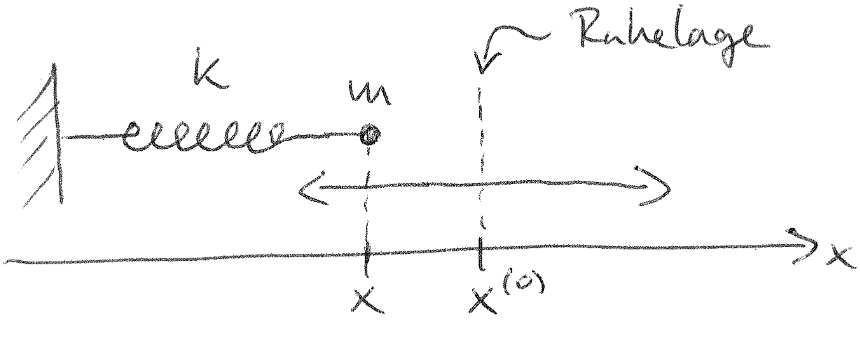
8 Dynamik von Feldern

Lagrange – Formalismus für

Punktteilchen	Felder
↓	↓
Mechanik (von Massenpunkten)	Elektrodynamik (auch: Kontinuumsmechanik)
↓	↓
Quantenmechanik	Quantenfeldtheorie

8.1 Gekoppelte harmonische Oszillatoren

A) eindimensionaler, harmonischer Oszillator



m : Masse

k : Federkonstante $(k > 0)$

lineare Rückstellkraft

$$\vec{F} = -k\,(x - x^{(0)})\,\vec{e}_x = -\vec{\nabla} V$$

harmonisches Potenzial

$$V = V(x) = \frac{1}{2} k\,(x - x^{(0)})^2$$

N II:

$$m\ddot{x} = -k\,(x - x_0) = -\frac{dV(x)}{dx}$$

Auslenkung als generalisierte Koordinate

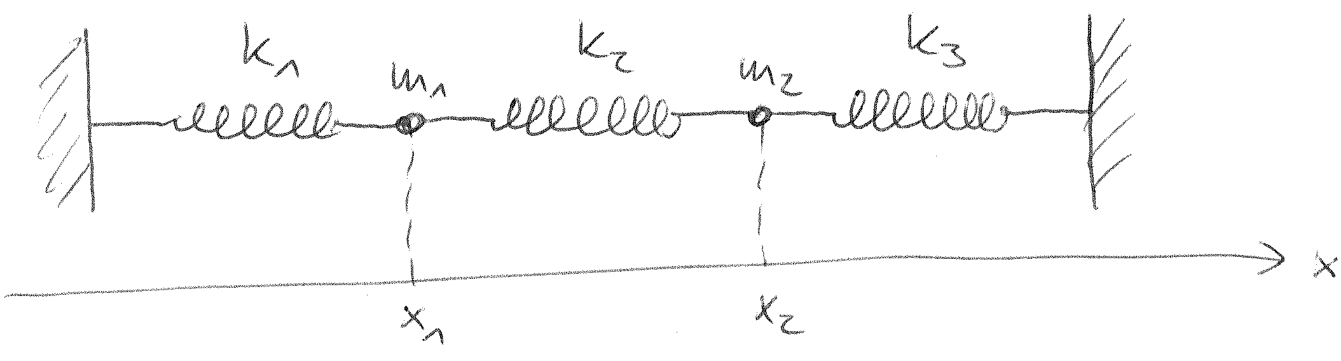
$$q = x - x^{(0)}$$

Lagrange-Funktion

$$L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - \frac{k}{2} q^2$$

L II: $\quad m\ddot{q} = -kq$

B) zwei gekoppelte harmonische Oszillatoren



Auslenkungen $\quad q_i = x_i - x_i^{(0)} \qquad i = 1,2$

Rückstellkräfte

$$\vec{F}_1 = (-k_1 q_1 - k_2 q_1 + k_2 q_2)\,\vec{e}_x$$

$$= (-k_1 q_1 - k_2 (q_1 - q_2))\,\vec{e}_x$$

$$= -\frac{\partial}{\partial q_1}\left(\frac{1}{2} k_1 q_1^2 + \frac{1}{2} k_2 (q_1 - q_2)^2\right)\vec{e}_x$$

$$\vec{F}_2 = -\frac{\partial}{\partial q_2}\left(\frac{1}{2} k_3 q_2^2 + \frac{1}{2} k_2 (q_2 - q_1)^2\right)\vec{e}_x$$

Lagrange-Funktion

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{m_1}{2}\dot{q}_1^2 + \frac{m_2}{2}\dot{q}_2^2$$

$$+ \frac{1}{2} k_1 q_1^2 + \frac{1}{2} k_2 (q_1 - q_2)^2 + \frac{1}{2} k_3 q_3^2$$

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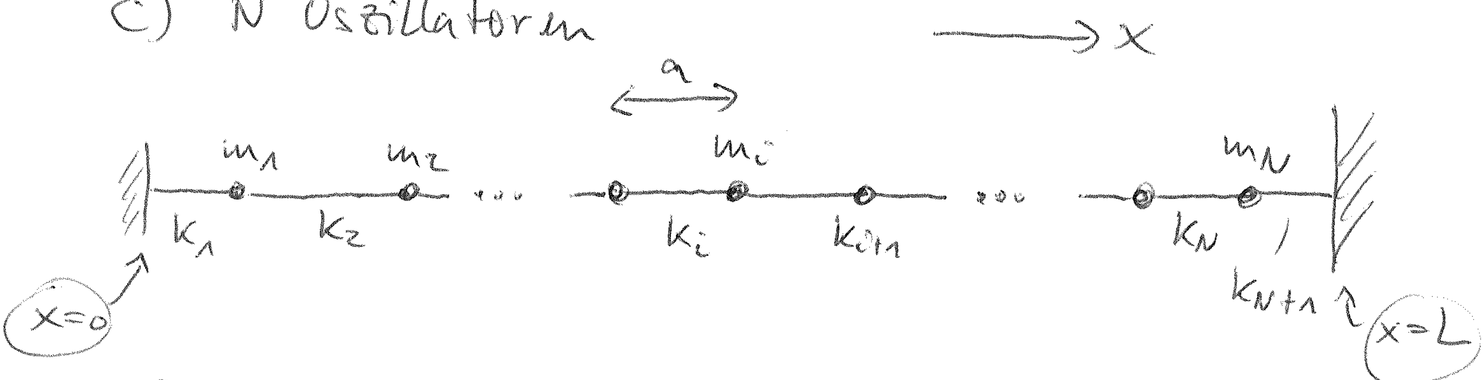
c) N Oszillatoren



$$\vec{F}_i = (-k_i (q_i - q_{i-1}) + k_{i+1} (q_{i+1} - q_i))\,\vec{e}_x$$

für $i = 1, .., N$ mit $\boxed{q_0 := 0\ ,\ q_{N+1} := 0}$

a: Gitterkonstante

$$F_i = -\frac{\partial}{\partial q_i} U(q_0, \dots, q_{N+1})$$

$\Leftarrow$ Beiträge für $j = i$ und $j = i+1$

mit

$$U(q_0, \dots, q_{N+1}) = \sum_{j=1}^{N+1} \frac{1}{2} k_j (q_j - q_{j-1})^2$$

Lagrange-Funktion:

$$\boxed{L_N(q, \dot{q}) = \sum_{i=1}^{N} \frac{m_i}{2} \dot{q}_i^2 - \sum_{i=1}^{N+1} \frac{k_i}{2} (q_i - q_{i-1})^2}$$

$$q_0 = q_{N+1} = 0$$

8.2 Kontinuumslimes und Lagrange-Dichte

betrachte den Limes

$$N \to \infty \;, \quad a \to 0 \;, \quad N \cdot a = L = \text{const}$$

(Gummiband)

definiere

$$\frac{m_i}{a} \longrightarrow \mu = \mu(x) \qquad \text{Massendichte}$$

$$k_i \cdot a \longrightarrow Y = Y(x) \qquad \text{Young-Modul}$$

(kürzere Feder ist härter bei gleicher Rückstellkraft) im Kontinuumslimes ist

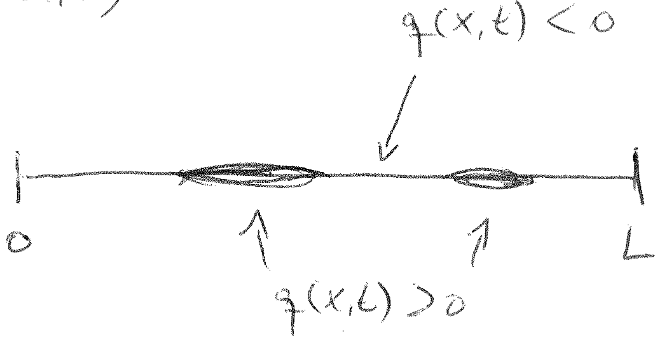
$$q_i \longrightarrow q(x) \qquad x \in [0, L]$$

$$q_i(t) \longrightarrow q(x,t)$$

$$\dot q_i(t) \longrightarrow \frac{\partial}{\partial t} q(x,t)$$

t fest

Gummiband



es folgt:

$$\sum_{i=1}^{N} \frac{m_i}{2} \dot q_i^2 = \sum_i a \, \frac{1}{2} \frac{m_i}{a} \dot q_i^2 \longrightarrow \int_0^L dx \, \frac{1}{2} \mu(x) \left(\frac{\partial q(x,t)}{\partial t} \right)^2$$

$$\sum_{i=1}^{N+1} \frac{k_i}{2} (q_i - q_{i-1})^2 = \sum_i a \, \frac{1}{2} (k_i a) \left(\frac{q_i - q_{i-1}}{a} \right)^2$$

$$\longrightarrow \int_0^L dx \, \frac{1}{2} Y(x) \left(\frac{\partial q(x,t)}{\partial x} \right)^2$$

und

$$L_N(q, \dot q) \longrightarrow L = \int_0^L dx \, \mathcal{L}$$

mit der <u>Lagrange-Dichte</u>

$$\mathcal{L} = \mathcal{L}\left(\cancel{q(x,t)}, \frac{\partial q(x,t)}{\partial x}, \frac{\partial q(x,t)}{\partial t}, x, \cancel{t} \right)$$

$$= \frac{1}{2} \mu(x) \left(\frac{\partial q(x,t)}{\partial t} \right)^2 - \frac{1}{2} Y(x) \left(\frac{\partial q(x,t)}{\partial x} \right)^2$$

Kontinuumslimes:

$$L_N \xrightarrow{N\to\infty} L$$

$S_N = \int dt\, L_N$

Wirkungsprinzip

$\delta S_N = 0$

$\downarrow$

L II – Gleichungen für Punktmassen $q_i(t)$

$$0 = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

$\xrightarrow{N\to\infty}$ (1)

$S = \int dt\, L = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L}$

Wirkungsprinzip $\delta S = 0$

$\downarrow$ (2)

Lagrange – Gleichungen für Feld $q(x,t)$

(Feldgleichungen) ?

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1) $$0 = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

$$= -k_i(q_i - q_{i-1}) - k_{i+1}(q_i - q_{i+1}) - m_i \ddot{q}_i$$

$\Leftrightarrow$

$$\frac{m_i}{a}\ddot{q}_i = \frac{k_{i+1}a\,\frac{q_{i+1}-q_i}{a} - k_i a\,\frac{q_i - q_{i-1}}{a}}{a}$$

für $N \to \infty$, $a \to 0$, $Na = L$:

$$\mu(x) \cdot \frac{\partial^2 q(x,t)}{\partial t^2} = \left(Y(x+dx) \frac{\partial q}{\partial x}(x+dx,t) - Y(x) \frac{\partial q}{\partial x}(x,t) \right) \Big/ dx$$

$$= \frac{d}{dx}\left(Y(x) \frac{\partial q}{\partial x}(x,t) \right) \qquad \left(f'(x) = \frac{f(x+dx) - f(x)}{dx} \right)$$

also:

$$\mu(x) \frac{\partial^2 q}{\partial t^2}(x,t) = Y'(x) \frac{\partial q}{\partial x}(x,t) + Y(x) \frac{\partial^2 q}{\partial x^2}(x,t)$$

für homogenes Gummiband, $\mu(x) = \mu$, $Y(x) = Y$, gilt:

$$\mu \frac{\partial^2 q}{\partial t^2} = Y \frac{\partial^2 q}{\partial x^2}$$

bzw.

$$\boxed{\frac{\mu}{Y} \frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} = 0}$$

Wellengleichung

Wellengeschwindigkeit c mit $c^2 = Y/\mu \leftarrow \frac{ka}{m/a}$

$$= a^2 \frac{k}{m} = a^2 \omega^2, \quad c = a\omega \quad \left(\omega = \sqrt{k/m} \right)$$

2) Herleitung der Wellengleichung aus $\delta S = 0$:

Wirkungsprinzip:

$$\delta \int_{t_1}^{t_2} dt \int_0^L dx\ \mathcal{L}\left(q, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial t}, x, t\right) \stackrel{!}{=} 0$$

Randbedingungen:

$$q(0,t) = q(L,t) = 0 \qquad \forall t$$

$$q(x,t_1) = q_1(x) \quad q(x,t_2) = q_2(x) \qquad \forall x$$

es gilt

$$0 = \delta S = \int dt \int dx\ \delta\mathcal{L}$$

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left[\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial(\partial q/\partial x)} \delta\left(\frac{\partial q}{\partial x}\right) + \frac{\partial \mathcal{L}}{\partial(\partial q/\partial t)} \delta\left(\frac{\partial q}{\partial t}\right) \right]$$

δ: Variation des Felds:

$$q(x,t) \mapsto q(x,t) + \delta q(x,t)$$

$$\frac{\partial q}{\partial x}(x,t) \mapsto \frac{\partial q}{\partial x}(x,t) + \delta \frac{\partial q}{\partial x}(x,t) = \frac{\partial}{\partial x}\delta q(x,t)$$

$$\frac{\partial q}{\partial t}(x,t) \mapsto \frac{\partial q}{\partial t}(x,t) + \delta \frac{\partial q}{\partial t}(x,t) = \frac{\partial}{\partial t}\delta q(x,t)$$

also:

$$0 = \int_{t_1}^{t_2} dt \int_0^L dx \left[\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial (\partial q/\partial x)} \frac{\partial}{\partial x} \delta q + \frac{\partial \mathcal{L}}{\partial (\partial q/\partial t)} \frac{\partial}{\partial t} \delta q \right]$$

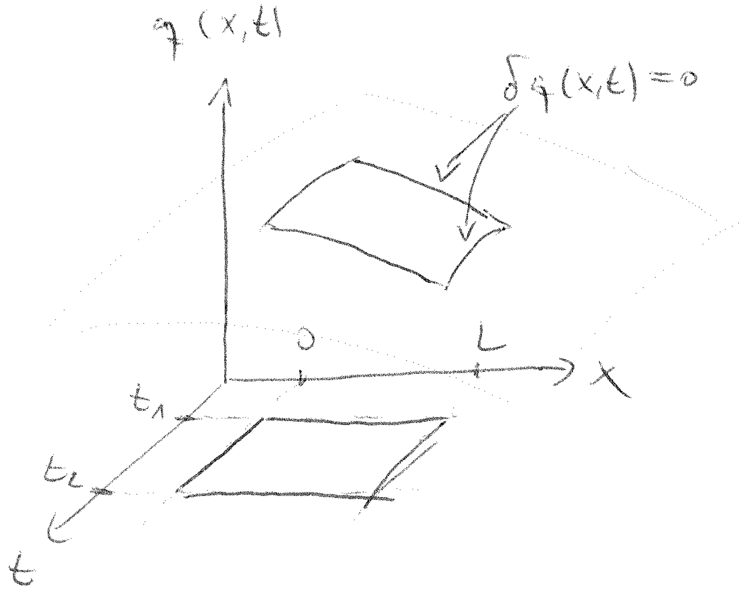
partielle Integration:

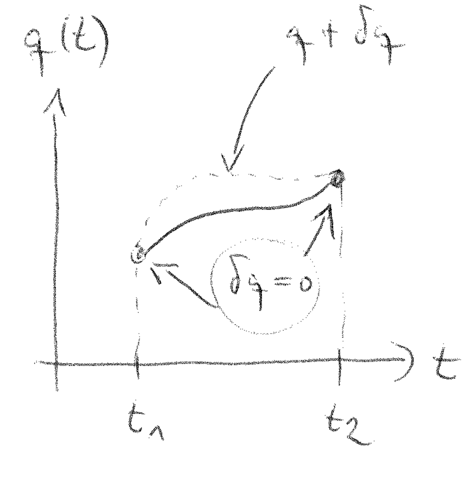
$$= \int_{t_1}^{t_2} dt \int_0^L dx \left[\frac{\partial \mathcal{L}}{\partial q} \delta q - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial (\partial q/\partial x)} \delta q - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial (\partial q/\partial t)} \right]$$

Randterme

$$\frac{\partial \mathcal{L}}{\partial (\partial q/\partial x)} \delta q(x,t) \Big|_{x=0}^{x=L} = 0 \qquad \delta q(0,t) = \delta q(L,0) = 0 \quad \forall t$$

$$\frac{\partial \mathcal{L}}{\partial (\partial q/\partial t)} \delta q(x,t) \Big|_{t=t_1}^{t=t_2} = 0 \qquad \delta q(x,t_1) = \delta q(x,t_2) = 0 \quad \forall x$$





(zum Vergleich)

Kurzschreibweise

$$\partial_x := \frac{\partial}{\partial x} \qquad \partial_t := \frac{\partial}{\partial t}$$

damit ist

$$0 = \int dt \int dx \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial(\partial_x q)} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial(\partial_t q)} \right] \delta q$$

$\delta q(x,t)$ beliebige Variation $\Rightarrow$

$$\boxed{\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial(\partial_x q)} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial(\partial_t q)} = 0}$$

neuer Term!

Feldgleichung
(Euler–Lagrange-Gleichung)

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hier: $\mathcal{L}(\cancel{q}, \partial_x q, \partial_t q, x, \cancel{t}) = \frac{\mu(x)}{2}(\partial_t q)^2 - \frac{Y(x)}{2}(\partial_x q)^2$

also

$$0 = 0 - \frac{d}{dx}\left(-\frac{Y(x)}{2} 2 \partial_x q\right) - \frac{d}{dt}\left(\frac{\mu(x)}{2} 2 \partial_t q\right)$$

$$= Y'(x) \frac{\partial q}{\partial x}(x,t) + Y(x) \frac{\partial^2 q}{\partial x^2}(x,t) - \mu(x) \frac{\partial^2 q}{\partial t^2}(x,t)$$

homogener Fall:

$$0 = \frac{\mu}{Y} \frac{\partial^2 q}{\partial t^2}(x,t) - \frac{\partial^2 q}{\partial x^2}(x,t) \qquad \checkmark$$

8.3 Klassische Theorie für ein skalares Feld

skalares Feld $\varphi = \varphi(\vec{r}, t)$

<u>Bsp:</u> elektrostatisches Potenzial

Temperaturfeld

$D=2$ Trommelmembran $\varphi = \varphi(x, y, t)$

Auslenkung

Dynamik des Felds:

partielle DGL in $\vec{r}, t, \partial_x \varphi, \partial_y \varphi, \partial_z \varphi, \partial_t \varphi, \partial_x^2 \varphi,$

$\partial_y^2 \varphi, \partial_x \partial_y \varphi, \ldots$ etc

<u>Postulat der klassischen Feldtheorie:</u>

Die raumzeitlichen Änderungen (die "Dynamik") eines skalaren Felds $\varphi(\vec{r}, t)$ werden durch das Wirkungsprinzip beschrieben:

$$\delta S = 0$$

mit der Wirkung

$$S = \int dt\, L = \int_{t_1}^{t_2} dt \int_V d^3r\, \mathcal{L}$$

und einer Lagrange-Dichte der Form

$$\mathcal{L} = \mathcal{L}(\varphi, \vec{\nabla}\varphi, \partial_t\varphi, \vec{r}, t) \qquad \left(\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial\vec{r}}\right)$$

- Dimension von $\mathcal{L}$: Energiedichte = $\frac{\text{Energie}}{\text{Volumen}}$
- alle phys. Grundgleichungen sind von dieser Form (d.h. z.B. $\mathcal{L}(\cancel{\vec{\nabla}^2\varphi}, \cancel{\partial_t^2\varphi}, \dots)$)

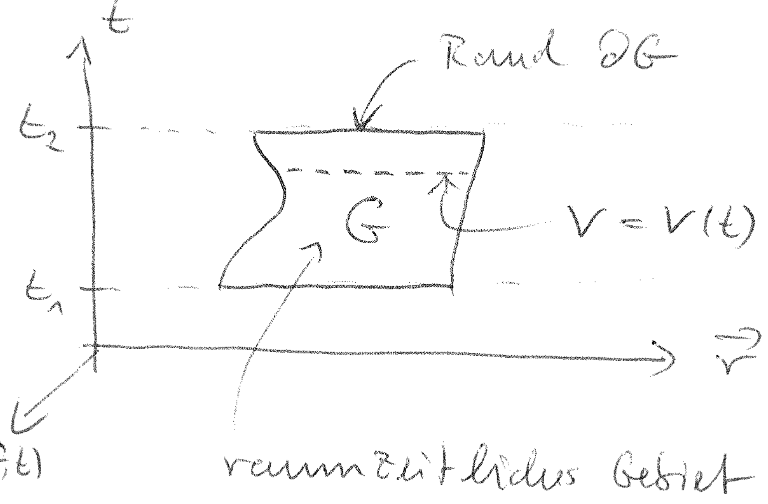
für die Variation des Felds $\delta\varphi(\vec{r},t)$ gelten dabei die Randbedingungen

$$\delta\varphi(\vec{r},t) = 0 \quad \text{für } \vec{r} \in \partial V \quad (\forall t)$$
$$\delta\varphi(\vec{r},t) = 0 \quad \text{für } t = t_1, t = t_2 \quad (\forall \vec{r})$$

(∂V ist der Rand / die Oberfläche des Volumens V)

<u>oder:</u>

$$\delta\varphi(\vec{r},t) = 0 \quad \text{für } (\vec{r},t) \in \partial G$$



beachte: meist ist $V = \mathbb{R}^3$ und $\delta\varphi(\vec{r},t) = 0$ für "$\vec{r} \to \infty$"

Herleitung der Lagrange-Gleichung (Feldgleichung)

$$0 = \delta S = \int dt \int d^3r \, \delta\mathcal{L}(\varphi, \vec{\nabla}\varphi, \partial_t\varphi, \vec{r}, t)$$

$$= \int dt \int d^3r \Big[\frac{\partial\mathcal{L}}{\partial\varphi(\vec{r},t)} \delta\varphi(\vec{r},t) + \frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\varphi(\vec{r},t))} \underbrace{\delta(\vec{\nabla}\varphi(\vec{r},t))}_{\vec{\nabla}\delta\varphi(\vec{r},t)}$$

$$+ \frac{\partial\mathcal{L}}{\partial(\partial_t\varphi(\vec{r},t))} \underbrace{\delta(\partial_t\varphi(\vec{r},t))}_{\partial_t\delta\varphi(\vec{r},t)} \Big]$$

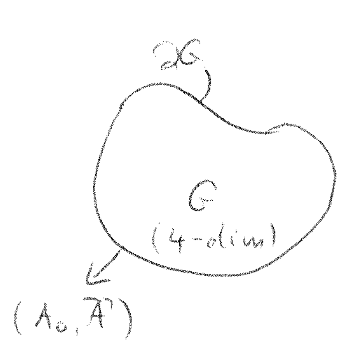
es gilt

$$\frac{\partial\mathcal{L}}{\partial\vec{\nabla}\varphi}\vec{\nabla}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}\partial_t\delta\varphi = \underbrace{\frac{d}{d\vec{r}}\left(\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\varphi)}\delta\varphi\right) + \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}\delta\varphi\right)}_{\text{ohne Beitrag zu } \delta S}$$

$$-\left(\frac{d}{d\vec{r}}\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\varphi)}\right)\cdot\delta\varphi - \left(\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}\right)\delta\varphi$$

denn:

$$\int_G dt\, d^3r \begin{pmatrix}\partial_t\\ \partial_x\\ \partial_y\\ \partial_z\end{pmatrix} \cdot \left[\begin{pmatrix}\partial\mathcal{L}/\partial(\partial_t\varphi)\\ \partial\mathcal{L}/\partial(\vec{\nabla}\varphi)\end{pmatrix}\cdot\delta\varphi\right]$$



$$\overset{\text{Gauß}}{=} \oint_{\partial G} d\begin{pmatrix}A_0\\ \vec{A}\end{pmatrix}\begin{pmatrix}\partial\mathcal{L}/\partial(\partial_t\varphi)\\ \partial\mathcal{L}/\partial(\vec{\nabla}\varphi)\end{pmatrix}\delta\varphi = 0$$, denn $\delta\varphi = 0$ auf ∂G

also:

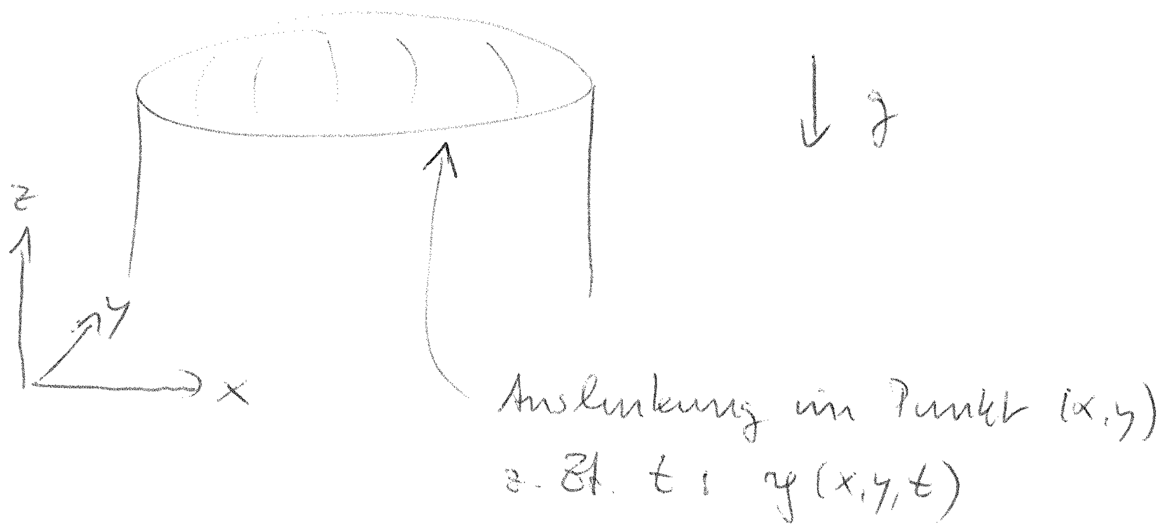
$$0 = \delta S = \int_{t_1}^{t_2} dt \int_V d\vec{r} \left[\frac{\partial \mathcal{L}}{\partial y(\vec{r},t)} - \frac{d}{d\vec{r}} \frac{\partial \mathcal{L}}{\partial(\vec{\nabla} y(\vec{r},t))} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial(\partial_t y(\vec{r},t))} \right] \delta y(\vec{r},t)$$

mit $\delta y(\vec{r},t)$ auf $G \setminus \partial G$ beliebig folgt

$$\boxed{\frac{\partial \mathcal{L}}{\partial y(\vec{r},t)} - \frac{d}{d\vec{r}} \frac{\partial \mathcal{L}}{\partial(\vec{\nabla} y(\vec{r},t))} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial(\partial_t y(\vec{r},t))} = 0}$$

- in $\vec{r}$ und t symmetrische Formulierung
  (→ relativistische Verallgemeinerung)

Bsp.: homogene Membran (Trommel)



Kontinuumsmechanik liefert

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial y^2} = -\frac{g}{c^2}$$

mit $c^2 = Y/\rho$ (c: Wellengeschwindigkeit)

Feldgleichung kann als $\delta S = 0$ geschrieben werden mit

$$\mathcal{L}\left(y, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial y}, \frac{\partial y}{\partial t}\right) = \underbrace{\frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2}_{\text{Kinetische Energie}} - \underbrace{\frac{Y}{2}\left(\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial y}\right)^2\right)}_{\text{Potenzial der inneren Kräfte}} \underbrace{- g\mu \cdot y}_{\text{äußeres Potenzial}}$$

es ist

$$\frac{\partial \mathcal{L}}{\partial y} = -g\mu \qquad \frac{\partial \mathcal{L}}{\partial(\vec{\nabla} y)} = -Y\vec{\nabla} y \qquad \left(\text{mit } \vec{\nabla} = \begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix}\right)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_t y)} = \mu\, \partial_t y$$

also

$$0 = -g\mu - \frac{d}{d\vec{r}}\left(-Y\vec{\nabla} y\right) - \frac{d}{dt}\left(\mu\, \partial_t y\right)$$

$$= -g\mu + Y\vec{\nabla}^2 y - \mu\, \partial_t^2 y$$

$$\Rightarrow \frac{1}{c^2}\partial_t^2 y - \vec{\nabla}^2 y = -g/c^2$$

Die Feldgleichungen sind forminvariant unter der Eichtransformation der Lagrange-Dichte

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \frac{d\Lambda_0}{dt} + \frac{d}{d\vec{r}}\vec{\Lambda}(\vec{r},t)$$

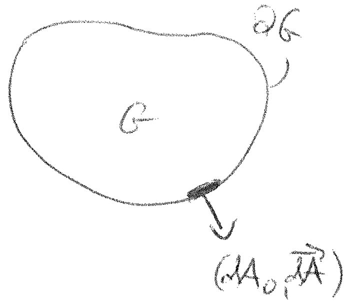
mit beliebigem $\Lambda_0 = \Lambda_0(\varphi(\vec{r},t),\vec{r},t)$
und $\vec{\Lambda} = \vec{\Lambda}(\varphi(\vec{r},t),\vec{r},t)$

<u>Beweis:</u>

$$\begin{pmatrix} d/dt \\ d/d\vec{r} \end{pmatrix}$$

$$S' = \int_G dt \int d^3r\, \mathcal{L}' = \int_G dt \int d^3r\, \mathcal{L} + \int_G dt \int d^3r\, \mathrm{div}_4 \begin{pmatrix} \Lambda_0 \\ \vec{\Lambda} \end{pmatrix}$$

$$= S + \int_{\partial G} d\begin{pmatrix} A_0 \\ \vec{A} \end{pmatrix} \cdot \begin{pmatrix} \Lambda_0 \\ \vec{\Lambda} \end{pmatrix}$$



also folgt

$$\delta S' = 0 \iff \delta S = 0$$

denn

$$\delta \int_{\partial G} d\begin{pmatrix} A_0 \\ \vec{A} \end{pmatrix} \begin{pmatrix} \Lambda_0 \\ \vec{\Lambda} \end{pmatrix} = 0$$

wegen $\delta\Lambda(\varphi(\vec{r},t),\vec{r},t) = 0$
und $\delta\varphi(\vec{r},t) = 0$ auf ∂G

8.4 Symmetrien und Erhaltungsgrößen

es sei $\mathcal{L} = \mathcal{L}(\varphi, \vec{\nabla}\varphi, \partial_t \varphi, \vec{r}, \cancel{t})$ nicht explizit zeitabhängig:

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

Energieerhaltung? Def. der Feldenergie?

es gilt

$$\frac{d\mathcal{L}}{dt} = \underbrace{\frac{\partial \mathcal{L}}{\partial \varphi} \frac{\partial \varphi}{\partial t}} + \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} \frac{\partial(\vec{\nabla}\varphi)}{\partial t} + \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \frac{\partial^2 \varphi}{\partial t^2}$$

$$\left(\frac{d}{d\vec{r}} \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} + \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \right) \cdot \frac{\partial \varphi}{\partial t}$$

$$= \frac{d}{d\vec{r}} \left(\frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} \frac{\partial \varphi}{\partial t} \right) + \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \frac{\partial \varphi}{\partial t} \right)$$

und somit

$$0 = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \cdot \frac{\partial \varphi}{\partial t} - \mathcal{L} \right) + \frac{d}{d\vec{r}} \left(\frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} \frac{\partial \varphi}{\partial t} \right)$$

Def: kanonisch konjugiertes Feld

$$\pi(\vec{r}, t) := \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \qquad \left(\text{vergleiche } p_n = \frac{\partial L}{\partial \dot{q}_n}\right)$$

Def: Hamilton-Dichte, Energie-Dichte

$$\mathcal{H} = \mathcal{H}(\vec{r},t) = \pi(\vec{r},t) \cdot \frac{\partial \varphi(\vec{r},t)}{\partial t} - \mathcal{L}(\varphi(\vec{r},t), \dots, \cancel{t})$$

Def: Energiestromdichte

$$\vec{j}_{\mathcal{H}} = \vec{j}_{\mathcal{H}}(\vec{r},t) = \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} \frac{\partial \varphi}{\partial t}$$

es folgt:

$$\boxed{\frac{\partial \mathcal{H}}{\partial t} + \vec{\nabla} \vec{j}_{\mathcal{H}} = 0}$$

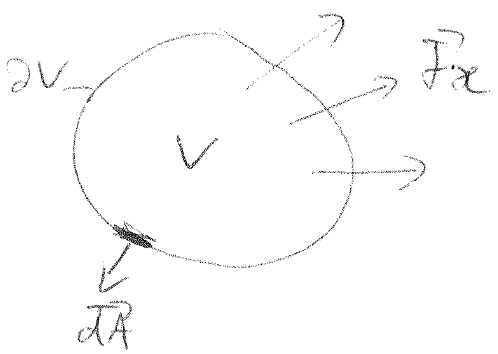
Energieerhaltung (differenziell)

integral:

$$H_V = \int_V d^3r \, \mathcal{H}$$ Energie im Volumen V

$$\frac{dH_V}{dt} = \int_V d^3r \, \frac{\partial \mathcal{H}}{\partial t} = -\int_V d^3r \, \vec{\nabla} \vec{j}_{\mathcal{H}} = -\oint_{\partial V} d\vec{A} \, \vec{j}_{\mathcal{H}}$$

Energie~~fluss~~ durch ∂V



$V \to \infty$:

$$\frac{dH}{dt} = 0 \qquad H = \text{const}$$

$$\pi \cdot \frac{\partial \varphi}{\partial t} - \mathcal{L} = \text{const}$$

sei jetzt $\mathcal{L} = \mathcal{L}(\varphi, \vec\nabla\varphi, \partial_t\varphi, \cancel{\vec r}, t)$, also

$$\frac{\partial \mathcal{L}}{\partial \vec r} = 0$$

es gilt:

$$\frac{d\mathcal{L}}{dx_i} = \underbrace{\frac{\partial\mathcal{L}}{\partial\varphi}\frac{\partial\varphi}{\partial x_i}} + \frac{\partial\mathcal{L}}{\partial(\vec\nabla\varphi)}\frac{\partial(\vec\nabla\varphi)}{\partial x_i} + \frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}\frac{\partial(\partial_t\varphi)}{\partial x_i}$$

$$\left(\frac{d}{d\vec r}\frac{\partial\mathcal{L}}{\partial(\vec\nabla\varphi)} + \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}\right)\frac{\partial\varphi}{\partial x_i}$$

$$= \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}\frac{\partial\varphi}{\partial x_i}\right) + \frac{d}{d\vec r}\left(\frac{\partial\mathcal{L}}{\partial(\vec\nabla\varphi)}\frac{\partial\varphi}{\partial x_i}\right)$$

also

$$0 = \frac{d}{dt}\underbrace{\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}\frac{\partial\varphi}{\partial x_i}\right)}_{p_i} + \sum_j \frac{d}{dx_j}\underbrace{\left(\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x_j)}\frac{\partial\varphi}{\partial x_i} - \delta_{ij}\mathcal{L}\right)}_{T_{ij}}$$

p_i kanonische Impulsdichte (Richtung: i)

T_{ij} } liefert kanonische Impulsstromdichte $\sum_j T_{ij}\vec e_j$ (Richtung i)

vektorieller (Impuls-) Erhaltungssatz

$$\boxed{0 = \frac{\partial p_i}{\partial t} + \vec\nabla\left(\sum_j T_{ij}\vec e_j\right)}$$

beachte p_i und $\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}$ sind verschieden!

8.6 Lagrange-Formalismus für Vektorfelder

- Vektorfeld $\vec{\psi}(\vec{r},t)$ mit Komponenten $\psi_n(\vec{r},t)$
- mehrere skalare Felder $\psi_n(\vec{r},t)$

Zusammen:

$$\psi_n(\vec{r},t) \quad n = 1,\dots,f$$

Wirkungsprinzip

$$\delta S[\psi_1,\dots\psi_f] = \delta S = 0$$

$$S = \int dt \int d^3r \, \mathcal{L}$$

Lagrange-Dichte

$$\mathcal{L} = \mathcal{L}(\psi_1,\dots\psi_f, \vec{\nabla}\psi_1,\dots,\vec{\nabla}\psi_f, \partial_t\psi_1,\dots,\partial_t\psi_f, \vec{r},t)$$

Lagrange-Gleichungen:

$$0 = \delta S = \int dt \int d^3r \left[\sum_n \frac{\partial \mathcal{L}}{\partial \psi_n}\delta\psi_n + \sum_n \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\psi_n)}\delta\vec{\nabla}\psi_n + \dots\right]$$

$$= \int dt \int d^3r \sum_n \left(\frac{\partial \mathcal{L}}{\partial \psi_n} - \frac{d}{d\vec{r}}\frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\psi_n)} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t\psi_n)}\right)\delta\psi_n$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \psi_n(\vec{r},t)} - \frac{d}{d\vec{r}}\frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\psi_n(\vec{r},t))} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t\psi_n(\vec{r},t))} = 0}$$

$n = 1,\dots,f$

Feldgleichungen

Lagrange-Mechanik für Punktteilchen	Klassische Feldtheorie
(gen.) Koordinaten q_n $n = 1, \dots, f$	Felder $\varphi_n(\vec{r}, t)$ $n = 1, \dots, f$
Wirkungsprinzip $\delta S[q] = \delta \int dt\, L = 0$	$\delta S[\varphi] = \delta \int dt \int d^3r\, \mathcal{L} = 0$
Lagrange-Funktion $L = L(q, \dot{q}, t) = T - U$ $q = (q_1, \dots, q_f)$	Lagrange-Dichte $\mathcal{L} = \mathcal{L}(\varphi, \nabla\varphi, \partial_t \varphi, \vec{r}, t)$ $\varphi = (\varphi_1, \dots, \varphi_f)$
Lagrange-Gleichungen $\frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} = 0$	Feldgleichungen $\frac{\partial \mathcal{L}}{\partial \varphi_n} - \frac{d}{d\vec{r}} \frac{\partial \mathcal{L}}{\partial \nabla \varphi_n} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} = 0$
kanonisch konj. Impuls $p_n = \frac{\partial L}{\partial \dot{q}_n}$	kanonisch konj. Feld $\pi_n(\vec{r}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi_n(\vec{r}, t))}$
Hamilton-Funktion $H = \sum_n p_n \dot{q}_n - L = H(q, p, t)$	Hamilton-Dichte $\mathcal{H} = \sum_n \pi_n(\vec{r}, t) \frac{\partial \varphi_n(\vec{r}, t)}{\partial t} - \mathcal{L}$
Energie-Erhaltung $\frac{dH}{dt} = 0$ falls $\frac{\partial L}{\partial t} = 0$	$\frac{\partial \mathcal{H}}{\partial t} + \operatorname{div} \vec{j}_{\mathcal{H}} = 0$ falls $\frac{\partial \mathcal{L}}{\partial t} = 0$ $\vec{j}_{\mathcal{H}} = \sum_n \frac{\partial \mathcal{L}}{\partial(\nabla \varphi_n)} \frac{\partial \varphi_n}{\partial t}$

Noether-Theorem

$q_n = q_n(q', t, \alpha)$

$L'(q', \dot q', t, \alpha) = L(q(q', t, \alpha), \dot q(q', \dot q', t, \alpha), t)$

L invariant: $\frac{dL'}{d\alpha}\Big|_{\alpha=0} = 0$

$$\Rightarrow \sum_n \frac{\partial L}{\partial \dot q_n}\frac{\partial q_n}{\partial \alpha}\Big|_{\alpha=0} = \text{const}$$

Eichtransformation der Lagrange-Funktion

$$L \mapsto L + \frac{d\Lambda}{dt}$$

$\Lambda = \Lambda(\vec r, t)$

$\psi_n = \psi_n(\psi_n', \vec r, t, \alpha)$

$\mathcal{L}'(\psi', \vec\nabla\psi', \partial_t\psi', \vec r, t, \alpha) =$
$\mathcal{L}(\psi(\psi', \vec r, t, \alpha), \vec\nabla\psi(\psi', \vec\nabla\psi', \vec r, t, \alpha), \dots)$

$\mathcal{L}$ invariant: $\frac{d\mathcal{L}'}{d\alpha}\Big|_{\alpha=0} = 0$

$$\Rightarrow \frac{d}{dt}\left(\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_t \psi_n)}\frac{\partial \psi_n}{\partial \alpha}\Big|_{\alpha=0}\right) + \frac{d}{d\vec r}\left(\sum_n \frac{\partial \mathcal{L}}{\partial(\vec\nabla \psi_n)}\frac{\partial \psi_n}{\partial \alpha}\Big|_{\alpha=0}\right) = 0$$

Eichtransformation der Lagrange-Dichte

$$\mathcal{L} \mapsto \mathcal{L} + \frac{d\Lambda_0}{dt} + \frac{d}{d\vec r}\vec\Lambda$$

$\Lambda = \Lambda(\psi(\vec r, t), \vec r, t)$

8.5 Noether-Theorem

betrachte Transformation des Felds T_α

$$\psi(\vec{r},t) \longmapsto \psi'(\vec{r},t)$$

α: kontinuierlicher Parameter mit

$T_{\alpha=0} = \mathbb{1}$ (Identität)

schreibe:

$$\psi(\vec{r},t) = T_\alpha\left(\psi'(\vec{r},t), \vec{r}, t, \alpha\right)$$

$$\psi = T_\alpha(\psi', \vec{r}, t, \alpha) = T_\alpha \psi' \quad \text{(kurz)}$$

Lagrange-Dichte

$$\mathcal{L}(\psi, \vec{\nabla}\psi, \partial_t\psi, \vec{r}, t)$$

$$= \mathcal{L}(T_\alpha\psi', \vec{\nabla}T_\alpha\psi', \partial_t T_\alpha\psi', \vec{r}, t)$$

$$=: \mathcal{L}'(\psi'(\vec{r},t), \vec{\nabla}\psi'(\vec{r},t), \partial_t\psi'(\vec{r},t), \vec{r}, t, \alpha)$$

$\mathcal{L}$ ist invariant unter T_α, falls

$$\mathcal{L}'(\psi', \vec{\nabla}\psi', \partial_t\psi', \vec{r}, t, \alpha) = \mathcal{L}'(\psi', \vec{\nabla}\psi', \partial_t\psi', \vec{r}, t, \alpha=0)$$

$$= \mathcal{L}(T_\alpha\psi', \vec{\nabla}T_\alpha\psi', \ldots)\Big|_{\alpha=0}$$

$$= \mathcal{L}(\psi', \vec{\nabla}\psi', \partial_t\psi', \vec{r}, t)$$

d.h. funktionale Gestalt unabhängig von α (und gleich der von $\mathcal{L}$!)

jetzt gilt:

$$0 = \frac{d}{d\alpha} \mathcal{L}'(\varphi', \vec{\nabla}\varphi', \partial_t \varphi', \vec{r}, t, \alpha)\Big|_{\alpha=0}$$

$$= \frac{d}{d\alpha} \mathcal{L}(T_\alpha \varphi', \vec{\nabla} T_\alpha \varphi', \partial_t T_\alpha \varphi', \vec{r}, t)\Big|_{\alpha=0}$$

$$= \underbrace{\frac{\partial \mathcal{L}}{\partial \varphi} \frac{\partial \varphi}{\partial \alpha}}_{\left(\frac{d}{dr} \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)}\right)\frac{\partial \varphi}{\partial \alpha} + \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)}\right)\frac{\partial \varphi}{\partial \alpha}} + \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} \frac{\partial(\vec{\nabla}\varphi)}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \frac{\partial(\partial_t \varphi)}{\partial \alpha}\Big|_{\alpha=0}$$

$$= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \frac{\partial \varphi}{\partial \alpha}\right) + \frac{d}{d\vec{r}}\left(\frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} \frac{\partial \varphi}{\partial \alpha}\right)\Big|_{\alpha=0}$$

also

$$\boxed{\frac{\partial \rho}{\partial t}(\vec{r},t) + \operatorname{div} \vec{j}(\vec{r},t) = 0}$$

mit Dichte

$$\rho(\vec{r},t) := \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \frac{\partial \varphi}{\partial \alpha}\Big|_{\alpha=0} = \pi(\vec{r},t) \frac{\partial \varphi(\vec{r},t)}{\partial \alpha}$$

und Stromdichte

$$\vec{j}(\vec{r},t) = \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} \frac{\partial \varphi}{\partial \alpha}\Big|_{\alpha=0}$$

Bsp: s. u. (Schrödinger-Gleichung)

8.7 Komplexe Felder und Eichtransformation

Bsp: Schrödinger-Gleichung
(grundlegende dynamische Gleichung der QM,
hier: klassische Feldgleichung)

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r},t) = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \psi(\vec{r},t) + V(\vec{r})\, \psi(\vec{r},t)$$

i: imaginäre Einheit
$\hbar$: Plancksches Wirkungsquantum
$\psi(\vec{r},t) \in \mathbb{C}$! ($V(\vec{r})$ reell)
m: Teilchenmasse (QM), hier: Parameter

beachte: $\psi = \mathrm{Re}\,\psi + i\,\mathrm{Im}\,\psi$
$\psi^* = \mathrm{Re}\,\psi - i\,\mathrm{Im}\,\psi$

2 unabhängige reelle Felder $\mathrm{Re}\,\psi$, $\mathrm{Im}\,\psi$

oder

2 unabhängige Felder ψ, ψ^*

$\rightarrow$ gewöhnlicher Formalismus für f-komponentige Felder

Wirtinger - Kalkül:

eine beliebige Funktion

$$f : \mathbb{C} \to \mathbb{C} \quad z \mapsto f(z) \quad z = x + iy \quad x, y \in \mathbb{R}$$

kann immer eindeutig auch als Funktion von $z, z^* \in \mathbb{C}$ aufgefasst werden:

$$f(z) = f(x, y) = f(x+iy, x-iy) = f(z, z^*)$$

Bsp:

$$f(z) = |z|^2 = x^2 + y^2 = (x+iy)(x-iy) = z z^* = f(z, z^*)$$

es gilt $(z = x+iy,\ z^* = x - iy)$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial z^*}\frac{\partial z^*}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z^*}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial z^*}\frac{\partial z^*}{\partial y} = i\frac{\partial f}{\partial z} - i\frac{\partial f}{\partial z^*}$$

und somit

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial z^*} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

beachte: in $f(z, z^*)$ sind z, z^* als unabhängig aufzufassen, $f(z)$ ergibt sich aber nur, falls z und z^* komplex konjugiert sind

Lagrange-Dichte zur Schrödinger-Gleichung:

$$\mathcal{L} = \mathcal{L}(\psi, \psi^*, \vec{\nabla}\psi, \vec{\nabla}\psi^*, \partial_t\psi, \partial_t\psi^*, \vec{r}, t)$$

$$\boxed{\mathcal{L} = \frac{1}{2} i\hbar \left(\psi^* \partial_t \psi - \psi \partial_t \psi^*\right) - \frac{\hbar^2}{2m} \vec{\nabla}\psi^* \vec{\nabla}\psi - V(\vec{r})\, \psi^* \psi}$$

$\mathcal{L}$ ist reell

Herleitung der Feldgleichung

$$0 = \frac{\partial \mathcal{L}}{\partial \psi} - \frac{d}{d\vec{r}} \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\psi)} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} \qquad (*)$$

$$= \frac{1}{2} i\hbar\, \partial_t \psi^* - V\psi^* - \frac{d}{d\vec{r}}\left(-\frac{\hbar^2}{2m} \vec{\nabla}\psi^*\right) - \frac{d}{dt}\left(\frac{1}{2} i\hbar\, \psi^*\right)$$

$$= -i\hbar\, \partial_t \psi^* - V\psi^* + \frac{\hbar^2}{2m} \vec{\nabla}^2 \psi^*$$

also

$$i\hbar\, \partial_t \psi = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \psi + V\psi \qquad \checkmark$$

---

aus (*) folgt (da $\mathcal{L}$ reell) durch komplexe Konjugation:

$$0 = \frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{d}{d\vec{r}}\left(\frac{\partial \mathcal{L}}{\partial \vec{\nabla}\psi^*}\right) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial(\partial_t \psi^*)}$$

Feldgleichungen für ψ und ψ^* sind äquivalent (falls ψ und ψ^* komplex konjugiert sind) es gilt $\frac{\partial \mathcal{L}}{\partial t} = 0 \rightarrow$ Energieerhaltung

kanonisch konjugierte Felder:

$$\pi(\vec{r},t) = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} = \frac{1}{2} i\hbar \psi^*(\vec{r},t)$$

$$\pi^*(\vec{r},t) = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi^*)} = -\frac{1}{2} i\hbar \psi(\vec{r},t)$$

Hamilton-Dichte:

$$\mathcal{H}(\vec{r},t) = \pi(\vec{r},t)\frac{\partial \psi(\vec{r},t)}{\partial t} + \pi^*(\vec{r},t)\frac{\partial \psi^*(\vec{r},t)}{\partial t} - \mathcal{L}$$

$$= \frac{1}{2} i\hbar \psi^* \partial_t \psi - \frac{1}{2} i\hbar \psi \partial_t \psi^* - \mathcal{L}$$

$$= + \frac{\hbar^2}{2m} \vec{\nabla}\psi^* \vec{\nabla}\psi + V\psi^*\psi \qquad (*)$$

Energiestromdichte

$$\vec{J}_{\mathcal{H}}(\vec{r},t) = \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\psi)}\frac{\partial \psi}{\partial t} + \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\psi^*)}\frac{\partial \psi^*}{\partial t}$$

$$= -\frac{\hbar^2}{2m}\left(\vec{\nabla}\psi^* \cancel{\vec{\nabla}\psi}\, \partial_t \psi + \vec{\nabla}\psi\, \partial_t \psi^*\right) \qquad (**)$$

damit

$$\frac{\partial \mathcal{H}}{\partial t} + \operatorname{div} \vec{J}_{\mathcal{H}} = 0$$

(kann auch aus (*) und (**) und der Schr.-Glg. direkt verifiziert werden)

Gesamtenergie

$$E = \int_{\mathbb{R}^3} d^3r\, \mathcal{H} = \underbrace{\int d^3r \left(\frac{\hbar^2}{2m} \vec{\nabla}\psi^+ \vec{\nabla}\psi + V\psi^+\psi \right)}$$

$$\int d^3r \left(\underbrace{\vec{\nabla}(\psi^+ \vec{\nabla}\psi)}_{\text{Gauß}} - \psi^+ \vec{\nabla}^2\psi \right)$$

$$\oint_{\partial V} d\vec{A}\; \psi^+ \vec{\nabla}\psi = 0$$

falls $\psi(\vec{r},t) \to 0$ für "$\vec{r} \to \infty$"

also:

$$E = \int d^3r\, \psi^+(\vec{r},t) \underbrace{\left(-\frac{\hbar^2}{2m} \vec{\nabla}^2 + V(\vec{r}) \right)} \psi(\vec{r},t)$$

QM: Hamilton-Operator
(in Ortsdarstellung)

$\mathcal{L}$ ist offensichtlich invariant unter

$$\psi(\vec{r},t) \mapsto \psi'(\vec{r},t) = e^{\frac{i}{\hbar}\Lambda} \psi(\vec{r},t) \qquad \Lambda \in \mathbb{R}$$

$$\psi(\vec{r},t) = e^{-\frac{i}{\hbar}\Lambda} \psi'(\vec{r},t) \qquad \psi^+(\vec{r},t) = e^{\frac{i}{\hbar}\Lambda} \psi'(\vec{r},t)^*$$

<u>Eichtransformation der Felder</u>

(dies muss zwingend der Fall sein, da $\mathcal{L}$ reell ist)

$\mathcal{L}$ invariant, $\Lambda \in \mathbb{R}$ kontinuierlicher Parameter

$\to$ Erhaltungsgröße! welche?

Noether: $\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0$ mit

$$\rho(\vec{r},t) = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} \frac{\partial \psi}{\partial \Lambda} + \frac{\partial \mathcal{L}}{\partial(\partial_t \psi^*)} \frac{\partial \psi^*}{\partial \Lambda} \Bigg|_{\Lambda=0}$$

$$\vec{j}(\vec{r},t) = \frac{\partial \mathcal{L}}{\partial(\vec{\nabla} \psi)} \frac{\partial \psi}{\partial \Lambda} + \frac{\partial \mathcal{L}}{\partial(\vec{\nabla} \psi^*)} \frac{\partial \psi^*}{\partial \Lambda} \Bigg|_{\Lambda=0}$$

es ist:

$$\rho(\vec{r},t) = \frac{1}{2} i\hbar \psi^* \left(-\frac{i}{\hbar}\right) e^{-\frac{i}{\hbar}\Lambda} \psi'$$
$$- \frac{1}{2} i\hbar \psi \left(+\frac{i}{\hbar}\right) e^{\frac{i}{\hbar}\Lambda} \psi'^* \Big|_{\Lambda=0}$$
$$= \frac{1}{2} \psi^* \psi + \frac{1}{2} \psi \psi^* = |\psi|^2$$

also $\int_{\mathbb{R}^3} d^3r\, |\psi|^2 = \text{const}$

und $\rho(\vec{r},t) = |\psi(\vec{r},t)|^2 \geq 0$

$\Big\}$ geeignet für Wahrscheinlichkeits-interpretation!
ρ = WK-Dichte
(nach Normierung)

Invarianz unter Eichtransformation

$\downarrow$

Wahrsch.-Interpretation möglich

also: QM <u>muss</u> mit <u>komplexem</u> Feld $\psi(\vec{r},t) \in \mathbb{C}$ formuliert werden!

Wahrscheinlichkeitsstromdichte:

$$\vec{j}(\vec{r},t) = -\frac{\hbar^2}{2m}\vec{\nabla}\psi^* \left(-\frac{i}{\hbar}\right) e^{-\frac{i}{\hbar}\Lambda}\psi' - \frac{\hbar^2}{2m}\vec{\nabla}\psi \left(+\frac{i}{\hbar}\right) e^{\frac{i}{\hbar}\Lambda}\psi'^* \Big|_{\Lambda=0}$$

$$= \frac{\hbar}{m}\frac{1}{2i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

$$= \frac{\hbar}{m}\,\mathrm{Im}\left(\psi^*\vec{\nabla}\psi\right)$$

beachte:

$$\vec{j} = \rho\cdot\vec{v} = \psi^*\psi\cdot\frac{\vec{p}}{m} = \mathrm{Re}\left(\psi^*\frac{\vec{p}}{m}\psi\right)$$

$$= \mathrm{Re}\left(\psi^*\frac{-i\hbar\vec{\nabla}}{m}\psi\right) = \frac{\hbar}{m}\mathrm{Re}(-i)\,\psi^*\vec{\nabla}\psi$$

$$= \frac{\hbar}{m}\,\mathrm{Im}\left(\psi^*\vec{\nabla}\psi\right)$$

QM Impulsoperator

$\vec{p} = -i\hbar\vec{\nabla}$