Wdh:

- $f : \mathbb{D}_f \to \mathbb{W}_f \qquad g : \mathbb{D}_g \to \mathbb{W}_g$

  $x \mapsto f(x) = y \qquad y \mapsto g(y)$

  Falls $f(\mathbb{D}_f) \subset \mathbb{D}_g$ ist $g \circ f$ definiert:

  $g \circ f : \mathbb{D}_f \to \mathbb{W}_g$

  $x \mapsto (g \circ f)(x) = g(f(x)) = g(y)$

Bsp:

$\cos((x+3)^2) = (h \circ g \circ f)(x)$

mit

$h(x) = \cos x$

$g(x) = x^2$

$f(x) = x + 3$

$a(bc) = (ab)c$

- $(g \circ f)(x) \neq (f \circ g)(x)$ (i. allg.) nicht kommutativ

- $g \circ f$ kann mit $h : \mathbb{D}_h \to \mathbb{W}_h$ verkettet werden, falls

  $(g \circ f)(\mathbb{D}_f) \subset \mathbb{D}_h$

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$$

$$\|$$

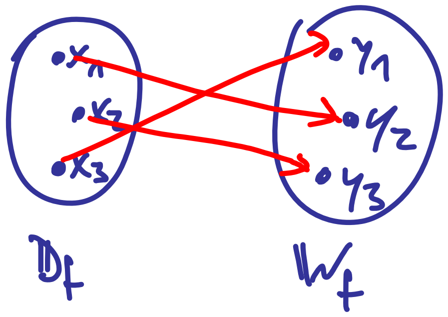
$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$$

$$\Rightarrow \quad h \circ (g \circ f) = (h \circ g) \circ f$$

assoziativ

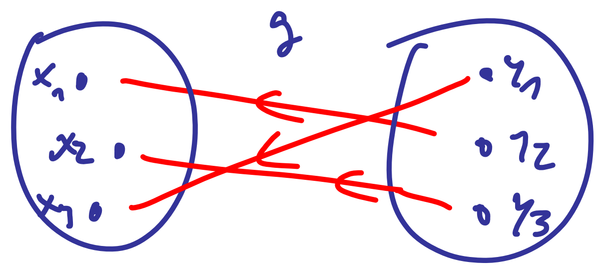
Umkehrfunktion

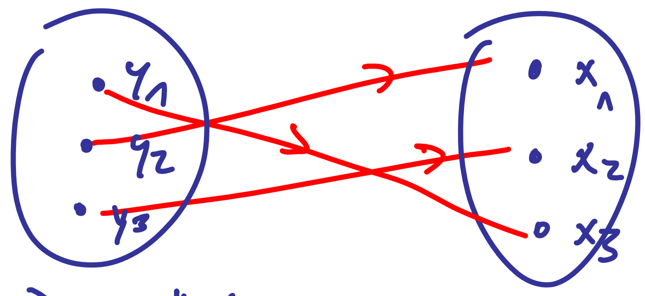
$f : \mathbb{D}_f \to \mathbb{W}_f$ bijektiv



Definiere eine Funktion $g : \mathbb{D}_g \to \mathbb{W}_g$ durch "Umdrehung der Pfeile"

$$\mathbb{D}_g = \mathbb{W}_f \qquad \mathbb{W}_g = \mathbb{D}_f$$

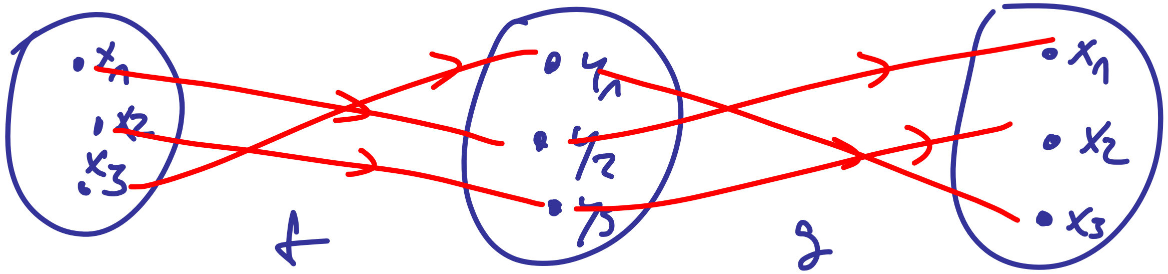


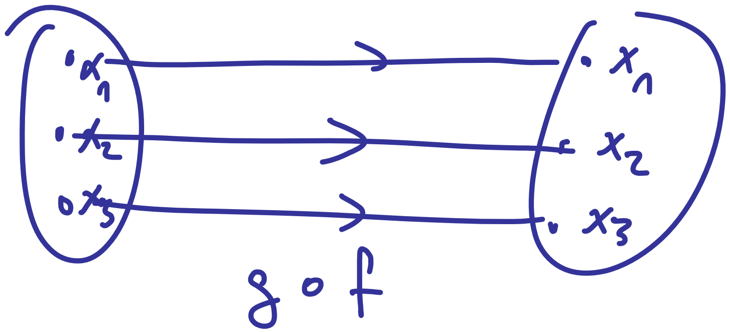


$$\mathbb{D}_g = \mathbb{W}_f \qquad \mathbb{W}_g = \mathbb{D}_f$$

Die Verkettung $g \circ f$ ist die Identitätsabbildung!

$g \circ f : \mathbb{D}_f \longrightarrow \mathbb{D}_f$



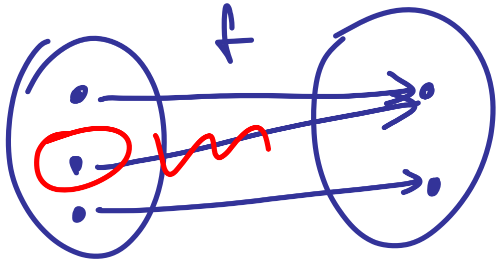


$(g \circ f)(x) = x$

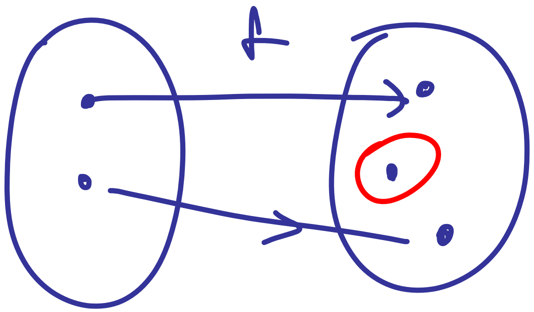
Warum f bijektiv?

Falls f

(i) nicht injektiv



(ii) nicht surjektiv



f ist nicht umkehrbar!

f kann umkehrbar gemacht werden, durch

(i) Einschränkung des Definitionsbereichs

(ii) Einschränkung des Wertebereichs

Definition:

Für eine bijektive Funktion $f: D_f \to W_f$ gibt eine Umkehrfunktion $g: W_f \to D_f$ so dass

$$(g \circ f)(x) = x \qquad \forall\, x \in D_f$$

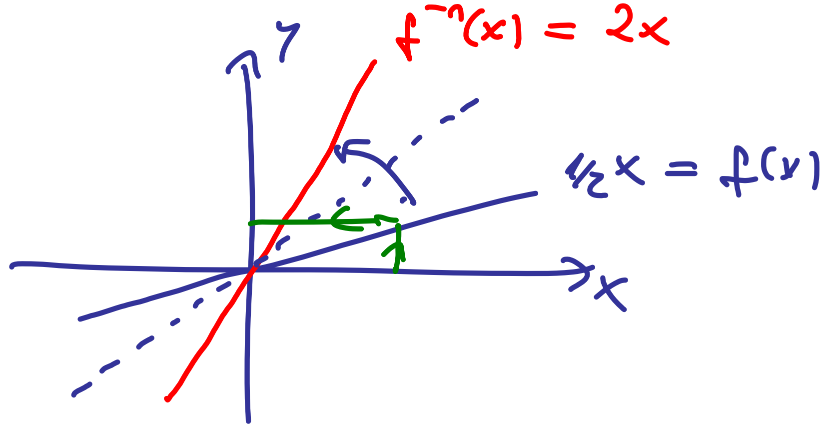
Diese Funktion g heißt die Umkehrfunktion von f und sie wird mit f^{-1} $(g = f^{-1})$ bezeichnet.

- $f^{-1}(x) \neq \dfrac{1}{f(x)}$ (i. allg.)

Bsp: $\underline{f(x) = x} \Rightarrow \underline{f^{-1}(x) = x}$, denn $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ ✓

$$\frac{1}{f(x)} = \frac{1}{x}$$

- Der Graph von f^{-1}, also $\Gamma_{f^{-1}}$ entsteht aus dem Graph von f, also Γ_f, durch Spiegelung an der Diagonalen.

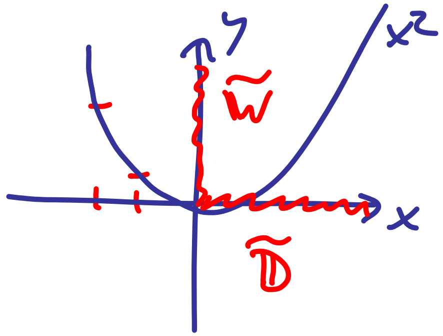


$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}\left(\tfrac{1}{2}x\right) = 2 \cdot \tfrac{1}{2}x = x$$

- $f : \mathbb{R} \to \mathbb{R}$

  $f(x) = x^2$

  nicht bijektiv

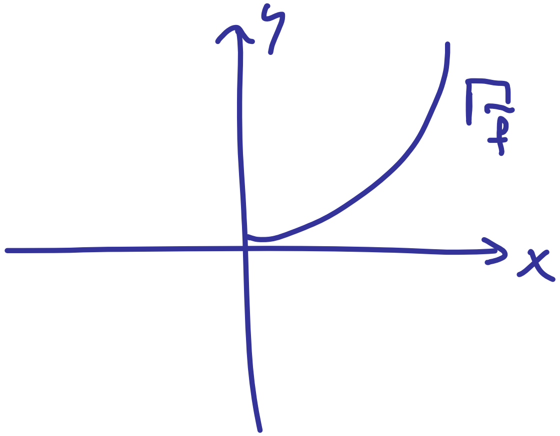


Einschränkung von Definitions- und Wertebereich von f:

$\tilde{f} : \tilde{D} \to \tilde{W}$

$$\tilde{D} = \{x \in \mathbb{R} \mid x \geq 0\} = \mathbb{R}_{+,0}$$

$$\tilde{W} = \mathbb{R}_{+,0}$$



$$\tilde f^{-1}(x) = \sqrt{x}$$

Test:

$$(\tilde f^{-1} \circ \tilde f)(x) = \tilde f^{-1}(\tilde f(x))$$
$$= \tilde f^{-1}(x^2) = \sqrt{x^2} = x$$

Die Umkehrfunktion einer bijektiven Funktion $f : D \to W$ bestimmt wie folgt

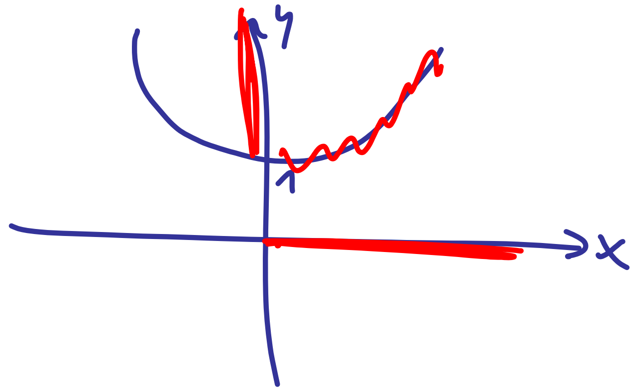
1) Funktionsgleichung hinschreiben $y = f(x)$
2) Auflösen der Gleichung nach x
3) Überprüfen, dass $(f^{-1} \circ f)(x) = x$

Bsp.: $\boxed{f(x) = 1 + x^2}$

$D_f = \mathbb{R}_{+,0} = [0, \infty[$

$W_f = \{ y \in \mathbb{R} \mid y \geq 1 \}$

$= [1, \infty[$

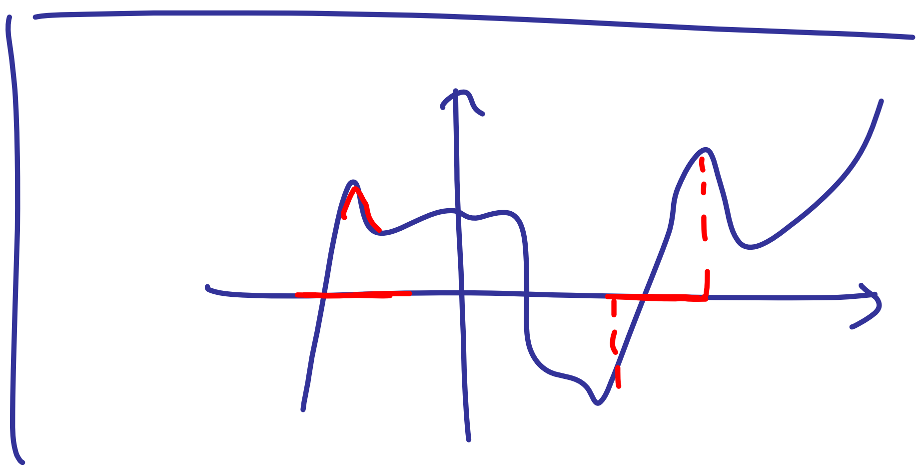


$y = 1 + x^2 \iff x^2 = y - 1 \iff x = \sqrt{y-1} \Leftarrow$ ✓

<u>oder</u> $x = -\sqrt{y-1}$

$f^{-1}(y) = \sqrt{y-1}$

$\boxed{f^{-1}(x) = \sqrt{x-1}}$

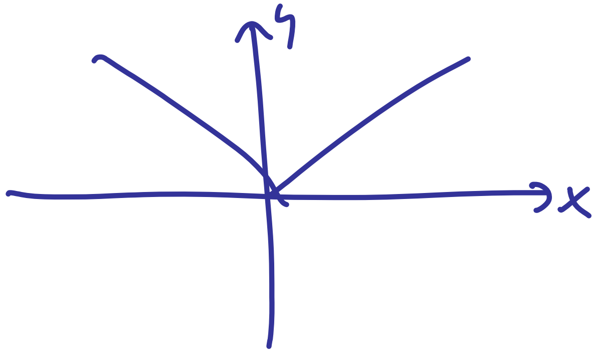


1.4 Spezielle Funktionen

Funktionen mit "Knicken" oder "Sprüngen"

(i) Betragsfunktion

$f: \mathbb{R} \to \mathbb{R} \qquad f(x) = |x| = \begin{cases} x & \text{falls } x \geq 0 \\ -x & \text{falls } x \leq 0 \end{cases}$



$|x| = \sqrt{x^2}$

Beweis:

$x \geq 0 \Rightarrow |x| = x$, $\sqrt{x^2} = x$ ✓

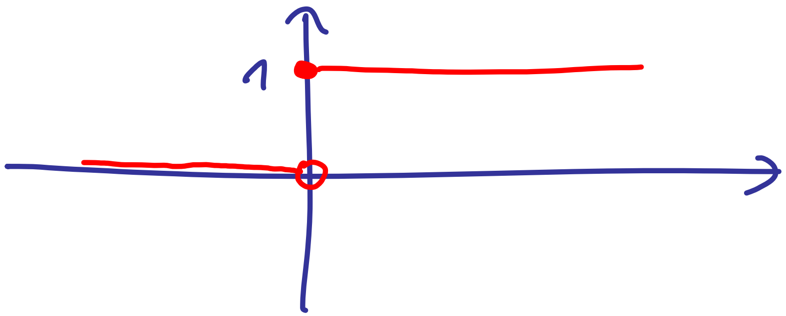
$x \leq 0 \Rightarrow |x| = -x$, $\sqrt{x^2} = -x$ ✓

$\underline{|-x| = |x|} = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$

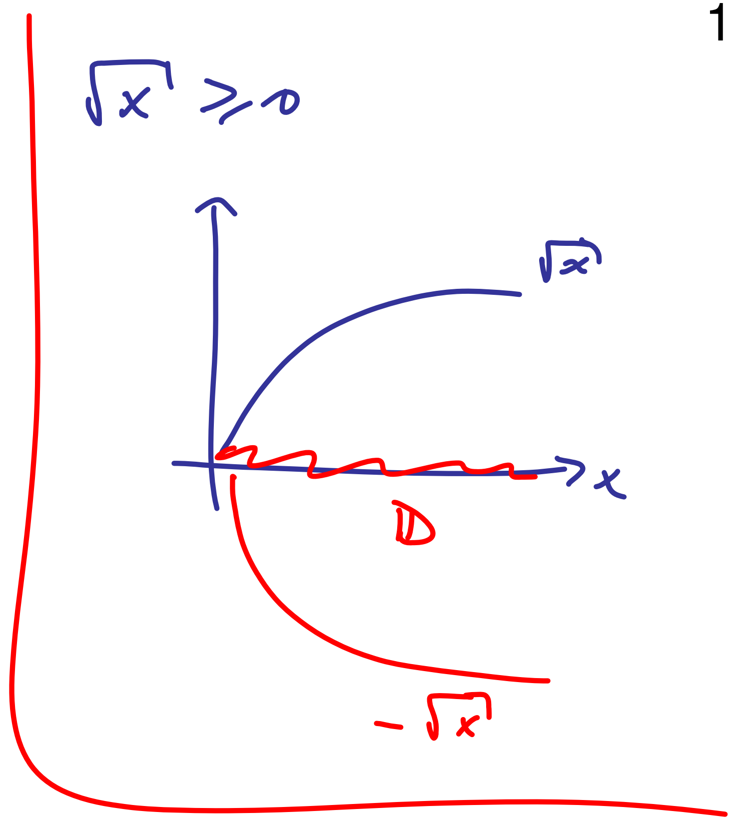
(ii) Heavysidesche Sprungfunktion

$$\Theta(x) : \mathbb{R} \to \mathbb{R}$$

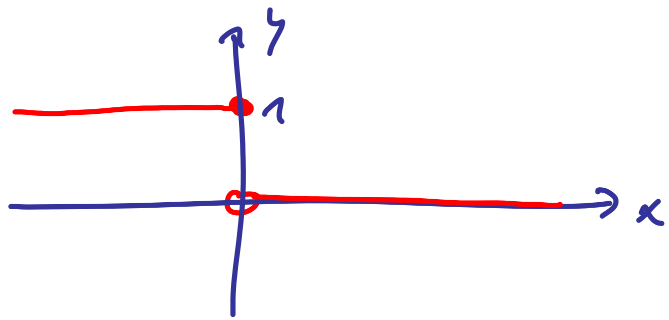
$$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



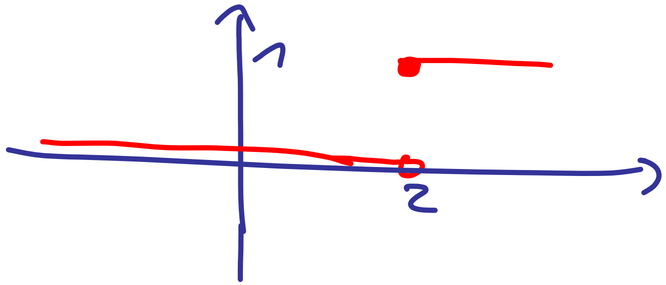
$\Theta(0)$ wird „oft“ anders definiert (manchmal)



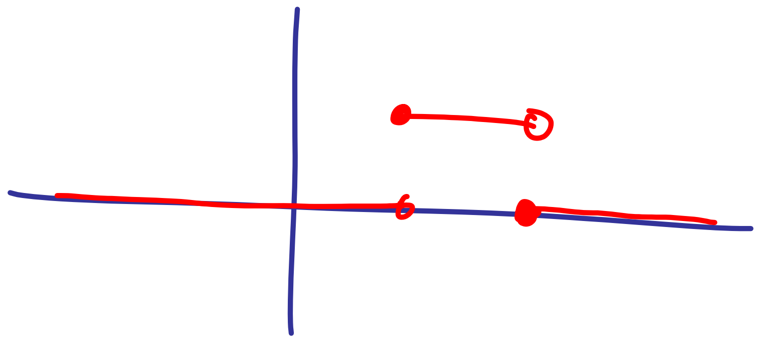
$$f(x) = \Theta(-x)$$



$$f(x) = \Theta(x-2)$$



$$f(x) = \Theta(x-1) - \Theta(x-2)$$



für $x=2$

$$\Theta(2-1) - \Theta(2-2)$$

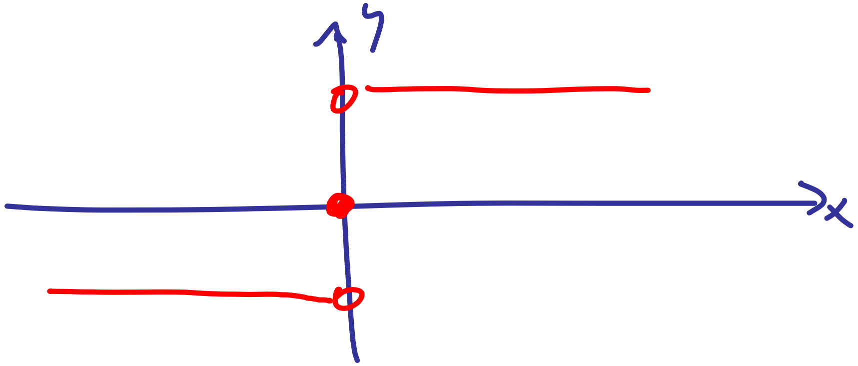
$$= \Theta(1) - \Theta(0)$$

$$= 1 - 1 = 0$$

(iii) Vorzeichenfunktion

$\mathrm{sign} : \mathbb{R} \to \mathbb{R}$

$$\mathrm{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



$\mathrm{sign}(x) = \dfrac{x}{|x|}$ falls $x \neq 0$

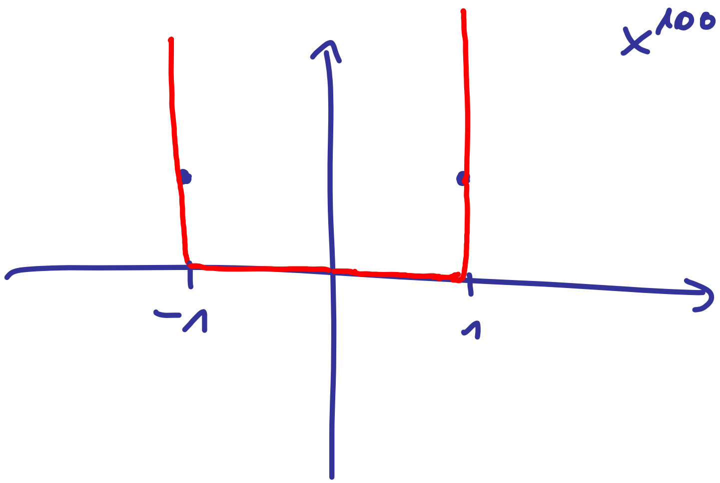
Potenzfunktionen

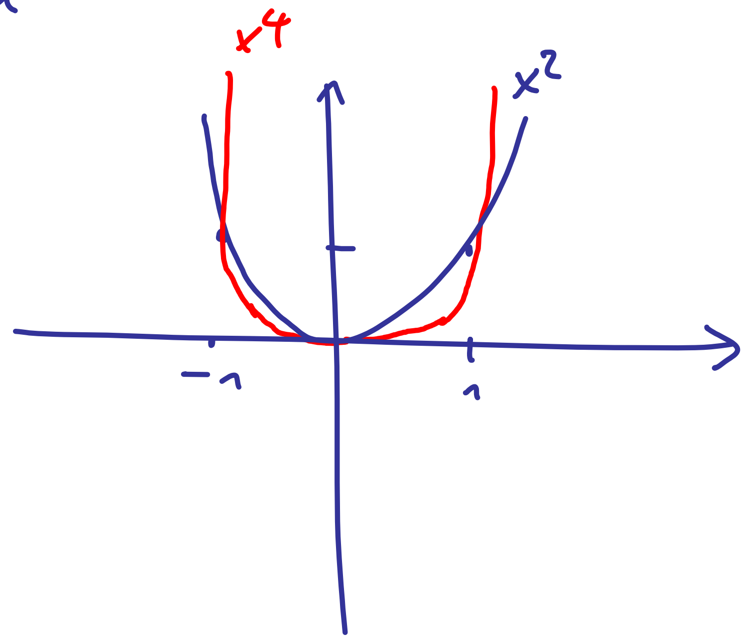
(i) Natürliche Exponenten

$$f: \mathbb{R} \to \mathbb{R} \qquad f(x) = x^n \quad \text{mit} \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\}$$

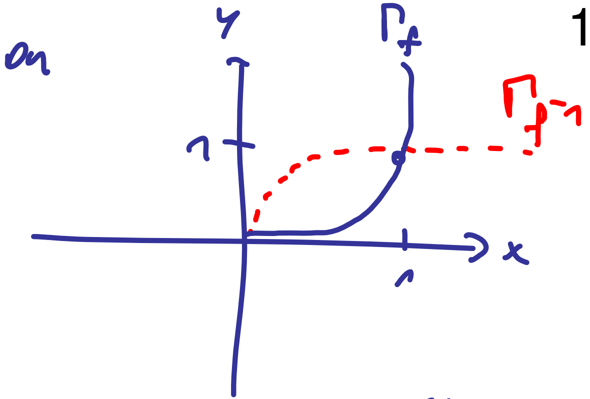
$$f(x) = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n\text{-mal}}$$

Fall 1: n gerade





standardmäßig $D = \mathbb{R}_{+,0}$ zur Definition der Umkehrfunktion



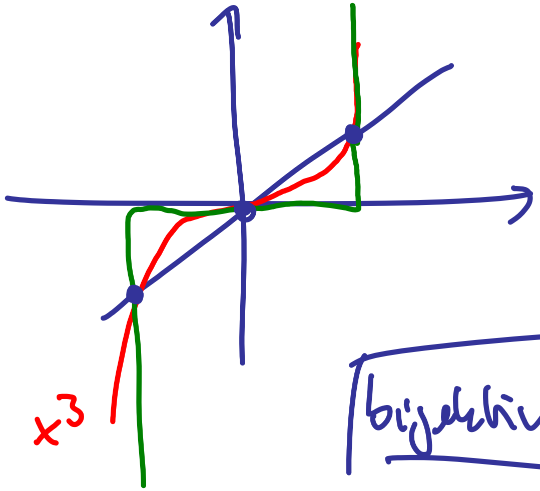
$f^{-1}(x) = \sqrt[n]{x} = x^{1/n}$ „n-te Wurzel aus x“

$f(x) = x^n$

Probe: $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(x^n) = \sqrt[n]{x^n} = (x^n)^{1/n}$

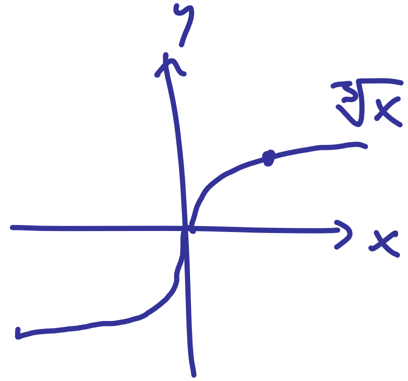
$= x^{(n \cdot \frac{1}{n})} = x^1 = x$ ✓

Fall 2: n ungerade $\quad f(x) = x^n \quad (-1)^5 = (-1) \cdot \ldots \cdot (-1) = -1$



$f(x) = x^{101}$

$f^{-1}(x) = \sqrt[n]{x} = x^{1/n}$

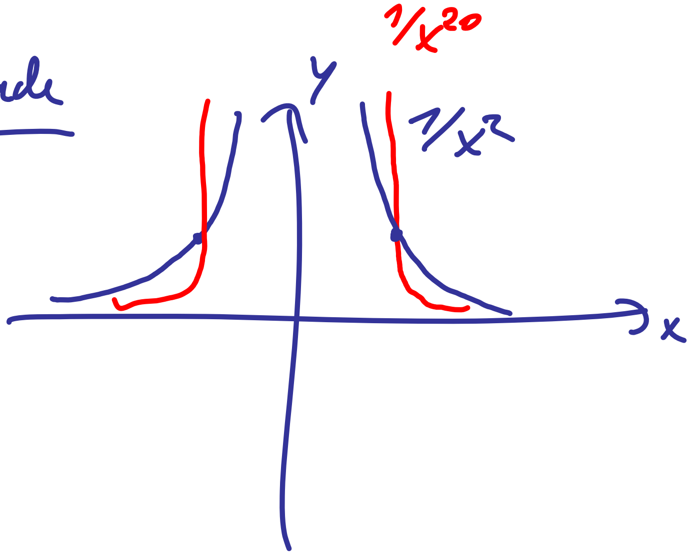


bijektiv? umkehrbar auf $\mathbb{R}$

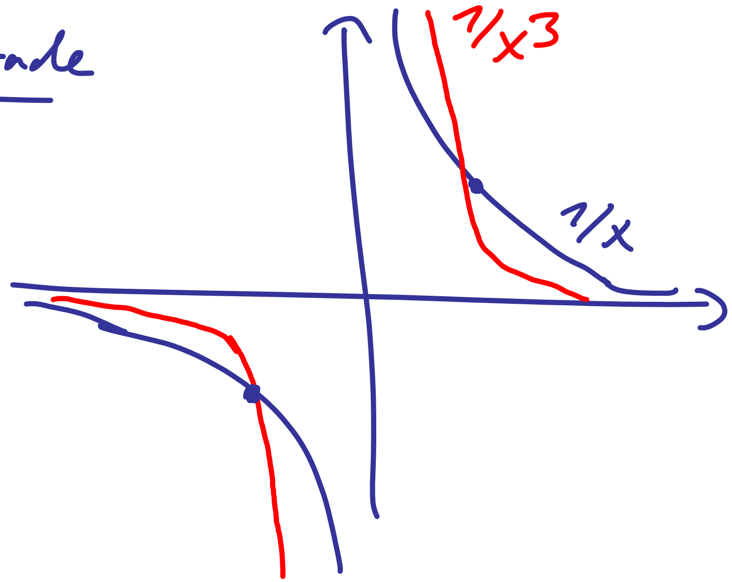
(ii) Negative Potenzen

$$f: \mathbb{R}^* \to \mathbb{R} \qquad f(x) = x^{-n} := \frac{1}{x^n} \qquad n \in \mathbb{N}$$

n gerade



n ungerade



(iii) Gebrochene Potenzen

Vom Freitag

$$f(x) = x^n$$

$$f^{-1}(x) = \sqrt[n]{x} = x^{1/n}$$