, dx = \int_0^{2\pi} \underset{u'(x)}{\cos x} \cdot \underset{v(x)}{\cos x} \, dx$

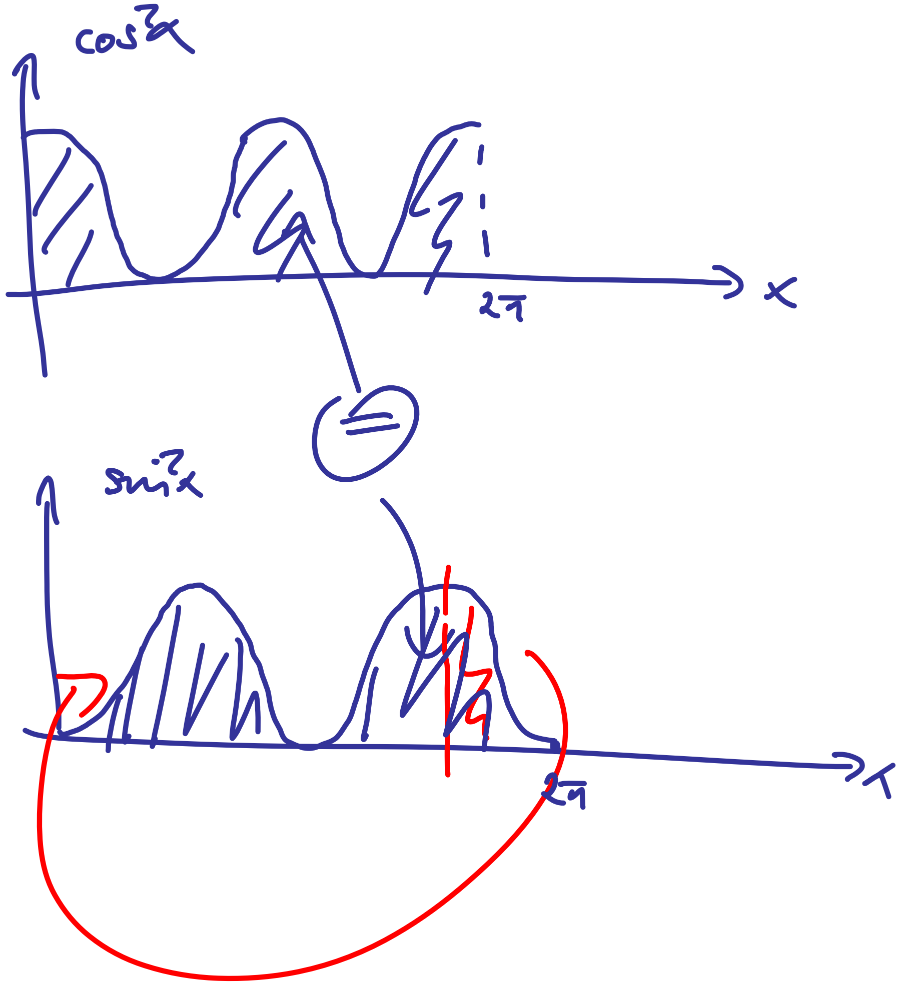
$$= \underbrace{\sin x \cdot \cos x \Big|_0^{2\pi}}_{0-0} - \int_0^{2\pi} \sin x \cdot (-\sin x) \, dx = \int_0^{2\pi} \sin^2 x \, dx$$

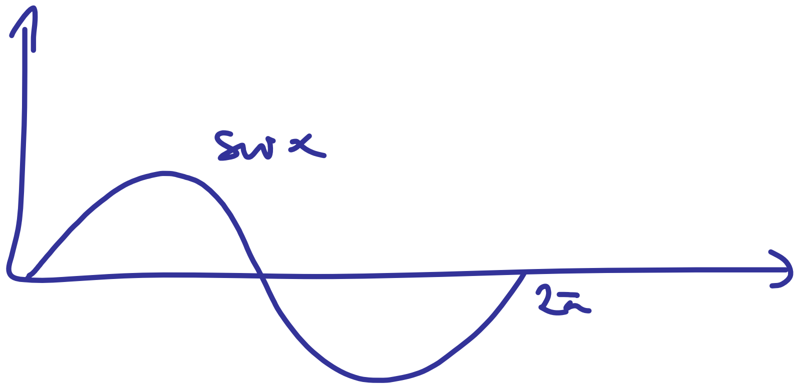
$$= \int_0^{2\pi} (1 - \cos^2 x) \, dx = \underbrace{\int_0^{2\pi} 1 \, dx}_{2\pi} - \int_0^{2\pi} \cos^2 x \, dx$$

$\sin^2 x + \cos^2 x = 1$

$$2 \cdot \int_0^{2\pi} \cos^2 dx = 2\pi \Rightarrow \boxed{\int_0^{2\pi} \cos^2 x \, dx = \pi}$$







$$\int_0^{2\pi} \cos^2 x \, dx = \int_0^{2\pi} \sin^2 dx$$

$$\Rightarrow \int_0^{2\pi} \cos^2 dx = \frac{1}{2}\int_0^{2\pi} \cos^2 x \, dx + \frac{1}{2}\int_0^{2\pi} \overset{\sin^2 x}{\cancel{\cos^2 x}} \, dx = \int_0^{2\pi} \frac{1}{2} \, 1 \, dx = \pi$$

Bsp:

$$\int_a^b \ln x \, dx = \int_a^b \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v} \, dx = x \cdot \ln x \Big|_a^b - \int_a^b x \cdot \frac{1}{x} \, dx$$

$$= \underline{b \ln b - a \ln a - (b-a)}$$

schreibe: $y = b$

$$\int_a^y \ln x \, dx = F(y) \Rightarrow \frac{d}{dy} F(y) = \ln y$$

$$F(y) = y \ln y - a \ln a - y + a$$

$$F(y) = y \ln y - y + c \qquad (c \in \mathbb{R} \text{ beliebig})$$

$$\boxed{F(x) = x \ln x - x + c}$$

$$\frac{d}{dx} F(x) = \frac{d}{dx}\left(x \ln x - x + c\right) = \ln x + x \frac{1}{x} - 1 + 0 = \ln x$$

Substitutionsregel

basiert auf der Umkehrung der Kettenregel

$$\boxed{\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(y)\, dy}$$

Beweis:

links: $$\int_a^b \underbrace{f(g(x)) \cdot g'(x)}\, dx = \int_a^b (F \circ g)'(x)\, dx = \underline{F(g(b)) - F(g(a))}$$

$$\frac{d}{dx}\left(F \circ g(x)\right) = \frac{d}{dx} F(g(x)) \underset{\text{Kettenregel}}{=} F'(g(x)) \cdot g'(x) = \underline{\underline{f(g(x)) \cdot g'(x)}}$$

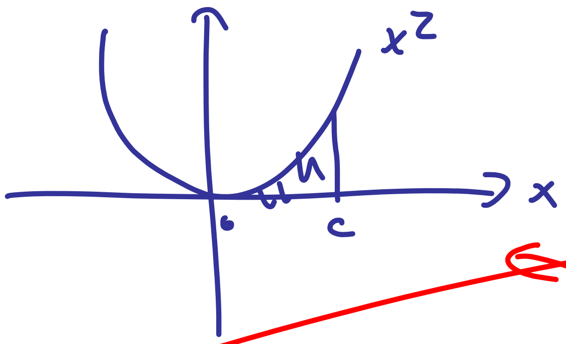
rechts: $$\int_{g(a)}^{g(b)} f(y)\, dy = \underline{F(g(b)) - F(g(a))}$$

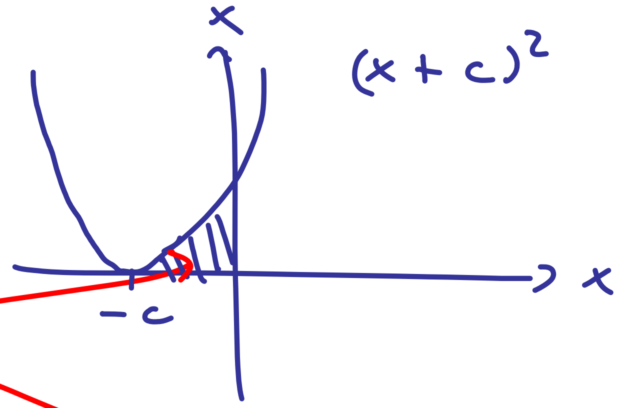
Bsp:

$$\int_a^b f(x+c)\,dx = \int_a^b f(\underbrace{x+c}_{g(x)})\cdot \underbrace{1}_{g'(x)}\cdot dx = \int_{a+c}^{b+c} f(y)\,dy$$

$g(x) = x+c$

$g'(x) = 1$





$(x+c)^2$ $\quad c>0$

$$\int_{-c}^{0} (x+c)^2\,dx = \quad \cdot \; - \; - \; \cdot \quad = \int_{-c+c}^{0+c} x^2\,dx = \int_0^c x^2\,dx$$

Bsp:

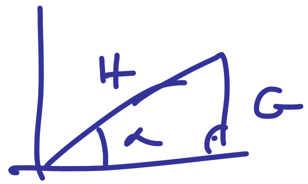
$$\int_0^{\sqrt{\pi/2}} \underbrace{2x}_{g'(x)} \cdot \underbrace{\cos x^2}_{f(g(x))}\, dx = \int_0^{\pi/2} \cos y\, dy = \sin x \Big|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0$$

mit $g(x) = x^2$

$$= 1 - 0 = 1$$

$$g(0) = 0$$

$$g\left(\sqrt{\pi/2}\right) = \pi/2$$



$$\int_{x_1}^{x_2} f(g(x)) \cdot g'(x)\, dx = \int f(\underbrace{g(x)}_{y}) \cdot \frac{dy}{\cancel{dx}} \cdot \cancel{dx} = \int_{y_1}^{y_2} f(y)\, dy$$

dx kürzen

mit dx erweitern

$$y = g(x)$$

$$g'(x) = \frac{d}{dx} g(x) = \frac{dg}{dx}(x) = \frac{dy}{dx}$$

$$y_2 = g(x_2)$$

$$y_1 = g(x_1)$$

$$\int_0^{\sqrt{\pi/2}} \underline{2x} \cdot \cos x^2 \, dx = \int_{y_1}^{y_2} \underline{2 \cdot \sqrt{y}} \cdot \underline{\cos y} \, \frac{dx}{dy} \, \underline{dy}$$

Substitution: $y = x^2$

$$\Rightarrow x = \sqrt{y}$$

?

$$\frac{dx}{dy} = \frac{d\sqrt{y}}{dy} = \frac{1}{2\sqrt{y}}$$

$$\frac{d}{dy} y^{1/2} = \frac{1}{2} y^{-1/2} = \frac{1}{2} \frac{1}{\sqrt{y}}$$

$$= \int \cancel{2} \cancel{\sqrt{y}} \cos y \, \cancel{\frac{1}{2}} \cancel{\frac{1}{\sqrt{y}}} \, dy = \int_{0^2}^{\sqrt{\pi/2}^2} \cos y \, dy = \int_0^{\pi/2} \cos y \, dy$$

Bsp: $\displaystyle\int_0^{\pi/\omega} \sin(\underbrace{\omega t})\, dt = \int_{y_1}^{y_2} \sin y \,\frac{dt}{dy}\, dy = \int_0^{\pi} \sin y \cdot \frac{1}{\omega}\, dy$

$\displaystyle = \frac{1}{\omega}(-\cos y)\Big|_0^{\pi} = \frac{1}{\omega}(1+1) = \frac{2}{\omega}$

substituiere: $y = g(t) = \omega t$

$y_1 = g(0) = 0$

$y_2 = g(\pi/\omega) = \pi$

$y = \omega t \qquad \dfrac{dy}{dt} = \omega \qquad \dfrac{dt}{dy} = \dfrac{1}{\frac{dy}{dt}} = \dfrac{1}{\omega}$

$\Downarrow$

$t = \dfrac{1}{\omega}\cdot y \qquad \dfrac{dt}{dy} = \dfrac{1}{\omega}$

Bsp:

$$\int_a^b x^3 e^{-ax^2} dx = \int \left(-\frac{y}{a}\right)^{3/2} e^y \frac{dx}{dy} dy$$

$$f(x) = \underline{-ax^2 = y}$$

$$\Rightarrow \quad x = \sqrt{-y/a}$$

$$\Rightarrow \quad x^3 = \left(-y/a\right)^{3/2}$$

$$\frac{dx}{dy} = \frac{d}{dy}\sqrt{-\frac{y}{a}} = \frac{1}{2\sqrt{-y/a}} \cdot \left(-\frac{1}{a}\right)$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{-2ax} = \underline{\frac{1}{-2a}\,\frac{1}{\sqrt{-y/a}}}$$

$$= \int \left(-\frac{y}{a}\right)^{3/2} e^y \frac{1}{-2a} \left(-\frac{y}{a}\right)^{-1/2} dy$$

$$= \int -\frac{y}{a} e^y \frac{1}{-2a} dy$$

$$= \frac{1}{2a^2} \int_{-a^3}^{-ab^2} y e^y dy$$

← wende part. Int. an!