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The Road to Holography in Asymptotically Flat Spacetimes

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based upon:

1. C. D., V. Moretti and N. Pinamonti,
Rev. Math. Phys. **18** (2006) 349-415 and arXiv:0712.1770 [gr-qc].

2. C. D. "Projecting massive scalar fields to null infinity," Ann. Henri Poincaré **9** (2008) 35

Motivations

Starting point:

Question: What is really holography?

Main answers:

- String Theory: It is a 1:1 correspondence between a type IIB superstring theory in $AdS_5 \times S^5$ and a $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills field theory on ∂AdS_5 (plus theme variations).
- AQFT: it exists a duality between any AQFT on AdS_{d+1} and a conformal AQFT on ∂AdS_{d+1} and viceversa[a].

A few "natural" questions:

1. Can we implement a similar correspondence in asymptotically flat backgrounds?[b]
2. Does an holographic correspondence in asymptotically flat spacetime really provides useful physical insights for bulk theories?

[a] K. H. Rehren, Annales Henri Poincaré **1** (2000) 607, M. Dütsch and K. H. Rehren, Annales Henri Poincaré **4** (2003) 613

[b] G. Arcioni and C. D., Nucl. Phys. B **674** (2003) 553, Class. Quant. Grav. **21** (2004) 5655, C. D., JHEP **0411** (2004) 011, Phys. Lett. B **615** (2005) 291.

Geometrical Setup

First issue: the notion of conformal boundary

Def: A $4D$ future time oriented spacetime $(M, g_{\mu\nu})$ solving Einstein vacuum equations is asymptotically flat with future time infinity[a] at null infinity - $\Im^+$ - if $\exists\ (\widehat{M}, \widehat{g}_{\mu\nu})$, with a preferred point i^+ and a diffeomorphism $\lambda : M \longrightarrow \lambda(M) \subset \widehat{M}$ and a conformal factor $\Omega \geq 0$ such that:

- $\Omega^2 g_{\mu\nu} = \lambda^*(\widehat{g}_{\mu\nu})$ in M
- $\lambda(M) = J^-(i^+) \setminus \partial J^-(i^+)$ and $\partial(\lambda(M)) = \Im^+ \cup i^+$,
- $\Im^+ = \partial\left(J^-(i^+)\right) \setminus i^+$ and $\lambda(M)$ is strongly causal,
- $\Omega \in C^\infty(\widehat{M})$ and $\Omega = 0$ on $\Im^+ \cup i^+$,
- $d\Omega \neq 0$ on $\Im^+ \cup i^+$ but $\widehat{\nabla}_\mu \widehat{\nabla}_\nu \Omega = -2\widehat{g}_{\mu\nu}$ on i^+,
- $\exists \omega > 0$ such that the integral curves of $\omega^{-1}\widehat{\nabla}^\mu \Omega$ are null complete geodesics and $\widehat{\nabla}_\mu \left(\omega^4 \widehat{\nabla}^\mu (\Omega)\right) = 0$ on $\Im^+$.

The structure of $\Im^+$ is the union of complete null geodesics and a differentiable manifold topologically $\mathbb{S}^2 \times \mathbb{R}$.

[a] H. Friedrich: Comm. Math. Phys. **119** (1988) 51.

An example: Minkowski spacetime

Suppose $M = \mathbb{R}^4$ and $ds^2 = -dudv + \frac{(v-u)^2}{4} d\mathbb{S}^2(\theta, \varphi)$ with $u = t - r$, $v = t + r$. and $d\mathbb{S}^2(\theta, \varphi) = d\theta^2 + \sin^2\theta d\varphi^2$

Choose $\Omega^2 = \frac{4}{(1+u^2)(1+v^2)}$ and λ defined by the coordinate change

$$u = \tan\frac{U}{2}, \quad v = \tan\frac{V}{2}.$$

Hence $\lambda(M) = (-\pi, \pi) \times (-\pi, \pi) \times \mathbb{S}^2$ and $\widehat{ds^2} = -dUdV + \sin^2(\frac{V-U}{4}) d\mathbb{S}^2(\theta, \varphi)$.

Therefore $\lambda(M) \subset \mathbb{R} \times \mathbb{S}^3$ *i.e.* Einstein static Universe!

Here $\Im^+ = \left\{V = \frac{\pi}{2}\right\}$, $i^+ = \left\{V = \frac{\pi}{2},\ U = -\frac{\pi}{2}\right\}$, $i_0 = \left\{V = \frac{\pi}{2},\ U = \frac{\pi}{2}\right\}$, whereas $\Im^- = \left\{U = \frac{\pi}{2}\right\}$ and $i_- = \left\{V = -\frac{\pi}{2},\ U = \frac{\pi}{2}\right\}$. But where is $\widehat{M}$?

The natural choice is $\widehat{M} \equiv \mathbb{R} \times \mathbb{S}^3 \setminus J^-\left(\Im^- \cup i^0 \cup i^-\right)$

This set and the metric $\widehat{ds^2}$ satisfy Friedrich hypothesis!

Three Important Properties

Fix an AF spacetime $(M, g_{\mu\nu})$, $\Im^+$ is intrinsically and universally characterised by $\mathcal{C} = (\Im^+, h_{\mu\nu} \doteq \widehat{g}_{\mu\nu}|_{\Im}, n^\mu \doteq \widehat{\nabla}^\mu \Omega\Big|_{\Im})$.

Intrinsic $\Longrightarrow$ no physical principle allows to select a preferred $\mathcal{C}$ under the gauge transformation

$$\Omega \to \omega\Omega, \quad \Im^+ \to \Im^+, \quad h_{\mu\nu} \to \omega^2 h_{\mu\nu}, \quad n^\mu \to \omega^{-1} n^\mu.$$

Universal $\Longrightarrow$ given any two AF spacetimes $(M_1, g_{1\mu\nu})$ and $(M_2, g_{2\mu\nu})$ and any two $\mathcal{C}$-structures $\mathcal{C}_1 \doteq (\Im^+_1, h_{1\mu\nu}, n^\mu_1)$ and $\mathcal{C}_2 \doteq (\Im^+_2, h_{2\mu\nu}, n^\mu_2)$, $\exists \gamma \in Diff(\Im^+_1, \Im^+_2)$:

$$\gamma(\Im^+_1) = \Im^+_2 \quad \gamma_* h_{1\mu\nu} = h_{2\mu\nu} \quad \gamma_* n^\mu_1 = n^\mu_2.$$

The set of $\gamma \in Diff(\Im^+, \Im^+)$, such that $(\gamma(\Im^+), \gamma_* n^\mu, \gamma_* g_{\mu\nu}) = \left(\Im^+, \omega_\gamma^{-1} n^\mu, \omega_\gamma^2 h_{\mu\nu}\right)$ for some ω_γ, is the BMS group[a].

In a **Bondi frame** $(u, \Omega = 0, z, \bar{z})$

$$u \longrightarrow u' = K_\Lambda(z, \bar{z})\,(u + f\,(z, \bar{z}))$$

$$z \longrightarrow z' = \Lambda z = \frac{az + b}{cz + d} \quad ad - bc = 1 \ \wedge \ a, b, c, d \in \mathbb{C}$$

where

$$K_\Lambda(z, \bar{z}) = \frac{1 + \mid z \mid^2}{\mid az + b \mid^2 + \mid cz + d \mid^2}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \Pi^{-1}(\Lambda),$$

where Π is the surjective covering homomorphism from $SL(2, \mathbb{C})$ onto $SO(3, 1)_\uparrow$ and $f(z, \bar{z}) \in C^\infty(\mathbb{S}^2)$.

$$BMS_4 = SL(2, \mathbb{C}) \ltimes C^\infty(\mathbb{S}^2),$$

[a] R. Sachs, “Asymptotic symmetries in gravitational theory”, Phys. Rev. **128** 2851 (1962)

Field Theory on $\Im^+$

First approach: Projecting bulk scalar fields to the boundary

Assumption: Both $(M, g_{\mu\nu})$ and $(\widehat{M}, \widehat{g}_{\mu\nu})$ are globally hyperbolic.

Proposition: Let us fix both $\widehat{M}$ and the conformal factor Ω. If $\phi : M \to \mathbb{R}$ solves $\left(\Box - \frac{R}{6}\right)\phi = 0$ with compactly supported Cauchy data then

- $\phi \in C^\infty(M)$
- $\Phi \doteq (\omega\Omega)^{-1}\phi$ is a smooth solution in $(\widehat{M}, \widehat{g}_{\mu\nu})$ with $\widehat{g}_{\mu\nu} = (\omega\Omega)^2 g_{\mu\nu}$ for

$$\widehat{\Box}\Phi - \frac{\widehat{R}}{6}\Phi = 0,$$

for any choice of the gauge factor ω,

- $\psi \doteq \Phi|_{\Im^+} \in C^\infty(\Im^+)$.
- the BMS group acts on ψ as

$$\left(A_{(\Lambda,f)}\psi\right)(u', z', \bar{z}') = K_\Lambda^{-1}(z,\bar{z})\psi(u,z,\bar{z}).$$

Aim: Construct from ψ a BMS invariant Quantum Field Theory.

Symplectic step: What is the space of wavefunctions on $\Im^+$?

Def: The symplectic space of real wavefunctions is

$$\mathcal{S}(\Im^+) = \left\{\psi : \Im^+ \to \mathbb{R} \mid \psi \quad \text{and } \partial_u \psi \in L^2(\mathbb{R}\times\mathbb{S}^2, dudS^2(z,\bar{z}))\right\}.$$

endowed with the nondegenerate symplectic form $\sigma : \mathcal{S}(\Im^+) \times \mathcal{S}(\Im^+) \to \mathbb{R}$

$$\sigma(\psi_1,\psi_2) = \int\limits_{\mathbb{R}\times\mathbb{S}^2} \left(\psi_1 \frac{\partial\psi_2}{\partial u} - \psi_2 \frac{\partial\psi_1}{\partial u}\right) dudS^2(z,\bar{z}),$$

- the BMS representation $A_{(\Lambda,f)}$ acts as a symplectomorphism,
- We associate to $(\mathcal{S}(\Im^+),\sigma)$ a Weyl algebra $\mathcal{W}(\Im^+)$, whose generators satisfy

$$a)\ W(-\psi) = W(\psi),^* \qquad b)\ W(\psi)W(\psi') = e^{\frac{i}{2}\sigma(\psi,\psi')}W(\psi+\psi'), \quad \psi \in \mathcal{S}(\Im^+)$$

- We associate to $\mathcal{W}(\Im^+)$ a quasi-free state $\lambda : \mathcal{W}(\Im^+) \to \mathbb{C}$ unambiguously defined as

$$\lambda\left(W(\psi)\right) = e^{-\frac{<\psi_+,\psi_+>}{2}},$$

being $<\psi_{1+},\psi_{2+}> \doteq -i\sigma(\overline{\psi_{1+}},\psi_{2+})$ and ψ_+ the positive frequency part of the Fourier transform along $\mathbb{R}$ of any $\psi \in \mathcal{S}(\Im^+)$.

The state λ enjoys the following remarkable properties:

- λ is pure, invariant under the BMS action.
- it is uniquely defined by a positive BMS-energy requirement with respect to any smooth one-parameter subgroup of the BMS constructed out of future directed timelike or null generators lying in T^4.
- The state λ is quasi-free and, in the folium of λ there are no further BMS-invariant pure states.

N.B. T^4 is the set of real combinations of the first four real spherical harmonics on $\mathbb{S}^2$. It is an Abelian subgroup of $C^\infty(\mathbb{S}^2)$ homomorphic to the four-dimensional translation group.

Holographic Issues

Goal: Investigate an holographic correspondence between a QFT of a scalar field in the bulk and a QFT for a BMS scalar field in the boundary

Quantum: it exists an injective $*$-homomorphism between the bulk algebra of observables and a (sub)algebra of the boundary counterpart.

Step 1: Assume $(M, g_{\mu\nu})$ globally hyperbolic and pick a Cauchy surface (Σ, σ) with

$$\sigma(\phi_1, \phi_2) = \int_\Sigma \phi_1 \nabla_N \phi_2 - \phi_2 \nabla_N \phi_1 d\mu(\Sigma), \quad \forall \phi_1, \phi_2 \in \mathcal{S}(M)$$

being $\mathcal{S}(M)$ the space of solutions of the equation $\left(\Box - \frac{R}{6}\right)\phi = 0$ with compactly supported initial data.

Step 2: Construct the usual Weyl algebra $\mathcal{W}(M)$ (usual properties, nothing fancy)

Step 3: Construct the projection map $\Gamma_M : \mathcal{S}(M) \to \mathcal{S}(\Im^+)$ such that $\phi \mapsto \lim_{\Im} \left[(\omega\Omega)^{-1}\phi\right]$ with $(\omega\Omega)^2 g_{\mu\nu} \to (\Im, h_{\mu\nu}, n^a)$.

Proposition: If $\Gamma_M(\mathcal{S}(M)) \subset \mathcal{S}(\Im^+)$ and $\sigma_M(\phi_1, \phi_2) = \sigma(\Gamma_M \phi_1, \Gamma_M \phi_2)$, then

- $i\,(\mathcal{W}(M))$ is a Weyl $*$-algebra of $\mathcal{W}(\Im^+)$ such that

$$i\,(W(\phi)) = W(\Gamma_M \phi), \quad \forall \phi \in \mathcal{S}(M)$$

Consequence: Any state $\lambda : \mathcal{W}(\Im^+) \longrightarrow \mathbb{C}$ can be pulled back to a state in M i.e. $\lambda_M : \mathcal{W}(M) \longrightarrow \mathbb{C}$ such that

$$\lambda_M(a) = \lambda\,(i(a))\,. \quad \forall a \in \mathcal{W}(M)$$

If we choose λ as the quasi free pure and unique state, then λ_M is

- quasi-free and Hadamard[a]
- invariant under the component connected to the identity of the Lie group of isometries in the bulk
- fulfills a suitable energy-positivity condition with respect to any notion of Killing time in the bulk.
- coincident with the Minkowski vacuum if $(M, g_{\mu\nu}) = (\mathbb{R}^4, \eta_{\mu\nu})$.

λ_M is a **natural candidate** to define massless (scalar) elementary particles in the bulk of any asymptotically flat spacetime!

a) Oh no! Time is running out!

b) Just of few minutes left!

c) There is still plenty of time!

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[a] V. Moretti: Comm. Math. Phys. **278** (2006) 727 and gr-qc/0610143, to appear on CMP

The problem of mass

Consider the Cauchy problem in Minkowski spacetime

$$\begin{cases} \left(\Box - m^2\right)\phi(x^\mu) = 0 \\ \left(\phi(0,\vec{x}), \frac{\partial\phi}{\partial t}(0,\vec{x})\right) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3) \end{cases},$$

then $\phi \in C^\infty(\mathbb{R}^4)$, but if we compactify Minkowski in Einstein universe then

$$\lim_{\Im} \Omega^{-1}\phi \to 0.$$

Proposition[a]: The space of sections of any vector bundle on $\Im^+$ which is homogeneous for the action of the Poincaré group carries only massless representations.

Problem: Can we save the best of both worlds?

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[a] A. D. Helfer: J. Math. Phys. **34** (1993) 3478

A possible solution - Part I

Consider norm finite solutions ϕ for the K.G. equation:

$$\begin{cases} \phi(x^\mu) \longrightarrow \widehat{\phi}(p_\mu)|_{\mathbb{H}_m} \in L^2(\mathbb{H}_m), \quad \mathbb{H}_m = \{\, p_\mu \mid \eta^{\mu\nu} p_\mu p_\nu = m^2 \} \\ \Lambda\widehat{\phi}(p_\mu) = \widehat{\phi}(\Lambda^{-1} p_\mu) \quad \forall \Lambda \in SO(3,1) \end{cases}$$

Harmonic analysis over hyperboloids[a] grants us

Proposition Given $\mathcal{C} = \{p_\mu \mid \eta^{\mu\nu} p_\mu p_\nu = 0\}$, it exists $T : L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m) \longrightarrow L^2(\mathcal{C})$ which is a unitary intertwiner between the $SO(3,1)$ quasi-regular representations.

N.B. Hence, for any $\hat{\phi}(p_\mu)|_{\mathbb{H}_m}$, we can introduce

$$\widehat{\psi}(p_\mu) \doteq T(\widehat{\phi}(p_\mu)|_{\mathbb{H}_m}, \widehat{\phi}(p_\mu)|_{\mathbb{H}_m}) \in L^2(\mathcal{C}).$$

[a] R. S. Strichartz: J. Func. Anal. **12** (1973) 341.

A possible solution - Part II

How can we interpret $\widehat{\psi}(p_\mu)$?

Global norm estimates grant us that[a]

Proposition: Any $\widehat{\psi}(p_\mu) \in L^2(\mathcal{C})$ is the restriction on the light cone of the Fourier transform of a solution for:

$$\begin{cases} \Box\psi(x^\mu) = 0 \\ \psi(0,\vec{x}) = f_1(\vec{x}),\ \frac{\partial\psi}{\partial t}(0,\vec{x}) = f_2(\vec{x}) \end{cases},$$

with $K^{\frac{1}{2}} f_1(\vec{x}) \in L^2(\mathbb{R}^3)$ and $K^{-\frac{1}{2}} f_2(\vec{x}) \in L^2(\mathbb{R}^3)$ where $K = \sqrt{-\triangle}$. Furthermore $\psi(x^\mu) \in L^4(\mathbb{R}^4)$ and $\exists C > 0$

$$||\psi(x^\mu)||_{L^4} \leq C \left[||K^{\frac{1}{2}} f_1(\vec{x})||_{L^2} + ||K^{-\frac{1}{2}} f_2(\vec{x})||_{L^2} \right].$$

[a] R. S. Strichartz: Duke Math. Jour. **44** (1977) 705.

A possible solution - Part III

Can we project $\Omega^{-1}\psi$ on $\Im^+$?

Proposition: Assume Minkowski spacetime is compactified in Einstein static universe $\widehat{M}$. Then, for any solution of the wave equation $\psi \in L^4(\mathbb{R}^4)$,

- $\Omega^{-1}\psi$ solves $\left[\widehat{\Box} - \frac{\widehat{R}}{6}\right]\psi = 0$ in the image of Minkowski in $\widehat{M}$.
- $\Omega^{-1}\psi \in L^4(\widehat{M}, \Omega^4 d^4x)$ and, whenever $\Omega^{-1}\psi \in W^{1,4}(\widehat{M})$, then it exists

$$\rho : W^{1,4}(\widehat{M}) \to L^2(\Im^+)$$

N.B. We call the projection on $\Im^+$ of a solution ϕ for the Klein-Gordon equation, the set (Ψ, U, T) where

1) $\Psi \doteq \rho(\Omega^{-1}\psi)$,

2) U is the quasi-regular $SO(3,1)$ representation,

3) T is the unitary intertwiner.

Boundary analysis

How do we interpret (Ψ, U, T) on $\Im^+$?

Proposition: The field Ψ transforming under the quasi-regular $SO(3,1)$ representation corresponds to a massless scalar BMS field à la Wigner-Mackey. Furthermore T maps Ψ into a BMS free massive scalar fields à la Wigner-Mackey.

N.B.: the proof is an application of

a) the definition for $\Im^+$ as the light cone with i^+ as tip,
b) Mackey induction-reduction theorem
c) the definition of the intertwiner T.

N.B.: the notion of BMS mass coincides with the Poincaré counterpart.

Conclusions

- Can we find a similar construction for **massive fields** on a generic curved background?
- Can we recast the bulk to boundary correspondence at a quantum level for massive fields?
- Can we recast the bulk to boundary correspondence if the bulk spacetime is an homogeneous and isotropic solution of Einstein equations?
- Can we implement bulk interactions? Hopefully because at $\Im^+$ they vanish
- Can we implement gauge fields in the boundary? Hopefully à la Weinstein!

What lies beyond?

The bright side

1) How do I construct a BMS free field theory without any reference to bulk fields?

- Classify all the unitary and irreducible rep. of the BMS group (Mackey's theory of induction)
- Complete the Wigner programme, *i.e.*, construct all the possible canonical (dynamical configurations) and induced wave functions

2) What happens if one considers massive real scalar fields?

Proposition[a]: The space of sections of any vector bundle on $\Im^+$ which is homogeneous for the action of the Poincaré group carries only massless representations.

We can circumvent the problem in **Minkowski** spacetime by means of

- Harmonic analysis of hyperboloids,
- Strichartz norm estimates.

[a] A. D. Helfer: J. Math. Phys. **34** (1993) 3478

What lies beyond?

The dark side

1) How happens if one considers massive scalar fields on a curved background?

- There is no coherent notion of Fourier transform, hence harmonic analysis on hyperboloids becomes useless,
- There is no counterpart of a global norm estimate à la Strichartz.

2) What happens if I consider “non scalar” fields, *e.g.*, Dirac spinors?

- They are much less studied than scalar field.

3) How can I describe an interacting field theory in terms of its bulk counterpart?

- They are very difficult to analyse in the framework of algebraic quantum field theory.

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