7. Effects of radiation

In this chapter consequences of synchrotron radiation on longitudinal und transverse beam dynamics are investigated. The radius of the particle's trajectory depends on its energy. Therefor one expresses in the relation for the radiated power

$$P_{\perp} = \frac{e^2 c \beta^4}{6\pi \varepsilon_0 \left(m_0 c^2\right)^4} \cdot \frac{E^4}{R^2}$$

the radius R by the magnetic field and the energy of the particle

$$\frac{1}{R} = \frac{e}{p} B = \frac{e c}{\beta E} B \qquad \Rightarrow \qquad \frac{E^2}{R^2} = \frac{e^2 c^2}{\beta^2} B^2$$

and obtains

$$P_{\perp} = \frac{e^4 c^3 \beta^2}{6\pi \varepsilon_0 \left(m_0 c^2\right)^4} \cdot E^2 \cdot B^2$$

From here on we will put $\beta \approx 1$ for simplification purposes.

7.1. Damping of synchrotron oscillations

The equation of motion of the longitudinal oscillation is in linear approximation

$$\Delta\ddot{\varphi} + 2\alpha_S \cdot \Delta\dot{\varphi} + \Omega_S^2 \cdot \Delta\varphi = 0,$$

what can also be written for the energy deviation as

$$\Delta\ddot{E} + 2\alpha_S \cdot \Delta\dot{E} + \Omega_S^2 \cdot \Delta E = 0$$

because of $\Delta E \sim \Delta\dot{\varphi}$. The damping constant of the longitudinal oscillation is

$$\alpha_S = \frac{1}{2T_0} \cdot \frac{dW(E_0)}{dE}.$$

Calculating the radiated energy W per turn one has to consider that a particle with energy deviation moves on a dispersion orbit with the beam displacement

$$\Delta x = D \cdot \frac{\Delta p}{p_0} \overset{\beta\approx 1}{\overbrace{=}} D \cdot \frac{\Delta E}{E_0}.$$

In a bending magnet, it covers a distance of

$$ds' = \left(1 + \frac{\Delta x}{R}\right) \cdot ds \, .$$

Hence, the following equation holds true for the radiated energy:

$$W = \int_0^{T_0} P_\perp dt = \oint P_\perp \frac{ds'}{c} = \frac{1}{c} \oint P_\perp \left(1 + \frac{\Delta x}{R}\right) ds = \oint P_\perp \left(1 + \frac{D}{R} \frac{\Delta E}{E_0}\right) ds \, .$$

For the radiated power, it applies in linear approximation

$$P_\perp = P_0 + \frac{dP_0}{dE} \Delta E + \frac{dP_0}{dB} \Delta B$$

and with

$$\Delta B \approx \frac{dB}{dx} \Delta x = D \cdot \frac{\Delta E}{E_0} \frac{dB}{dx}$$

as well as

$$\left.\begin{aligned} \frac{1}{R} &= \frac{e\,c}{E_0} B \\ k &= -\frac{e\,c}{E_0} \frac{dB}{dx} \end{aligned}\right\} \rightarrow \quad \frac{1}{B} \frac{dB}{dx} = -k\,R$$

it yields for W again in linear approximation (in ΔE):

$$W = \frac{1}{c}\oint\left(P_0 + P_0 \frac{D}{R}\frac{\Delta E}{E_0} + \frac{dP_0}{dE}\Delta E - \frac{dP_0}{dB} D\,B\,k\,R\frac{\Delta E}{E_0}\right) ds\,.$$

Since $d/dE = d/d\Delta E$ and $P_\perp \sim E^2 \cdot B^2$ one obtains:

$$\frac{dW(E_0)}{dE} = \frac{1}{c}\oint\left(\frac{2\,P_0}{E_0} - \frac{2\,P_0\,k\,R\,D}{E_0} + \frac{P_0\,D}{R\,E_0}\right) ds\,.$$

The energy W_0 is radiated along the design orbit:

$$\frac{1}{c}\oint P_0\,ds = W(E_0) \equiv W_0\,,$$

and thus it yields

$$\frac{dW(E_0)}{dE} = \frac{2W_0}{E_0} + \frac{1}{c\,E_0}\oint P_0\,D\cdot\left(-2\,k\,R + \frac{1}{R}\right)\cdot ds$$

Thus, we obtain for the damping constant:

$$\alpha_S = \frac{1}{2T_0}\frac{dW(E_0)}{dE} = \frac{1}{2T_0}\frac{W_0}{E_0}\cdot\left(2+\mathcal{D}\right),$$

whereas the variable $\mathcal{D}$ represents the additional optics contribution:

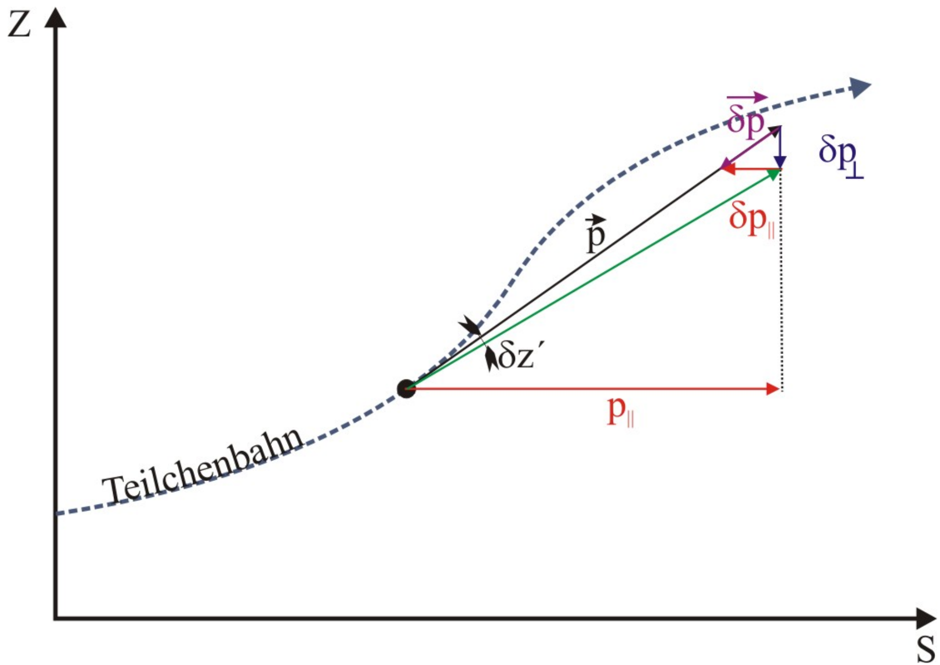
$$\mathcal{D} = \frac{1}{c\,W_0}\oint P_0\,D\left(-2k\,R+\frac{1}{R}\right)ds = \frac{1}{c}\frac{\oint P_0\,D\left(-2k\,R+\frac{1}{R}\right)ds}{\oint P_0\,\frac{ds}{c}}$$

and with $P_0 = \text{const}\cdot\frac{1}{R^2}$ we obtain for the optics expression

$$\mathcal{D} = \frac{\oint D/R\cdot\left(1/R^2-2k\right)ds}{\oint 1/R^2\,ds}$$

7.2. Damping of betatron oscillations

As the synchrotron radiation is emitted in course, the momentum of the particle is decreased by emission of a photon. However, the post-acceleration in a cavity restores only the longitudinal momentum:



So, the small angle z' of the particle's momentum is decreased with respect to the *s* axis for the absolute value

$$|\delta z'| = \left| \frac{p_\perp}{p_{||}} - \frac{p_\perp - \delta p_\perp}{p_{||}} \right| = \frac{\delta p_\perp}{p_{||}}.$$

The dispersion must be taken into consideration for a detailed treatment. We will first concentrate on the vertical plane where D = 0 and later derive its additional contribution in the horizontal plane.

7.2.1. Vertical plane

The vertical betatron oscillation of a single particle is described by:

$$z = \sqrt{\varepsilon\,\beta(s)} \cdot \cos\left(\phi(s) + \phi_0\right)$$

$$z' = -\sqrt{\frac{\varepsilon}{\beta(s)}} \left\{ \sin\left(\phi(s) + \phi_0\right) + \alpha(s)\cos\left(\phi(s) + \phi_0\right) \right\}$$

With the definition of the oscillation amplitude $A = \sqrt{\varepsilon\,\beta(s)}$

and the well-known relation for the emittance

$$\varepsilon_z = \gamma z^2 + 2\alpha z z' + \beta z'^2$$

we have

$$\boxed{A^2 = \beta\gamma z^2 + 2\alpha\beta z z' + \beta^2 z'^2}.$$

After one turn, one yields with $\delta p_\perp / \delta p_\parallel \approx z'$ as well as $c \cdot \delta p_\parallel \approx W_0$:

$$\delta z = 0, \qquad \delta z' = -z' \frac{W_0}{E_0}.$$

This results in an amplitude variation per turn of

$$\delta A^2 = 2A \cdot \delta A = \frac{\partial A^2}{\partial z}\delta z + \frac{\partial A^2}{\partial z'}\delta z' = -\left(2\alpha\beta z z' + 2\beta^2 z'^2\right)\frac{W_0}{E_0}$$

Averaging over all possible oscillation phases, one yields

$$\langle z z' \rangle_{\text{rev}} = -\frac{1}{2}\alpha\varepsilon, \qquad \langle z'^2 \rangle_{\text{rev}} = \frac{1}{2}\varepsilon\gamma.$$

and inserting this into the relation for δA^2

$$\left\langle \delta A^2 \right\rangle_{\text{rev}} = \left(-\alpha^2\beta\varepsilon + \beta^2\gamma\varepsilon\right)\frac{W_0}{E_0} = \beta^2\varepsilon\left(-\frac{\alpha^2}{\beta} + \frac{1+\alpha^2}{\beta}\right)\frac{W_0}{E_0} = \beta\varepsilon\frac{W_0}{E_0} = A^2\frac{W_0}{E_0}$$

one gets a medium amplitude variation per turn of

$$\frac{\left\langle \delta A \right\rangle_{\text{rev}}}{\text{A}} = -\frac{1}{2}\frac{W_0}{E_0}.$$

Divided by the revolution time, one obtains the damping constant

$$\alpha_z = -\frac{1}{T_0}\frac{\left\langle \delta A \right\rangle_{\text{rev}}}{\text{A}} = \frac{W_0}{2E_0T_0}$$

7.2.2. Horizontal plane

In the following, we replace the coordinate *z* by *x*. Thus we obtain firstly the same expression for α_X as in the vertical plane. Because of the not vanishing dispersion the dispersion orbit changes additionally during the emission of a photon of the energy

$\delta\varepsilon$ and by this way, it creates a variation of the position δx and of the angle $\delta x'$ of the betatron oscillation:

$$\delta x = -\delta x_\varepsilon = -D\frac{\delta\varepsilon}{E_0}, \qquad \delta x' = -\delta x_\varepsilon' = -D'\frac{\delta\varepsilon}{E_0}$$

The additional variation ΔA of the oscillation amplitude $A = \sqrt{\varepsilon\beta}$ can be calculated out of (see above)

$$A^2 = \beta\gamma x^2 + 2\alpha\beta x x' + \beta^2 x'^2$$

to

$$2A \cdot \Delta A = -\left\{\left(2\beta\gamma x + 2\alpha\beta x'\right)D + \left(2\alpha\beta x + 2\beta^2 x'\right)D'\right\} \cdot \frac{\delta\varepsilon}{E_0}.$$

Since the emitted energy $\delta\varepsilon$ depends on the beam orbit, this effect has to be explicitly considered. Hence, we again expand $P_\perp$ as a function of x and obtain in linear approximation

$$P_\perp = P_0 + \frac{2P_0}{B_0}\frac{dB}{dx}\cdot x = P_0\cdot\left(1-2kRx\right).$$

The energy loss in a small track segment ds' is

$$d\varepsilon = -\frac{P_\perp}{c}\cdot ds'$$

and with the geometric relation (see above)

$$ds' = \left(1+\frac{x}{R}\right)\cdot ds$$

one yields

$$\delta\varepsilon = -\frac{P_0}{c}\cdot\left(1-2kRx+\frac{x}{R}\right)\cdot\delta s.$$

thus giving

$$2A\Delta A = \frac{2P_0}{cE_0}\left(\frac{1}{R}-2kR\right)\left\{\left(\beta\gamma x^2+\alpha\beta xx'\right)D+\left(\alpha\beta x^2+\beta^2 xx'\right)D'\right\}$$

After averaging over the betatron phase, we get with $\left\langle xx' \right\rangle_{\text{rev}} = -\frac{1}{2}\alpha\varepsilon$, $\left\langle x^2 \right\rangle_{\text{rev}} = \frac{1}{2}\varepsilon\beta$

$$\begin{aligned}\left\langle \Delta A^2 \right\rangle_{\text{rev}} &= \frac{P_0}{cE_0}\left(\frac{1}{R} - 2kR\right)\left\{\left(\varepsilon\beta^2\gamma - \varepsilon\alpha^2\beta\right)D + \left(\varepsilon\alpha\beta^2 - \varepsilon\alpha\beta^2\right)D'\right\}\cdot\delta s \\ &= \frac{P_0}{cE_0}\left(\frac{1}{R} - 2kR\right)\varepsilon\beta D\end{aligned}$$

and after the integration over one turn it yields:

$$\frac{\left\langle \Delta A \right\rangle_{\text{rev}}}{\text{A}} = \frac{1}{2cE_0}\oint P_0 D\left(\frac{1}{R} - 2kR\right)\cdot ds = \frac{W_0}{2E_0}\mathcal{D}.$$

Using this additional term, one gets the following damping constant of the betatron oscillation:

$$\alpha_x = -\frac{1}{T_0}\left(\frac{\left\langle \delta A \right\rangle_{\text{rev}}}{\text{A}} + \frac{\left\langle \Delta A \right\rangle_{\text{rev}}}{\text{A}}\right) = \frac{W_0}{2E_0T_0}\cdot\left(1 - \mathcal{D}\right)$$

7.3. The Robinson theorem

For a plane accelerator we have obtained:

$$\alpha_S = \frac{W_0}{2E_0T_0}\cdot J_S \qquad \text{with} \qquad J_S = 2+\mathcal{D}$$

$$\alpha_X = \frac{W_0}{2E_0T_0}\cdot J_X \qquad \text{with} \qquad J_X = 1-\mathcal{D}$$

$$\alpha_Z = \frac{W_0}{2E_0T_0}\cdot J_Z \qquad \text{with} \qquad J_Z = 1$$

From this results $J_S + J_X = 3$. Neglecting the constraint on plane accelerators, the important theorem evolved by *K.W. Robinson* still holds true quite generally:

$$J_X + J_Z + J_S = 4$$

In the following, the consequences shall be investigated exemplary:

7.3.1. Radiation damping at weak focusing

Introducing the field index

$$n = -\frac{R}{B}\cdot\frac{dB}{dx} \qquad \Rightarrow \qquad n = k\cdot R^2$$

one yields for the $\mathcal{D}$ -parameter

$$\mathcal{D} = \frac{\oint \frac{D}{R^3}(1-2n)\cdot ds}{\oint \frac{ds}{R^2}} \underbrace{=}_{R=\text{const.}} \frac{1}{2\pi R}\cdot\oint(1-2n)\cdot\frac{D}{R}\cdot ds\,.$$

Out of the definition equation of the dispersion function we obtain

$$D'' + \left(\frac{1}{R^2}-k\right)\cdot D = \frac{1}{R}$$

and by integration because of the periodic boundary condition $\oint D''ds = 0$

$$\oint\left(\frac{1}{R^2}-k\right)\cdot D\cdot ds = \oint\frac{ds}{R}\,.$$

Hence, it applies for an isomagnetic magnet structure with weak focusing

$$\oint (1-n)\frac{D}{R}\cdot ds \;=\; 2\pi R\,.$$

Since the field index remains constant within the bending magnets, one yields

$$\boxed{\mathcal{D} \;=\; \frac{1-2n}{1-n}}\,.$$

Thus, we obatin for the damping constants of a weak focusing circular accelerator:

$$J_x \;\approx\; \frac{n}{1-n} \qquad J_z \;\approx\; 1 \qquad J_s \;\approx\; \frac{3-4n}{1-n}$$

$$\Rightarrow \qquad \boxed{0 \;<\; n \;<\; 0{,}75}$$

7.3.2. Radiation damping at combined function accelerators

Again for simplification purposes, we only consider the isomagnetic case, express the $\mathcal{D}$ - parameter in dependence of the field index

$$\mathcal{D} = \frac{1}{2\pi R} \cdot \oint (1-2n) \cdot \frac{D}{R} \cdot ds$$

and use deduced above

$$\oint (1-n) \frac{D}{R} \cdot ds = 2\pi R .$$

As the field index does not remain constant in the bending magnets at strong focusing any longer, using the momentum compaction factor we get

$$\boxed{\mathcal{D} = 2 - \frac{\oint \frac{D}{R} \cdot ds}{2\pi R} = 2 - \frac{\alpha_C L}{2\pi R}}$$

Since typically $\alpha_C \ll 1$, then $\mathcal{D} \approx 2$, and for the damping constants holds true:

$$J_X \approx -1 \qquad J_Z \approx 1 \qquad J_S \approx 4$$

$\Rightarrow$ The horizontal betatron oscillation is dedamped!!!

In a classical synchrotron (combined function) the beam remains stable because of the short dwell time and the adiabatic damping. However suchlike accelerators are not suited as storage rings!

7.3.3. Radiation damping at separated function accelerators

At accelerators with separated dipole and quadrupole magnets, $k/R = 0$ holds true. It applies for the $\mathcal{D}$ - parameter again in isomagnetic approximation

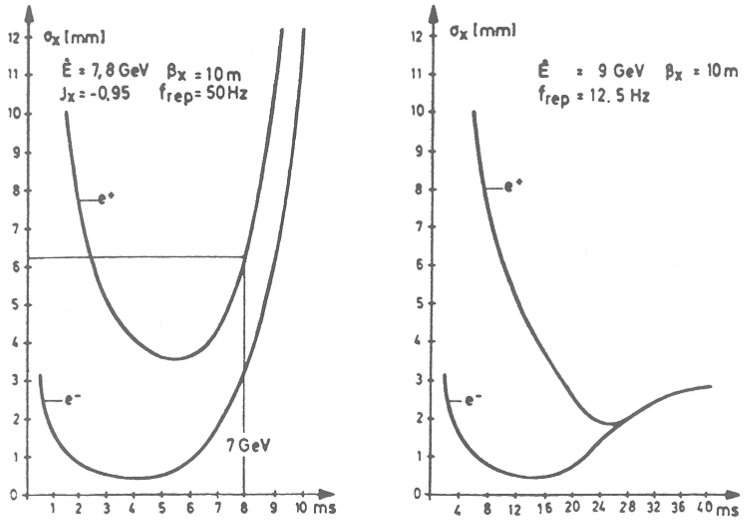
$$\boxed{\mathcal{D} = \frac{\oint \frac{D}{R} \cdot ds}{2\pi R} = \frac{\alpha_C L}{2\pi R}}$$

Usually, $\alpha_C \ll 1$ is also in this case and therefore $\mathcal{D} \approx 0$. Thus one gets a "natural damping distribution"

$$J_X \approx 1 \qquad J_Z \approx 1 \qquad J_S \approx 2$$

$\Rightarrow$ Damping in all 3 oscillation planes

Because of this advantage not only storage rings but also synchrotrons are built as seperated function accelerators in the meantime:



Variation of horizontal beam size with time during the acceleration cycle in the DESY I (left) and DESY II (right) synchrotrons [24]

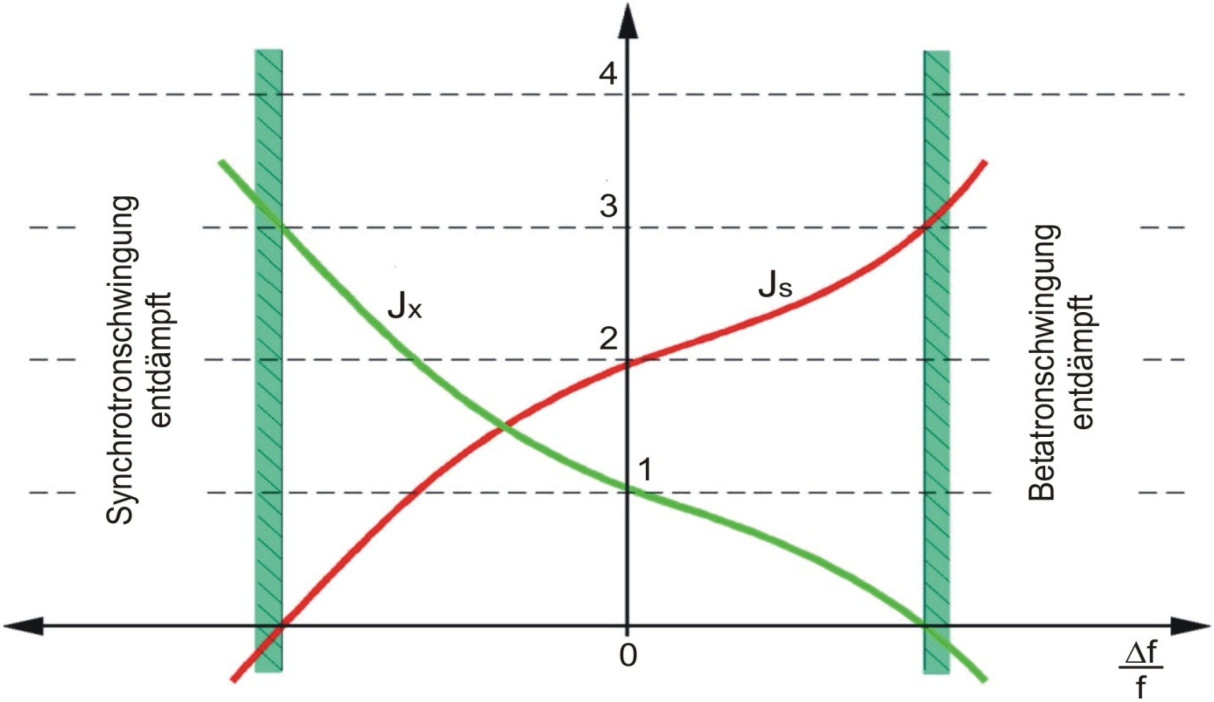
At electron accelerators the damping distribution can be varied by shifting the beam to a dispersion orbit. This can happen by variation of the radio frequency. Because of

$$\frac{\Delta L}{L} = -\frac{\Delta f}{f} = \alpha_C \frac{\Delta E}{E}$$

a horizontal beam displacement arises at a frequency variation Δf

$$\boxed{x_D = -\frac{D}{\alpha_C} \cdot \frac{\Delta f}{f}}.$$

The beam does not proceed on the magnetic axis of the quadrupoles anymore, so that the quadrupole magnets operate as a superposition of a quadrupole and a dipole and that $k/R \neq 0$. Thus the accelerator passes increasingly into a combined function machine. Hence the $\mathcal{D}$ - parameter and the damping distribution vary.


4
3
2
1
0
Synchrotronschwingung
entdämpft
Betatronschwingung
entdämpft
Jx
Js
Δf
f

7.4. Energy distribution in longitudinal phase space

On the one hand the longitudinal energy oscillation is damped by the emission of synchrotron light und the restoration of the longitudinal momentum, on the other hand it is also excited by the stochastic process of the radiation of energy quanta. In the following we examine the temporal variation of the medium quadratic energy:

Radiation:

Because of the damping the amplitude of the oscillation decays exponentially:

$$\Delta\hat{E}(t) = \Delta\hat{E}(t_0)\cdot e^{-\alpha_S\cdot(t-t_0)} \quad \Rightarrow \quad \boxed{\frac{d\,\Delta\hat{E}^2}{d\,t} = -2\alpha_S\cdot\Delta\hat{E}^2}$$

Excitation:

With the emission of a photon of the energy ε the mean quadratic amplitude of the energy oscillation varies:

$$\left\langle \delta\,\Delta\hat{E}^2 \right\rangle = \varepsilon^2$$

Per time unit $\dot{n}(\varepsilon)\cdot d\varepsilon$ photons are emitted in the energy interval $\left[\varepsilon,\varepsilon+d\varepsilon\right]$. Hence the resulting medium amplitude variation per time unit is

$$d_{\varepsilon}\left\{\frac{d\left\langle\Delta\hat{E}^{2}\right\rangle}{dt}\right\} = \varepsilon^{2}\cdot\dot{n}(\varepsilon)\cdot d\varepsilon$$

And after integration over all photon energies of the spectrum one yields

$$\boxed{\frac{d\left\langle\Delta\hat{E}^{2}\right\rangle}{dt} = \int_{0}^{\infty}\varepsilon^{2}\cdot\dot{n}(\varepsilon)\cdot d\varepsilon = \dot{N}\cdot\left\langle\varepsilon^{2}\right\rangle}$$

Excitation and damping of the longitudinal oscillations compensate each other in equilibrium:

$$\boxed{\dot{N}\left\langle\varepsilon^{2}\right\rangle - 2\alpha_{S}\left\langle\Delta\hat{E}^{2}\right\rangle = 0}$$

For the calculation of $\dot{N}\left\langle \varepsilon^2 \right\rangle = \int \varepsilon^2 \, \dot{n}(\varepsilon)\, d\varepsilon$ we revert to the results of chapter 6. With the dwell time T in the bending magnets and

$$\dot{n}(\varepsilon) = \frac{1}{\varepsilon} \cdot \frac{d\,P_\perp}{d\,\varepsilon} = \frac{1}{\hbar\varepsilon T} \cdot \frac{d\,I}{d\,\omega} = \frac{1}{\hbar\varepsilon} \cdot \frac{c}{2\pi R} \cdot \frac{d\,I}{d\,\omega}$$

and execution of the integration one gets

$$\dot{N}\left\langle \varepsilon^2 \right\rangle = \frac{9\sqrt{3}}{8\pi} \frac{P_0}{\left(\hbar\omega_c\right)^2} \int_0^\infty d\varepsilon \cdot \varepsilon^2 \int_{\varepsilon/\hbar\omega_c}^\infty dx \cdot K_{5/3}(x) = \frac{55}{24\sqrt{3}} \hbar\omega_c P_0 \sim \frac{\gamma^7}{R^3} .$$

For this reason, the mean quadratic amplitude of the energy oscillation is

$$\left\langle \Delta E^2 \right\rangle = \frac{1}{2\alpha_S} \dot{N}\left\langle \varepsilon^2 \right\rangle = \frac{55}{48\sqrt{3}\,\alpha_S} \hbar\omega_c P_0 .$$

The damping constant α_S is linked with the energy radiation W_0 per turn, and with $W_0 = \left\langle P_0 \right\rangle \cdot T_0$ one yields

$$\alpha_S = \frac{W_0 J_S}{2T_0 E_0} = \frac{\left\langle P_0 \right\rangle J_S}{2E_0} = \frac{\left\langle P_0 \right\rangle}{2\gamma m_0 c^2} J_S .$$

For this purpose, the radiated power must be averaged over one turn:

$$P_0 = \frac{\langle P_0 \rangle}{\langle 1/R^2 \rangle} \cdot \frac{1}{R^2}.$$

The energy width of the beam adds up at sinusoidal energy oscillations to the half of the quadratic amplitude ($\sigma_E{}^2 = \langle \Delta\hat{E}^2 \rangle / 2$), and after inserting the critical frequency $\omega_c = 3c\gamma^3/2R$ and average determination one gets for the **relative energy width:**

$$\frac{\sigma_E{}^2}{E^2} = \frac{55}{32\sqrt{3}} \cdot \frac{\hbar c \gamma^2}{J_S\, m_0\, c^2} \cdot \frac{\langle 1/R^3 \rangle}{\langle 1/R^2 \rangle}$$

If all bending magnets are equal (i.e. same length and same bending radius), it yields

$$\left(\frac{\sigma_E}{E}\right)^2 = \frac{55}{32\sqrt{3}} \cdot \frac{\hbar c \gamma^2}{J_S\, m_0\, c^2} \cdot \frac{1}{R},$$

what can be expressed (for electrons) by

$$\left(\frac{\sigma_E}{E}\right)^2 = C_q \cdot \frac{\gamma^2}{J_S\, R} \qquad \text{with} \qquad C_q \approx 3.84 \cdot 10^{-13}\,\mathrm{m}$$

approximately.

Thus the relative energy width grows linearly with increasing energy!

The order of magnitude is typically ≈ 0.1 %.

Since energy width and bunch length are directly linked with each other (see chapter 5.3.2), one gets for the standard deviation (FWHM/2)

$$\sigma_S = \frac{c}{\omega_0}\sqrt{\frac{2\pi\eta E_0}{heU_0\cos\varphi_0}} \cdot \left(\frac{\sigma_E}{E}\right)$$

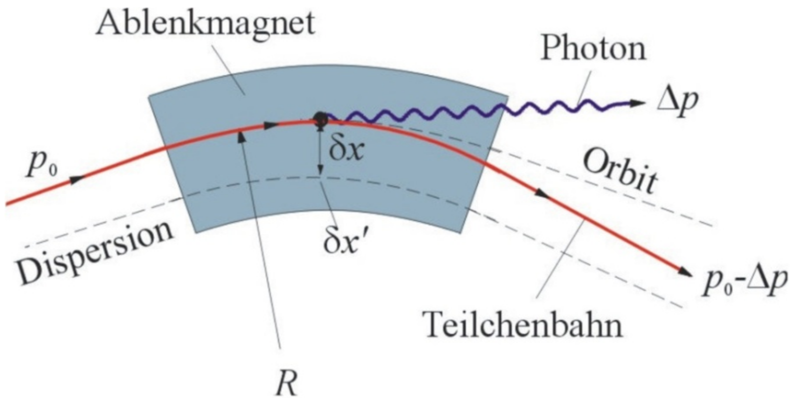
The resulting natural energy and intensity distributions are Gauss-shaped!

7.5. Natural beam emittance

By the emission of synchrotron radiation, the transverse betatron oscillation is damped but also excited simultaneously because

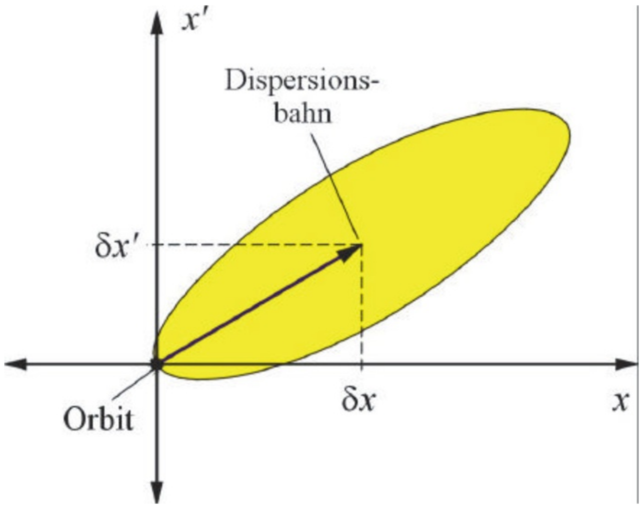
1. the photons are not emitted exactly into the course of the electrons, but in a cone with the angle of aperture $2/\gamma$,
2. the dispersion orbit and therewith the betatron amplitude varies at emission in dispersive sections.

The 1st effect can be neglected compared to the 2nd one.



As a result of the variation of the particle's energy the betatron amplitude increases. The easiest way for its determination is the determination for one particle circling before the emission with design momentum on the design orbit. After the emission it is resided on a dispersion orbit with

$$\delta x = D\frac{\Delta p}{p_0}, \qquad \delta x' = D'\frac{\Delta p}{p_0}.$$



The excitation results from the increase of the horizontal oscillation amplitude

$$\delta \hat{A}_x^{\,2} = \gamma \cdot \delta x^2 + 2\alpha \cdot \delta x \delta x' + \beta \cdot \delta x'^2$$
$$= \left(\frac{\delta p}{p_0}\right)^2 \cdot \left(\gamma \cdot D^2 + 2\alpha \cdot D D' + \beta \cdot D'^2\right)$$
$$= \left(\frac{\delta p}{p_0}\right)^2 \cdot \mathcal{H}(s)$$

Since $\Delta p / p_0 = \Delta E / E_0$ holds true for ultra-relativistic particles, one obtains analogously to the determination of the longitudinal energy distribution with

$$\varepsilon_x = \left\langle \Delta \hat{A}_x^{\,2} \right\rangle / 2 \quad \text{and} \quad \delta \hat{A}_x^{\,2} = \frac{\varepsilon^2}{E_0^{\,2}} \cdot \mathcal{H}(s) \;\Rightarrow\; \left\langle \Delta \hat{A}_x^{\,2} \right\rangle = \frac{1}{2\alpha_x} \cdot \frac{\dot{N} \left\langle \varepsilon^2 \right\rangle}{E_0^{\,2}} \cdot \mathcal{H}(s)$$

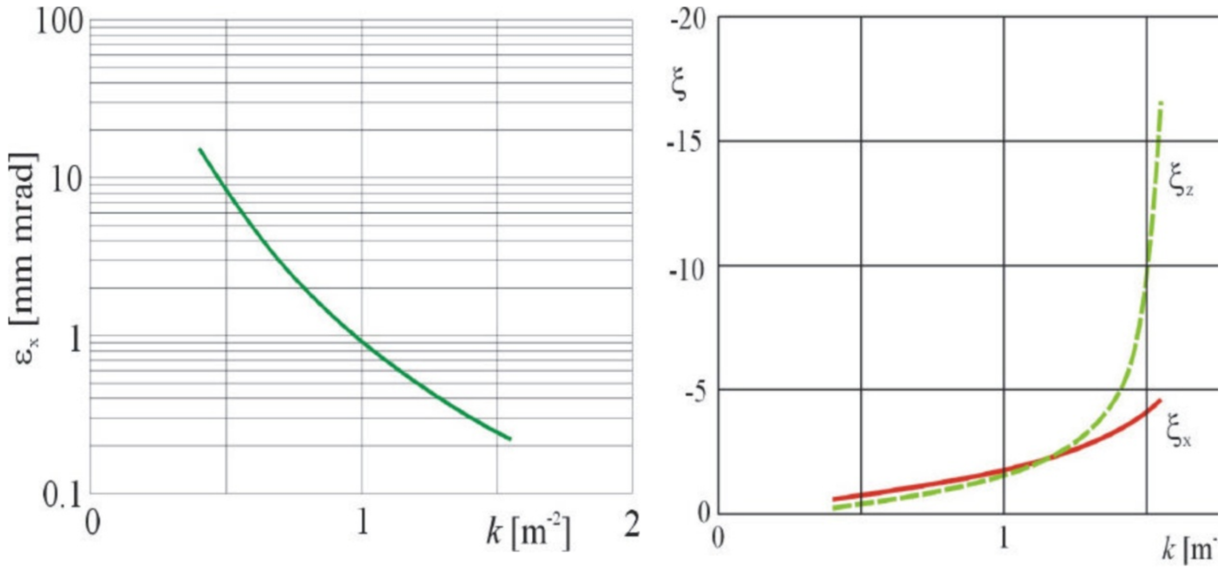
$$\varepsilon_x = \left\langle \frac{\left\langle \Delta E^2 \right\rangle}{2 E_0^{\,2}} \cdot \frac{\alpha_S}{\alpha_X} \cdot \mathcal{H}(s) \right\rangle = \frac{55}{32\sqrt{3}} \cdot \frac{\hbar c \gamma^2}{J_X m_0 c^2} \cdot \frac{\left\langle 1/R^3 \cdot \mathcal{H}(s) \right\rangle}{\left\langle 1/R^2 \right\rangle}$$

If all bending magnets have the same bending radius R and the same length l the following equation applies approximately

$$\varepsilon_x\left[\mathrm{m}\cdot\mathrm{rad}\right] = 1{,}47\cdot 10^{-6}\cdot\frac{E^2\left[\mathrm{GeV}\right]}{R\left[\mathrm{m}\right]\cdot l\left[\mathrm{m}\right]}\cdot\int_0^l \mathcal{H}(s)\cdot ds$$

Therefore, a small natural emittance requires small beta functions and small dispersions (design of the magnetic structure!)

The emittance can also be decreased by variation of the optics. By way of example we examine here a FODO structure consisting of 16 dipoles (length 1.5 m) and 16 quadrupoles (length 0.4 m) with drift spaces (length 0.55 m) between the dipoles and the quadrupoles (cf. Wille). Stable solutions exist for $0.4\,\mathrm{m}^{-2} < |k| < 1.6\,\mathrm{m}^{-2}$:



Increasing k the values of β_x, D decrease and therefore the emittance, too.

But also the chromaticity increases (especially the vertical one).

7.6. Synchrotron integrals

The relations deduced in the precedent paragraphs can also be expressed in a general form by the so called **synchrotron integrals** $\mathfrak{I}_i$:

$$\mathfrak{I}_1 = \oint \frac{D}{R} ds \qquad \mathfrak{I}_2 = \oint \frac{ds}{R^2} \qquad \mathfrak{I}_3 = \oint \frac{ds}{R^3}$$

$$\mathfrak{I}_4 = \oint \frac{D}{R}\left(\frac{1}{R^2} - 2k\right) ds = \oint \frac{(1-2n)D}{R^3} ds$$

$$\mathfrak{I}_5 = \oint \frac{\mathcal{H}}{R^3} ds \qquad \mathfrak{I}_6 = \oint k^2 D^2 \, ds$$

Hence one obtains with

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{m_0 c}$$

the following relations (where $r_e = 1/\left(4\pi\varepsilon_0 \cdot m_0 c^2\right)$, the class. electron radius):

Damping parameters:

$$J_x = 1 - \frac{\Im_4}{\Im_2} \qquad J_z = 1 \qquad J_s = 2 + \frac{\Im_4}{\Im_2}$$

Damping times:

$$\tau_x = \frac{3T_0}{r_e \gamma^3} \cdot \frac{1}{\Im_2 - \Im_4} \qquad \tau_z = \frac{3T_0}{r_e \gamma^3} \cdot \frac{1}{\Im_2} \qquad \tau_s = \frac{3T_0}{r_e \gamma^3} \cdot \frac{1}{2\Im_2 + \Im_4}$$

Relative energy width:

$$\left(\frac{\sigma_E}{E_0}\right)^2 = C_q \cdot \gamma^2 \cdot \frac{\Im_3}{2\Im_2 + \Im_4} = \frac{C_q \cdot \gamma^2}{J_s} \cdot \frac{\Im_3}{\Im_2}$$

Natural emittance:

$$\varepsilon_x = C_q \cdot \gamma^2 \cdot \frac{\Im_5}{\Im_2 - \Im_4} = \frac{C_q \cdot \gamma^2}{J_x} \cdot \frac{\Im_5}{\Im_2}$$

7.7. Beam lifetime

Because of limitations due to the aperture or due to the overvoltage factor one expects a finite lifetime of a stored beam:

7.7.1. Limitation due to betatron oscillations

A Gauss-shaped intensity distribution in the transverse phase space results from radiation processes. In the following we investigate only the horizontal phase space because $\varepsilon_z \xrightarrow{\text{ideal}} 0$. The equilibrium emittance ε_x is the area in the phase space normalized to π for 1σ - deviations in the displacement $\sigma_x = \sqrt{\beta\,\varepsilon_x}$ and in the angle $\sigma_{x'} = \sqrt{\gamma\,\varepsilon_x}$:

$$\boxed{\varepsilon_x = \gamma\, x^2 + 2\alpha\, x\, x' + \beta\, x'^2}\,.$$

A particle is characterized by its phase space trajectory and thus by its individual “emittance“ $\in$. The maximum feasible displacement is then

$$x_{\max}{}^{2} = \beta \cdot \in$$

The distribution function in the horizontal phase space reads (c.f. Hinterberger):

$$\rho(x,x') = \frac{1}{2\pi\varepsilon_x} \cdot e^{-\frac{\vec{x}^T\left[\sigma_x{}^{-1}\right]\vec{x}}{2}} = \frac{1}{2\pi\varepsilon_x} \cdot e^{-\frac{\gamma x^2 + \alpha x x' + \beta x'^2}{2\varepsilon_x}},$$

which can be transformed with $\pi \cdot d\in = dx \cdot dx'$ into the following distribution function:

$$\boxed{\rho(\in) \cdot d\in = \frac{1}{2\varepsilon_x} \cdot e^{-\frac{\in}{2\varepsilon_x}} \cdot d\in}.$$

In the case of equilibrium, it applies for the particle flux

$$\dot{N}\left(\in_0 - \delta\in \rightarrow \in_0 + \delta\in\right) = \dot{N}\left(\in_0 + \delta\in \rightarrow \in_0 - \delta\in\right).$$

The latter can be calculated out of the oscillation damping:

$$\dot{N}\left(\in_0 + \delta\in \rightarrow \in_0 - \delta\in\right) = \left.\frac{\partial N}{\partial\in} \cdot \frac{\partial\in}{\partial t}\right|_{\in=\in_0}$$

and with

$$\frac{\partial N}{\partial \in} = N \cdot \rho(\in) = \frac{N}{2\varepsilon_x} \cdot e^{-\frac{\in}{2\varepsilon_x}}$$

as well as

$$\frac{\partial \in}{\partial t} = \frac{\partial}{\partial t}\left\{\hat{\in} \cdot e^{-\frac{2t}{\tau_x}}\right\} = -\frac{2\in}{\tau_x}$$

one gets:

$$\boxed{\dot{N} = -\frac{1}{\tau_x}\frac{\in_0}{\varepsilon_x} e^{-\frac{\in_0}{2\varepsilon_x}} \cdot N = -\frac{1}{\tau} \cdot N}.$$

If the horizontal phase space is limited by the aperture $x_{\max}$, it applies

$$\frac{\in_0}{\varepsilon_x} = \frac{\beta \in_0}{\beta \varepsilon_x} = \frac{x_{\max}{}^2}{\sigma_x{}^2}$$

and we obtain for the **lifetime-time constant:**

$$\tau = \tau_x \cdot \frac{e^{\xi/2}}{\xi} \qquad \text{with} \qquad \xi = \left(\frac{x_{\max}}{\sigma_x}\right)^2 .$$

For a typical damping time of a betatron oscillation of $\tau_x = 10$ ms one gets:

$x_{\max}/\sigma_x$	5.0	5.5	6.0	6.5	7.0
lifetime τ	1.8 min	20.4 min	5.1 h	98.3 h	103 days

Herefrom follows the **golden rule for high lifetimes**:

$$\boxed{\frac{x_{\max}}{\sigma_x} > 6.5} .$$

In practice, the available aperture has to be slightly larger in order to have some safety margins for beam steering and dispersion effects.

7.7.2. Limitation due to energy oscillations

In the longitudinal phase space, a Gauss-shaped intensity distribution arises, too. Here we examine the two parameters $\Delta\varphi$ und ΔE, whereas they are linked each other by (here β is the Lorentz factor!)

$$\Delta\dot{\varphi} = \frac{\eta \cdot \omega_{RF}}{\beta^2 \cdot E_0} \cdot \Delta E \overbrace{\approx}^{\beta \approx 1} -\frac{\alpha_c \cdot \omega_{RF}}{E_0} \cdot \Delta E$$

(cf. chapter 5.3.).

In linear approximation it holds for the synchrotron oscillation in the case of equilibrium

$$\Delta\ddot{\varphi} + \Omega_S{}^2 \cdot \Delta\varphi = 0 \qquad \Rightarrow \qquad \frac{1}{2}\Delta\dot{\varphi}^2 + \frac{1}{2}\Omega_S{}^2 \cdot \Delta\varphi^2 = \text{const.}$$

Expressing $\Delta\dot{\varphi}$ by ΔE, an invariant oscillation amplitude (and thus also the longitudinal emittance) may be defined:

$$\varepsilon_S = \Delta E^2 + \left(\frac{\Omega_S \cdot E_0}{\alpha_c \cdot \omega_{RF}}\right)^2 \cdot \Delta\varphi^2 \overset{!}{=} \sigma_E{}^2 = \left(\frac{\Omega_S \cdot E_0}{\alpha_c \cdot \omega_{RF}} \cdot \sigma_\varphi\right)^2$$

Here a particle can also be characterized by its phase space trajectory und thus its individual “emittance“ $\in_S$. Here the maximum feasible energy deviation adds simply up to

$$\Delta E_{\max}{}^2 = \in_S .$$

For the distribution function we can write a duplex Gauss distribution

$$\rho(\Delta\varphi, \Delta E) = \frac{1}{\sqrt{2\pi\sigma_\varphi{}^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_E{}^2}} \cdot e^{-\frac{\Delta\varphi^2}{2\sigma_\varphi{}^2}} \cdot e^{-\frac{\Delta E^2}{2\sigma_E{}^2}}$$

and with $\pi \cdot d \in_S = \frac{\Omega_S \cdot E_0}{\alpha_c \cdot \omega_{RF}} \cdot d\left(\Delta\varphi\right) \cdot d\left(\Delta E\right) = \frac{\sigma_E}{\sigma_\varphi} \cdot d\left(\Delta\varphi\right) \cdot d\left(\Delta E\right)$ we obtain:

$$\rho(\in_S) = \frac{1}{2\varepsilon_S} \cdot e^{-\frac{\in_S}{2\varepsilon_S}} .$$

Totally analogous to the transverse phase space one gets for the **lifetime time constant**

$$\tau = \tau_S \cdot \frac{e^{\xi/2}}{\xi} \qquad \text{with} \qquad \xi = \frac{\epsilon_S^{\ 2}}{\varepsilon_S^{\ 2}} = \left(\frac{\Delta E_{\max}}{\sigma_E}\right)^2$$

In isomagnetic approximation one can express this with the handier parameter of the overvoltage factor q as

$$\xi = \frac{32\sqrt{3}}{55\pi\hbar c} \cdot \frac{eU_0 \sin\varphi_0}{h\alpha_c} \cdot \frac{J_S \cdot R}{\gamma^3} \cdot \underbrace{2\left\{\sqrt{q^2-1} - \arccos\frac{1}{q}\right\}}_{\equiv F(q)},$$

whereas $F(q)$ is denoted as **energy aperture function**.

Since the energy loss per turn must be compensated by the acceleration, it is necessary $eU_0 \sin\varphi_0 = \frac{e^2}{3\varepsilon_0} \cdot \frac{\gamma^4}{R}$, and one obtains finally

$$\tau = \tau_S \cdot \frac{e^{\xi/2}}{\xi} \qquad \text{with} \qquad \xi = \underbrace{\frac{32\,e^2}{55\sqrt{3}\,\pi\,\varepsilon_0\,\hbar\,c}}_{\approx 0.01} \cdot \frac{J_S \cdot \gamma}{\alpha_c \cdot h} \cdot F(q).$$

7.8. Diminution of the beam emittance

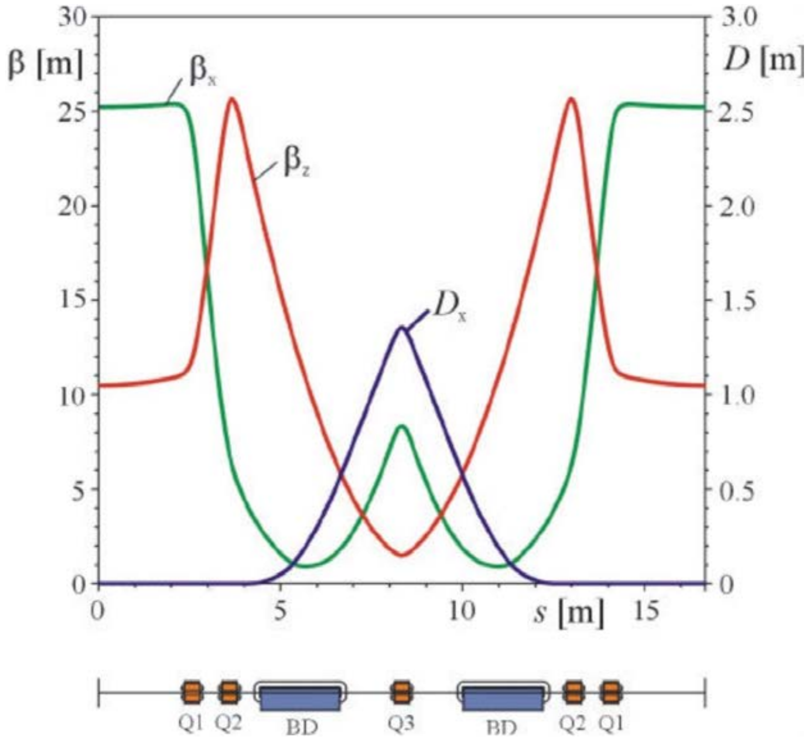
Small beam emittances are vitally important designing synchrotron light sources as the users are interested in a high beam brilliance:

$$\boxed{B = \frac{\dot{N}_\gamma / (0{,}1\%\mathrm{BW})}{4\pi^2\,\varepsilon_x\,\varepsilon_z\,I}}.$$

Planning a magnet structure one has to pay attention on a preferably small dispersion. In the following, we will discuss suitable achromatic systems, mainly based on the double bend achromat approach used for third-generation synchrotron radiation sources

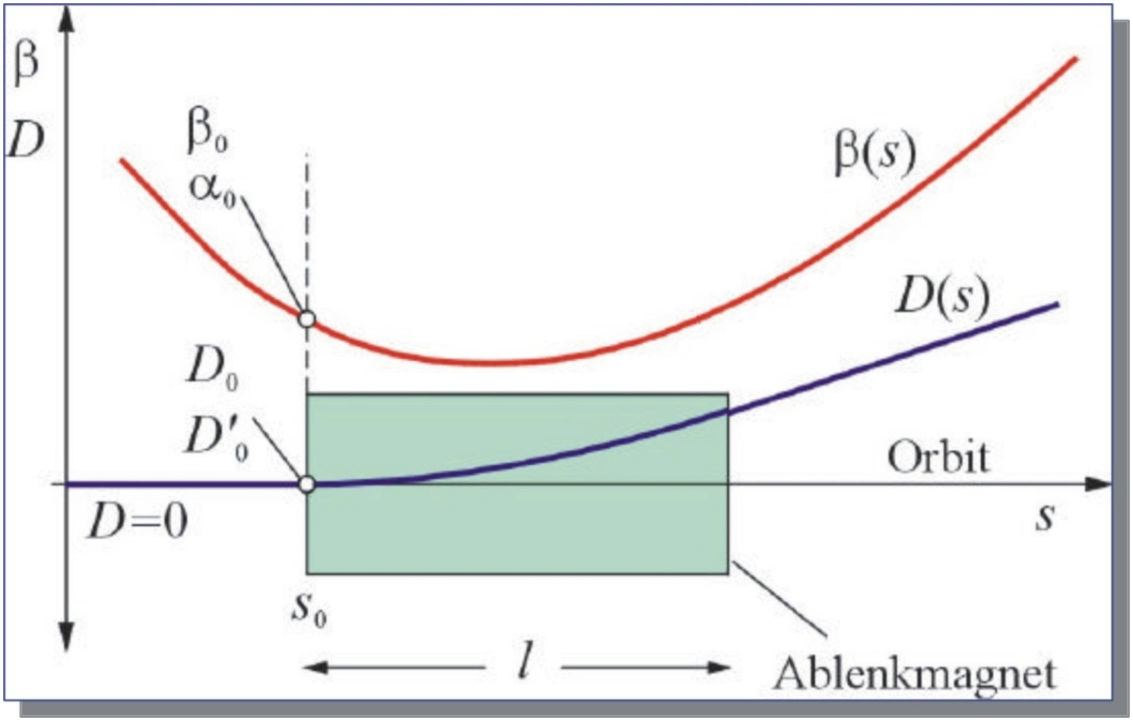
7.8.1. Dispersion suppression

It makes sense (also for the installation of wigglers / undulators) to have a preferably large number of dispersion-free sections, optimally behind every 2nd dipole magnet:



The beta function should be preferably small (preferably high quadrupole gradients), but not at the expense of the dispersion!

Because of symmetry reasons it is sufficient to examine only one bending magnet:



We obtain of the horizontal transfer matrix with

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{s}{R} & R\cdot\sin\frac{s}{R} & R\cdot\left(1-\cos\frac{s}{R}\right) \\ -\frac{1}{R}\cdot\sin\frac{s}{R} & \cos\frac{s}{R} & \sin\frac{s}{R} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

and the initial conditions $D_0 = 0,\ D_0' = 0$ the trend of D in a bending magnet:

$$\boxed{D(s) = R\cdot\left(1-\cos\frac{s}{R}\right) \approx \frac{s^2}{2\,R}, \qquad D'(s) = \sin\frac{s}{R} \approx \frac{s}{R}}.$$

Out of it one can only find out that the bending magnets should have a preferably short length. Thus the influence of the beta function seems to be essential. Falling back to the concept of the beta matrix we obtain with

$$\mathbf{B}_1 = \mathbf{M}\cdot\mathbf{B}_0\cdot{}^{T}\mathbf{M}$$

and the approximation $R \cdot \sin s/R \approx s$ in the transfer matrix $\mathbf{M}$ of the rectangular magnets

$$\begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

the expansion of the Twiss parameters in the bending magnet:

$$\begin{aligned} \alpha(s) &= \alpha_0 - \gamma_0\, s \\ \beta(s) &= \beta_0 - 2\alpha_0\, s + \gamma_0\, s^2 \\ \gamma(s) &= \gamma_0 \end{aligned}$$

For the horizontal emittance the following equation held in isomagnetic approximation:

$$\boxed{\varepsilon_x = \frac{55}{32\sqrt{3}} \cdot \frac{\hbar}{m_0\, c} \cdot \frac{\gamma^2}{R \cdot l} \cdot \int_0^l \mathcal{H}(s) \cdot ds}\ .$$

and by inserting the $\mathcal{H}$ - function

$$\mathcal{H}(s) = \gamma(s)\cdot D^2(s) + 2\alpha(s)\cdot D(s)D'(s) + \beta(s)\cdot D'^2(s)$$
$$= \frac{1}{R^2}\cdot\left(\frac{\gamma_0}{4}\cdot s^4 - \alpha_0\cdot s^3 + \beta_0\cdot s^2\right)$$

one gets with the angle of deflection $\theta = l/R$ of a dipole magnet:

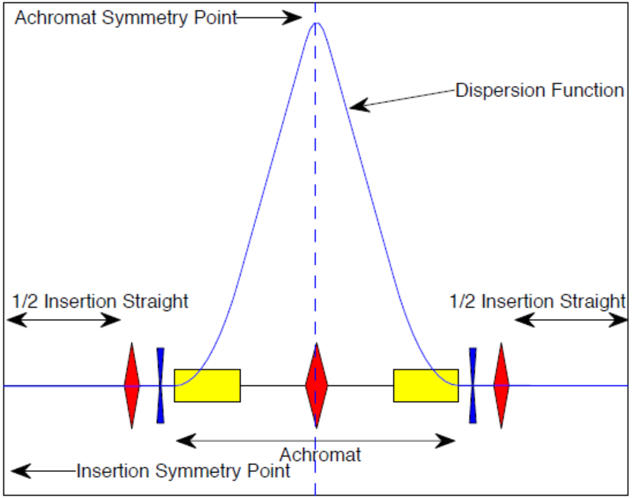
$$\varepsilon_x = \frac{55}{32\sqrt{3}}\cdot\frac{\hbar}{m_0\,c}\cdot\gamma^2\cdot\theta^3\cdot\left(\frac{\gamma_0\,l}{20} - \frac{\alpha_0}{4} + \frac{\beta_0}{3l}\right)$$

Thus the emittance only depends significantly on the initial values yet and becomes minimum $\partial\varepsilon_x/\partial\alpha_0 = 0,\ \partial\varepsilon_x/\partial\beta_0 = 0$. This results:

$$\varepsilon_x = \min \qquad \Leftrightarrow \qquad \begin{cases} \beta_0 = 2\sqrt{\frac{3}{5}}\cdot l \approx 1{,}549\cdot l \\ \alpha_0 = \sqrt{15} \approx 3{,}873 \end{cases}$$

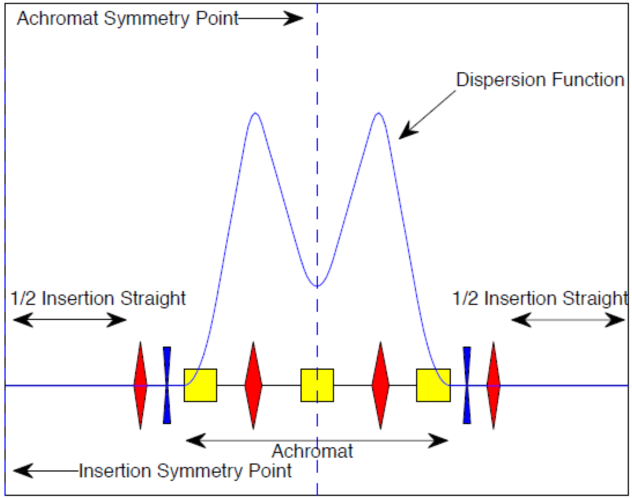
In practice one deviates marginally from these values, since the chromaticity would trend to extremely high values in case of the optimum initial values $\alpha_0,\ \beta_0$.

7.8.2. Common lattice options



DBA:

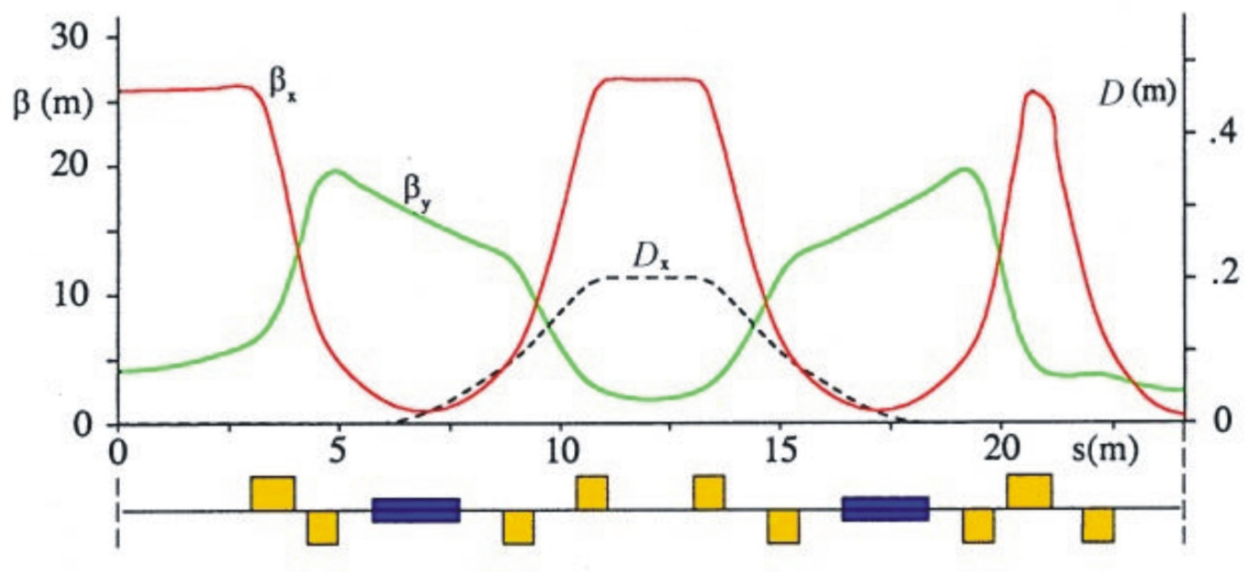
double-bend achromat



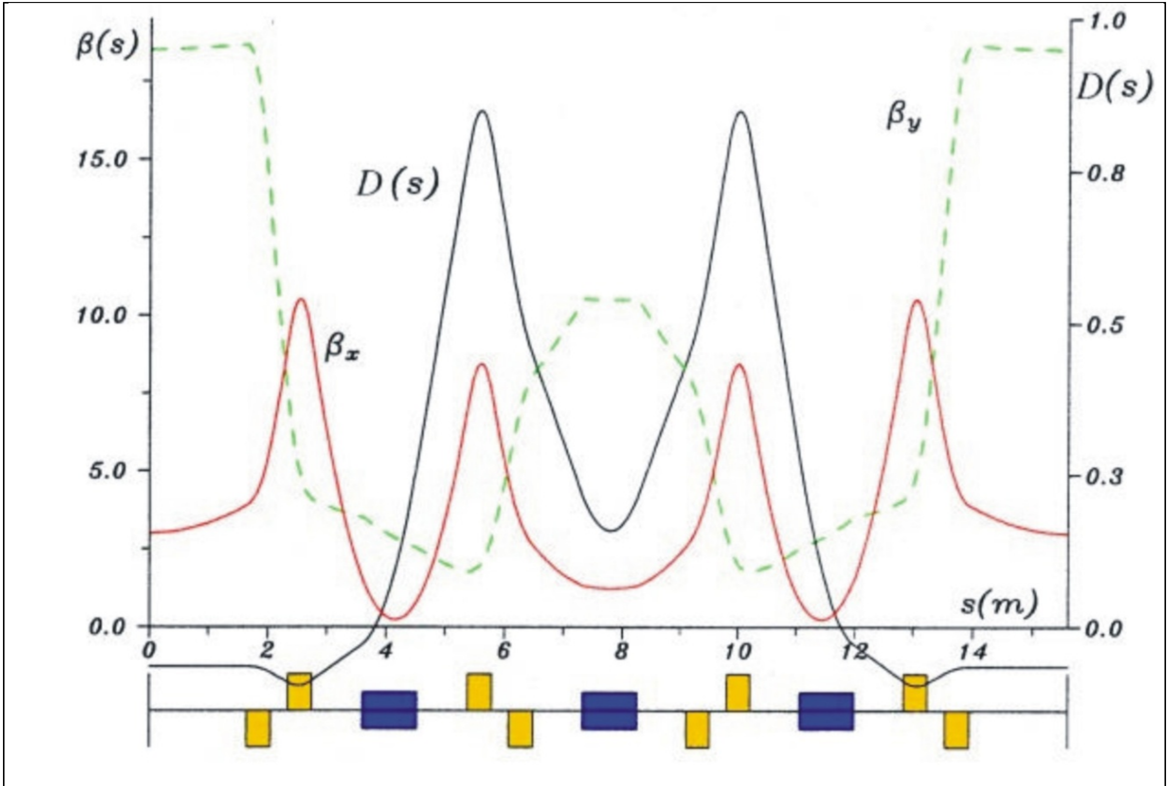
TBA:

triple-bend achromat

In the following some examples of magnetic structures of synchrotron light sources are presented:



Magnetstruktur der „European Synchrotron Radiation Facility“ in Grenoble



Magnetstruktur von „BESSY II“ in Berlin

3rd Generation Rings (Current and Future)

SLS (2002) 2.4GeV
ε_x= 3.9 nm, ε_y=72 pm, I=300 mA

ALS (1993) 1.9GeV
ε_x= 6.3 (2.2) nm, ε_y=30 pm, I=500 mA

MAX-4 (2016) 3GeV
ε_x= 0.2-0.3 nm, ε_y= 8 pm, I=500 mA

NSLS-II (2013) 3GeV
ε_x= 0.6-1.1 nm, ε_y= 8 pm, I=500 mA

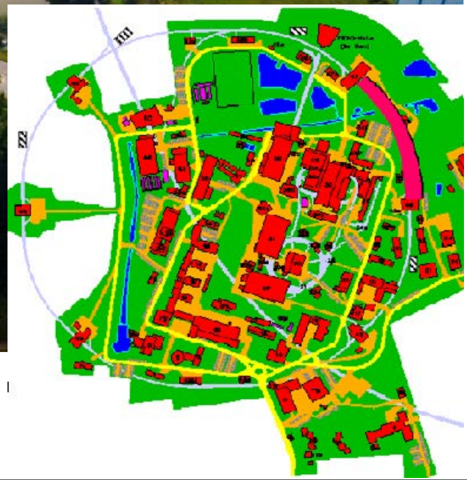
Soleil (2006) 2.75 GeV
ε_x= 3.7/5.6 nm, ε_y=37 pm, I=400(500) mA

APS (1995) 7GeV
ε_x= 2.5/3 nm, ε_y=25 pm, I=100 mA

Diamond (2007)
3.0 GeV
ε_x= 3.0 nm,
ε_y= 30 pm,
I=300(500) mA

Advanced Light Source C. Steier, FLS 2012, Ultimate Rings Comparison, 2012-03-06 5

PETRA III @ DESY



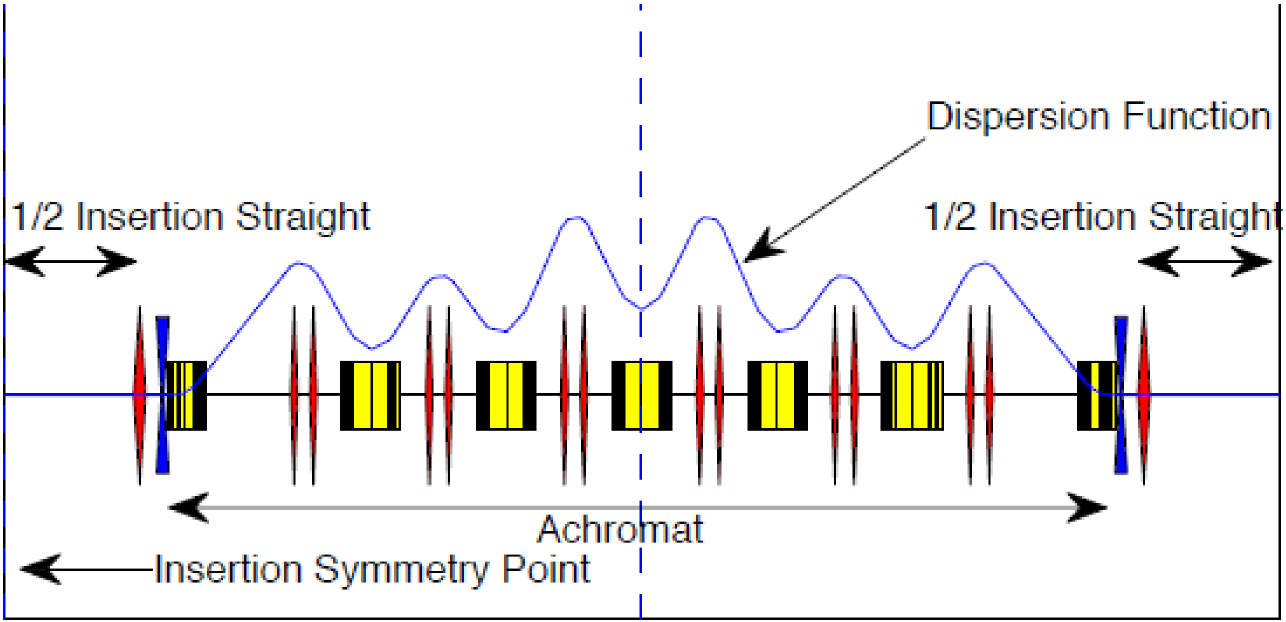
PETRA III (2009) 6 GeV

ε_x = 1.0 nm, ε_y = 10 pm damping wigglers!

I = 100 mA

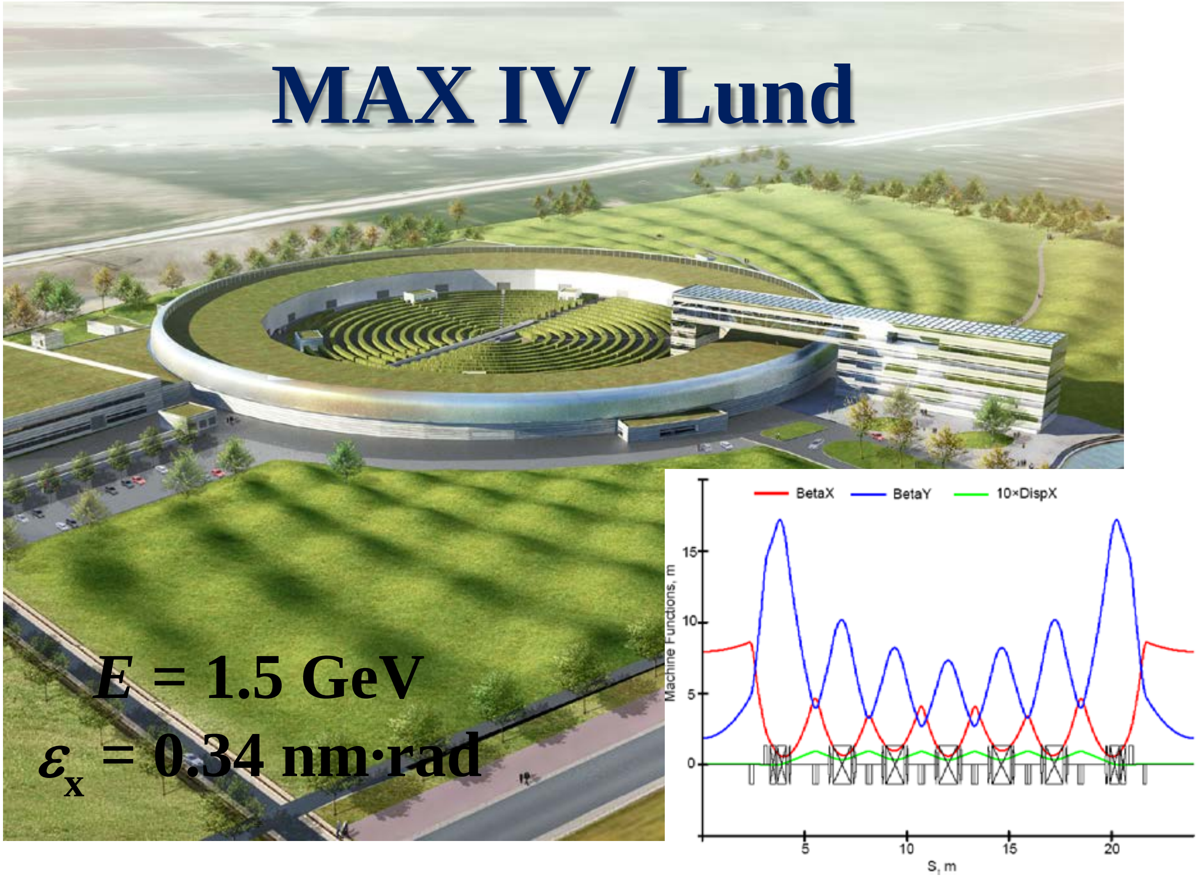
7.8.3. Multi-bend achromat concept

The MBA concept is based on the idea to break the dipoles into shorter magnets separated by focusing elements. Matching sections assure zero dispersion in the straights:



In order to safe space, the unit cells have defocusing bending dipoles!

The first source of this kind, which came to operation last year is:

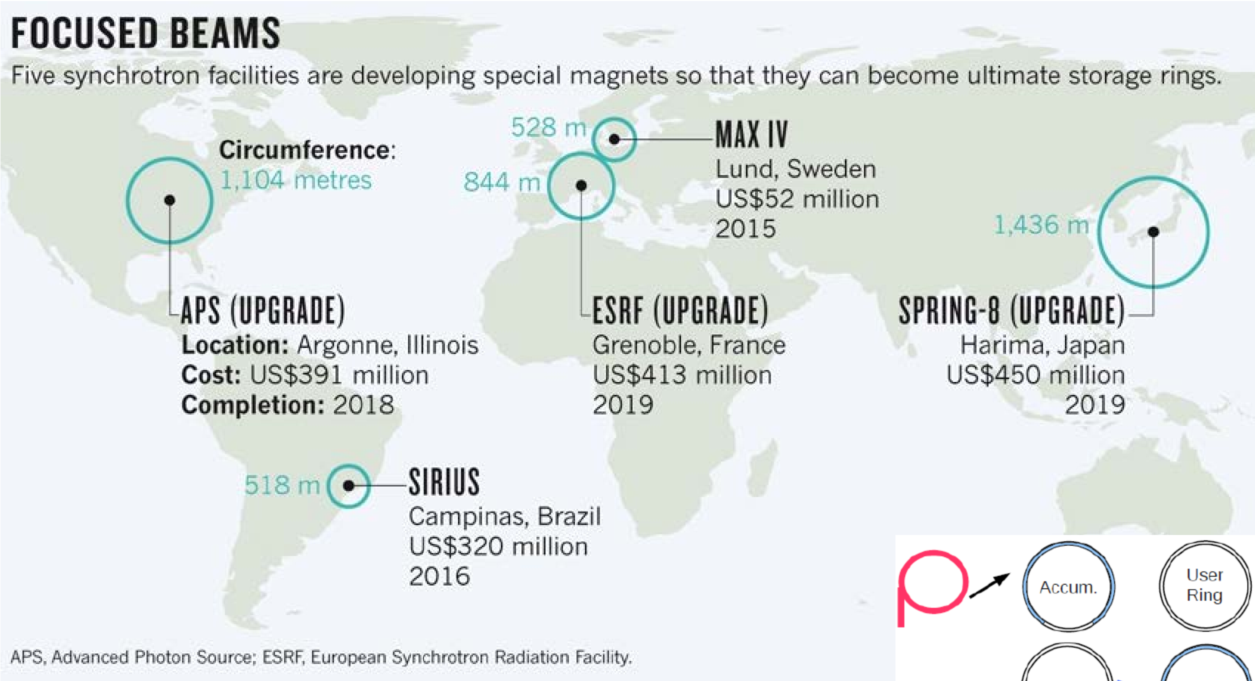


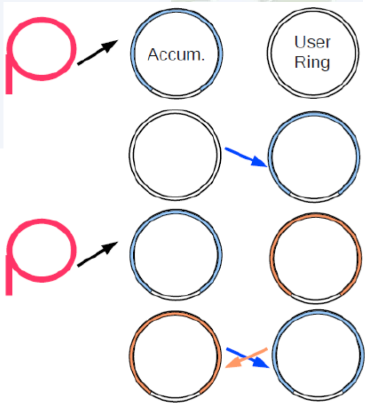
An overview over existing storage ring light sources and planned upgrades gives the following figure and table and figure:

Table 1. Summary of various present and next-generation storage ring light source designs, without intrabeam scattering. $M = \epsilon_0 C^3/E^2$ is given in units of pm km^3/GeV^2

Name	Energy GeV	Structure	C km	ϵ_0 pm	M	σ_δ %	Comments
ESRF	6	2-BA×32	0.845	4000	67	0.11	In operation
APS	7	2-BA×40	1.1	3100	84	0.096	In operation
PETRA III[9]	6	FODO/2-BA	2.3	1000	338	0.1	In operation
DIFL[7]	3	7-BA×12	0.4	500	3.6	0.08	
NSLS-II[6]	3	2-BA×30	0.792	500	28	0.099	Eight wigglers
MAX IV[8]	3	7-BA×20	0.528	263	4.3	0.096	Four wigglers
USRLS[23]	7	4-BA×50	2.0	300	49	?	No nonlin. optim.
XPS7[24]	7	6-BA×40	1.1	78	2.1	0.176	Poor nonlin. dyn.
Tsumaki 2006[25]	6	10-BA×32	2.0	35	7.8	0.089	Accumulation possible
USR7[21]	7	10-BA×40	3.16	30	19	0.079	On-axis injection
PEP-X ultimate[29]	4.5	7-BA×48	2.2	24	12	0.13	
IU ring[30]	5	10-BA×40	2.66	9.1	6.9	0.038	
τUSR[31]	9	7-BA×180	6.21	2.9	8.6	0.096	~Size of Tevatron
SPring-8 II[32]	6	6-BA×48	1.4	67	5.1	0.096	Replaces SPring-8

(from Borland, J. Phys. Conf. Ser. 2013)





Generally, the following challenges have to be addressed:

- small dynamic aperture → swap out concept
- intra-beam scattering
- lifetimes, instabilities, …